RATIONAL APPROXIMATION TO EULER'S CONSTANT AT A GEOMETRIC RATE OF CONVERGENCE

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ABSTRACT. We give a rational approximation to Euler's constant at a geometric rate of convergence, which is easy to compute. Moreover, such an approximation is completely monotonic. The approximants are built up in terms of expectations of the harmonic numbers acting on the standard Poisson process.

1. INTRODUCTION

The Euler-Mascheroni constant γ, first introduced by Leonhard Euler (1707–1783), can be defined in various equivalent ways, for instance,

$$(1.1) \qquad \gamma = -\int_0^\infty e^{-x} \log x \, dx.$$

Since γ appears in many different mathematical branches (see in this respect the survey paper by Lagarias [16]), a lot of effort has been devoted to compute such a constant. The original definition by Euler, that is,

$$\gamma = \lim_{n \to \infty} (H(n) - \log n) = 0.5772156\cdots,$$

where

$$(1.2) \qquad H(n) = \sum_{k=1}^n \frac{1}{k}, \quad n = 1, 2, \ldots, \qquad H(0) = 0,$$

is the nth harmonic number, gives us poor rates of convergence. In fact, it is well known that

$$\gamma = H(n) - \log n + O(n^{-1}).$$

For this reason, many authors have considered sequences of the form

$$(1.3) \qquad H(n) - \log(n + a_n) - Q_n, \qquad \lim_{n \to \infty} a_n = \lim_{n \to \infty} Q_n = 0,$$

where a_n and Q_n are quotients of polynomials, in order to improve rates of convergence (see, for instance, Yang [22], Chen and Mortici [5], Lu [17], Lu *et al.* [18], Hu *et al.* [14], Wu and Bercu [21], and the references therein). In any case, the resulting rate of convergence is of polynomial order n^{-s}, for some $s \geq 1$, depending on the choice of the sequences a_n and Q_n.

In contrast with (1.3), we choose in this paper a function of the form

$$(1.4) \qquad f(\lambda) = \mathbb{E}H(N_\lambda) - \log \lambda, \quad \lambda > 0,$$

where $\mathbb{E}$ stands for mathematical expectation and N_λ is a random variable having the Poisson law with mean λ, i.e.,

$$(1.5) \qquad P(N_\lambda = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, 2, \ldots$$

It turns out that the function $f(\lambda)$ in (1.4) is simple to compute and satisfies

$$(1.6) \qquad \gamma = \mathbb{E}H(N_\lambda) - \log \lambda + O\left(e^{-\lambda}/\lambda\right),$$

as shown in Lemma 2.1 and Theorem 3.1 below. As observed by the referee, formula (1.6) is the probabilistic reformulation of the well known exponential integral method for approximating Euler's constant used by Sweeny [20] in 1963 to compute 3566 decimal places of γ. Further improvements in calculation of γ are based on the Bessel function method introduced by Brent and McMillan [4].

Apart from the probabilistic approach outlined above, we point out the following two main features. In first place, we provide a precise rational approximation to γ with a geometric rate of convergence in Corollary 3.4, which is the main result of this paper (unfortunately, the speed of convergence is not fast enough to prove the irrationality of γ). This is possible because the approximating function $f(\lambda)$ in (1.4) only contains one single logarithm, which can be evaluated by a fast converging series (see formula (3.10) below). In second place, the function $f(\lambda)$ is completely monotonic (see Theorem 3.1). In particular, $f(\lambda)$ converges to γ in a decreasing and convex way.

Certainly, we find in the literature different results on exponential convergence to γ at the price of using logarithms at different arguments. For instance, Sondow [19] showed that

$$\gamma = \frac{A_n - L_n}{\binom{2n}{n}} + O\left(2^{-6n}\, n^{-1/2}\right), \quad \text{as } n \to \infty,$$

where

$$A_n = \sum_{i=0}^{n} \binom{n}{i}^2 H(n+i), \; L_n = 2 \sum_{0 \le i < j \le n} \sum_{k=1}^{j-i} \frac{(-1)^{i+j-1}}{j-i} \binom{n}{i}\binom{n}{j} \log(n+i+k),$$

whereas Coffey [6] gave the formula

$$(1.7) \qquad \gamma = \frac{\log 2}{2} - \frac{2}{3 \log 2} \sum_{k=1}^{\infty} \frac{1}{3^k} \sum_{j=1}^{k} (-1)^j \binom{k}{j} 2^j \frac{\log(j+1)}{j+1},$$

where the series in (1.7) has actually the geometric rate $1/3$, as shown in [2].

