A unified approach to higher-order convolutions within a certain subset of Appell polynomials

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Abstract. We consider the subset $\mathcal{R}$ of Appell polynomials whose exponential generating function is given in terms of the moment generating function of a certain random variable Y. This subset contains the Hermite, Bernoulli, Apostol-Euler, and Cauchy type polynomials, as well as various kinds of their generalizations, among others. We obtain closed form expressions for higher-order convolutions of Appell polynomials in the subset $\mathcal{R}$. We give a unified approach mainly based on random scale transformations of Appell polynomials, as well as on a probabilistic generalization of the Stirling numbers of the second kind. Different illustrative examples, including reformulations of convolution identities already known in the literature, are discussed in detail. In such examples, the convolution identities involve the classical Stirling numbers.

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1. Introduction

In recent years, a lot of attention has been devoted to obtaining explicit formulas for higher-order convolutions of the form

$$\sum_{j_1+\cdots+j_m=n} \binom{n}{j_1,\ldots,j_m} C(j_1,\ldots,j_m) A_{j_1}^{(1)}(x_1)\cdots A_{j_m}^{(m)}(x_m), \tag{1.1}$$

where $\boldsymbol{A}^{(k)}(x) = (A_n^{(k)}(x))_{n\geq 0}$ is a sequence of Appell polynomials, $x_k \in \mathbb{R}$, $k = 1,\ldots,m$, and $C(j_1,\ldots,j_m)$ are properly chosen constants. One instance

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of these formulas is the expression of the second-order Bernoulli polynomials $\boldsymbol{B}(x) = (B_n(x))_{n\geq 0}$ shown by Nörlund [1], i.e.,

$$\sum_{k=0}^{n} \binom{n}{k} B_k(x)B_{n-k}(y) = -n(x+y-1)B_{n-1}(x+y) - (n-1)B_n(x+y). \quad (1.2)$$

Since the pioneering work by Dilcher [2], many authors have provided explicit expressions for the sums in (1.1) using different methodologies. We mention the papers by Gessel [3], Zhao [4],Wang [5], Agoh and Dilcher [6], He and Araci [7], Wu and Pan [8], He [9], and Dilcher and Vignat [10], among many others.

The aim of this paper is to give a unified approach to obtain closed form expressions for higher-order convolutions of Appell polynomials in the set $\mathcal{R}$ defined below. This approach can be summarized as follows (see Section 2 for more precise definitions). Following Ta [11], we consider the set $\mathcal{R}$ of Appell polynomials $\boldsymbol{A}(x) = (A_n(x))_{n\geq 0}$ whose exponential generating function is given by

$$G(\boldsymbol{A}(x), z) = \frac{e^{xz}}{\mathbb{E}e^{zY}},$$

for a certain random variable Y, where $\mathbb{E}$ stands for mathematical expectation. For any $w \in \mathbb{R}$ and $\boldsymbol{A}(x) \in \mathcal{R}$, we consider the scale transformation $T_w\boldsymbol{A}(x) = (T_wA_n(x))_{n\geq 0}$ defined by

$$T_wA_n(x) = w^nA_n(x/w) = \sum_{k=0}^{n} \binom{n}{k} w^kA_k(0)x^{n-k}, \quad n = 0, 1, \dots.$$

In the first place, we give closed form expressions for

$$\begin{aligned} &\sum_{j_1+\cdots+j_m=n} \binom{n}{j_1, \dots, j_m} T_{w_1}A^{(1)}_{j_1}(w_1x_1)\cdots T_{w_m}A^{(m)}_{j_m}(w_mx_m) \\ &= \sum_{j_1+\cdots+j_m=n} \binom{n}{j_1, \dots, j_m} w_1^{j_1}\cdots w_m^{j_m} A^{(1)}_{j_1}(x_1)\cdots A^{(m)}_{j_m}(x_m) \end{aligned} \quad (1.3)$$

in terms of the moments of the random variables Y_k associated to each $\boldsymbol{A}^{(k)}(x) \in \mathcal{R}$, $k = 1, \dots, m$ (see Theorem 3.4 in Section 3). In the second place, we replace $(w_1, \dots, w_m)$ by a random vector $\mathbb{W} = (W_1, \dots, W_m)$ in (1.3) and then take expectations, so that we obtain closed form expressions for

$$\sum_{j_1+\cdots+j_m=n} \binom{n}{j_1, \dots, j_m} \mathbb{E}\left(W_1^{j_1}\cdots W_m^{j_m}\right) A^{(1)}_{j_1}(x_1)\cdots A^{(m)}_{j_m}(x_m). \quad (1.4)$$

This is done in Theorem 3.5 in Section 3, which is the main result of this paper. The comparison between (1.1) and (1.4) reveals the probabilistic meaning of the constant $C(j_1, \dots, j_m)$, that is,

$$C(j_1, \dots, j_m) = \mathbb{E}\left(W_1^{j_1}\cdots W_m^{j_m}\right).$$

The approach outlined above is general enough for two reasons. First, this is so because the Hermite, Bernoulli, Apostol-Euler, and Cauchy type polynomials, as well as various kinds of their generalizations belong to the set $\mathcal{R}$ (c.f. [11] and [12]). Of course, there are Appell sequences that do not belong to $\mathcal{R}$ (see the comments below following formula (3.1) in Section 3). Secondly, this is a general approach because each choice of the random vector $\mathbb{W}$ in (1.4) leads us to a different type of convolution identity. This is illustrated in Section 4 in the particular case of the Bernoulli polynomials. Actually, it will be shown there that if $\boldsymbol{W}$ is a deterministic vector, we obtain generalizations of convolution identities already proved by Dilcher [2], Wang [5], and Chu and Zhou [13]. If $\boldsymbol{W}$ has the multivariate Dirichlet distribution, we obtain a generalization of Miki's identity (see Miki [14], Gessel [3], and Dilcher and Vignat [10]). If $\boldsymbol{W}$ has independent and identically distributed components, each one having the exponential density, then we obtain a generalization of an identity proposed by Matiyasevich (see Agoh [15] and Agoh and Dilcher [6]). Finally, if $\boldsymbol{W}$ has the multivariate normal density, we obtain a new identity involving the Hermite polynomials.

