Painlevé equations, integrable systems and the stabilizer set of Virasoro orbit

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Abstract

We study a geometrical formulation of the nonlinear second-order Riccati equation (SORE) in terms of the projective vector field equation on S^1, which in turn is related to the stability algebra of Virasoro orbit. Using Darboux integrability method we obtain the first-integral of the second-order Riccati equation and the results are applied to the study of its Lagrangian and Hamiltonian descriptions. Using these results we show the existence of a Lagrangian description for second-order Riccati equation, and the Painlevé II equation is analysed.

Keywords: Riccati, projective vector field, Darboux polynomial, master symmetry, bi-Lagrangian system, Painlevé II, Chazy equation, Bures equation.

MSC Classification: 34A26; 34A34; 35Q53;

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1 Introduction

The study of the Riccati differential equations goes back to the early days of modern mathematical analysis, since such equations represent one of the simplest types of nonlinear ordinary differential equations (hereafter ODEs) and consequently Riccati equations play an important rôle in physics, mathematics and engineering sciences. The usual first-order Riccati equation appears as a reduction from a linear second-order ordinary differential equation when taking into account its invariance under dilations according to Lie recipe (the inverse property is called the Cole–Hopf transformation). Such correspondence between first-order Riccati equations and second-order linear ODEs can be beautifully manifested through their solutions. For example, if one solution of a linear second-order ODE is known, then any other solution can be obtained by means of a quadrature (i.e. a simple integration), and the same property holds true for the usual first-order Riccati equation, but with two quadratures. Moreover, when three solutions are known, no quadrature is necessary. The first-order Riccati equation is a prototypical example of (systems of) differential equations admitting a superposition rule, also called Lie systems [1, 2]. This is why some authors [3] have considered it as the first step in the study of (systems of) nonlinear differential equations.

The second-order Riccati equation has also become of a paramount importance in recent years, mainly because of its connection with integrable ODEs. It is known that the second-order Riccati equations are related to a large class of integrable ordinary differential equations of anharmonic oscillators, *viz.* the Ermakov–Pinney, Normalised Ermakov–Pinney, Neumann type systems, etc. [4, 5, 6, 7, 8, 9, 10]. The second-order Riccati equation has been studied in [11] from a geometric perspective and it has been proved to admit two alternative Lagrangian formulations, both Lagrangians being of a non-natural class (neither potential nor kinetic term). A more geometric approach can be found in [12] where the theory of Darboux polynomials, the extended Prelle-Singer methods, (pre-)symplectic forms, or Jacobi multipliers, are used.

Higher-order Riccati equations can be obtained by reduction from Matrix Riccati equations [13, 14]. Not only first-order Riccati equation but also higher-order Riccati differential equations can be linearised via Cole–Hopf transformation (i.e. they appear as a reduction from linear differential equations when taking into account the dilation symmetry).

The study of higher-order Riccati equations [12] was also carried out from the perspective of the theory of Darboux polynomials and the extended Prelle-Singer methods. Higher-order Riccati equations play the rôle of Bäcklund transformations for integrable partial differential equations of order higher than that of the KdV equation. In fact Grundland and Levi [15, 16] constructed the Bäcklund transformations of several integrable systems from higher-order Riccati equations. Such Bäcklund transformations are closely related to many important integrability properties such as the inverse scattering method, the Painlevé property, Lax pairs, and an infinite number of conservation laws [17].

In the study of differential equations one finds cases that are in some sense solvable, or integrable, and this enables one to study their dynamical behaviours using Lie theoretic methods [18, 19]. They often admit different geometric and Hamiltonian formulations. Integrable systems are a fundamental class of *explicitly solvable* dynamical systems of current interest in mathematics and physics. One notable example is the so called Ermakov–Milne–Pinney equation [20, 21, 22, 23, 24, 25]: in 1950 Pinney presented in a one-page paper [25] the solution of the differential equation

$$y'' + \omega^2(x) y = \frac{1}{y^3}, \tag{1.1}$$

where the symbol y'' denotes the second derivative with respect to the independent variable x. He gave the general solution of (1.1) in the form

$$y(x) = (A\,\phi_1^2 + 2B\,\phi_1\,\phi_2 + C\,\phi_2^2)^{1/2}, \tag{1.2}$$

where $y_1 = \phi_1(x)$ and $y_2 = \phi_2(x)$ are any two linearly independent solutions of the associated linear differential equation $y'' + \omega^2(x)y = 0$, and A, B and C are related according to $B^2 - AC = 1/W^2$ with W being the constant Wronskian of the two linearly independent solutions ϕ_1 and ϕ_2. Ermakov [23] introduced the above linear differential equation as an additional auxiliary equation to the second-order differential equation (1.1) to define a system of two second-order differential equations, and found an invariant when multiplying by an integrating factor and obtained an invariant after integration with respect to x.

The solutions of Painlevé's Second Equation (or Painlevé II)

$$y'' = \alpha + xy + 2y^3, \quad \alpha \in \mathbb{R}, \tag{1.3}$$

are meromorphic functions on the plane [26, 27]. It is known (see e.g. the papers by Gromak [28] and [29], and also [30]) that every transcendental solution w of (1.3) has infinitely many poles with residue $+1$ and also with residue -1, except when $\alpha = \pm 1/2$ and w is also a solution of the Riccati equation (see [27, 30])

$$\frac{dw}{dx} = \pm\frac{x}{2} \pm w^2.$$

In this paper we derive the Painlevé II, as well the second-order second degree Painlevé II equation [31, 32] using Riccati hierarchy, which is in turn connected to the Virasoro stabilizer set or projective connection. The second-order second degree Painlevé II is also known as Jimbo-Miwa equation [32] and plays an important rôle in random matrix theory [33].

1.1 Motivation and plan

As indicated above, our main goal is to study higher-order Riccati equations in various directions. In particular, we focus onto the geometrical aspects of the Riccati sequence. At first we explore its connection to the stabilizer set of the Virasoro orbit, also known as the projective vector field equation. There are many papers devoted to Virasoro algebra and projective connection on S^1, but very few papers discuss about their relations to nonlinear oscillator-like equations, Riccati chains and (second-order) Painlevé equations. This paper provides a relationship between the stabilizer set of the Virasoro orbit and second-order Riccati equations. Many papers (for example, [34, 35, 36]) have been written to elucidate the interaction between the geodesic flow on the Bott–Virasoro group (see for example, [37, 38, 39]) and integrable partial differential equations, especially the KdV type systems, but very few articles deal with finite-dimensional systems.

Some of the main questions to be discussed in this paper can be summarized in the following points:

- Relation of Riccati equation with the Virasoro algebra.

  The Virasoro group plays a very important rôle in integrable systems. It is known that the Virasoro group serves as the configuration space of the KdV and the Camassa-Holm equations and these equations can be regarded as equations of the geodesic flows related to different right-invariant metrics on this group. In other words, they have the same symmetry group. Recall that the Virasoro group is a one-dimensional central extension of the group of smooth transformations of the circle. In this work, we focus on the integrable dynamical systems related to the stabilizer set of the Virasoro orbit. In fact the Ermakov–Milne–Pinney equation, which describes the time-evolution of an isotonic oscillator (also sometimes called pseudo-oscillator) – i.e. an oscillator with inverse quadratic potential – is the first and foremost example of this class. It is also known [34, 35] that the entire (coupled) KdV family is connected to the Euler-Poincaré formalism of the (extended)

Bott–Virasoro group. This connection can be extended to the super Bott–Virasoro group [40].

The second-order Riccati equation has a nice geometric interpretation in terms of Virasoro orbit [7, 8]. Vector fields $f(x)\,d/dx \in \mathfrak{X}(S^1)$ associated to the stabilizer set of Virasoro orbit are called projective vector fields and the corresponding differential equation satisfied then by the function f is called the projective vector field equation [41, 42]. The second-order Riccati equations are associated to these differential equations. In fact, solutions of a large class of 0+1 dimensional integrable systems can be expressed in terms of the global and local projective vector fields.

- Analysis of the infinitesimal symmetries of the standard Riccati equation.

  This study is presented by making use of a geometric approach. It is proved that these symmetries are related to the solutions of the projective vector field equation.

- Study of the relation of the second-order Riccati equation with Painlevé equations.

  There are a certain number of properties relating the second-order Riccati equation with some other equations of the mathematical physics as the Airy equation [43] or the Ermakov–Milne–Pinney equation [23, 24, 25]. Moreover, the second-order Riccati equation is also related to some of the Painlevé-Gambier equations and this can be used to find the solutions of Painlevé's Second Equation (or Painlevé II) for special values of the parametres.

The paper is organised as follows. Section 2 is a short presentation from a geometric perspective of properties of first- and higher-order Riccati equations and their relations with linear equations and nonlinear superposition rules and to show many properties among solutions of Hill equation [44] and related ones, in particular the projective vector field equation, and to point out the possibility of of extending such results to connect Hill equation to solutions of Reid type equations [45], Thomas equations [46], Gambier equation [47] and Kummer–Schwarz equations [48, 49]. In Section 3 we give a concise introduction to Virasoro orbit, stabilizer set and projective connections. Section 4 is devoted to study the infinitesimal symmetries of standard Riccati equation and their relations with the solutions of the projective vector field equation. In particular we relate the projective vector field equation to the second-order Riccati equation and show how both equations admit a Lax formulation. We elucidate in Section 5 the connection between the ordinary Painlevé II and second-order second degree Painlevé II equations with the Virasoro orbit. Special examples as Airy equation giving rise to a second-order Riccati equation and a Painlevé equation are given.

2 First-order and higher-order Riccati equations

Our main goal is to explore the higher-order Riccati equations, but before giving the formal description of higher-order Riccati equations and Riccati chain, a short introduction may be helpful. So we start our journey by giving a short introduction to first-order Riccati equation.

The usual Riccati equation

$$u' = f(x) + g(x)u + h(x)u^2, \tag{2.1}$$

is a first-order nonlinear differential equation with a quadratic non-linearity. The solutions of the Riccati equation are free from movable branch points and can have only movable poles [47, 50, 51]. This Riccati equation is the simplest nonlinear differential equation admitting a nonlinear superposition rule for expressing the general solution in terms of three particular solutions. The linear fractional transformation (or change of variables from the passive viewpoint)

$$\bar{u} = \frac{a(x)u + b(x)}{c(x)u + d(x)}\,, \qquad ad - bc = 1\,, \tag{2.2}$$

transforms each Riccati equation into another one (see e.g. [52]). In order to solve a Riccati equation by quadratures, it is enough to know one particular solution, because this allows us to reduce the problem to carry out two quadratures, while when two particular solutions are known the problem can be reduced to a new differential equation solvable by just one quadrature, and finally, if three particular solutions $u_1(x), u_2(x), u_3(x)$, are known, we can construct all other solutions u without use of any further quadrature [2]. This is carried out by using the property that the cross-ratio of four solutions of (2.1) is a constant, i.e. for any other solution $u(x)$, there is a real number k such that:

$$\frac{u(x) - u_1(x)}{u(x) - u_2(x)} = k\,\frac{u_3(x) - u_1(x)}{u_3(x) - u_2(x)}, \tag{2.3}$$

where k is an arbitrary constant characterising each particular solution. For instance $u_1(x)$ corresponds to $k = 0$, u_3 corresponds to $k = 1$ and $u_2(x)$ is obtained in the limit of $k \to \infty$.

It was shown by Lie and Scheffers [53] that the Riccati equation is essentially the only first-order nonlinear ordinary differential equation of which possesses a nonlinear superposition rule which comes from the preceding relation:

$$u = \Phi(u_1, u_2, u_3; k) = \frac{u_1(u_3 - u_2) + k\,u_2(u_1 - u_3)}{u_3 - u_2 + k\,(u_1 - u_3)},$$

i.e. if $u_1(x), u_2(x)$ and $u_3(x)$ are solutions of (2.1), then for any real value k,

$$u(x) = \frac{u_1(x)(u_3(x) - u_2(x)) + k\,u_2(x)(u_1(x) - u_3(x))}{u_3(x) - u_2(x) + k\,(u_1(x) - u_3(x))},$$

is another solution and each solution is of this form for an appropriate choice of k. The particular solution u_2 appears as the limit $k \to \infty$.

Therefore the general solution $u(x)$ is non-linearly expressed in terms of three generic particular solutions, $u_1(x)$, $u_2(x)$ and $u_3(x)$, and a constant parameter k.

Remark also that a first-order Riccati differential equation (2.1) with $h(x)$ of constant sign on an open set, is related by means of the relation

$$u = -\frac{1}{h}\frac{y'}{y} \Longleftrightarrow y = \exp\left(-\int^x h(\zeta)\,u(\zeta)\;d\zeta\right) \tag{2.4}$$

to a homogeneous linear second-order differential equation

$$y'' + a_1(x)y' + a_0(x)y = 0, \tag{2.5}$$

with

$$a_1 = -g - \frac{h'}{h}, \qquad a_0 = f\,h. \tag{2.6}$$

This relation (2.4) is not a change of coordinates or transformation, but for any real number λ, the functions y and $\lambda\, y$ have associated the same function u. Conversely, given the linear second-order differential equation (2.5), for any nonvanishing function h, (2.4) reduces (2.5) to the Riccati equation (2.1) with coefficients determined by (2.6).

In a similar way, consider the following nonlinear second-order differential equation on an open set I [3]:

$$u'' + \big[\,\beta_0(x) + \beta_1(x)u\,\big]u' + \alpha_0(x) + \alpha_1(x)u + \alpha_2(x)u^2 + \alpha_3(x)u^3 = 0\,, \quad x \in I \subset \mathbb{R}, \tag{2.7}$$

where we suppose that $\alpha_3(x) > 0, \forall x \in I$, and the two functions β_0, β_1, are not functionally independent of α's functions, but satisfy

$$\beta_0 = \frac{\alpha_2}{\sqrt{\alpha_3}} - \frac{\alpha_3'}{2\alpha_3}\,, \quad \beta_1 = 3\sqrt{\alpha_3}\,.$$

In particular, the case $\beta_0 = \alpha_1 = \alpha_2 = 0$, $\alpha_3 = 1$, $\beta_1 = 3$, i.e. $u'' + 3u\,u' + \alpha_0(x) + u^3 = 0$, was studied in Davis and Ince books [3, 47] and will be more carefully considered in next sections, while that of $\beta_0 = \alpha_0 = \alpha_1 = \alpha_2 = 0$, $\beta_1 = 1$, i.e. $u'' + u\,u' + \alpha_3(x)u^3 = 0$, was studied by Leach and coworkers [9, 54] who proved that only the case $\alpha_3(x) = 1/9$ is linearisable. . Moreover, it possesses an eight-dimensional Lie algebra of (infinitesimal) symmetries and is fully integrable.

An important property is that, as indicated in [11], the nonlinear second-order differential equation (2.7) can be related to a linear third-order differential equation

$$y''' + a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{2.8}$$

by means of

$$u(x) = \frac{1}{\sqrt{\alpha_0(x)}}\,\frac{y'(x)}{y(x)}\,. \tag{2.9}$$

The equation (2.7) is a second-order generalisation of the usual first-order Riccati equation and, because of this, is usually known as second-order Riccati equation [11]: it is a nonlinear equation whose general solution can be expressed in terms of solutions of a linear third-order differential equation. Conversely, if we start with the linear third-order differential equation (2.8), then we can get an associate nonlinear second-order differential equation by introducing the appropriate reduction of order. Although the complete induction proof has already been given in an unpublished paper by one of us [8], we prefer to narrate it for the completeness of this survey.