On the other hand, Elsner [7] showed the following result on exponential convergence to γ in terms of linear combinations of logarithms

$$\left| \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \binom{2n+k-1}{n} (H_{k+n-1} - \log(k+n)) - \gamma \right| \le \frac{1}{2n^2 \binom{2n}{n}},$$

for $n = 1, 2, \ldots$. This result was further improved in the works by Elsner [8] and Kh. Hessami Pilehrood and T. Hessami Pilehrood [11, 12], among others. For

instance, the following inequality for $n = 1, 2, \ldots$, is proved in [12]:

$$\left| \gamma - \frac{(4n)!^2}{(8n)!} \sum_{k=0}^{8n} (-1)^k \binom{8n}{k} \binom{6n+k}{4n}^2 (H_{k+4n} - \log(k + 4n + 1)) \right| < \frac{4n}{(2^4 3^{12})^n}.$$

Finally, Kh. Hessami Pilehrood and T. Hessami Pilehrood [13] have provided a rational approximation p_n/q_n converging to γ subexponentially. In fact, these authors have shown that

$$\frac{p_n}{q_n} - \gamma = e^{-4\sqrt{n}} \left(2\pi + O\left(n^{-1/2}\right) \right), \qquad \text{as } n \to \infty,$$

where

$$q_n = \sum_{k=0}^{n} \binom{n}{k}^2 k!, \qquad p_n = \sum_{k=0}^{n} \binom{n}{k}^2 k! \left(2H(n-k) - H(k)\right), \quad n = 1, 2, \ldots$$

From a methodological point of view, we use a probabilistic approach based on expectations of functions of well known random variables. In particular, we use in many occasions Fubini's theorem in the following probabilistic form (see, for instance, Gut [10, p. 65]). Let X and Y be two independent random variables and let $g : \mathbb{R}^2 \to \mathbb{R}$ be a measurable function. If

$$h(y) = \mathbb{E}g(X, y), \quad y \in \mathbb{R},$$

then

$$\mathbb{E}h(Y) = \mathbb{E}g(X, Y),$$

whenever the preceding expectations exist.

2. Probabilistic tools

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We will consider the following random variables which are assumed to be mutually independent: U and V have the uniform distribution on $[0, 1]$, T has the exponential density with unit mean, that is,

$$(2.1) \qquad \rho(\theta) = e^{-\theta}, \quad \theta > 0,$$

and $(N_\lambda)_{\lambda \geq 0}$ is the standard Poisson process, i.e., a stochastic process having independent stationary increments, starting at the origin and such that N_λ, $\lambda > 0$, has the Poisson law defined in (1.5).

We will also use the following simple facts. In first place, the probability density of UV is given by (cf. Feller [9, p. 26] or Adell and Lekuona [3])

$$(2.2) \qquad \rho^\star(\theta) = -\log \theta, \quad 0 < \theta \leq 1.$$

Secondly, with respect to the standard Poisson process $(N_\lambda)_{\lambda \geq 0}$, we have the well known Poisson-gamma relation (see, for instance, Johnson $et\ al.$ [15, p. 190])

$$(2.3) \qquad P(N_\lambda > k) = \int_0^\lambda P(N_\theta = k)\, d\theta = \lambda P(N_{\lambda U} = k), \quad k \in \mathbb{N}_0, \quad \lambda \geq 0.$$

Also, it follows from (1.5) that

$$(2.4) \qquad \mathbb{E}(1 - v)^{N_\lambda} = e^{-\lambda v}, \quad v \in \mathbb{R}.$$