The main tool to give explicit expressions for the sums in (1.3) is a probabilistic generalization of the Stirling numbers of the second kind recently introduced in [16] (see also Theorems 3.1 and 3.3 in Section 3). Sections 4 and 5 are devoted to illustrate Theorem 3.5. To keep the paper to a moderate size, we restrict our attention to the case in which every $\boldsymbol{A}^{(j)}(x)$ in (1.4) is a Bernoulli or a certain type of Cauchy polynomials, whereas other polynomials in $\mathcal{R}$, such as the generalized Bernoulli polynomials of order m and the generalized Apostol-Euler polynomials of order m (cf. [12]) are not considered here. As a counterpart, we consider different choices of the random vector $\mathbb{W}$ and make a comparison with similar results already known in the literature. It turns out that the identities obtained in Sections 4 and 5 are computable in terms of the classical Stirling numbers.

2. Preliminaries

In this section, we collect some definitions and properties, already shown in [12] and [16], which are necessary to state our main results.

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Unless otherwise specified, we assume from now on that $n \in \mathbb{N}_0$, $m \in \mathbb{N}$, $x \in \mathbb{R}$, and $z \in \mathbb{C}$ with $|z| \le r$, where $r > 0$ may change from line to line. Denote by $\mathcal{G}$ the set of all real sequences $\boldsymbol{u} = (u_n)_{n \ge 0}$ such that $u_0 \neq 0$ and

$$\sum_{n=0}^{\infty} |u_n| \frac{r^n}{n!} < \infty,$$

for some radius $r > 0$. If $\boldsymbol{u} \in \mathcal{G}$, we denote its generating function by

$$G(\boldsymbol{u}, z) = \sum_{n=0}^{\infty} u_n \frac{z^n}{n!}.$$

If $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{G}$, the binomial convolution of $\boldsymbol{u}$ and $\boldsymbol{v}$, denoted by $\boldsymbol{u} \times \boldsymbol{v} = ((u \times v)_n)_{n \geq 0}$, is defined as

$$(u \times v)_n = \sum_{k=0}^{n} \binom{n}{k} u_k v_{n-k}.$$

It turns out that this definition is characterized in terms of generating functions (c.f. [12, Proposition 2.1]) as

$$G(\boldsymbol{u} \times \boldsymbol{v}, z) = G(\boldsymbol{u}, z) G(\boldsymbol{v}, z).$$

In addition (cf. [12, Corollary 2.2]), $(\mathcal{G}, \times)$ is an abelian group with identity element $\boldsymbol{e} = (e_n)_{n \geq 0}$, where $e_0 = 1$ and $e_n = 0$, $n \in \mathbb{N}$. Observe that if $\boldsymbol{u}^{(j)} = (u_n^{(j)})_{n \geq 0} \in \mathcal{G}$, $j = 1, \ldots, m$, then

$$(u^{(1)} \times \cdots \times u^{(m)})_n = \sum_{j_1 + \cdots + j_m = n} \binom{n}{j_1, \ldots, j_m} u_{j_1}^{(1)} \cdots u_{j_m}^{(m)}, \tag{2.1}$$

where

$$\binom{n}{j_1, \ldots, j_m} = \frac{n!}{j_1! \cdots j_m!}, \quad j_1, \ldots, j_m \in \mathbb{N}_0, \quad j_1 + \cdots + j_m = n$$

is the multinomial coefficient.

On the other hand, let $\boldsymbol{A}(x) = (A_n(x))_{n \geq 0}$ be a sequence of polynomials such that $\boldsymbol{A}(0) \in \mathcal{G}$. Recall that $\boldsymbol{A}(x)$ is called an Appell sequence if one of the following equivalent conditions is satisfied:

$$A_n(x) = \sum_{k=0}^{n} \binom{n}{k} A_k(0) x^{n-k}, \tag{2.2}$$

or

$$G(\boldsymbol{A}(x), z) = G(\boldsymbol{A}(0), z) e^{xz}.$$

Denote by $\mathcal{A}$ the set of all Appell sequences. The binomial convolution of $\boldsymbol{A}(x), \boldsymbol{C}(x) \in \mathcal{A}$, denoted by $(\boldsymbol{A} \times \boldsymbol{C})(x) = ((A \times C)_n(x))_{n \geq 0}$, is defined as (cf. [12, Section 3])

$$(\boldsymbol{A} \times \boldsymbol{C})(x) = \boldsymbol{A}(0) \times \boldsymbol{C}(x) = \boldsymbol{A}(x) \times \boldsymbol{C}(0) = \boldsymbol{A}(0) \times \boldsymbol{C}(0) \times \boldsymbol{I}(x),$$

where $\boldsymbol{I}(x) = (x^n)_{n \geq 0}$, or equivalently, by

$$(A \times C)_n(x) = \sum_{k=0}^{n} \binom{n}{k} A_k(0) C_{n-k}(x) = \sum_{k=0}^{n} \binom{n}{k} C_k(0) A_{n-k}(x)$$
$$= \sum_{j_1 + j_2 + j_3 = n} \binom{n}{j_1, j_2, j_3} A_{j_1}(0) C_{j_2}(0) x^{j_3}.$$

As shown in [12, Theorem 3.1], $(\mathcal{A}, \times)$ is an abelian group with identity element $\boldsymbol{I}(x)$. Also, $(\boldsymbol{A} \times \boldsymbol{C})(x)$ is characterized by its generating function

$$G((\boldsymbol{A} \times \boldsymbol{C})(x), z) = G(\boldsymbol{A}(0), z) G(\boldsymbol{C}(0), z) e^{xz}. \tag{2.3}$$

For any $\boldsymbol{A}^{(j)}(x) \in \mathcal{A}$ and $x_j \in \mathbb{R}$, $j = 1, \ldots, m$, with $x_1 + \cdots + x_m = x$, formula (2.3) implies that

$$\boldsymbol{A}^{(1)}(x_1) \times \cdots \times \boldsymbol{A}^{(m)}(x_m) = (\boldsymbol{A}^{(1)} \times \cdots \times \boldsymbol{A}^{(m)})(x), \tag{2.4}$$

because both sides in (2.4) have the same generating function.