One can go further and define j-order Riccati equations as those appearing as a reduction of a linear $(j+1)$-order differential equation [12]. More specifically, given the differential equation

$y^{(n)} = d^n y/dx^n = 0$, its invariance under dilations suggests, according to Lie recipe, to look for a new variable z such that the dilation vector field $y\,\partial/\partial y$ becomes $(1/k)\partial/\partial z$, the factor k being purely conventional (we often put $k = 1$). Then $y = e^{kz}$, up to an irrelevant factor, and with this change of variable, the linear differential equation $y^{(n)} = 0$, for $n > 1$, becomes $R^{n-1}(u) = 0$ with $u =' z$ and R the differential operator $R = (D + k\,u)$. In fact, when $D = d/dx$, we have that $e^{-kz}De^{kz} = D+k\,u$, and then we can prove by complete induction that $y^{(n)} = k\,e^{kz}R^{n-1}(u)$ (see [12]): the property is true for $n = 1$, because $y' = k\,z'\,e^{kz} = k\,e^{kz}\,u = k\,e^{kz}\,R^0(u)$, and if the property is true till $n = j$, it also holds for $n = j + 1$, because

$$D^{j+1}y = D(y^{(j)}) = D(k\,e^{kz}R^{j-1}(u)) = k\,e^{kz}(D + k\,u)R^{j-1}(u) = k\,e^{kz}\,R^j(u).$$

Therefore, Lie recipe applied to the invariant under dilations differential equation $y^{(n)} = 0$ transforms such an equation into $R^{(n-1)}(u) = 0$. The first terms of such sequence, called Riccati sequence, are

$$R^0(u) = u, \qquad R^1(u) = u' + k\,u^2, \qquad R^2(u) = u'' + 3k\,u\,u' + k^2u^3, \tag{2.10}$$

$$R^3(u) = u''' + 4k\,u\,u'' + 3k\,u'^2 + 6k^2\,u^2\,u' + k^3\,u^4, \tag{2.11}$$

$$R^4(u) = u^{(iv)} + 5k\,u\,u''' + 10k\,u'u'' + 15k^2\,uu'^2 + 10k^2\,u^2u'' + 10k^3\,u^3u' + k^4u^5, \ldots \tag{2.12}$$

and for the general linear differential equation of order n,

$$a_0(x)y + \sum_{j=1}^{n} a_j(x)\,y^{(j)} = 0, \quad a_n(x) = 1, \tag{2.13}$$

the change of variable $y = e^{kz}$ leads to the reduced equation for $u = z'$

$$a_0(x) + \sum_{j=1}^{n} a_j(x)\,R^{j-1}(u) = 0, \quad a_n(x) = 1.$$

The explicit solutions of the second, third and fourth order terms of the Riccati sequence can be found in [55].

3 Projective vector field, Virasoro algebra, and second-order Riccati equation

Let $\Omega^1 = T^*S^1$ be the cotangent bundle of a circle S^1. Since S^1 is diffeomorphic to the 1-dimensional Lie group $U(1)$, if we remove the point corresponding to the neutral element of $U(1)$, and the corresponding one on S^1, then there exists a natural coordinate x, defined up to a factor by the canonical coordinate, which we can fix such that the domain of the chart is $(0, 2\pi)$. This Ω^1 is a trivial real line bundle on S^1. Similarly, let Ω^m denote the m-fold tensor product of Ω^1. The local coordinate expression of a section of Ω^m, usually called tensor density, is given by $s(x) = g(x)\,(dx)^m$, where $(dx)^m$ is a shorthand notation for the tensorial product of m times dx. The set $\Gamma(\Omega^m)$ of such sections is a free $C^\infty(S^1)$-module, a natural local basis being given by $(dx)^m$, and in this sense such sections are described in such a natural coordinates by functions on S^1, that however do not transform as functions under a change of coordinates (see e.g. [56] for more details. A vector field on S^1 is of the form

$$X_f = f(x)\frac{d}{dx} \in \mathfrak{X}(S^1), \tag{3.1}$$

where $f(x)$ is a function on S^1 (i.e. a 2π–periodic function on the real line). Note however that taking into account that $\mathcal{L}_{X_f} dx^m = m\, f'\, dx^m$, where f' denotes the derivative of f and $m \in \mathbb{N}$, we see that the infinitesimal action of X_f on a section $s = g(x)\,(dx)^m$ of Ω^m is given by (see e.g. [41])

$$\mathcal{L}_{X_f} s = (fg' + mf'g)(dx)^m, \tag{3.2}$$

where $\mathcal{L}_{X_f}$ is the Lie derivative with respect to the vector field X_f given by (3.1).

Sometimes it is convenient to use the above mentioned identification of a section with its component, a function, and then the Lie derivative when acting on sections of Ω^m expressed in terms of functions, denoted $\mathcal{L}^{(m)}_{X_f}$, is written, according to (3.2) as

$$\mathcal{L}^{(m)}_{X_f} = f(x)\frac{d}{dx} + mf'(x). \tag{3.3}$$

This really means that if $s = \psi(x)\,(dx)^m$, then

$$\mathcal{L}_{X_f} s = \left(f(x)\frac{d\psi}{dx} + mf'(x)\,\psi(x)\right)\,(dx)^m = (\mathcal{L}^{(m)}_{X_f}\psi)\,\,(dx)^m.$$

There is a natural symmetric bilinear map $\Gamma(\Omega^{m_1}) \times \Gamma(\Omega^{m_2}) \to \Gamma(\Omega^{m_1+m_2})$ given by the usual commutative product of tensor densities,

$$(f(x)(dx)^{m_1}, g(x)(dx)^{m_2}) \mapsto f(x)g(x)(dx)^{m_1+m_2}.$$

In particular, for $m = 0$, $\Gamma(\Omega^0) = C^\infty(S^1)$, and (3.3) remains valid for $m = 0$, $\mathcal{L}^{(0)}_{X_f} = \mathcal{L}_{X_f}$. We can, at least formally, extend the values of the index m to the set of integer numbers, $m \in \mathbb{Z}$, or even to rational numbers, $m \in \mathbb{Q}$, and then the product by elements of $\Gamma(\Omega^0)$ is the external composition law of the module structure. Note that the elements of $\Gamma(\Omega^{-1})$ when multiplied by those of $\Gamma(\Omega^m)$ look like an inner contraction with a vector field and in this sense $\Gamma(\Omega^{-1})$ is to be identified with $\mathfrak{X}(S^1)$ (i.e. we can denote the tangent bundle $\tau_{S^1} : TS^1 \to S^1$ as Ω^{-1} ($\equiv TS^1$)). The transformation property under changes of coordinates of elements of Ω^{-1} coincides with that of elements of $\mathfrak{X}(S^1)$.

Note also that when $X_g = g(x)d/dx \in \Omega^{-1}$, then putting $m = -1$ in (3.3) we get the usual expression:

$$\mathcal{L}_{X_f} X_g = [X_f, X_f] = \left[f(x)\frac{d}{dx}, g(x)\frac{d}{dx}\right] = (f\,g' - g\,f')\frac{d}{dx}. \tag{3.4}$$

We will denote $\Omega^{\pm 1/2}$ the 'square root' of the tangent and cotangent bundle of S^1 respectively. The respective sections of such bundles will be of the form $s = \psi(x)dx^{\pm\frac{1}{2}} \in \Gamma(\Omega^{\pm\frac{1}{2}})$.

Another remarkable property is that the $C^\infty(S^1)$-module $\Gamma(\Omega^m)$ of sections of the m-fold tensor product bundle Ω^m has a natural structure of a Poisson algebra with the commutative product given by the above usual product of tensor densities and the Poisson algebra commutator given by the Rankin–Cohen bracket [57, 58]

$$\big\{f(x)(dx)^{m_1}, g(x)(dx)^{m_2}\big\} = \big(m_1\, f(x)g'(x) - m_2\, f'(x)g(x)\big)(dx)^{m_1+m_2+1},$$

which is also known as first transvectant.

The group Diff (S^1) acts on $\Gamma(\Omega^m)$, as given by

$$\Phi^*(f(x)(dx)^m) = f(\Phi(x))(\Phi'(x))^m(dx)^m,$$

for $\Phi \in$ Diff (S^1). The 'Lie algebra' of such a group is identified with the Lie algebra given in $\mathfrak{X}(S^1)$ by the commutator of vector fields, i.e. $[X_{f_1}, X_{f_2}] = (f_1 f_2' - f_2 f_1')d/dx$, and acts on the $C^\infty(S^1)$-module $\Gamma(\Omega^m)$ by the Lie derivative given in (3.2).

The theory of projective connections on the circle has much to do with the theory of Riccati equations we are dealing with. We will use the following definition of a projective connection [56] and corresponds to the case $n = 2$ in [59]:

Definition 1 *A projective connection on the circle is a* $\mathbb{R}$*-linear second-order differential operator*

$$\Delta : \Gamma(\Omega^{-\frac{1}{2}}) \longrightarrow \Gamma(\Omega^{\frac{3}{2}}) \tag{3.5}$$

such that:

1. *The principal symbol of* Δ *is the identity.*
2. $\displaystyle\int_{S^1} (\Delta s_1)s_2\, dx = \int_{S^1} s_1(\Delta s_2)\, dx,$ *for all pairs of sections* $s_1, s_2 \in \Gamma(\Omega^{-\frac{1}{2}})$.

Let us take $s = \psi(x)(dx)^{-\frac{1}{2}} \in \Gamma(\Omega^{-\frac{1}{2}})$, and then a linear second-order differential operator on such space of sections is such that $\Delta s \in \Gamma(\Omega^{3/2})$ is locally described by [56]:

$$\Delta s \ = \ (a\,\psi'' + b\,\psi' + c\,\psi)\,(dx)^{\frac{3}{2}}, \tag{3.6}$$

where a, b and c are real functions. The first condition in the definition of projective connection implies that for Δ to be a projective connection it must be $a = 1$, and, on the other hand, the second condition implies that $b = 0$. Hence each projective connection can be identified with a Hill operator [44, 60]

$$\Delta_v \ = \ \frac{d^2}{dx^2} + v(x), \tag{3.7}$$

where v is an arbitrary function on S^1 (i.e. a periodic function on $\mathbb{R}$). The differential equation

$$\Delta_v\psi = 0 \Longleftrightarrow \frac{d^2\psi}{dx^2} + v(x)\,\psi = 0, \tag{3.8}$$

is called Hill equation.

Definition 2 *Given a projective connection on the circle with associated Hill operator*

$$\Delta_v = \frac{d^2}{dx^2} + v(x), \tag{3.9}$$

a vector field $X_f \in \mathfrak{X}(S^1)$ *is called projective vector field with respect to the projective connection* Δ_v *when it leaves invariant the projective connection, i.e.*

$$\mathcal{L}^{(3/2)}_{X_f} \Delta_v s \;=\; \Delta_v(\mathcal{L}^{(-1/2)}_{X_f} s), \tag{3.10}$$

for all $s \in \Gamma(\Omega^{-\frac{1}{2}})$, *where* $\mathcal{L}^{(m)}_{X_f}$ *is the expression for the Lie derivative with respect to* X_f *given by (3.3).*

The characterisation of projective vector fields with respect to a given projective connection is given in the following theorem (see [41]) which we repeat here for the sake of completeness:

Theorem 1 $X_f \in \Gamma(\Omega^{-1})$ *is a projective vector field with respect to the projective connection (3.9) if and only if (see [41])* f *is a solution of the third-order linear differential equation called projective vector field equation:*

$$y''' + 4v\, y' + 2v'\, y \;=\; 0. \tag{3.11}$$

Proof.- In fact, if $s = \psi(x)(dx)^{-\frac{1}{2}} \in \Gamma(\Omega^{-\frac{1}{2}})$, then

$$\Delta_v s = (\psi'' + v\,\psi)\,(dx)^{3/2},$$

and therefore,

$$\mathcal{L}^{(3/2)}_{X_f} \Delta_v s = \left[f(\psi''' + v'\,\psi + v\,\psi') + \frac{3}{2} f'(\psi'' + v\,\psi) \right]\,(dx)^{3/2}.$$

On the other side,

$$\mathcal{L}^{(-1/2)}_{X_f} s = \left(f\,\psi' - \frac{1}{2} f'\,\psi \right)\,(dx)^{-\frac{1}{2}},$$

and thus,

$$\begin{aligned} \Delta_v(\mathcal{L}^{(-1/2)}_{X_f} s) \;&=\; \left[\left(\tfrac{d^2}{dx^2} + v \right) (f\,\psi' - \tfrac{1}{2} f'\,\psi) \right]\,(dx)^{3/2} \\ &=\; \left(f\,\psi''' + \tfrac{3}{2} f'\,\psi'' + v\,f\,\psi' - \tfrac{1}{2}(v\,f' + f''')\psi \right)\,(dx)^{3/2}. \end{aligned}$$

Consequently,

$$\mathcal{L}^{(3/2)}_{X_f} \Delta_v s - \Delta_v(\mathcal{L}^{(-1/2)}_{X_f} s) = \left(v'\,f + \frac{3}{2} v\,f' + \frac{1}{2} f''' + \frac{1}{2} v\,f' \right)\psi\,(dx)^{3/2},$$

from where we see that the invariance condition (3.10) implies that f is a solution of the differential equation (3.11). This is the reason why equation (3.11) is called projective vector field equation.

□

Each solution of the projective vector field equation provides a local projective vector field, while a global solution of the equation with the mentioned periodicity condition defines a global projective vector field.

It will be proved in next subsection that these projective vector fields can alternatively be seen to be the elements generating the stability subalgebra of the point $(-1, v\,(dx)^2)$ of the Virasoro orbit of the corresponding v (see next Subsection).

Sometimes v in (3.9) is replaced by $k\,v$, where $k \in \mathbb{R}$, and then equation (3.11) becomes:

$$y''' + 4k\,v\,y' + 2k\,v'\,y \;=\; 0. \tag{3.12}$$

It is also to be remarked that the projective vector field equation (3.11) also appears in the search for first–order differential operators $Q = \alpha(x) + \beta(x)d/dx$ leaving invariant the set of solutions of Hill equation (3.8). In fact it was shown in [61] that β must be a solution of (3.11), and then α must be equal to $\frac{1}{2}\beta' + C$ where C is any constant.

3.1 Virasoro algebra and projective vector field equation

Let $\mathrm{Diff}_+(S^1)$ be the group of orientation preserving diffeomorphisms of the circle S^1. We represent an element of $\mathrm{Diff}_+(S^1)$ as a diffeomorphism $\Phi(e^{i\,x}) = e^{if(x)}$ where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a function such that (a) $f \in C^\infty(\mathbb{R})$, (b) $f(x + 2\pi) = f(x) + 2\pi$, (c) $f'(x) > 0$. Therefore the tangent space $T_{\mathrm{id}}\mathrm{Diff}_+(S^1)$ is the set of elements $f(x)\,d/dx \in \mathfrak{X}(S^1)$ with f a periodic differentiable function such that $f'(x) > 0$. Note however that under a change of coordinates f does not change as a function but as the coordinate of a vector field.