On the other hand, let $w \in \mathbb{R}$ and $\boldsymbol{A}(x) \in \mathcal{A}$. We define the scale transformation $T_w \boldsymbol{A}(x) = (T_w A_n(x))_{n \geq 0}$ as

$$T_w A_n(x) = w^n A_n(x/w) = \sum_{k=0}^{n} \binom{n}{k} w^k A_k(0) x^{n-k}, \quad w \neq 0, \tag{2.5}$$

and

$$T_0 A_n(x) = A_0(0) x^n. \tag{2.6}$$

As shown in [12, Proposition 4.1], $T_w \boldsymbol{A}(x)$ is an Appell sequence characterized by its generating function

$$G(T_w \boldsymbol{A}(x), z) = G(\boldsymbol{A}(0), wz) e^{xz}. \tag{2.7}$$

In addition, the map $T_w : \mathcal{A} \to \mathcal{A}$ is an isomorphism, whenever $w \neq 0$.

From now on, we will always consider random variables Y satisfying the integrability condition

$$\mathbb{E} e^{r|Y|} < \infty, \tag{2.8}$$

for some $r > 0$. Also, let $(Y_j)_{j \geq 1}$ be a sequence of independent copies of Y and denote by

$$S_k = Y_1 + \cdots + Y_k, \quad k \in \mathbb{N} \quad (S_0 = 0). \tag{2.9}$$

In [16], we have introduced the Stirling polynomials of the second kind associated to Y as

$$S_Y(n, r; x) = \frac{1}{r!} \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} \mathbb{E}(x + S_k)^n, \quad r = 0, 1, \ldots, n, \tag{2.10}$$

as well as the Stirling numbers of the second kind associated to Y as

$$S_Y(n, r) = S_Y(n, r; 0), \quad r = 0, 1, \ldots, n. \tag{2.11}$$

Equivalently (c.f. [16, Theorem 3.3]), the polynomials $S_Y(n, r; x)$ are defined via their generating function as

$$\frac{e^{zx}}{r!} (\mathbb{E} e^{zY} - 1)^r = \sum_{n=r}^{\infty} \frac{S_Y(n, r; x)}{n!} z^n, \quad r \in \mathbb{N}_0. \tag{2.12}$$

Note that if $Y = 1$, we have from (2.11) or (2.12),

$$S_1(n, r) = S(n, r), \quad r = 0, 1, \ldots, n,$$

$S(n, r)$ being the classical Stirling numbers of the second kind. Explicit expressions for $S_Y(n, r; x)$ for various choices of the random variable Y can be found in [16, Section 4].

3. Main results

We consider the subset $\mathcal{R} \subseteq \mathcal{A}$ of Appell sequences $\boldsymbol{A}(x)$ whose generating function is given by

$$G(\boldsymbol{A}(x), z) = \frac{e^{xz}}{\mathbb{E}e^{zY}}, \tag{3.1}$$

for a certain random variable Y satisfying (2.8). Note that if X is another random variable satisfying (3.1), then X and Y have the same law (see, for instance, Billingsley [17, p. 346]). For this reason, we say that $\boldsymbol{A}(x)$ has associated random variable Y.

In [12, Section 6], we showed the following properties: if $\boldsymbol{A}(x), \boldsymbol{C}(x) \in \mathcal{R}$, then $(\boldsymbol{A} \times \boldsymbol{C})(x) \in \mathcal{R}$. However, $(\mathcal{R}, \times)$ is not an abelian subgroup of $(\mathcal{A}, \times)$. More precisely, let $\boldsymbol{A}(x) \in \mathcal{R}$ with nonconstant associated random variable Y. Then the inverse of $\boldsymbol{A}(x)$ in the abelian group $(\mathcal{A}, \times)$ does not belong to $\mathcal{R}$.

It turns out that any Appell sequence $\boldsymbol{A}(x)$ in $\mathcal{R}$ with associated random variable Y can be written in terms of the Stirling polynomials of the second kind associated to Y or in terms of the moments of the random variables S_k defined in (2.9), as the following result shows.

Theorem 3.1. *Let $\boldsymbol{A}(x) \in \mathcal{R}$ with associated random variable Y. Then,*

$$\begin{aligned} A_n(x) &= \sum_{r=0}^{n} (-1)^r r! S_Y(n, r; x) = \sum_{k=0}^{n} \binom{n+1}{k+1} (-1)^k \mathbb{E}(x + S_k)^n \\ &= \sum_{r=0}^{n} \binom{n}{r} x^{n-r} \sum_{k=0}^{r} \binom{r+1}{k+1} (-1)^k \mathbb{E}S_k^r. \end{aligned} \tag{3.2}$$

Proof. It follows from assumption (2.8) and the dominated convergence theorem that

$$|\mathbb{E}e^{zY} - 1| < 1, \quad |z| \le s,$$

for some $s > 0$. Whenever $|z| \le s$, we have from (2.12) and (3.1),

$$\begin{aligned} G(\boldsymbol{A}(x), z) &= \frac{e^{xz}}{1 + (\mathbb{E}e^{zY} - 1)} = \sum_{r=0}^{\infty} (-1)^r e^{xz} (\mathbb{E}e^{zY} - 1)^r \\ &= \sum_{r=0}^{\infty} (-1)^r r! \sum_{n=r}^{\infty} \frac{S_Y(n, r; x)}{n!} z^n = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{r=0}^{n} (-1)^r r! S_Y(n, r; x), \end{aligned}$$

thus showing the first equality in (3.2). The second one readily follows from (2.10) and the elementary combinatorial identity

$$\sum_{r=k}^{n} \binom{r}{k} = \binom{n+1}{k+1}.$$

Finally, using (2.2) and the second equality in (3.2), we obtain

$$A_n(x) = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} A_r(0) = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} \sum_{k=0}^{r} \binom{r+1}{k+1} (-1)^k \mathbb{E}S_k^r.$$

This shows the third equality in (3.2) and completes the proof. □

Remark 3.2. Assume that $\boldsymbol{A}(x) \in \mathcal{R}$. By Theorem 3.1, we have $A_0(x) = 1$. This implies, by virtue of (2.6), that $T_0\boldsymbol{A}(x) = \boldsymbol{I}(x)$.

In the following result, we show that binomial convolutions of scale transformations of Appell sequences in the subset $\mathcal{R}$ also belong to $\mathcal{R}$.

Theorem 3.3. *Let $\boldsymbol{w} = (w_1, \dots, w_m) \in \mathbb{R}^m$ and let $\boldsymbol{A}^{(j)}(x) \in \mathcal{R}$ with associated random variable $Y^{(j)}$, $j = 1, \dots, m$. Suppose that the random vector $\boldsymbol{Y} = (Y^{(1)}, \dots, Y^{(m)})$ has mutually independent components. Then, the Appell sequence $(T_{w_1}\boldsymbol{A}^{(1)} \times \cdots \times T_{w_m}\boldsymbol{A}^{(m)})(x)$ belongs to $\mathcal{R}$ with associated random variable $\boldsymbol{w} \cdot \boldsymbol{Y} = w_1Y^{(1)} + \cdots + w_mY^{(m)}$.*

Proof. By Remark 3.2, we can assume without loss of generality that $w_j \neq 0$, $j = 1, \dots, m$. By assumption,

$$G(\boldsymbol{A}^{(j)}(x), z) = \frac{e^{xz}}{\mathbb{E}e^{zY^{(j)}}}, \quad |z| \leq r_j, \tag{3.3}$$

for some $r_j > 0$, $j = 1, \dots, m$. Denote

$$r = \min\left(\frac{r_1}{|w_1|}, \dots, \frac{r_m}{|w_m|}\right) > 0.$$

For $|z| \leq r$, we see from (2.3), (2.7), and (3.3) that

$$\begin{aligned}
&G((T_{w_1}\boldsymbol{A}^{(1)} \times \cdots \times T_{w_m}\boldsymbol{A}^{(m)})(x), z) \\
&= G(T_{w_1}\boldsymbol{A}^{(1)}(0), z) \cdots G(T_{w_m}\boldsymbol{A}^{(m)}(0), z)e^{xz} \\
&= G(\boldsymbol{A}^{(1)}(0), zw_1) \cdots G(\boldsymbol{A}^{(m)}(0), zw_m)e^{xz} \\
&= \frac{e^{xz}}{\mathbb{E}e^{zw_1Y^{(1)}} \cdots \mathbb{E}e^{zw_mY^{(m)}}} = \frac{e^{xz}}{\mathbb{E}e^{z\boldsymbol{w}\cdot\boldsymbol{Y}}},
\end{aligned}$$

where the last equality holds because the random vector $\boldsymbol{Y}$ has mutually independent components. Hence, the result follows from (3.1). □

In the setting of Theorem 3.3, we use from now on the following notations. Denote

$$w_1x_1 + \cdots + w_mx_m = x, \quad x_1, \dots, x_m \in \mathbb{R}. \tag{3.4}$$

Let $(Y_l^{(j)})_{l\geq 1}$ be a sequence of independent copies of $Y^{(j)}$ and assume that the sequences $(Y_l^{(j)})_{l\geq 1}$, $j = 1, \dots, m$, are mutually independent. Set

$$S_k^{(j)} = Y_1^{(j)} + \cdots + Y_k^{(j)}, \quad k \in \mathbb{N}, \quad S_0^{(j)} = 0. \tag{3.5}$$

Observe that for any $k \in \mathbb{N}_0$ the sums $S_k^{(j)}$, $j = 1, \dots, m$, are mutually independent.

With these notations, we state our first main result on higher-order convolution identities for scale transformations of Appell sequences in $\mathcal{R}$.

Theorem 3.4. *In the setting of Theorem* 3.3*, we have*

$$\begin{aligned}&\sum_{j_1+\cdots+j_m=n}\binom{n}{j_1,\ldots,j_m}w_1^{j_1}\cdots w_m^{j_m}A_{j_1}^{(1)}(x_1)\cdots A_{j_m}^{(m)}(x_m)\\&=\sum_{r=0}^{n}(-1)^r r!S_{\boldsymbol{w}\cdot\boldsymbol{Y}}(n,r;x)\\&=\sum_{r=0}^{n}\binom{n}{r}x^{n-r}\sum_{k=0}^{r}\binom{r+1}{k+1}(-1)^k\mathbb{E}\left(w_1S_k^{(1)}+\cdots+w_mS_k^{(m)}\right)^r.\end{aligned}$$

Proof. By (2.4) and (3.4), we have the basic identity

$$\begin{aligned}&T_{w_1}\boldsymbol{A}^{(1)}(w_1x_1)\times\cdots\times T_{w_m}\boldsymbol{A}^{(m)}(w_mx_m)\\&=(T_{w_1}\boldsymbol{A}^{(1)}\times\cdots\times T_{w_m}\boldsymbol{A}^{(m)})(x).\end{aligned}\tag{3.6}$$

Again by Remark 3.2, we can assume without loss of generality that $w_j\neq 0$, $j=1,\ldots,m$, in (3.6). From (2.1) and (2.5), we see that

$$\begin{aligned}&(T_{w_1}A^{(1)}(w_1x_1)\times\cdots\times T_{w_m}A^{(m)}(w_mx_m))_n\\&\sum_{j_1+\cdots+j_m=n}\binom{n}{j_1,\ldots,j_m}T_{w_1}A_{j_1}^{(1)}(w_1x_1)\cdots T_{w_m}A_{j_m}^{(m)}(w_mx_m)\\&=\sum_{j_1+\cdots+j_m=n}\binom{n}{j_1,\ldots,j_m}w_1^{j_1}\cdots w_m^{j_m}A_{j_1}^{(1)}(x_1)\cdots A_{j_m}^{(m)}(x_m).\end{aligned}\tag{3.7}$$