The group $\mathrm{Diff}_+(S^1)$ is endowed with a smooth manifold structure based on the Fréchet space $C^\infty(S^1)$. Its Lie algebra $\mathfrak{g} = \mathfrak{diff}(S^1)$ is the real linear space $\mathfrak{X}(S^1)$ endowed with the usual bracket of vector fields: If $X_{f_i} = f_i(x)d/dx \in \mathfrak{X}(S^1)$ for $i = 1, 2$, then

$$[X_{f_1}, X_{f_2}] = (f_1\, f_2' - f_2\, f_1')\frac{d}{dx}.. \tag{3.13}$$

The natural action of $\mathrm{Diff}_+(S^1)$ induces its adjoint action on $\mathfrak{X}(S^1)$, and correspondingly $\mathfrak{diff}(S^1)$ acts on $\mathfrak{X}(S^1)$ by the adjoint representation of the Lie algebra $\mathfrak{diff}(S^1)$ as indicated in (3.4) and (3.13).

The complexification $\mathfrak{X}^{\mathbb{C}}(S^1)$ of $\mathfrak{X}(S^1)$ has elements $V = f(\theta)\frac{\partial}{\partial\theta}$, with f a complex periodic function. The linear hull of the set generated by the the element of $\mathfrak{X}^{\mathbb{C}}(S^1)$ is

$$\epsilon^k = i\,e^{ikx}\frac{\partial}{\partial x}, \quad k \in \mathbb{Z},$$

for which the following commutation relations hold:

$$[\epsilon^k, \epsilon^l] = \left[ie^{ikx}\frac{\partial}{\partial x}, ie^{ilx}\frac{\partial}{\partial x}\right] = (k - l)\epsilon^{k+l},$$

is a real Lie algebra called Witt algebra.

One can build a non-trivial 2-cocycle corresponding to Gelfand-Fuks by integration, and known as the Bott-Virasoro cocycle. Consider an orientation-preserving diffeomorphism $\sigma : S^1 \to S^1$, and let S^1 be endowed with the flat volume form $\mu = dx$. Under a diffeomorphism $x \mapsto \sigma(x)$ we get

$$(\sigma^*(\mu))_x = d(\sigma(x)) = \sigma'(x)dx = e^{\log(\sigma'(x))}\mu \equiv e^{T[\sigma^{-1}]}\mu.$$

This yields a $C^\infty(S^1)$-valued twisted derivative, also known as 1-cocycle

$$T[\sigma^{-1}](x) = \log(\sigma'(x)). \tag{3.14}$$

We define the Bott-Thurston-Virasoro 2-cocycle [62] using the twisted derivative.

Definition 3 *The Bott-Thurston-Virasoro 2-cocycle on* $\mathfrak{Diff}(S^1)$ *is defined as*

$$c(\sigma_1, \sigma_2) = \frac{1}{2}\int_{S^1} T[\sigma_1]\, dT[\sigma_1 \circ \sigma_2], \tag{3.15}$$

where d *denotes the differential of functions on the circle.*

Using two important properties, namely,

$$T[\sigma_1 \circ \sigma_2] = T[\sigma_1] + T[\sigma_2] \circ \sigma_2^{-1}, \tag{3.16}$$

$$T[\sigma_1] \circ \sigma_1 = -T[\sigma_1^{-1}], \tag{3.17}$$

we can prove that

$$\int_{S^1} T[\sigma_1] dT[\sigma_1 \circ \sigma_2] = \int_{S^1} T[(\sigma_1 \circ \sigma_2)^{-1}] dT[\sigma_2^{-1}]. \tag{3.18}$$

Hence we can write this 2-cocycle in a more known form

$$B(\sigma_1, \sigma_2) = \frac{1}{2}\int_{S^1} \log(\sigma_1 \circ \sigma_2)'\, d\log(\sigma_2') = \frac{1}{2}\int_{S^1} T[(\sigma_1 \circ \sigma_2)^{-1}]\, dT[\sigma_2^{-1}]. \tag{3.19}$$

We call this 2-cocycle Bott-Thurston-Virasoro cocycle, Bott-Virasoro cocycle, or simply Bott cocycle.

3.1.1 Bott-Virasoro cocycle to Gelfand-Fuchs cocycle and Virasoro algebra

The group $\mathrm{Diff}_+(S^1)$ has non-trivial central extensions by the Abelian group $U(1)$, the Bott–Virasoro group $\widehat{\mathrm{Diff}}_+(S^1)$. Recall that a central extension is given by an exact sequence of groups

$$1 \longrightarrow U(1) \xrightarrow{j} \widehat{\mathrm{Diff}}_+(S^1) \xrightarrow{\pi} \mathrm{Diff}_+(S^1) \longrightarrow 1,$$

where $j(U(1))$ lies in the centre of $\widehat{\mathrm{Diff}}_+(S^1)$. Choosing a normalised (local) section ξ for π we see that if $\sigma_1, \sigma_2 \in \mathrm{Diff}_+(S^1)$, then $\xi(\sigma_1 \circ \sigma_2)$ differs from the product of $\xi(\sigma_1)$ and $\xi(\sigma_2)$ in an element $e^{ic(\sigma_1,\sigma_2)}$ of $j(U(1))$. The associativity property is equivalent to the condition:

$$c(\sigma_1 \circ \sigma_2, \sigma_3) + c(\sigma_1, \sigma_2) = c(\sigma_1, \sigma_2 \circ \sigma_3) + c(\sigma_2, \sigma_3).$$

which corresponds to the cocycle condition for the map $e^{ic} : \mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1) \to U(1)$.

Changing the section ξ amounts to modify the cocycle e^{ic} by a coboundary $e^{i\zeta}$ (i.e. a cocycle for which there exists a map $e^{i\tau} : \mathrm{Diff}_+(S^1) \to U(1)$ such that $\zeta(\sigma_1, \sigma_2) = \tau(\sigma_1 \circ \sigma_2) - \tau(\sigma_1) - \tau(\sigma_2))$). Then, c is replaced by $\bar{c}(\sigma_1, \sigma_2) = c(\sigma_1, \sigma_2) + \zeta(\sigma_1, \sigma_2)$.

More explicitly, the Bott–Virasoro group $\widehat{\mathrm{Diff}}_+(S^1)$ is defined by the central extension of $\mathrm{Diff}_+(S^1)$ by $U(1)$, determined by the *Bott cocycle* e^{ic} [63, **?**] where the map

$$c : \ \mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1) \longrightarrow \mathbb{R}$$

is given by [35]:

$$c(\sigma_1, \sigma_2) = \frac{1}{2} \int_{S^1} \log[(\sigma_1 \circ \sigma_2)'] \, d \log |\sigma_2'|,$$

for $\sigma_i \in \mathrm{Diff}_+(S^1)$. This cocycle satisfies the normalization condition $c(\sigma_1, \sigma_1^{-1}) = 0$. The cocycle can also be written as

$$c(\sigma_1, \sigma_2) = \frac{1}{2} \int_{S^1} \log(\sigma_1' \circ \sigma_2) \ d \log |\sigma_2'|,$$

because using the chain rule we have $(\sigma_1 \circ \sigma_2)' = (\sigma_1' \circ \sigma_2) \, \sigma_2'$ and then,

$$\frac{1}{2} \int_{S^1} \log(\sigma_1 \circ \sigma_2)' \ d \log |\sigma_2'| = \frac{1}{2} \int_{S^1} \log(\sigma_1' \circ \sigma_2) \ d \log |\sigma_2'| + \frac{1}{2} \int_{S^1} \log(\sigma_2') \ d \log |\sigma_2'|,$$

and an integration by parts shows that the last term vanishes as a consequence of the periodicity.

Note that

$$c(\sigma_1 \circ \sigma_2, \sigma_3) = \frac{1}{2} \int_{S^1} \log(\sigma_1 \circ \sigma_2 \circ \sigma_3)' \ d \log |\sigma_3'| = \frac{1}{2} \int_{S^1} \log(\sigma_1' \circ \sigma_2 \circ \sigma_3) \ d \log |\sigma_3'| + c(\sigma_2, \sigma_3),$$

and similarly,

$$c(\sigma_1, \sigma_2 \circ \sigma_3) = \frac{1}{2} \int_{S^1} \log(\sigma_1 \circ \sigma_2 \circ \sigma_3)' \ d \log |\sigma_2 \circ \sigma_3'| = \frac{1}{2} \int_{S^1} \log(\sigma_1' \circ \sigma_2 \circ \sigma_3) \ d \log |\sigma_3'| + c(\sigma_1, \sigma_2),$$

from where the cocycle condition follows. If (t, σ) denotes the product $j(t)\xi(\sigma)$ the composition law in $\widehat{\mathrm{Diff}}(S^1)$ is

$$(t_1, \sigma_1) \cdot (t_2, \sigma_2) = (t_1 + t_2 + c(\sigma_1, \sigma_2), \sigma_1 \circ \sigma_2).$$

There is a corresponding non-trivial central extension of the Lie algebra $\mathfrak{X}(S^1)$ by the trivial Lie algebra $\mathbb{R}$ that is called the Virasoro algebra and denoted $\mathfrak{vir}$ – i.e. we have a central extension

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{vir} \longrightarrow \mathfrak{X}(S^1) \longrightarrow 0.$$

More specifically, the elements of $\mathfrak{vir}$ can be identified with pairs (real number, 2π-periodic function). In other words, the 2π-periodic function is the component of an element of the set

$\mathfrak{X}(S^1)$, which is known to be endowed with a Lie algebra structure. The corresponding cocycle $\zeta : \mathfrak{X}(S^1) \times \mathfrak{X}(S^1) \to \mathbb{R}$ is given by

$$\zeta(X_f, X_g) = \int_{S^1} f' g''\, dx,$$

where $X_f = f(x)\,\partial/\partial x$ and $X_g = g(x)\,\partial/\partial x$. Obviously, $\zeta(X_f, X_g) = -\zeta(X_g, X_f)$ as a simple integration by parts shows, because of the periodicity of functions f and g and $f'\,g'' = \dfrac{d}{dx}(f'\,g') - g'\,f''$.

Proposition 1 *The Bott-Virasoro cocycle is the integral of the Gelfand-Fuchs cocycle in the sense*

$$\zeta(X_f, X_g) = \frac{d^2}{dt\,ds}\,(c(\phi_t, \varphi_s))_{|s=t=0} - \frac{d^2}{dt\,ds}\,(c(\varphi_s, \phi_t))_{|s=t=0}\,, \tag{3.20}$$

where ϕ_t and φ_s are the flows of the vector fields X_f and X_g, respectively.

Proof.- Taking into account that

$$\frac{d}{dt}\,(c(\phi_t, \varphi_s))_{|t=0} = \frac{1}{2}\int_{S^1} (\log'(\phi_0 \circ \psi_s)')(f \circ \varphi_s)' d \log \varphi_s' = \frac{1}{2}\int_{S^1} (f' \circ \varphi_s) d \log \varphi_s',$$

and

$$\frac{d}{ds}\left(\frac{1}{2}\int_{S^1} (f' \circ \varphi_s) d \log \varphi_s'\right)_{s=0} = \frac{1}{2}\int_{S^1} f'\, dg',$$

and similarly for the other term, we find that

$$\zeta(X_f, X_g) = \frac{1}{2}\int_{S^1} (f'\, dg' - g'\, df') = \int_{S^1} f' g''\, dx.$$

□

Therefore the commutator in $\mathfrak{vir}$ takes the form

$$\Big[(a, X_f), (b, X_g)\Big] = (\zeta(X_f, X_g), [X_f, X_g]) = \left(\int_{S^1} f' g''\, dx, (f g' - g f')\frac{d}{dx}\right).$$

The dual linear space $\mathfrak{vir}^*$ can be identified to the set $\{(\mu, v\,(dx)^2) \mid \mu \in \mathbb{R}, v \in C^\infty(S^1)\}$. In fact, an element $(\mu, v\,(dx)^2)$ maps linearly $\mathfrak{vir}$ into the set of the real numbers as follows:

$$\langle(\mu, v\ (dx)^2), (a, X_f)\rangle = a\,\mu + \int_{S^1} f(x)\, v(x)\ dx,$$

and conversely, each linear map from $\mathfrak{vir}$ into $\mathbb{R}$ can be represented as such a pair $(\mu, v\ (dx)^2)$.

The remarkable point is that if $\mathrm{ad}_{(a,X_f)}$ is the image under the adjoint representation of the element (a, X_f) of the Virasoro algebra $\mathfrak{vir}$ and $\mathrm{ad}^*_{(a,X_f)}$ is the adjoint element, then

$$\mathrm{ad}^*_{(a,X_f)}(1, v\ (dx)^2) = \left(0, \frac{1}{2} f''' + 2v\, f' + v'\, f\right).$$

In fact, it follows from the definition

$$\begin{aligned}\langle \mathrm{ad}^*_{(a,X_f)}(\mu, v\ (dx)^2), (b, X_g)\rangle &= \langle (\mu, v\ (dx)^2), \mathrm{ad}_{(a,X_f)}(b, X_g)\rangle \\ &= \left\langle (\mu, v\ (dx)^2), \left(\int_{S^1} f'g''dx, [X_f, X_g]\right)\right\rangle \\ &= \mu\int_{S^1} f'g'' + \int_{S^1} v(fg' - f'g)dx = \int_{S^1}(\mu f''' - 2v\, f' - v' f)g(x)dx,\end{aligned}$$

and then

$$\mathrm{ad}^*_{(a,X_f)}(\mu, v\ (dx)^2) = (0, \mu\, f''' - 2v\, f' - v'\, f)\,.$$

Hence we see that the stability Lie algebra of the point $(-1/2, v\ (dx)^2)$ relative to the action defined by ad^* on its dual is given by a vector field $X_f \in \mathfrak{X}(S^1)$ such that f is a solution of the third-order differential equation (3.11), which is the projective vector field equation. We have recovered the projective vector field equation from the point of view of the coadjoint orbit of the Virasoro algebra.

3.2 Projective vector field equation and its structure

The projective vector field equation (3.11) is a linear differential equation and therefore invariant under the dilation vector field. Lie recipe amounts to introduce a new dependent variable z instead of y in such a way that the dilation vector field $D = y\,\partial/\partial y$ has the form $D = \partial/\partial z$ (i.e. $y = Ce^z$, for any constant C), and then

$$y' = Cz'\,e^z\,,\quad y'' = C(z'' + z'^2)e^z\,,\quad y''' = C(z''' + 3z'z'' + z'^3)e^z\,,\qquad C \neq 0. \tag{3.21}$$

Note that $z' = y'/y$, no matter of the value of $C \neq 0$. The important point is that the differential equation transformed from (3.11) does not depend on the variable z but on its derivatives and then the order of the differential equation (3.11) can be reduced by one just by introducing the new variable $u = z'$ and so we obtain the second-order differential equation for u,

$$u'' + 3u\,u' + u^3 + 4vu + 2v' = 0\,, \tag{3.22}$$

which is a second-order Riccati equation, called projective second-order Riccati equation.

The purpose of this section is to set up a geometrical formulation of such nonlinear second-order ODEs (3.22) in terms of the projective vector field equation (3.11) (see e.g. [59, 65]).