By Theorem 3.3, the Appell sequence on the right-hand side of (3.6) belongs to $\mathcal{R}$ with associated random variable $\boldsymbol{w}\cdot\boldsymbol{Y}$, and $(w_1Y_l^{(1)}+\cdots+w_mY_l^{(m)})_{l\geq 1}$ is a sequence of independent copies of $\boldsymbol{w}\cdot\boldsymbol{Y}$. We therefore have from (3.5) and Theorem 3.1,

$$\begin{aligned}&(T_{w_1}A^{(1)}\times\cdots\times T_{w_m}A^{(m)})_n(x)=\sum_{r=0}^{n}(-1)^r r!S_{\boldsymbol{w}\cdot\boldsymbol{Y}}(n,r;x)\\&=\sum_{r=0}^{n}\binom{n}{r}x^{n-r}\sum_{k=0}^{r}\binom{r+1}{k+1}(-1)^k\mathbb{E}\left(w_1S_k^{(1)}+\cdots+w_mS_k^{(m)}\right)^r.\end{aligned}$$

This, together with (3.6) and (3.7), completes the proof. □

Let $\boldsymbol{W}=(W_1,\ldots,W_m)$ be a random vector whose components satisfy (2.8). We denote

$$C_s(j_1,\ldots,j_m;\boldsymbol{x})=\mathbb{E}\left(W_1^{j_1}\cdots W_m^{j_m}(\boldsymbol{x}\cdot\boldsymbol{W})^s\right),\quad s\in\mathbb{N}_0,\tag{3.8}$$

where $j_\nu\in\mathbb{N}_0$, $x_\nu\in\mathbb{R}$, $\nu=1,\ldots,m$, and $\boldsymbol{x}=(x_1,\ldots,x_m)$. We simply set

$$C(j_1,\ldots,j_m)=C_0(j_1,\ldots,j_m;\mathbf{0})=\mathbb{E}(W_1^{j_1}\cdots W_m^{j_m}).\tag{3.9}$$

The second main result, which generalizes Theorem 3.4, is the following.

Theorem 3.5. *Let $\boldsymbol{W} = (W_1, \ldots, W_m)$ be a random vector whose components satisfy* (2.8)*. In the setting of Theorem* 3.4*, we have*

$$\sum_{j_1+\cdots+j_m=n} \binom{n}{j_1, \ldots, j_m} C(j_1, \ldots, j_m) A^{(1)}_{j_1}(x_1) \cdots A^{(m)}_{j_m}(x_m)$$
$$= \sum_{r=0}^{n} \binom{n}{r} \sum_{k=0}^{r} \binom{r+1}{k+1} (-1)^k \sum_{i_1+\cdots+i_m=r} \binom{r}{i_1, \ldots, i_m} \times$$
$$\times\, C_{n-r}(i_1, \ldots, i_m; \boldsymbol{x}) \mathbb{E}(S_k^{(1)})^{i_1} \cdots \mathbb{E}(S_k^{(m)})^{i_m}.$$

Proof. By assumption (2.8),

$$\mathbb{E} e^{r_j |W_j|} < \infty, \quad j = 1, \ldots, m,$$

for some $r_j > 0$. This implies that each W_j has finite moments of any order, $j = 1, \ldots, m$. Thus, by Hölder's inequality, all the expectations involving the random vector $\boldsymbol{W}$ in Theorem 3.5 are finite.

From (3.4) and Theorem 3.4, we have

$$\sum_{j_1+\cdots+j_m=n} \binom{n}{j_1, \ldots, j_m} w_1^{j_1} \cdots w_m^{j_m} A^{(1)}_{j_1}(x_1) \cdots A^{(m)}_{j_m}(x_m)$$
$$= \sum_{r=0}^{n} \binom{n}{r} \sum_{k=0}^{r} \binom{r+1}{k+1} (-1)^k \sum_{i_1+\cdots+i_m=r} \binom{r}{i_1, \ldots, i_m} \times$$
$$\times\, w_1^{i_1} \cdots w_m^{i_m} (x_1 w_1 + \cdots + x_m w_m)^{n-r} \mathbb{E}(S_k^{(1)})^{i_1} \cdots \mathbb{E}(S_k^{(m)})^{i_m}.$$

Recalling (3.8) and (3.9), the conclusion follows by replacing $(w_1, \ldots, w_m)$ by the random vector $\boldsymbol{W} = (W_1, \ldots, W_m)$ and then taking expectations. □

We emphasize that the random vector $\boldsymbol{W}$ in Theorem 3.5 does not necessarily have independent components. In fact, we will consider in Corollary 4.3 in Section 4 a random vector $\boldsymbol{W}$ such that $W_1 + \cdots + W_m = 1$.

4. Bernoulli polynomials

As said in the Introduction, Theorems 3.4 and 3.5 can be applied when $\boldsymbol{A}^{(j)}(x)$ are the Bernoulli, Apostol-Euler, and Cauchy type polynomials, among many others. In this section, we will restrict our attention to the case in which the $\boldsymbol{A}^{(j)}(x)$ are the classical Bernoulli polynomials. As a counterpart, we will consider different choices of the random vector $\boldsymbol{W}$.

In this section, Y is a random variable having the uniform distribution on $[0, 1]$, $(Y_j)_{j\geq 1}$ is a sequence of independent copies of Y and

$$S_k = Y_1 + \cdots + Y_k, \quad k \in \mathbb{N}, \qquad S_0 = 0. \tag{4.1}$$

Recall that the Bernoulli polynomials $\boldsymbol{B}(x) = (B_n(x))_{n\geq 0}$ are defined via their generating function as

$$G(\boldsymbol{B}(x), z) = \frac{z e^{xz}}{e^z - 1} = \frac{e^{xz}}{\mathbb{E} e^{zY}}, \tag{4.2}$$

where the last equality in (4.2) was already noticed by Ta [11]. On the other hand, Sun [18] showed the following probabilistic representation for the classical Stirling numbers of the second kind $S(n,k)$:

$$S(n,k) = \binom{n}{k} \mathbb{E} S_k^{n-k}, \quad k = 0,1,\dots,n. \tag{4.3}$$

Corollary 4.1. *We have*

$$\begin{aligned}&\sum_{j_1+\cdots+j_m=n} \binom{n}{j_1,\dots,j_m} C(j_1,\dots,j_m) \prod_{\nu=1}^m B_{j_\nu}(x_\nu)\\ &= \sum_{r=0}^n \binom{n}{r} \sum_{k=0}^r \binom{r+1}{k+1} (-1)^k \sum_{i_1+\cdots+i_m=r} \binom{r}{i_1,\dots,i_m} C_{n-r}(i_1,\dots,i_m;\boldsymbol{x}) \times\\ &\times \prod_{\nu=1}^m \frac{S(k+i_\nu,k)}{\binom{k+i_\nu}{k}}.\end{aligned}$$

Proof. It suffices to apply Theorem 3.5 to the case $\boldsymbol{A}^{(j)}(x) = \boldsymbol{B}(x)$, $j = 1,\dots,m$. Note that, in such a case, the random sums $S_k^{(j)}$, $j = 1,\dots,m$ defined in (3.5) have the same law as that of the random sum S_k defined in (4.1), thus having from (4.3)

$$\mathbb{E}(S_k^{(j)})^i = \frac{S(k+i,k)}{\binom{k+i}{k}}, \quad j = 1,\dots,m,$$

for any $k, i \in \mathbb{N}_0$. The proof is complete. □

We point out that the right-hand side in Corollary 4.1 only depends on the random vector $\boldsymbol{W}$ and on the Stirling numbers $S(n,k)$.

Different particular cases of Corollary 4.1 are obtained for each choice of $\boldsymbol{W}$. The first one is the following.

Corollary 4.2. *Let $w_j, x_j \in \mathbb{R}$, $j = 1,\dots,m$, as in* (3.4)*. Then,*

$$\begin{aligned}&\sum_{j_1+\cdots+j_m=n} \binom{n}{j_1,\dots,j_m} \prod_{\nu=1}^m w_\nu^{j_\nu} B_{j_\nu}(x_\nu)\\ &= \sum_{r=0}^n \binom{n}{r} x^{n-r} \sum_{k=0}^r \binom{r+1}{k+1} (-1)^k \sum_{i_1+\cdots+i_m=r} \binom{r}{i_1,\dots,i_m} \times\\ &\times \prod_{\nu=1}^m w_\nu^{i_\nu} \frac{S(k+i_\nu,k)}{\binom{k+i_\nu}{k}}.\end{aligned}$$

Proof. Choose in Corollary 4.1 the deterministic vector $\boldsymbol{W} = (w_1,\dots,w_m)$. Recalling (3.8) and (3.9), it is enough to observe that

$$C(j_1,\dots,j_m) = w_1^{j_1} \cdots w_m^{j_m},$$

as well as

$$C_{n-r}(i_1,\dots,i_m;\boldsymbol{x}) = w_1^{i_1} \cdots w_m^{i_m} x^{n-r}.$$

□

In his classical result, Dilcher [2] considered the case $w_1 = \cdots = w_m = 1$ and obtained an identity involving the products $s(m,k)B_j(x)$, where $s(m.k)$ are the Stirling numbers of the first kind. Wang [5] (see also Chu and Zhou [13]) provided identities when at most two of the numbers w_j, $j = 1,\dots,m$ are different from 1 and the product of Bernoulli polynomials is replaced by a product involving both the Bernoulli and Euler polynomials.

Denote by $\langle x\rangle_n$ the rising factorial, i.e.,

$$\langle x\rangle_n = \frac{\Gamma(x+n)}{\Gamma(x)},$$

$\Gamma(\cdot)$ being Euler's gamma function. For any $\alpha > 0$, denote by X_α a random variable having the gamma density

$$\rho_\alpha(\theta) = \frac{\theta^{\alpha-1}e^{-\theta}}{\Gamma(\alpha)}, \quad \theta > 0. \tag{4.4}$$

Let $m = 2,3,\dots$ and $\alpha_j > 0$, $j = 1,\dots,m$. Suppose that $(X_{\alpha_j},\ j = 1,\dots,m)$ are mutually independent random variables such that each X_{α_j} has the gamma density $\rho_{\alpha_j}(\theta)$. We consider the random vector $\boldsymbol{W} = (W_1,\dots,W_m)$ defined as

$$W_j = \frac{X_{\alpha_j}}{X_{\alpha_1} + \cdots + X_{\alpha_m}}, \quad j = 1,\dots,m. \tag{4.5}$$

Observe that $W_1 + \cdots + W_m = 1$. It is well known (cf. Kotz *et al.* [19] or Dilcher and Vignat [10]) that $\boldsymbol{W}$ has the multivariate Dirichlet distribution and

$$\mathbb{E}\left(W_1^{j_1}\cdots W_m^{j_m}\right) = \frac{\langle\alpha_1\rangle_{j_1}\cdots\langle\alpha_m\rangle_{j_m}}{\langle\alpha_1+\cdots+\alpha_m\rangle_{j_1+\cdots+j_m}}, \quad j_l,\dots,j_m \in \mathbb{N}_0. \tag{4.6}$$

With these ingredients, we enunciate the following result.

Corollary 4.3. *Let $m = 2,3,\dots$, $\alpha_j > 0$, $j = 1,\dots,m$, and $x \in \mathbb{R}$. Then,*

$$\begin{aligned}
&\frac{1}{\langle\alpha_1+\cdots+\alpha_m\rangle_n}\sum_{j_1+\cdots+j_m=n}\binom{n}{j_1,\dots,j_m}\prod_{\nu=1}^m\langle\alpha_\nu\rangle_{j_\nu}B_{j_\nu}(x)\\
&=\sum_{r=0}^n\binom{n}{r}x^{n-r}\sum_{k=0}^r\binom{r+1}{k+1}\frac{(-1)^k}{\langle\alpha_1+\cdots+\alpha_m\rangle_r}\sum_{i_1+\cdots+i_m=r}\binom{r}{i_1,\dots,i_m}\times\\
&\times\prod_{\nu=1}^m\langle\alpha_\nu\rangle_{i_\nu}\frac{S(k+i_\nu,k)}{\binom{k+i_\nu}{k}}.
\end{aligned}$$

Proof. Choose $x_j = x$ and W_j as in (4.5) in Corollary 4.1, $j = 1,\dots,m$. Since $W_1 + \cdots + W_m = 1$, we have from (3.8)

$$C_{n-r}(i_1,\dots,i_m;\boldsymbol{x}) = x^{n-r}\mathbb{E}\left(W_1^{i_1}\cdots W_m^{i_m}\right) = \frac{\langle\alpha_1\rangle_{i_1}\cdots\langle\alpha_m\rangle_{i_m}}{\langle\alpha_1+\cdots+\alpha_m\rangle_r}x^{n-r},$$

because $i_1 + \cdots + i_m = r$. Similarly,

$$C(j_1,\dots,j_m) = \frac{\langle\alpha_1\rangle_{j_1}\cdots\langle\alpha_m\rangle_{j_m}}{\langle\alpha_1+\cdots+\alpha_m\rangle_n}.$$

Therefore, the conclusion follows from Corollary 4.1. □

Dilcher and Vignat [10] have recently given a similar result to Corollary 4.3. This result generalizes Miki's identity (see Miki [14]), as well as an extension of it shown by Gessel [3].