Proposition 2 *Let ψ_1 and ψ_2 be two linearly independent solutions of the second-order differential equation*

$$\psi'' + k\,v\,\psi = 0\,. \tag{3.23}$$

Then, the three-dimensional linear space of solutions of the linear third-order differential equation

$$y''' + 4k\,v\,y' + 2k\,v'\,y = 0 \tag{3.24}$$

is spanned by the functions ψ_1^2, ψ_2^2, and $\psi_1\psi_2$.

Proof.- If ψ_1 and ψ_2 are two linearly independent solutions of the equation (3.23) then taking derivatives we obtain that

$$\psi_1''' + kv\,\psi_1' + kv'\,\psi_1 = 0\,, \qquad \psi_2''' + kv\,\psi_2' + kv'\,\psi_2 = 0\,.$$

Now, if we make use of these two equations, then the following third-order derivative

$$D^3(\psi_i\psi_j) = \psi_i'''\psi_j + 3\psi_i''\psi_j' + 3\psi_i'\psi_j'' + \psi_i\psi_j'''\,,$$

can be rewritten as follows

$$D^3(\psi_i\psi_j) = -(kv\,\psi_i' + kv'\,\psi_i)\psi_j - 3kv\,\psi_i\psi_j' + 3\psi_i'(-kv\,\psi_j) - \psi_i(kv\,\psi_j' + kv'\,\psi_j),$$

that after simplification it becomes

$$D^3(\psi_i\psi_j) = -k[2v'\,\psi_i\psi_j + 4v(\psi_i'\psi_j + \psi_i\psi_j')]\,.$$

We have therefore obtained

$$D^3(\psi_i\psi_j) + 4kv(\psi_i'\psi_j + \psi_i\psi_j') + 2k\,v'(\psi_i\psi_j) = 0\,,$$

what proves that the three functions $f_{ij} = \psi_i\psi_j$, $i, j = 1, 2$, are solutions of (3.24). Finally, the Wronskian of these three functions is given by

$$W[\psi_1^2, \psi_1\psi_2, \psi_2^2] = 2(\psi_1\,\psi_2' - \psi_2\,\psi_1')^3\,.$$

Therefore, as $\{\psi_1, \psi_2\}$ is a fundamental set of solutions of the linear second-order equation (3.23), the functions ψ_1^2, ψ_2^2 and $\psi_1\psi_2$ are linearly independent and they span the linear space of solutions of (3.24). commen

□

Remark: The result of the preceding Proposition 1 is an example of a particular case of those in which the general solution of a linear third-order ODE can be obtained from those of a linear second-order ODE. This problem has been studied, for example, by Singer in [66]: Example 1.4 therein, Airy equation, is a special case of Proposition 1.

Remark: The usual Schrödinger equation for the determination of stationary states is of the same type as Hill equation where the function v is not periodic but a generic function $v(x) = E - V(x)$, but the symmetry properties of both equations are the same ones.

Proposition 3 *Suppose ψ is a solution of a second-order linear differential equation*

$$\psi'' + v_1\psi' + v_2\psi = 0. \tag{3.25}$$

Then $y = \psi^2$ is a solution to the third-order linear differential equation

$$y''' + 3v_1y'' + (v_1' + 4v_2 + 2v_1^2)y' + (2v_2' + 4v_1v_2)y = 0, \tag{3.26}$$

where v_1 and v_2 are functions of x.

Proof.- The proof is essentially based on the fact that y, y', y'', y''' are all linear combinations of ψ^2, $\psi\psi'$ ψ'^2. We obtain

$$y'' = 2\big((\psi')^2 - v_1\psi\psi' - v_2\psi^2\big),$$

$$y''' = 2\big(v_1^2\psi\psi' + v_1v_2\psi^2 - v_1'\psi\psi' - v_1'\psi^2 - 3v_1\psi'^2 - 4v_2\psi\psi'\big).$$

One can obtain the result using all these expressions.

□.

Corollary 1 *Let $v_2 = -\frac{1}{2}v_1' - \frac{1}{4}v_1^2$. Then the coefficients of equation (3.26) are expressed in terms of (higher-order) Riccati equations*

$$v_1' + 4v_2 + 2v_1^2 = -v_1' + v_1^2 = \left(-\frac{d}{dx} + v_1\right) v_1, \tag{3.27}$$

$$2v_2' + 4v_1v_2 = -\big(v_1'' + 3v_1v_1' + v_1^3\big) = -\left(\frac{d}{dx} + v_1\right)^2 v_1. \tag{3.28}$$

Proof.- By direct computation.

□

Proposition 4 *1. Let f be a solution of the projective vector field equation (3.11). Then the function u such that $k\,u = f'/f$ (where $k \neq 0$) is a particular solution of the following second-order Riccati equation:*

$$u'' + 3k\,u\,u' + k^2u^3 + 4kv\,u + 2k\,v' = 0\,. \tag{3.29}$$

2. Suppose that $u_1(x)$ is a solution of the Riccati equation

$$\zeta' + k\zeta^2 + k\,v = 0\,. \tag{3.30}$$

Then the function $u = 2u_1$ is a solution of the second-order Riccati equation (3.29).

Proof.- 1. The equation (3.12) is invariant under dilations, and then we can choose an adapted coordinate for which dilations generator is $(1/k)\partial/\partial z$ (i.e. we use a function z instead of y such that $y = e^{kz}$), and then using the expressions analogous to (3.21)

$$y' = kz'\,e^{kz}\,, \quad y'' = (kz'' + k^2z'^2)e^{kz}\,, \quad y''' = (kz''' + 3k^2z'z'' + k^3z'^3)e^{z}\,, \tag{3.31}$$

the differential equation that we obtain does not depend on z but on its derivatives, z', z'' and z''', and if we define $u = z'$, it becomes (3.29), which is a generalization of the nonlinear oscillator equation. Here the coefficients are fixed by the projective vector field equation. In other words, the new reduced equation can be rewritten as

$$R^2(u) + 4vR^0(u) + 2v' = 0,$$

where the functions R^k are defined in Section 2.

2. First, we note that if u_1 satisfies (3.30), then the derivative of relation

$$u_1' + ku_1^2 + kv = 0, \tag{3.32}$$

leads to the following second-order relation

$$u_1'' + 2ku_1\, u_1' + kv' = 0\,. \tag{3.33}$$

On the other side, putting $u = 2u_1$ in the left hand side of (3.29) we obtain

$$u'' + 3ku\, u' + k^2u^3 + 4kv\, u + 2k\, v' = 2u_1'' + 12ku_1\, u_1' + 8k^2u_1^3 + 8ku_1\, v + 2kv',$$

and therefore taking into account (3.33):

$$u'' + 3ku\, u' + k^2u^3 + 4kv\, u + 2k\, v' = -2(2ku_1u_1' + kv') + 12ku_1u_1' + 8k^2u_1^3 + 8ku_1v + 2k\, v',$$

which simplifying terms reduces to

$$u'' + 3ku\, u' + k^2u^3 + 4kv\, u + 2k\, v' = 8ku_1(u_1' + k\, u_1^2 + k\, v),$$

from where using (3.32) on the right hand side we obtain the result of the Proposition.

This shows a relationship among solutions of the ordinary Riccati equation (3.30) and those of the related second-order Riccati equation (3.29).

3.3 Global projective vector field and integrable ODEs

Since the solution space of the projective vector field equation is spanned by ψ_1^2, ψ_2^2, and $\psi_1\psi_2$, an arbitrary solution of the projective vector field equation is given by [59]

$$\Psi = A\psi_1^2 + 2B\psi_1\psi_2 + C\psi_2^2, \tag{3.34}$$

an arbitrary linear combination of basis vectors. This is periodic when ψ_1 and ψ_2 are periodic, and hence it is a global solution of the projective vector field equation as a consequence of the existence of global solutions of the Hill equation. This Ψ is called the *global projective vector field* [56].

Milne–Pinney equation is a second-order differential equation that can be written, together with a harmonic oscillator equation, as a system of two first-order differential equations and it turns out to be a Lie system (see e.g. [22]). The same is true for the Hill equation (or equivalently for the harmonic oscillator with a time-dependent frequency). As the Vessiot-Guldberg Lie algebra of both equations is the same we can determine a mixed superposition rule allowing to write the general solution of Pinney equation in terms of two independent solutions of Hill equation (see [22]). More explicitly, the following superposition rule was proved in [24, 25]:

Proposition 5 *If ψ_1 and ψ_2 satisfy Hill's equation – i.e. they are periodic solutions of (3.8), i.e.*

$$\frac{d^2\psi_i}{dx^2} + v(x)\psi_i = 0, \quad i = 1, 2, \tag{3.35}$$

then the square root ψ of the function Ψ given by (3.34), which, as indicated above, is a solution of the projective vector field equation, that is, $\psi = \sqrt{A\psi_1^2 + 2B\psi_1\psi_2 + C\psi_2^2}$, with A, B and C arbitrary real numbers, is a periodic function satisfying the Milne–Pinney equation

$$\psi'' + v(x)\psi = \frac{\sigma}{\psi^3}, \tag{3.36}$$

with $\sigma = AC - B^2$.

Note also that we can consider the nonlinear second-order differential equation [67]

$$f\,f'' - \frac{1}{2}\,f'^2 + 2v\,f^2 - \frac{\sigma}{2} = 0, \tag{3.37}$$

which with the change of variable $f = -\psi^2/2$ becomes (3.36). As it will be pointed out later on, this is so because the function $\Phi(f, f', f'') = f\,f'' - \frac{1}{2}\,f'^2 + 2v\,f^2$ is a first integral of (3.11). Moreover, taking derivative with respect to x at equation (3.37) we see that a solution of such equation is a solution of the projective vector field equation (3.11).

In a similar way, we can consider the second-order Kummer-Schwarz equation:

$$\frac{1}{2}\frac{f''}{f} - \frac{3}{4}\left(\frac{f'}{f}\right)^2 + \sigma f^2 + v = 0,$$

which is a particular case of the second-order Gambier equation [68] and it was recently analysed in [48] from the perspective of Lie theory. It has been proved to be a Lie system associated with a Vessiot-Guldberg Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, and therefore admitting a nonlinear superposition rule [1]. But as in the case of Milne-Pinney equation we can also find a mixed superposition rule in terms of solutions of Hill equation. More explicitly, the solution is given by

$$f(x) = (A\psi_1^2 + 2B\psi_1\psi_2 + C\psi_2^2)^{-1}, \tag{3.38}$$

where ψ_1 and ψ_2 satisfy the Hill's equation and $B^2 = AC - \sigma W^{-2}$, where W is the Wronskian determinant of both solutions, $W = W[\psi_1, \psi_2]$ [48].

It has also been proved in [45] that if ψ_1 and ψ_2 satisfy Hill's equation, with $\psi_2(x_0) \neq 0$, and W denotes the Wronskian, $W = W[\psi_1, \psi_2]$, then for each nonzero real number m, such that $0 \neq m \neq 1$, the function

$$\Psi = \left(\psi_1^m + \frac{c}{(m-1)W^2}\,\psi_2^m\right)^{1/m}, \qquad c \in \mathbb{R},$$

satisfies the differential equation

$$y'' + v(x)y = c\,\frac{(\psi_1\psi_2)^{m-2}}{y^{2m-1}}.$$

There is a clear difference with respect to the preceding case of Milne–Pinney equation because now the right-hand side of equation depends on the functions ψ_1 and ψ_2.

This result was generalised in [69] where it is shown that if ψ_1 and ψ_2 are linearly independent solutions of the linear homogeneous second-order differential equation

$$y'' + r(x)\, y' + q(x)\, y = 0,$$

then the function

$$\Psi = (A\psi_1^m + B\psi_2^m)^{1/m}$$

is a solution of the Reid type equation

$$y'' + r(x)\, y' + q(x)\, y = AB(m-1)(\psi_1\psi_2)^{m-2}\frac{W^2}{y^{2m-1}},$$

where W is as before the Wronskian of the two functions ψ_1 and ψ_2.

Moreover, if we set $\psi = (\psi_1\psi_2)^{k/2}$ then ψ satisfies Thomas equation [46]

$$y'' + r(x)y' + kq(x)y = (1-l)\frac{y'^2}{y} - \frac{1}{4}kW^2y^{1-4l}, \qquad kl = 1, \tag{3.39}$$

where W denotes the Wronskian of the two solutions. We can recover reduced Gambier equation (or Painlevé-Gambier XXVII equation) [47] for special values of m.

One must note that for $r(x) = 0$, Thomas equation (3.39) is reduced to

$$y'' + \frac{q(x)}{l}y = (1-l)\frac{y'^2}{y} - \frac{1}{4l}W^2y^{1-4l}, \tag{3.40}$$

this becomes the Pinney equation (1.1) for $l = 1$ and $W^2 = -4k$. Setting, instead, $l = -1/2$, $q = 2c$, where c is some constant, then (3.40) reduces to Kummer-Schwarz (KS2) equation

$$y'' = \frac{3}{2}\frac{y'^2}{y} + \frac{W^2}{2}y^3 - 2c\, y. \tag{3.41}$$

The KS2 equation is of interest mainly on account of its relationship with other differential equations of physical and mathematical interest. For instance, when $y > 0$ the change of variables $z = 1/\sqrt{y}$ transforms the KS2 equation into an Ermakov–Milne–Pinney equation

$$z'' - c\, z = -\frac{W}{4z^3}.$$

Moreover, it can be shown that the non-local transformation $dz/dt = \zeta$ maps the KS2 equation to a particular variant of the third-order Kummer-Schwarz equation which is closely related with Schwarzian derivatives. Indeed if $\{u_1, u_2\}$ is a basis of the linear space of solutions of the second-order equation $\ddot{u} - 2cu = 0$, then the Kummer-Schwarz equation has a general solution of the form [48]

$$\zeta(t) = (Au_2^2 + Bu_1u_2 + Cu_1^2)^{-1}, \qquad B^2 - AC = W^2. \tag{3.42}$$

4 Invariants, prolongations and Riccati equation

For a geometric approach to a partial differential equation in m independent variables and one dependent variable u, we must consider the space $M \times V = \{(\mathbf{x}, u) \mid \mathbf{x} \in M, u \in V\}$, where $M = \mathbb{R}^m$ and $V = \mathbb{R}$. Suppose that G is a Lie group acting on some open subset $N \subseteq M \times V$, by $\Phi : G \times N \to \Phi(N) \subseteq M \times V$. Then the transformation $\Phi_g : N \to \Phi(N) \subseteq M \times V$, for $g \in G$, is

$$\Phi_g(\mathbf{x}, u) = (\bar{\mathbf{x}}, \bar{u}), \qquad g \in G,$$

and a hypersurface given by $u = u(\mathbf{x})$ in N is transformed into another one, $\bar{u} = \bar{u}(\bar{\mathbf{x}})$, where $(\mathbf{x}, u) \in \Phi_g(N)$.

An infinitesimal transformation is given by (see e.g. [70])

$$\begin{aligned} \bar{\mathbf{x}} &= \mathbf{x} + \epsilon\, \mathbf{f}(\mathbf{x}, u) + \vartheta(\epsilon^2), \\ \bar{u} &= u + \epsilon\, \eta(\mathbf{x}, u) + \vartheta(\epsilon^2), \end{aligned}$$

wich can be understood as the infintesimal flow of the vector field in $M \times V$

$$X = f^i(\mathbf{x}, u)\frac{\partial}{\partial x^i} + \eta(\mathbf{x}, u)\frac{\partial}{\partial u}. \tag{4.1}$$

We restrict ourselves to the case of ordinary differential equations (i.e. $m = 1$) and then $M = \mathbb{R}$. A (may be local) smooth section σ for the projection $\pi : M \times V \to M$ defines a smooth function $u = u(x)$, by means of $\sigma(x) = (x, u(x))$ and then for each natural number $k \in \mathbb{N}$ it induces a function $u^{(k)} = \mathrm{pr}^{(k)}u$, called the k-th prolongation of u, where $\mathrm{pr}^{(k)}u : \mathbb{R} \longrightarrow \mathbb{R}^{k+1}$ is the curve whose components are the derivatives of u of orders from 0 to k. The total space $M \times V^{(k+1)} \subseteq \mathbb{R}^{k+2}$, the coordinates of which represent the independent variable x, the dependent variable u and the derivatives of u to order k, is called the *k-th order jet space* of the underlying space $M \times V$, sometimes denoted $J^k\pi$.

Similarly, a vector field on $N \subset M \times V$ given in local coordinates by

$$X = f(x, u)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u}, \tag{4.2}$$

admits a k-th-order prolongation. We are only interested in the first-order prolongation [70]. Note that if we consider the corresponding infinitesimal transformation (4.1), then

$$\frac{d\bar{u}}{d\bar{x}} = (D\eta - u_x Df) = \eta_x + (\eta_u - f_x)u_x - f_u u_x^2, \tag{4.3}$$

where $\eta_x = \partial\eta/\partial x$, $\eta_u = \partial\eta/\partial u$, $u_x = du/dx = u^{(1)}$, and similarly, for f_x and f_u, with D being given by

$$D = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u}.$$

It follows directly from

$$\begin{aligned} \frac{d\bar{u}}{d\bar{x}} &= \frac{d(u + \epsilon\,\eta + \vartheta(\epsilon^2))}{d(x + \epsilon f + \vartheta(\epsilon^2))} = \frac{u_x + (\eta_x + \eta_u u_x)\epsilon + \vartheta(\epsilon^2)}{1 + (f_x + f_u u_x)\epsilon + \vartheta(\epsilon^2)} \\ &= u_x + \epsilon\,(D\eta - u_x Df) + \vartheta(\epsilon^2) = u_x + (\eta_x + (\eta_u - f_x)u_x - f_u u_x^2)\,\epsilon + \vartheta(\epsilon^2). \end{aligned}$$

This provides us with the following well-known definition of first-order prolongation of X [70]:

Definition 4 *The prolongation* $\mathrm{pr}^{(1)}(X)$ *of the vector field* $X \in \mathfrak{X}(N)$ *with local expression (4.2) is the vector field on* $J^1\pi$ *given by*

$$\mathrm{pr}^{(1)}(X) = X^{(1)} = f(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u} + (D\eta - u_x Df)\frac{\partial}{\partial u_x}. \tag{4.4}$$

These prolongations will play a relevant rôle in the search for symmetries of differential equations as it is shown in next Subsection, where the first-order Riccati equation is used as an example.

The vector field (4.2) is projectable when $f_u \equiv 0$, and then (4.4) reduces to

$$\mathrm{pr}^{(1)}(X) = X^{(1)} = f(x)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u} + (D\eta - u_x f'(x))\frac{\partial}{\partial u_x}. \tag{4.5}$$

4.1 Infinitesimal symmetries of standard Riccati equation and projective vector field equation

The first-order differential equation for the function u, $(D-u)u = v(x)$, namely,

$$u' = u^2 + v(x), \tag{4.6}$$

where $u' = u_x$ and v is now a given function, usually called standard Riccati equation, is a particular case of the general Riccati equation:

$$u' = a_2(x)u^2 + a_1(x)u + a_0(x)\,, \tag{4.7}$$

for $a_2(x) = 1$, $a_1(x) = 0$ and $a_0(x) = v(x)$. The solutions of such non-autonomous differential equation (4.6) are given by the integral curves of the vector field on $\mathbb{R}^2$

$$X = \frac{\partial}{\partial x} + (u^2 + v(x))\frac{\partial}{\partial u}\,. \tag{4.8}$$

An infinitesimal Lie symmetry of such differential equation is represented by a projectable vector field on $\mathbb{R}^2$,

$$Y = f(x)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u}, \tag{4.9}$$

such that there exists a function τ satisfying

$$[Y, X] = \tau\, X, \tag{4.10}$$

because then the solutions of the differential equation (4.6) are transformed under the flow of the vector field Y into solutions, up to a reparametrisation. In other words, the vector field Y preserves the 1-dimensional distribution generated by X.

Taking into account that

$$[f\partial_x + \eta\partial_u, \partial_x + (u^2+v)\partial_u] = -(Xf)\partial_x + \big(f v_x + 2\eta\, u - \partial_x\eta - (u^2+v)\partial_u\eta\big)\,\partial_u,$$

we find that the symmetry condition implies that $\tau = -Xf = -f_x$, and that f and η are related as follows:

$$\partial_x\eta + (u^2+v)\partial_u\eta - f v_x - 2\eta\, u = (u^2+v)\, f_x\,. \tag{4.11}$$

We can alternatively consider the given differential equation (4.6) as defining a 2-dimensional submanifold in $\mathbb{R}^3$, with local coordinates $(x, u, u^{(1)})$, defined by zero level set of the constant rank map $\phi : \mathbb{R}^3 \to \mathbb{R}$, $\phi(x, u, u^{(1)}) = u^{(1)} - u^2 - v(x)$, that is,

$$\Sigma = \phi^{-1}(0) = \{(x, u, u^{(1)}) \mid u^{(1)} - u^2 - v(x) = 0\}. \tag{4.12}$$

The first-order prolongation of the vector field Y given by (4.9) is

$$Y^{(1)} = \mathrm{pr}^{(1)}(Y) = f(x)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u} + \left(\eta_x(x,u) + u^{(1)}\eta(x, u - u^{(1)} f_x(x)\right)\frac{\partial}{\partial u^{(1)}}\,, \tag{4.13}$$

and when we consider the symmetry condition, which is but the tangency condition $Y^{(1)}\phi_{|\Sigma} = 0$, we find

$$Y^{(1)}(u^{(1)} - (u^2 + v(x)))_{|u^{(1)}=u^2+v(x)} = 0\,, \tag{4.14}$$

and more explicitly,

$$\partial_x\eta(x,u) + (u^{(1)}\partial_u\eta(x,u) - f_x(x)u^{(1)})_{|u^{(1)}=u^2+v(x)} = f(x)v_x(x) + 2\eta(x,u)\,u\,, \tag{4.15}$$

which reproduces (4.11).

We can now establish the following relationship among symmetries of the standard Riccati equation and solutions of the projective vector field equation.

Proposition 6 *The standard Riccati equation (4.6) remains invariant with respect to the first prolongation* $\mathrm{pr}^{(1)}Y$ *of the vector field (4.9), provided that the function* f *satisfies the projective vector field equation.*

Proof.- Note first that if Y is an infinitesimal symmetry of X as indicated in (4.10), then for each function $g(x)$, $Y + g(x)X$ is also an infinitesimal symmetry of X, i.e. we can replace simultaneously f by $g + f$ and η by $\eta + g(u^2 + v(x))$, and then η_{uu} will change to $\eta_{uu} + 2g$ and we can restrict us to the case $\eta_{uu} = 0$. Consequently, once that f is fixed in (4.11) we can try to determine an affine function η (i.e. of the form $\eta(x,u) = k(x)u + h(x)$, satisfying such symmetry condition. Then the functions k and h must satisfy:

$$k_x(x)u + h_x(x) + (u^2 + v(x))(k(x) - f_x(x)) = f(x)v_x(x) + 2u(k(x)u + h(x)),$$

and therefore

$$u^2(k(x) + f_x(x)) + u(2h(x) - k_x(x)) + f(x)v_x(x) + v(x)(f_x(x) - k(x)) - h_x(x) = 0.$$

This shows first that we must choose $k(x) = -f_x(x)$ – i.e. the function η is such that

$$\eta(x,u) = -f_x(x)\,u + h(x),$$

and using such an expression for $\eta(x,u)$ in the symmetry condition (4.11) we find

$$u(f_{xx}(x)+2h(x))+f(x)\,v_x(x)+2v(x)f_x(x)-h_x(x)=0,$$

which shows that we should choose h such that $h=-\frac{1}{2}f_{xx}(x)$, and then replacing h by this value in the preceding equation we find that f must be such that

$$\frac{1}{2}f_{xxx}(x)+2v(x)f_x(x)+f(x)\,v_x(x)=0\,,$$

and hence, as indicated in [56], f is a solution of the projective vector field equation (3.11).

Remark: One would obtain the same result if one starts from a more general differential equation $u'=a_2(x)u^2+a_1(x)u+a_0(x)$. In this case u must be expressed in terms of a_2, a_1 and a_0 and their derivatives.

4.2 Integrals of motion and other dynamical features

A constant of the motion, or simply an integral of the motion, for a system of ODE's

$$\frac{dy_i}{dx}=X_i(x,y_1,\cdots,y_n)\quad i=1,\cdots,n \tag{4.16}$$

is a non-constant differentiable function $\Phi(x,y_1,\cdots,y_n)$ that retains a constant value on any integral curve of the system. This means that its derivative with respect to x vanishes on the solution curves:

$$\frac{d\Phi}{dx}=0\Rightarrow\frac{\partial\Phi}{\partial x}+\sum_i\frac{\partial\Phi}{\partial y_i}\frac{dy_i}{dx}=0\Longrightarrow\widetilde{D}[\Phi]=0, \tag{4.17}$$

where the vector field on $\mathbb{R}^{n+1}$

$$\widetilde{D}:=\frac{\partial}{\partial x}+\sum_i X_i\frac{\partial}{\partial y_i}$$

is called the material or total derivative. For an autonomous system and a first integral independent of x, this reduces to

$$D\Phi=\sum_i X_i\frac{\partial\Phi}{\partial y_i}=0 \tag{4.18}$$

where

$$D=\sum_i X_i\frac{\partial}{\partial y_i}$$

is just the vector field on $\mathbb{R}^n$ associated with the given autonomous system. The x-independent integrals of motion are usually called first-integrals.

Higher-order differential equations can be written as an associated system of first-order differential equations and then the constants of motion for such system are called constants of motion for the higher-order differential equation. As an instance the projective vector field equation (3.11) can be written as the linear system on $\mathbb{R}^3$

$$\left\{\begin{array}{rcl} \dfrac{dy}{dx} &=& w \\ \dfrac{dw}{dx} &=& a \\ \dfrac{da}{dx} &=& -4vw - 2v'y \end{array}\right.$$

and then the function $\Phi(x, y, w, a) = 2v(x)y^2 - \frac{1}{2}w^2 + ya$ is a constant of the motion, because the vector field

$$X = \frac{\partial}{\partial x} + w\frac{\partial}{\partial y} + a\frac{\partial}{\partial w} - (4vw + 2v'y)\frac{\partial}{\partial a}$$

is such that $X\Phi = 0$. This fact was pointed out in relation to (3.37). The constant of motion can also be rewritten as

$$\Phi(x, y, y', y'') = 2v(x)y^2 - \frac{1}{2}y'^2 + y\,y'',$$

which is a constant of the motion for the projective vector field equation (3.11). This shows that a nonvanishing function is solution of the projective vector field equation (3.11) if and only if there exists a constant C such that the function is solution of the second-order differential equation

$$y'' = -2v(x)\,y + \frac{y'^2}{2y} + C.$$

When $C = 0$ this equation is invariant under dilations and therefore its order can be reduced as indicated in Subection 3.2 by putting $u = y'/y$.

With an analogous procedure one can compute the first-integrals of similar type for a third-order differential equation. For example, for the first-integral of the stationary Calogero-Degasperis-Ibragimov-Shabat equation [71].

$$y''' + 3y^2\,y'' + 9y\,y'^2 + 3y^4y' = 0, \tag{4.19}$$

it has associated a vector field

$$X = w\frac{\partial}{\partial y} + a\frac{\partial}{\partial w} - (3y^2a + 9yw^2 + 3y^4w)\frac{\partial}{\partial a},$$

and the function

$$\Phi(y, w, a) = 2ay + 6y^3w + y^6 - w^2$$

is such that $X\Phi = 0$, and therefore is constant of motion. The corresponding function given by

$$\Phi(y, y', y'') = y\big(2y'' + (6yy' + y^4)y\big) - y'^2,$$

which leads to the family of differential equations

$$f\big(2f_{xx} + (6ff_x + f^4)f\big) - {f_x}^2 = C.$$

The remarkable point here is that one can interpret equation (4.19) in terms of the stabilizer set of the coadjoint action. The function $y = f(x)$ is a solution of (4.19) if and only if

$$\mathrm{ad}^*_{f\frac{d}{dx}}(3ff_x + (1/2)f^4)dx^2 = 0.$$

5 Second-order Riccati equation, Painlevé II and higher Painlevé type equations

Second-order second degree equations of Painlevé type appear in mathematical physics (for example, [72, 73]) and were studied mainly by Bureau [74]. The same problem was revisited much later by Cosgrove [75, 76], in collaboration with Scoufis [31], who, by restricting somewhat the scope, were able to produce a complete classification of integrable subcases. The question of the derivation of integrable second-order second-degree systems related to the Painlevé equations (a bottom-up approach, compared to the Cosgrove-Scoufis top-down one) was addressed by Sakka and Mugan [77, 78]. The second degree equations are also important in determining transformation properties of the Painlevé equations. In this section we elucidate its connection with the second-order Riccati equation.

Consider once again the linear third-order equation (3.11), i.e. the projective vector field equation, and the corresponding second-order Riccati equation (3.22).

Let us define for each function f the functions

$$u_1 = \frac{f'}{f}, \qquad u_2 = \frac{f''}{f} - \frac{1}{2}\left(\frac{f'}{f}\right)^2 + w, \tag{5.1}$$

where w is an arbitrary but fixed function. Then,

$$u_1' = \frac{d}{dx}\left(\frac{f'}{f}\right) = \frac{f''}{f} - \left(\frac{f'}{f}\right)^2 = u_2 - \frac{1}{2}u_1^2 - w. \tag{5.2}$$

Moreover,

$$u_2' = \frac{f'''}{f} - \frac{f''\,f'}{f^2} - u_1 u_1' + w',$$

and if we use the definition of u_2 given in (5.1) to write

$$\frac{f''}{f} = u_2 + \frac{1}{2}u_1^2 - w,$$

we see that when f is a solution of (3.11), then

$$\frac{f'''}{f} = -4v\,u_1 - 2v',$$

and

$$u_2' = -4v\,u_1 - 2v' - u_1\left(u_2 + \frac{1}{2}u_1^2 - w\right) - u_1 u_1' + w'$$

that is,

$$u_2' = -4v\,u_1 - 2v' + u_1(w - u_2) - u_1\left(u_1' + \frac{1}{2}u_1^2\right) + w'.$$

Consequently, using (5.2) we see that such functions u_1 and u_2 satisfy the system of differential equations

$$\begin{cases} u_1' &= u_2 - \frac{1}{2}u_1^2 - w\,, \\ \\ u_2' &= 2(w-2v)u_1 - 2u_1u_2 - 2v' + w'. \end{cases} \tag{5.3}$$

Conversely, if (u_1, u_2) is a solution of this system, then $f(x) = \exp\left(\int^x u_1(\zeta)\, d\zeta\right)$ is a solution of (3.11), because, by definition of f,

$$\frac{f'}{f} = u_1, \quad \frac{f''}{f} = \frac{d}{dx}\left(\frac{f'}{f}\right) + \left(\frac{f'}{f}\right)^2 = u_1' + u_1^2 = u_2 + \frac{1}{2}u_1^2 - w,$$

and

$$\frac{f'''}{f} = \frac{d}{dx}\left(\frac{f''}{f}\right) + \frac{f''\,f'}{f^2} = u_2' + u_1u_1' - w' + u_1\left(u_2 + \frac{1}{2}u_1^2 - w\right) = u_2' - w' + 2u_1(u_2 - w),$$

from here we see that

$$\frac{f''' + 4v\,f' + 2v'\,f}{f} = u_2' - w' + 2u_1(u_2 - w) + 4v\,u_1 + 2v'$$

and the second differential equation of the system (5.3) shows that f is solution of (3.11).