To prove the next result, we will need the following reformulation of the well known Chu-Vandermonde identity (see, for instance, Chang and Xu [20] or Vignat and Moll [21] for a probabilistic proof of this identity).

Lemma 4.4. *Let $t_1, \dots, t_m \in \mathbb{R}$ with $t_1 + \cdots + t_m = t$. Then,*

$$\sum_{l_1+\cdots+l_m=n} \binom{t_1+l_1}{l_1} \cdots \binom{t_m+l_m}{l_m} = \binom{t+m+n-1}{n}.$$

Proof. Use the formula

$$\binom{-\beta}{n} = (-1)^n \binom{\beta+n-1}{n}, \quad \beta \in \mathbb{R},$$

and apply the classical Chu-Vandermonde identity. □

Corollary 4.5. *Let $x \in \mathbb{R}$. Then,*

$$\begin{aligned}\sum_{j_1+\cdots+j_m=n} B_{j_1}(x) \cdots B_{j_m}(x) &= \sum_{r=0}^{n} \binom{m+n-1}{n-r} x^{n-r} \times \\ &\times \sum_{k=0}^{r} \binom{r+1}{k+1} (-1)^k \sum_{i_i+\cdots+i_m=r} \prod_{\nu=1}^{m} \frac{S(k+i_\nu, k)}{\binom{k+i_\nu}{k}}.\end{aligned} \tag{4.7}$$

Proof. Choose in Corollary 4.1 a random vector $\boldsymbol{W} = (W_1, \dots, W_m)$ whose components are independent and identically distributed random variables, each one having the exponential density $\rho_1(\theta)$ defined in (4.4). By (3.9), (4.4), and the independence assumption, we see that

$$C(j_1, \dots, j_m) = \mathbb{E}W_1^{j_1} \cdots \mathbb{E}W_m^{j_m} = \prod_{\nu=1}^{m} j_\nu!. \tag{4.8}$$

Also, choose $x_j = x$, $j = 1, \dots, m$. We have from (3.8), (4.8), and Lemma 4.4,

$$\begin{aligned}&C_{n-r}(i_1, \dots, i_m; \boldsymbol{x}) \\ &= x^{n-r} \sum_{l_1+\cdots+l_m=n-r} \binom{n-r}{l_1, \dots, l_m} \mathbb{E}W_1^{i_1+l_1} \cdots \mathbb{E}W_m^{i_m+l_m} \\ &= x^{n-r}(n-r)! \prod_{\nu=1}^{m} i_\nu! \sum_{l_1+\cdots+l_m=n-r} \binom{i_1+l_1}{l_1} \cdots \binom{i_m+l_m}{l_m} \\ &= x^{n-r}(n-r)! \prod_{\nu=1}^{m} i_\nu! \binom{m+n-1}{n-r},\end{aligned} \tag{4.9}$$

because $i_1 + \cdots + i_m = r$. The result readily follows from (4.8), (4.9), and Corollary 4.1. □

Agoh and Dilcher [6] considered the left-hand side in (4.7) and showed an identity in terms of products of the form $\boldsymbol{B}_{l_0}(x)\boldsymbol{B}_{l_1}(0)\cdots\boldsymbol{B}_{l_k}(0)$. This result is a generalization of an identity proposed by Matiyasevich (see Agoh [15] for a proof and further references on Matiyasevich identity).

Recall that the Hermite polynomials $\boldsymbol{H}(x) = (H_n(x))_{n\geq 0}$ are defined via the generating function

$$G(\boldsymbol{H}(x), z) = e^{xz-z^2/2} = \frac{e^{xz}}{\mathbb{E}e^{zZ}} = \sum_{n=0}^{\infty} H_n(x)\frac{z^n}{n!}, \quad z \in \mathbb{C},$$

where Z is a random variable having the standard normal density. Clearly, $\boldsymbol{H}(x) \in \mathcal{R}$. On the other hand, such polynomials can be represented in probabilistic terms as (cf. Withers [22] or Ta [11])

$$H_n(x) = \mathbb{E}(x + \boldsymbol{i}Z)^n, \tag{4.10}$$

where $\boldsymbol{i}$ is the imaginary unit. Representation (4.10) allows us to give the following convolution identities.

Corollary 4.6. *Let $x \in \mathbb{R}$. Then,*

$$\begin{aligned}
&\sum_{j_1+\cdots+j_m=n} \binom{n}{j_1,\ldots,j_m} \prod_{\nu=1}^{m} H_{j_\nu}(0)B_{j_\nu}(x) \\
&= \sum_{r=0}^{n} \binom{n}{r} x^{n-r} \sum_{k=0}^{r} \binom{r+1}{k+1} (-1)^{n-k} \times \\
&\times \sum_{\substack{i_1+\cdots+i_m=r \\ l_1+\cdots+l_m=n-r}} \binom{r}{i_1,\ldots,i_m}\binom{n-r}{l_1,\ldots,l_m} \prod_{\nu=1}^{m} \frac{S(k+i_\nu,k)}{\binom{k+i_\nu}{k}} H_{i_\nu+l_\nu}(0).
\end{aligned}$$