On the other side, the very essence of the differential equation (3.22) is that it is obtained as a system of two coupled first-order differential equations. One starts from a first-order Riccati equation $u_1' = u_2 - \frac{1}{2}u_1^2 - w$ like the first equation in the system (5.3), where w is an arbitrary differentiable function of the independent variable x, and then couple u_1 to u_2 through a linear differential equation on u_2 involving u_1, the second differential equation in (5.3).

We can also express these two equations in terms of u_2. After rearranging the second equation of (5.3 we obtain

$$u_1 = \frac{1}{2}\frac{u_2' + 2v' - w'}{w - 2v - u_2}. \tag{5.4}$$

Therefore, if we define a new function $K(x)$ by

$$K = \pm(u_2 - w + 2v), \tag{5.5}$$

we can write

$$u_1 = -\frac{1}{2}\frac{K'}{K}.$$

We can express the first equation of (5.3) in terms of K and substitute u_2 by $w - 2v \pm K$ and we obtain

$$K'' = \frac{5}{4}\frac{K'^2}{K} + 4vK \mp 2K^2. \tag{5.6}$$

A more amenable form is obtained when we substitute $K = -1/p$, and then

$$p'' = \frac{3}{4}\frac{p'^2}{p} - 4vp \mp 2. \tag{5.7}$$

This is a reduced Gambier equation or G5 equation in Gambier's classification [79].

Remarks: (a) If we assume v to be a constant, $v = \frac{\alpha}{4}$, then the second-order Riccati equation (3.22) reduces to the modified Emden equation

$$u'' + 3u\,u' + u^3 + \alpha\,u = 0, \tag{5.8}$$

which therefore can be obtained from the pair of differential equations

$$\begin{cases} u_1' &= u_2 - \frac{1}{2}u_1^2 - w, \\ u_2' &= (2w - \alpha)u_1 - 2u_1u_2 + w'. \end{cases}$$

Equation (5.8) is a very well-known integrable system.

(b) The projective second-order Riccati equation (3.22) can be related to the family of Ermakov–Milne–Pinney equations [23, 24, 25]

$$\Psi'' + v\Psi = \frac{\sigma}{\Psi^3}, \tag{5.9}$$

where σ is some constant. In fact, such a one-parameter family is described by the third-order differential equation obtained by elimination of the parameter σ,

$$(\Psi^3\Psi'' + v\Psi^4)' = 0. \tag{5.10}$$

This differential equation is invariant under dilations and Lie recipe for reduction amounts to define a new variable ζ such that $\Psi = e^{\frac{1}{2}\zeta}$ (i.e. $u = \zeta' = 2\Psi'/\Psi$), and then replacing

$$\frac{\Psi'}{\Psi} = \frac{1}{2}u, \qquad \frac{\Psi''}{\Psi} = \frac{1}{2}u' + \frac{1}{4}u^2, \qquad \frac{\Psi'''}{\Psi} = \frac{1}{2}u'' + \frac{3}{4}uu' + \frac{1}{8}u^3,$$

in the equation (5.10) of the family

$$\Psi^3\Psi''' + 3\Psi^2\Psi'\Psi'' + 4v\Psi^3\Psi' + v'\Psi^4 = 0,$$

we obtain the second-order differential equation (3.22). This immediately yields our result.

□

The projective vector field equation (3.11) admits a Lax formulation in the following sense. Given a function f we define a pair of matrices P and Q as follows:

$$P = \begin{pmatrix} f'/2 & -\int^x v(\zeta)f'(\zeta)\,d\zeta \\ -f & -f'/2 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & v \\ -1 & 0 \end{pmatrix}, \tag{5.11}$$

for which

$$QP = \begin{pmatrix} -vf & -vf'/2 \\ -f'/2 & \int^x v(\zeta)f'(\zeta)\,d\zeta \end{pmatrix}, \qquad PQ = \begin{pmatrix} \int^x v(\zeta)f'(\zeta)\,d\zeta & vf'/2 \\ f'/2 & -vf \end{pmatrix}.$$

Then, the matrices P and Q are a Lax pair, that is,

$$P' + [Q, P] = 0, \tag{5.12}$$

if and only if f is a solution of the integro-differential equation

$$f'' + 2v\, f + 2\int^x v(\zeta) f'(\zeta)\, d\zeta = 0,$$

and taking derivatives we find that f is a solution of the projective vector field equation (3.11).

The corresponding Lax formulation for the projective second-order Riccati equation (3.22) will be given by the same matrix Q and

$$P = \begin{pmatrix} \frac{1}{2} u e^{\int^x u(\zeta)\, d\zeta} & -\int^x v(\zeta) u(\zeta) e^{\int^\zeta u(\zeta')\, d\zeta'}\, d\zeta \\ -e^{\int^x u(\zeta)\, d\zeta} & -\frac{1}{2} u e^{\int^x u(\zeta)\, d\zeta} \end{pmatrix},$$

and then the matrices P and Q are a Lax pair if and only if u is a solution of the integro-differential equation

$$(u' + u^2 + 2v) \exp \int^x u(\zeta)\, d\zeta + 2\int^x v(\zeta) u(\zeta) \exp \int^\zeta u(\zeta')\, d\zeta'\, d\zeta = 0.$$

Now, taking derivative with respect to x we find

$$(u'' + 2uu' + 2v') \exp \int^x u(\zeta)\, d\zeta + u(u' + u^2 + 2v) \exp \int^x u(\zeta)\, d\zeta + 2uve^{\int^x u(\zeta)\, d\zeta} = 0,$$

and simplifying the common factor we obtain the projective second-order Riccati equation (3.22).

Remark: The equation (5.12) has a nice geometric interpretation in terms of the coadjoint action of loop algebra [80, 81]. The loop group $C^\infty(S^1, G)$ is the group of smooth functions on the circle S^1 with values on real semi-simple Lie group G. Its Lie algebra $C^\infty(S^1, \mathfrak{g})$ has a central extension provided by the Kac-Moody cocycle

$$\omega(A, B) = \int_{S^1} \left\langle A(x), \frac{dB(x)}{dx} \right\rangle dx.$$

The coadjoint representation of the extended loop algebra is given by

$$\mathrm{ad}^*(a, P)(b, Q) = \left(0, [P, Q] + b\frac{dP(x)}{dx}\right),$$

for any $P, Q \in C^\infty(S^1, \mathfrak{g})$ and $a, b \in \mathbb{R}$. The equation (5.12) is the *stabilizer set* of the coadjoint orbit confined to hyperplane $b = -1$. Thus once again we obtain an equation which can be interpreted in terms of stabilizer orbit.

5.1 Painlevé II, Bäcklund transformation and second-order Riccati equation

We wish to discuss the integrable class of Painlevé II equation; in other words, we are interested in the rational solutions for integer valued parameter α of the Painlevé II equation and their explicit characterisation in terms of the Airy function Ai. It is worth to note that half-integer valued parameter of the Riccati equation are also characterised by the Airy function. Here the Painlevé II equation (PII) is an ordinary nonlinear second-order differential equation with a parameter α,

$$u'' = 2u^3 + xu + \alpha. \tag{5.13}$$

This equation has exactly one rational solution for α being an arbitrary integer and has no rational solution if α is not an integer [28, 82, 83]. It admits a Bäcklund transformation $(u, \alpha) \mapsto (u, -\alpha)$, which clearly maps rational solutions into rational solutions.

Consider now the Airy differential equation

$$\psi'' + x\,\psi = 0. \tag{5.14}$$

It is clear that this Airy differential equation is a particular instance of the Hill's equation (3.8) for the special choice $v(x) = x$. The following proposition elucidates the relation between PII and second-order Riccati, which has appeared in [8, 84].

Proposition 7 *1.- If ψ is a solution of the Airy equation*

$$\psi'' + x\psi = 0,$$

then the function $u_1 = \psi'/\psi$ satisfies the Riccati equation $u' + u^2 + x = 0$.

2.- The second-order equation obtained by derivation from such Riccati equation for the function $u = \lambda\, u_1$ is the Painlevé II equation

$$u'' = \frac{2}{\lambda^2}u^3 + 2x\,u - \lambda, \tag{5.15}$$

which after a rescaling becomes the Painlevé II equation

Proof.- 1.- It suffices to use the expression (2.13) for reduction recipe with $n = 2$, $a_0(x) = x$, $a_1(x) = 0$ and $a_2(x) = 1$, (i.e. $R^1(u) + x\,R^0(u) = 0$), and the expressions (2.10), then $u_1 = \psi'/\psi$ is a solution of

$$u' + u^2 + x = 0.$$

2.- Note that the function $u = \lambda\, u_1$ satisfies the differential equation $u' = -\frac{u^2}{\lambda} - \lambda\, x$ and therefore, deriving with respect to x we find that the second-order Riccati differential equation satisfied by the function $u = \lambda\, u_1$ is

$$u'' = -\frac{2u}{\lambda}\left(-\frac{u^2}{\lambda} - \lambda\, x\right) - \lambda = \frac{2u^3}{\lambda^2} + 2x\,u - \lambda.$$

If we now consider a new independent variable $\bar{x} = \mu x$ with $0 \neq \mu \in \mathbb{R}$, the differential equation is transformed into

$$\frac{d^2u}{d\bar{x}^2} = \frac{2u^3}{\mu^2\lambda^2} + 2\frac{\bar{x}}{\mu^3} - \frac{\lambda}{\mu^2},$$

and by choosing the parameters λ and μ as $\mu^3 = 2$ and $\lambda = \mu^{-1}$ we see that the differential equation becomes a Painlevé II type differential equation (5.13) with $\alpha = 1/2$:

$$\frac{d^2u}{d\bar{x}^2} = 2u^3 + \bar{x}\,u - \frac{1}{2}.$$

This proves that the second-order Riccati equation can be brought back to 'standard Painlevé II' by an appropriate scaling.

□

Under the map $x \to -x$, it takes the form of a similar equation but with the opposite sign, that is, i.e. if ψ is a solution of the Airy differential equation

$$\psi'' - x\psi = 0,$$

then $u_1 = \lambda\,\psi'/\psi$ satisfies the differential equation

$$u'' = \frac{2}{\lambda^2}u^3 - 2x\,u - \lambda, \tag{5.16}$$

that can be transformed into a Painlevé II equation with an appropriate change of independent variable as before.

Remark: The Painlevé transcendents (P-II – P-VI) possess *Bäcklund transformations* which map solutions of a given Painlevé equation into solutions of the same Painlevé equation, but with different values of the parameters. Therefore two Painlevé equations for $\alpha = 2$ and $\alpha = -2$ are connected by Bäcklund transformations.

Remark: Considering $\psi'' \pm (x/2)\psi = 0$ we can also relate Painlevé II equations for parameters $\alpha = \mp 2$ and these are connected by the Bäcklund transformations.

Remark: If $y = f(x)$ is a solution of the differential equation (3.11), then for a given real number β the function $y = f(x)$ is a solution of the differential equation

$$y''' + 4\bar{v}\,y' + 2\bar{v}'\,y + 6\beta y\,y' = 0, \tag{5.17}$$

with $\bar{v} = v - \beta\,f$. Now if the function f is positive and define the positive function ψ by $f = \psi^2$, then as

$$f' = 2\psi\,\psi', \qquad f'' = 2\psi'^2 + 2\psi\,\psi'', \qquad f''' = 6\psi'\,\psi'' + 2\psi\,\psi''',$$

we see that ψ is then a solution of

$$2y\,y''' + 6y'\,y'' + 8\bar{v}\,y\,y' + 2\bar{v}'\,y^2 + 12\beta\,y^3y' = 0,$$

which can be rewritten as

$$(y''\,y^3 + \bar{v}\,y^4 + \beta\,y^6)' = 0,$$

i.e. there exists a constant σ such that ψ is a solution of the differential equation

$$y'' + \bar{v}\, y + \beta\, y^3 = \frac{\sigma}{y^3}.$$

For the particular case $\bar{v} = x/2$, that means that ψ is a solution of the Ermakov-Painlevé II equation. Conte showed how one can transform it to Painlevé II (see e.g. [85]).

Remark: Let us briefly describe the connection between Painlevé II hierarchy and our approach. We proved that the solutions of the the projective vector field equation generate the stability algebra of Virasoro orbit. In other words, instead of considering the symplectic structure defined by $\mathcal{O}_1 = \partial_x$ and the Hamiltonian function $\frac{1}{3!}u^3 - \frac{1}{2}u_x^2$, i.e.

$$u_t = \frac{\partial}{\partial x}\left(\frac{\delta}{\delta u}\left[\frac{1}{3!}u^3 - \frac{1}{2}u_x^2\right]\right) = \frac{\partial}{\partial x}\left(\frac{1}{2}u^2 + u_{xx}\right) = u\, u_x + u_{xxx},$$

we consider the second Hamiltonian structure of the KdV equation

$$\mathcal{O}_2 = \partial_x^3 + \frac{2}{3}u\, \partial_x + \frac{1}{3}u_x. \tag{5.18}$$

together with the Hamiltonian density $\frac{1}{2}u^2$. Using 'frozen Lie-Poisson structure' we can define the first Hamiltonian structure of the KdV equation too. This satisfies famous Lenard scheme

$$\partial_x \mathcal{H}_{n+1} = \left(\partial_x^3 + \frac{2}{3}u\, \partial_x + \frac{1}{3}u_x\right)\mathcal{H}_n, \tag{5.19}$$

where

$$\mathcal{H}_1 = u, \qquad \mathcal{H}_2 = \frac{u^2}{2}, \qquad \mathcal{H}_3 = \frac{1}{3!}u^3 - \frac{1}{2}u_x^2, \qquad \cdots$$

are the conserved densities. The mKdV hierarchy is obtained from the KdV hierarchy through Miura map $u = v_x - v^2$. The second Painlevé hierarchy is given recursively by Joshi from the modified KdV hierarchy [86]

$$P_{II}^n(u, \beta_n) \equiv \left(\frac{d}{dx} + 2v\right)\mathcal{J}_n(u_x - u^2) - x\, u - \beta_n = 0, \tag{5.20}$$

where β_n are constants and $\mathcal{J}_n$ is the operator defined by the first and second Hamiltonian structures of the KdV equation

$$\partial_x \mathcal{J}_{n+1}(u) = (\partial_x^3 + 4u\partial_x + 2u_x)\mathcal{J}_n(u),$$

with $\mathcal{J}_1(u) = u$.

We must recall that the solution of projective vector field equation is governed by the solutions of Hill's equation. Let (ψ_1, ψ_2) be the solution of the latter then the solution of the former equation is given by $(\psi_1^2, \psi_1\psi_2, \psi_2^2)$. This is called Veronese embedding, where (ψ_1, ψ_2) is the homogeneous coordinates of $\mathbb{P}_1$. The morphism $\varphi : \mathbb{P}_1 \to \mathbb{P}_d$ given by

$$\varphi([\psi_1 : \psi_2]) = [\psi_1^d : \psi_1^{d-1}\psi_2 : \cdots : \psi_1\psi_2^{d-1}\psi_2 : \psi_2^d,$$

is called the d-Veronese embedding.

Using the higher Veronese embedding we study the solutions of the higher-order projective vector field equations, which in turn yields higher-order Riccati equation.

Lemma 1 *Let ψ_1 and ψ_2 be two linearly independent solutions of Hill's equation. Then,*
(a) The differential equation

$$y^{(iv)} + 10vy'' + 10v'y' + (9v^2 + 3v'')y = 0 \tag{5.21}$$

traces out a four-dimensional space of solutions spanned by $\{\psi_1^3,\ \psi_1^2\psi_2,\ \psi_1\psi_2^2,\ \psi_2^3\}$.
(b) The differential equation

$$y^{(v)} + 20vy''' + 30v'y'' + 18v''y' + 64v^2y' + 4v'''y + +64vv'y = 0 \tag{5.22}$$

traces out a five-dimensional space of solutions spanned by $\{\psi_1^4,\ \psi_1^3\psi_2,\ \psi_1^2\psi_2^2,\ \psi_1\psi_2^3,\ \psi_2^4\}$.

Proof.- By a direct lengthy computation.

□

Using the standard Cole–Hopf transformation $u = y'/y$ we obtain the third-order and fourth-order Riccati equations associated to these equations, respectively, which are, according to (2.12) given by

$$u''' + 4u\,u'' + 3u'^2 + 6u^2\,u' + 10v\,u' + u^4 + 10v\,u^2 + 10v'\,u + 9v^2 + 3v'' = 0, \tag{5.23}$$

and

$$\begin{aligned} u^{(iv)} + 5u\,v''' &+ 10u'u'' + 15uu'^2 + 10u^2u'' + 10u^3u' + u^5 + 20v(u'' + 3uu' + u^3) \\ &+ 30v'(u' + u^2) + 18v''u + 64v^2u + 4v''' + 64vv' = 0. \end{aligned}$$

Note that if we put $v = 0$ then, all these equations form a Riccati hierarchy.

5.2 Higher Riccati equations and higher-order Painlevé class systems

One can easily check that the third-order Riccati equation (5.23) can be transformed by setting $z = u^2$ into a special case of the Chazy equation XII, given by

$$z''' + 10zz'' + 9z'^2 + 36z^2z' + 20z^4 = 0. \tag{5.24}$$

Recently Ablowitz *et al* [87, 88] studied a general class of Chazy equation, defined as

$$u''' - 2uu'' + 3u'^2 = \alpha(6u' - u^2)^2, \tag{5.25}$$

where α is a real number. The particular case $\alpha = 1/16$ was mentioned in [76] and that of $\alpha = \dfrac{4}{36-n^2}$, i.e.

$$u''' - 2uu'' + 3u'^2 = \frac{4}{36-n^2}(6u' - u^2)^2. \tag{5.26}$$

has a singled value general solution. More explicitly, this equation was first written down and solved by Chazy [89, 90] and is known today as the generalized Chazy equation. Clarkson and Olver showed that a necessary condition for the equation (52) to possess the Painlevé property is that the coefficient must be $\alpha = \frac{4}{36-n^2}$ with $1 < n \in \mathbb{N}$, provided that $n \neq 6$. It has been further shown in [91] that the cases $n = 2, 3, 4$ and 5, correspond to the dihedral triangle, tetrahedral, octahedral and icosahedral symmetry classes.

Proposition 8 *If $\bar{u}$ is a solution of the third Riccati equation*

$$u''' + 3u'^2 + 4uu'' + 6u^2u' + u^4 = 0, \tag{5.27}$$

then, using $\bar{x} = -x$ as independent variable, the function $u = 2\,u_1$ satisfies

$$u''' - 2uu'' + 3u'^2 = \frac{1}{8}(6u' - u^2)^2. \tag{5.28}$$

Proof.- In fact, if $\bar{u}$ is a solution (5.27) then $u = 2\,\bar{u}$ is solution of

$$u''' + \frac{3}{2}v'^2 + 2u\,u' + \frac{3}{2}u^2\,u' + \frac{1}{8}u^4 = 0,$$

and with the mentioned change of independent variable,

$$u''' - \frac{3}{2}v'^2 - 2u\,u' + \frac{3}{2}u^2\,u' - \frac{1}{8}u^4 = 0,$$

which can be rewritten as in (5.28).

□

Remark: The Chazy IV equation

$$u''' = -3uu'' - 3u'^2 - 3u^2u' \tag{5.29}$$

is a derivative of the second-order Riccati equation or second member of the Riccati chain given in (2.10) with $k = 1$.

We are able to construct fourth-order equations of the Painlevé class family, derived by Bureau [74]. The Painlevé classification of the class of differential equations of the 4-th-order first and second degree was studied by Cosgrove [75, 76]. The subcase which will be relevant here is the Bureau symbol P1.

Cosgrove presented the results of the Painlevé classification for fourth-order differential equations where the Bureau symbol is P1. He gave a long list of the equations F-VII – F-XVIII in this category. Six equations, denoted by F-I, F-II, ..., F-VI, have Bureau symbol P2.

It is worth to note that all the cases with symbols P3 and P4 were found to violate a standard Painlevé test, they admit non-integer resonances.

We derive the following different equations from the list of Cosgrove on the Painlevé classification of fourth-order equations with Bureau symbol P1.

Proposition 9 *The following two equations follow from the higher order Riccati equations*

$$\begin{array}{llcl} \textit{F-XII} & u^{(iv)} & = & -4uu''' - 6u^2u'' - 4u^3u' - 12uu'^2 - 10u'u'' \\ \textit{F-XVI} & u^{(iv)} & = & -5uu''' - 10u'uv'' - 15uu'^2 - 10u^2u'' - 10u^3u' - u^5 \\ & & + & A(x)(u''' + 4uu'' + 3u'^2 + 6u^2u' + u^4) + B(x)(u'' + 3uu' + u^3(x)) \\ & & + & C(x)(u^2 + u') + D(x)u + E(x) = 0. \end{array}$$

Proof.- A) The F-XII fourth-order equations with Bureau symbol P1 follows directly from the expressions (2.12) for the fourth-order Riccati $R^4(u) = 0$ and the third-order Riccati $R^3(u) = 0$ equations (also known as Burgers higher-order flows) by means of the relation

$$\text{F-XII } : R^4(u) - u\, R^3(u) = 0.$$

B) The F-XVI fourth-order equations with Bureau symbol P1 is the combination of all higher-order Riccati equations.

□

5.3 Second degree Painlevé II equation

We have seen in the earlier section how Painlevé II is connected to second-order Riccati or projective vector field equation. In this section we carry out this investigation further to incorporate the second degree Painlevé II equation [92, 93, 94].

The Hamiltonian of the standard Painlevé II equation $u'' = 2u^3 + x\,u + \alpha$ is given by

$$H(x, u, w) = \frac{w^2}{2} - \left(u^2 + \frac{x}{2}\right) w - \left(\alpha + \frac{1}{2}\right) u, \tag{5.30}$$

where u and w play the rôle of coordinate and momentum, i.e. the Poisson bivector field is given by $\dfrac{\partial}{\partial u} \wedge \dfrac{\partial}{\partial w}$. The Hamiltonian equations of motion yield a set of Riccati equations

$$\left\{ \begin{array}{lcl} u' & = & \dfrac{\partial H}{\partial w} = w - u^2 - \dfrac{x}{2}, \\ w' & = & -\dfrac{\partial H}{\partial u} = 2u\,w + \alpha + \dfrac{1}{2} \end{array} \right. . \tag{5.31}$$

This system was studied by Morales [95] who proved that for $\alpha \in \mathbb{Z}$ the system is not integrable by means of rational first integrals.

In fact, taking derivative with respect to x at the first equation, using for w the value obtained form it, i.e. $w = u' + u^2 + \dfrac{x}{2}$, and replacing w' by the value given by the second equation we obtain

$$u'' = 2u\left(u' + u^2 + \frac{x}{2}\right) + \alpha + \frac{1}{2} - 2u\,u' - \frac{1}{2} = 2u^3 + u\,x + \alpha,$$

and therefore the function u satisfies the Painlevé II type differential equation (5.13).

Now, if $u(x)$ and $w(x)$ are solutions of the system (5.31), the function $h(x)$ defined by $h(x) = H(x, u(x), w(x))$ is such that

$$h'(x) = \frac{\partial H}{\partial x}(x, u(x), w(x)) = -\frac{w(x)}{2}, \qquad h''(x) = -\frac{w'(x)}{2} = -\frac{1}{2}\left(2u(x)\,w(x) + \alpha + \frac{1}{2}\right),$$

and then one easily check that

$$(h''(x))^2 + 4\,(h'(x))^3 + 2\,h'(x)\,(x\,h'(x) - h(x)) - \frac{1}{4}\left(\alpha + \frac{1}{2}\right)^2 = 0. \tag{5.32}$$

Conversely, if a function $h(x)$ satisfies the relation (5.32), then the functions

$$u(x) = \frac{1}{2}\frac{h''(x)}{h'(x} + \left(\alpha + \frac{1}{2}\right)\frac{1}{4h'(x)}, \qquad w(x) = -2\,h'(x)$$

are solutions of the system (5.31).

Observe that the right hand side of the expression

$$-2\,h' = w = u' + u^2 + \frac{x}{2} \tag{5.33}$$

is the term appearing from the reduction of the linear second-order differential Hill equation $y'' + \frac{x}{2}y = 0$ when considering Lie recipe of invariance under dilations, i.e. $u = y'/y$. Therefore, the following Hill's equation

$$y'' + \left(2\,h'(x) + \frac{x}{2}\right)y = 0, \tag{5.34}$$

leads by Lie recipe reduction from dilation symmetry to the preceding equation.

Note that the solution u can be found directly from the function h, the expression using the second expression in (5.31):

$$u = \frac{2h'' + \alpha + \dfrac{1}{2}}{4h'}. \tag{5.35}$$

Thus we can extract several important information about the Painlevé II equation from the second-order projective Riccati and the projective vector field equation.

6 Outlook

It is well known that several celebrated integrable PDEs, like KdV equation, Camassa-Holm equation and many other integrable systems are connected to the coadjoint orbit of the Virasoro. In particular, their configuration space is the Virasoro group and these two integrable systems can be regarded as equations of the geodesic flow associated to different right-invariant metrics on this group. All these well known integrable PDEs have hierarchical structures and these are also associated to the Virasoto orbit.

Kirillov interpreted the stabilizer of a point in the coadjoint orbit of the Virasoro algebra. in terms of the second Hamiltonian operator of the KdV equation. Geometrically this turns out to be the projective vector field equation. Earlier we have seen that this stabilizer set harvest many well-known finite-dimensional integrable systems, like C. Neumann system, Ermakov-Pinney equation, Kummer-Schwarz equation etc. etc.

There are many papers addressing the connection between infinite-dimensional integrable systems and Virasoro orbit, but very few of them have discussed the finite-dimensional integrable systems. In this paper we discussed this subject quite elaborately. We showed that the higher-order Riccati plays a key role in this development.

The standard Riccati equation and its higher-order generalisations play a very important rôle in mathematical physics and dynamical systems. The second and higher-order Riccati equations also play a relevant rôle in integrable ODEs, Painlevé and Chazy equations. It has been shown [12] that the use of geometrical techniques to deal with the elements of the Riccati equation is very efficient to unveil some previously hidden aspects of such equations. The novelty of this article is to study higher-order Riccati equations and various other connected integrable ODEs using coadjoint orbit method of Virasoro algebra. In particular, we have given the geometric description of the higher-order Riccati equations using the stabilizer set of the Virasoro orbit or projective vector field equation. We have also explored the geometric connections between the higher-order Riccati equations and Painlevé type equations. It would be interesting to extend our study to coupled Riccati equations, how they are connected to the stabilizer set of the extended Virasoro orbit or superconformal orbit. All the coupled KdV equations and two-component Camassa-Holm equation's configuration space is the extended Virasoro group and all these systems can be regarded as equations of the geodesic flow associated to different right-invariant metrics. One can check that the stabilier set of the coadjoint orbit of the coextended Virasoro algebra is a coupled ordinary differential equation. We wish to investigate the coupled integrable ODEs associated to this stabilizer orbit.

The application of these differential geometric methods to deal with multicomponent systems and their integrability is a very interesting subject to be studied. Hence, several interesting issues connected to this paper ought to be addressed in future, we have only hit the tip of the iceberg. We hope to answer some of these questions in forthcoming papers.

Acknowledgements.

It is great pleasure to thank Valentin Ovsienko, Basil Grammaticos, Alfred Ramani, Anindya Ghose Choudhury and Peter Leach for numerous enlightening discussions, valuable suggestions and kind encouragement. PG thanks the Departamento de Física Teórica de la Universidad de Zaragoza for its hospitality and acknowledges support from IHES, Bures sur Yvette, France and CIB, EPFL, Lausanne, Switzerland. PG would also like to thank his previous institute, S. N. Bose National Centre, where part of the work has been carried out.

Declarations

JFC and MFR acknowledge support from research projects PGC2018-098265-B-C31 (MINECO, Madrid) and DGA-E48/20R (DGA, Zaragoza). Work by the author PG was supported by the Khalifa University of Science and Technology under grant number FSU-2021-014.

Conflicts of interest

The authors declare that they have no conflict of interest.

Data transparency

Not applicable.

References

[1] Cariñena J.F., de Lucas J., "Lie systems: theory, generalisations, and applications", *Dissertationes Mathematicae* **479**, 162 pp., Institute of Mathematics, Polish Academy of Sciences, Warszawa, 2011

[2] Cariñena J.F., Ramos A., "Integrability of Riccati equation from a group theoretical viewpoint", *Int. J. Mod. Phys. A* **14**, 1935–1951 (1999).

[3] Davis H.T., *Introduction to Nonlinear Differential and Integral Equations* (Dover, New York, 1962).

[4] Bouquet S.E., Feix M.R., Leach P.G.L., "Properties of second-order ordinary differential equations invariant under time-translation and self-similar transformation", *J. Math. Phys.* **32**, 1480–1490 (1991).

[5] Euler M., Euler N., Leach P.G.L., "The Riccati and Ermakov–Pinney hierarchies", *J. Nonlinear Math. Phys.* **14**, 290–310 (2007).

[6] Guha P., "Diffeomorphism, Periodic KdV and C. Neumann system", *Diff. Geom. Appl.* **12**, 1–8 (2000).

[7] Guha P., "Stabilizer orbit of Virasoro action and integrable systems", *Int. J. Geom. Methods Mod. Phys.* **2**, 1–12 (2005).