Proof. Choose in Corollary 4.1 $x_j = x$, $j = 1,\ldots,m$, and $\boldsymbol{W} = (W_1,\ldots,W_m)$ a random vector with independent and identically distributed components, each one having the same law as Z. By (3.8) and (4.10), we have

$$\begin{aligned}
C_{n-r}(i_1,\ldots,i_m;\boldsymbol{x}) &= x^{n-r} \sum_{l_1+\cdots+l_m=n-r} \binom{n-r}{l_1,\ldots,l_m} \mathbb{E}W_1^{i_1+l_1}\cdots\mathbb{E}W_m^{i_m+l_m} \\
&= (-\boldsymbol{i})^n x^{n-r} \sum_{l_1+\cdots+l_m=n-r} \binom{n-r}{l_1,\ldots,l_m} H_{i_1+l_1}(0)\cdots H_{i_m+l_m}(0),
\end{aligned}$$

because $i_1+\cdots+i_m = r$. Similarly, by (3.9) we have

$$C(j_1,\ldots,j_m) = \boldsymbol{i}^n H_{j_1}(0)\cdots H_{j_m}(0).$$

Hence, the conclusion follows from Corollary 4.1. $\square$

5. Cauchy type polynomials

The Cauchy polynomials of the first kind $\boldsymbol{c}(x) = (c_n(x))_{n\geq 0}$ are defined by the generating function

$$\frac{z}{(1+z)^x \log(1+z)} = \sum_{n=0}^{\infty} c_n(x) \frac{z^n}{n!}, \quad |z|<1,$$

$c_n(0) = c_n$, $n \in \mathbb{N}_0$, being the classical Cauchy numbers of the first kind (see, for instance, Comtet [23, Ch. VII] or Merlini *et al.* [24]). Generalizations of these polynomials can be found in Komatsu and Ramírez [25], Pyo *et al.* [26], and the references therein.

Observe that the polynomials $\boldsymbol{c}(x) = (c_n(x))_{n\geq 0}$ are no Appell sequences. For this reason, we consider here the Cauchy type polynomials $\boldsymbol{C}(x) = (C_n(x))_{n\geq 0}$ defined by means of the generating function

$$G(\boldsymbol{C}(x), z) = \frac{z e^{xz}}{\log(1+z)} = \sum_{n=0}^{\infty} C_n(x) \frac{z^n}{n!}, \quad |z|<1. \tag{5.1}$$

Observe that

$$C_n(0) = c_n(0) = c_n. \tag{5.2}$$

Let U and T be two independent random variables such that U is uniformly distributed on $[0,1]$ and T has the exponential density $\rho_1(\theta)$ defined in (4.4). In this section, we set $Y = -UT$ and denote by

$$S_k = Y_1 + \cdots + Y_k, \quad k \in \mathbb{N}, \qquad S_0 = 0, \tag{5.3}$$

where $(Y_j)_{j\geq 1}$ is a sequence of independent copies of Y. In [27], we showed that

$$\mathbb{E}e^{zY} = \frac{\log(1+z)}{z}, \quad |z|<1, \tag{5.4}$$

as well as

$$s(n,k) = \binom{n}{k} \mathbb{E}S_k^{n-k}, \quad k = 0, \ldots, n, \tag{5.5}$$

where $s(n,k)$ are the Stirling numbers of the first kind. We therefore have from (5.1) and (5.4)

$$G(\boldsymbol{C}(x), z) = \frac{e^{xz}}{\mathbb{E}e^{zY}}, \quad |z|<1,$$

thus showing that the Cauchy type polynomials $\boldsymbol{C}(x)$ belong to the set $\mathcal{R}$.

Keeping the same notations as in (3.8) and (3.9), we state the following result.

Corollary 5.1. *We have*

$$\sum_{j_1+\cdots+j_m=n}\binom{n}{j_1,\ldots,j_m}C(j_1,\ldots,j_m)\prod_{\nu=1}^{m}C_{j_\nu}(x_\nu)$$
$$=\sum_{r=0}^{n}\binom{n}{r}\sum_{k=0}^{r}\binom{r+1}{k+1}(-1)^k\sum_{i_1+\ldots+i_m=r}\binom{r}{i_1,\ldots,i_m}C_{n-r}(i_1,\ldots,i_m;\boldsymbol{x})\times$$
$$\times\prod_{\nu=1}^{m}\frac{s(k+i_\nu,k)}{\binom{k+i_\nu}{k}}.$$

Proof. The proof follows along the lines of that in Corollary 4.1, by using formula (5.5) instead of formula (4.3). □

By considering different choices of the random vector $\boldsymbol{W}$, we can give analogous results to Corollaries 4.2–4.6 for the Cauchy type polynomials $\boldsymbol{C}(x)$. Details are omitted.

To conclude the paper, we give a specific result for higher order convolutions of the Cauchy numbers $(c_n)_{n\geq 0}$ defined in (5.2).

Corollary 5.2. *We have*

$$\sum_{j_1+\cdots+j_m=n}\binom{n}{j_1,\ldots,j_m}c_{j_1}\cdots c_{j_m}=\sum_{k=0}^{n}\binom{n+1}{k+1}(-1)^k\frac{s(n+km,km)}{\binom{n+km}{n}}.$$

Proof. In Theorem 3.4, choose $x_j=0$, $w_j=1$, $\boldsymbol{A}^{(j)}(x)=\boldsymbol{C}(x)$,$j=1,\ldots,m$, thus having from (3.4) and (5.2)

$$\begin{aligned}&\sum_{j_1+\cdots+j_m=n}\binom{n}{j_1,\ldots,j_m}c_{j_1}\cdots c_{j_m}\\&=\sum_{k=0}^{n}\binom{n+1}{k+1}(-1)^k\mathbb{E}\left(S_k^{(1)}+\ldots+S_k^{(m)}\right)^n,\end{aligned}\tag{5.6}$$

where $S_k^{(j)}$, $j=1,\ldots,m$ are independent copies of S_k, as defined in (5.3). Since the random variables $S_k^{(1)}+\cdots+S_k^{(m)}$ and S_{km} have the same law, we have from (5.5)

$$\mathbb{E}\left(S_k^{(1)}+\cdots+S_k^{(m)}\right)^n=\mathbb{E}S_{km}^n=\frac{s(n+km,km)}{\binom{n+km}{n}}.$$

This, together with (5.6), shows the result. □

We finally point out that a similar result to Corollary 5.2 in terms of both Stirling numbers of first and second kinds was obtained by Zhao [4, Corollary 3.1]

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