[8] Guha P., "Riccati Chain, higher-order Painlevé Type Equations and Stabilizer Set of Virasoro Orbit", *MPI-MIS preprint 143/2006.*

[9] Leach P.G.L., "First-integrals for the modified Emden equation $\ddot{q}+\alpha(t)\dot{q}+q^n=0$", *J. Math. Phys.* **26**, 2510–2514 (1985).

[10] Zhang X., *Integrability of Dynamical Systems: Algebra and Analysis*, Developments in Mathematics **47**, Springer Nature (Singapore, 2017).

[11] Cariñena J.F., Rañada M.F., Santander M., "Lagrangian formalism for nonlinear second-order Riccati systems: one-dimensional integrability and two-dimensional superintegrability", *J. Math. Phys.* **46**, 062703 (2005).

[12] Cariñena J.F., Guha P., Rañada M.F., "A geometric approach to higher-order Riccati chain: Darboux polynomials and constants of the motion", *Journal of Physics: Conference Series* **175**, 012009 (2009).

[13] Harnad J., P. Winternitz P., Anderson R.L., "Superposition principles for matrix Riccati equations", *J. Math. Phys.* **24**, 1062–1072 (1983).

[14] del Olmo M.A., Rodríguez M., Winternitz P., "Integrability, chaos and nonlinear superposition formulas for differential matrix Riccati equations". In: *Quantum Chaos and Statistical Nuclear Physics*, edited by T.H. Seligman and H. Nishioka, Lecture Notes in Physics **263**, 372–378 (Springer, 1986).

[15] Grundland A.M., Levi D., "On higher-order Riccati equations as Bäcklund transformations", *J. Phys. A: Math. Gen.* **32**, 3931–3937 (1999).

[16] Grundland A.M., de Lucas J., "A Lie systems approach to the Riccati hierarchy and partial differential equations", *J. Differential Equations* **263**, 299–337 (2017)

[17] Weiss J., "The Painlevé property for partial differential equations. II: Bäcklund transformation, Lax pairs, and the Schwarzian derivative", *J. Math. Phys.* **24**, 1405–1403 (1983).

[18] Cariñena J.F., Falceto, J. Grabowski, M.F. Rañada, "Geometry of Lie integrability by quadratures", *J. Phys. A:Math. Theor.* **48**, 215206 (2015).

[19] Cariñena J.F., Falceto F., Grabowski J., "Solvability of a Lie algebra of vector fields implies their integrability by quadratures", *J. Phys. A:Math. Theor.* **49**, 425202 (2016).

[20] Cariñena J.F., "A new approach to Ermakov systems and applications in quantum physics", *Eur. Phys. J. Special Topics* **160**, 51–60 (2008).

[21] Cariñena J.F., de Lucas J., "A nonlinear superposition rule for solutions of Milne–Pinney equation", *Phys. Lett.* A **372**, 5385–5389 (2008).

[22] Cariñena J.F., de Lucas J., Rañada M.F., "Recent applications of the theory of Lie systems in Ermakov systems", *SIGMA* **4**, 031 (2008).

[23] Ermakov V.P., " Second order differential equations. Conditions of complete integrability, *Univ. Izv. Kiev Ser.* III9, 1-25 (1880).

[24] Milne W.E., "The numerical determination of characteristic numbers", *Phys. Rev.* **35**, 863–867 (1930).

[25] Pinney E., "The nonlinear differential equation $y''(x) + p(x)y + cy^{-3} = 0$", *Proc. Amer. Math. Soc.* **1**, 681 (1950).

[26] Hinkkanen A., Laine I., "Solutions of the first and second Painlevé equations are meromorphic", *J. Anal. Math.* **79**, 345–377 (1999).

[27] Steinmetz N., "On Painlevé's equations I, II and IV", *J. Anal. Math.* **82**, 363–377 (2000).

[28] Gromak V.I., "Bäcklund transformations of the higher-order Painlevé equations". In: *Bäcklund and Darboux transformations. The geometry of solitons*, CRM Proc. Lecture Notes **29**, 3–28, Amer. Math. Soc., Providence, RI, 2001.

[29] Gromak V.I., Laine I., Shimomura S., *Painlevé Differential Equations in the Complex Plane*, Walter de Gruyter, Berlin, 2002.

[30] Steinmetz N., "Airy Solutions of Painleve's Second Equation", *Computational Methods and Function Theory* **3**, 117-126 (2004)

[31] Cosgrove C.M., Scoufis G., "Painlevé classification of a class of differential equations of the second-order and second degree", *Stud. Appl. Math.* **88**, 25–87 (1993).

[32] Jimbo M., Miwa T., "Monodromy preserving deformations of ordinary differential equations with rational coefficients II", *Physica D* **2**, 407–448 (1981).

[33] Forrester P.J., Witte N.D., "Application of the τ-function theory of Painlevé equations to random matrices: PIV, PII and the GUE", *Comm. Math. Phys.* **219**, 357–398 (2001).

[34] Guha P., "Integrable geodesic flows on the (super) extension of the Bott–Virasoro group", *Lett. Math. Phys.* **52**, 311–328 (2000).

[35] Guha P., "Geodesic flows, biHamiltonian structure and complex KdV type systemsStabilizer orbit of Virasoro action and integrable systems", *Int. J. Math. Anal. Appl.* **310**, 45–56 (2005).

[36] Ovsienko V., Khesin B., "KdV super equation as an Euler equation", *Funct. Anal. Appl.* **21**, 329–331 (1987).

[37] Kirillov A., "Infinite dimensional Lie groups; their orbits, invariants and representations. The Geometry of Moments". In: *Twistor Geometry and Non-linear Systems*, edited by A. Dold and B. Eckmann, Lecture Notes in Mathematics **970** (Springer 1980).

[38] Khesin B., Wendt R., *The Geometry of Infinite-Dimensional Groups* (Springer 2008).

[39] Ovsienko V., Tabachnikov S., *Projective Differential Geometry Old and New: From the Schwarzian Derivative to the Cohomology of Diffeomorphism Groups*, Cambridge Tracts in Mathematics **165** (Cambridge University Press, Cambridge, UK, 2005).

[40] Guha P., Olver P.J., "Geodesic flow and two (super) component analog of the Camassa-Holm equation", *SIGMA* **2**, 054 (9 pages) (2006).

[41] Guha P., "Diffeomorphisms on S^1, Projective structures and integrable systems", *ANZIAM J.* **44**, 169–180 (2002).

[42] Guha P., "Projective and affine connections on S^1 and integrable systems", *J. Geom. Phys.* **46**, 231–242 (2003).

[43] Vallée O., Soares M., *Airy functions and applications to Physics* (Imperial College Press, 2004).

[44] Magnus W., Winkler, S., *Hill's equation* (Dover, 2004).

[45] Reid J.L., "An exact solution of the nonlinear differential equation $\ddot{y} + p(t)y = q_m(t)/y^{2m-1}$", *Proc. Am. Math. Soc.* **27**, 61–62 (1971).

[46] Thomas J.B., "Equations equivalent to a linear differential equations", *Proc. Amer. Math. Soc.* **3**, 899–903 (1952).

[47] Ince E.L., *Ordinary Differential Equations* (Dover Publications, 1956).

[48] de Lucas J., Sardón C., "On Lie systems and Kummer-Schwarz equations" *J. Math. Phys.* **54**, 033505 (2013).

[49] Cariñena J.F., Grabowski, J., de Lucas J., Sardón C., "Dirac–Lie systems and Schwarzian equations", J. Diff. Eqns. **257**, 2303–2340 (2014).

[50] Ablowitz M.J., Fokas A.S., *Complex Variables: Introduction and Applications*, Cambridge Texts in Applied Mathematics, (Cambridge University Press; 2nd edition 2003).

[51] Hille E., *Ordinary differential equations in the complex domain* (Dover Pub, New York, 1976).

[52] Cariñena J.F., Marmo G., Nasarre J., "The non-linear superposition principle and the Wei–Norman method", *Int. J. Mod. Phys. A* **13**, 3601–3627 (1998).

[53] Lie S., *Vorlesungen ber continuierliche Gruppen mit Geometrischen und anderen Anwendungen*, Edited and revised by G. Scheffers (Teubner, Leipzig, 1893).

[54] Leach P.G.L., Feix M.R., S. Bouquet, "Analysis and solution of a nonlinear second-order differential equation through rescaling and through a dynamical point of view", *J. Math. Phys.* **29**, 2563–2569 (1988).

[55] Bruzon, M.S. Gandarias M.L., Senthilvelan M., "Nonlocal symmetries of Riccati and Abel chains and their similarity reductions", *J. Math. Phys.* **53**, 023512 (2012).

[56] Hitchin N., "Vector fields on the circle. In: *Mechanics, Analysis and Geometry: 200 years after Lagrange*, 359–378, Edited by M. Francaviglia (Elsevier Science Pub., 1991).

[57] Ovsienko R., Redou P., "Generalized Transvectants-Rankin–Cohen Brackets", *Lett. M th. Phys.* **63**, 19–28 (2003).

[58] Pevzner M., "Rankin–Cohen Brackets and Associativity", *Lett. Math. Phys.* **85**, 195–202 (2008).

[59] Sil S., Guha P., "Symmetry reductions and exact solutions of two new generalized negative KdV type equations", *J. Geom. Phys.* **178**, 104558 (2022)

[60] Hill G.W., "On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and moon", *Acta Math.* **8**, 1–36 (1886).

[61] Matamala A.R., Salas C.A., Alid M.J., "On continuous symmetries of second-order homogeneous linear ordinary differential equations", *J. Physics: Conf. Series* **128**, 012057 (2008).

[62] Guieu, L., Roger, C., *L'algébre et le groupe de Virasoro*, Publications du CRM, (Université de Montréal, 2007).

[63] Bott, R., "On the characteristic classes of groups of diffeomorphisms", *Enseign. Math.* **XXIII**, fasc. 3-4. 209-220 (1977) .

[64] Lempert, L., "The Virasoro group as a complex manifold", *Mathematical Research Letters* **2**, 479–495 (1995)

[65] Babelon O., Bernard D., and Talon M., *Introduction to Classical Integrable Systems*, Cambridge University Press, 2003.

[66] Singer M., "Solving Homogeneous Linear Differential Equations in Terms of Second Order Linear Differential Equations", *Am. J. Math.* **107**, 663–696 (1985).

[67] Common A.K., Hone A.N.W., Musette M., "A new discrete Hénon–Heiles system", *J. Nonl. Math. Phys.* **10**, 27–40 (2003).

[68] Cariñena J.F., Guha P., de Lucas J., "A quasi-Lie schemes approach to second-order Gambier equations", *SIGMA* **9**, 026 (2013).

[69] Reid J.L., "Homogeneous solutions of a nonlinear differential equation", *Proc. Am. Math. Soc.* **38**, 532–536 (1973).

[70] Olver P., *Equivalence, Invariants, and Symmetry* (Cambridge University Press, Cambridge, UK, 1995).

[71] Euler M., Euler N., Leach P.G.L. , "Properties of the Calogero-Degasperis-Ibragimov-Shabat Differential Sequence", *Lobachevskii Journal of Mathematics* **32**, 61–70 (2011).

[72] Gordoa P. R., "Bäcklund transformations for the second member of the first Painlevé hierarchy", *Phys. Lett. A.* **287**, 365–370 (2001).

[73] Grammaticos B., Ramani A., Guha P., "Second-degree Painlevé equations and their contiguity relations", *Regular and Chaotic Dynamics* **19**, 37–47 (2014).

[74] Bureau F., "Differential equations with fixed critical points", *Annali di Matematica* **64**, 229–364 (1964).

[75] Cosgrove C., "Higher-order Painlevé equations in the polynomial class. I. Bureau symbol P2", *Stud. Appl. Math.* **104**, 1–65 (2000).

[76] Cosgrove C., "Higher-order Painlevé equations in the polynomial class. II. Bureau symbol $P1$", *Stud. Appl. Math.* **116**, 321–413 (2006).

[77] Sakka A., Mugan U., "Second-order second degree Painlevé equations related with Painlevé I, II, III equations", *J. Phys. A.* **30**, 5159–5177 (1997).

[78] Sakka A., Mugan U., "Second-order second-degree Painlevé equations related to Painlevé IV, V, VI equations", *J. Phys. A.* **30**, 2471–2490 (1998).

[79] Gambier B., "Sur les équations différentielles du second ordre et du premier degré dont l'integrale générale est á points critiques fixés", *Acta. Math.* **33**, 1–55 (1910).

[80] Pressley A., Segal G, *Loop groups*, Clarendon Press, (Oxford, 1986).

[81] Reiman A.G., Semenov-Tian-Shanskii M.A., "Current algebras and partial differential equations", *Soviet Math. Dokl.* **21**, 630–634 (1980).

[82] Clarkson P., "Painlevé equations–nonlinear special functions". In: *Orthogonal polynomials and special functions*, *Lecture Notes in Math.* **1883**, 331–411 (Springer, Berlin, 2006).

[83] Grammaticos B., Ramani A., "Integrability (and how to detect it)". In: *Integrability of Nonlinear Systems*, Edited by Y. Kosmann-Schwarzbach, B. Grammaticos, K.M. Tamizhmani, Lect. Notes in Phys. **638**, 31–94 (Springer-Verlag, Berlin-Heidelberg 2004).

[84] Joshi, N., Kajiwara, K., Mazzocco M., "Generating Function Associated with the Determinant Formula for the Solutions of the Painlevé II Equation", *Astérisque* **297**, 67–78 (2004).

[85] Rogers C., "Hybrid Ermakov-Painlevé IV Systems", *J. Nonl. Math. Phys* **21**, 628–642 (2014).

[86] Joshi N., "The second Painlevé hierarchy and the stationary KdV hierarchy", *Publ. RIMS, Kyoto Univ.* **40**, 1039–1061 (2004).

[87] Ablowitz M., Chakravarty S., Halburd R., "The generalized Chazy equation and Schwarzian triangle functions. Mikio Sato: a great Japanese mathematician of the twentieth century", *Asian J. Math.* **2**, 619–624 (1998).

[88] Ablowitz M., Chakravarty S., Halburd R., "The generalized Chazy equation from the self-duality equations", *Stud. Appl. Math.* **103**, 75–88 (1999).

[89] Chazy J., "Sur les équations différentielles dont l'intégrale générale est uniforme et admet des singularities essentielles mobiles", *C. R. Acad. Sci. Paris* **149**, 563–565 (1909).

[90] Chazy J., "Sur les équations différentielles dont l'intégrale générale posséde une coupure essentielle mobile", *C. R. Acad. Sci., Paris* **150**, 456–458 (1910).

[91] Clarkson P., Olver P., "Symmetry and the Chazy equation", *J. Differential Equations* **124**, 225–246 (1996).

[92] Okamoto K., "Polynomial Hamiltonians associated with Painlevé equations, I", *Proc. Japan Acad. Ser. A Math. Sci.* **56**, 264–268 (1980).

[93] Okamoto K., "Polynomial Hamiltonians associated with Painlevé equations, II", *Proc. Japan Acad. Ser. A Math. Sci.* **56**, 367–371 (1980).

[94] Okamoto K., "The Hamiltonians associated to the Painlevé equations". In: *The Painlevé property*, 735–787, R. Conte Ed., *CRM Ser. Math. Phys.,* (Springer, New York, 1999).

[95] Morales J.J., "A remark about the Painlevé transcendents. Théories asymptotiques et équations de Painlevé", *Sémin. Congr. Soc. Math. France* **14**, 229–235 (2006).