Denoting by $\mathbb{O} \subseteq \mathbb{N}$ the subset of odd positive integers, we have

$$(2.5) \qquad P(N_\lambda \in \mathbb{O}) = \frac{1 - e^{-2\lambda}}{2}, \quad \lambda \geq 0.$$

This last formula follows by setting $v = 2$ in (2.4), since

$$P(N_\lambda \in \mathbb{N}_0 \setminus \mathbb{O}) - P(N_\lambda \in \mathbb{O}) = \mathbb{E}(-1)^{N_\lambda} = e^{-2\lambda}, \quad \lambda \geq 0.$$

Finally, let X be an $\mathbb{N}_0$-valued random variable and let $g : \mathbb{N}_0 \to \mathbb{R}$ be a nondecreasing function. Denote by $\Delta g(k) = g(k+1) - g(k)$, $k \in \mathbb{N}_0$, the first forward difference of g. By Fubini's theorem, it can be easily seen that

$$(2.6) \qquad \mathbb{E}g(X) - g(0) = \sum_{k=0}^{\infty} \Delta g(k) P(X > k).$$

With these ingredients, we state the following auxiliary result which is crucial to compute γ.

Lemma 2.1. *For any $\lambda > 0$, we have*

$$\mathbb{E}H(N_\lambda) = \lambda \mathbb{E}e^{-\lambda UV} = \mathbb{E}\left(\frac{1 - e^{-\lambda U}}{U}\right) = -\sum_{k=1}^{\infty} \frac{(-\lambda)^k}{k!k}$$

$$(2.7)$$

$$= \sum_{k=0}^{\infty} \frac{2}{2k+1} P(N_{\lambda/2} > 2k).$$

Proof. By (1.2), we see that $\Delta H(k) = 1/(k+1)$, $k \in \mathbb{N}_0$. Also,

$$(2.8) \qquad \mathbb{E}V^k = \mathbb{E}(1-V)^k = \frac{1}{k+1}, \quad k \in \mathbb{N}_0.$$

We thus have from (2.3), (2.6), and Fubini's theorem

$$\mathbb{E}H(N_\lambda) = \sum_{k=0}^{\infty} \frac{1}{k+1} P(N_\lambda > k) = \lambda \sum_{k=0}^{\infty} \frac{1}{k+1} P(N_{\lambda U} = k)$$

$$= \lambda \sum_{k=0}^{\infty} \mathbb{E}(1-V)^k P(N_{\lambda U} = k) = \lambda \mathbb{E}e^{-\lambda UV},$$

where the last equality follows from (2.4). The second equality in (2.7) follows again by Fubini's theorem, since

$$\lambda \mathbb{E}e^{-\lambda UV} = \mathbb{E}\int_0^1 \lambda e^{-\lambda U\theta}\, d\theta = \mathbb{E}\left(\frac{1 - e^{-\lambda U}}{U}\right).$$

On the other hand, we have from (2.8)

$$\lambda \mathbb{E}e^{-\lambda UV} = \lambda \sum_{j=0}^{\infty} \frac{(-\lambda)^j}{j!} \mathbb{E}U^j \mathbb{E}V^j = -\sum_{j=0}^{\infty} \frac{(-\lambda)^{j+1}}{(j+1)!(j+1)},$$

which shows the third equality in (2.7). Finally, using (2.3), (2.5), and Fubini's theorem, we get

$$\mathbb{E}\left(\frac{1 - e^{-\lambda U}}{U}\right) = 2\mathbb{E}\left(\frac{1}{U} P(N_{\lambda U/2} \in \mathbb{O})\right) = \sum_{k=0}^{\infty} \frac{1}{2k+1} \lambda P(N_{\lambda U/2} = 2k)$$

$$= \sum_{k=0}^{\infty} \frac{2}{2k+1} P(N_{\lambda/2} > 2k).$$

This shows the last equality in (2.7) and completes the proof. $\qquad\square$

3. Main results

Together with Lemma 2.1, the following result is the key point to compute the Euler-Mascheroni constant.

Theorem 3.1. *The function f defined in (1.4) is completely monotonic on $(0, \infty)$ and satisfies the identity*

$$(3.1) \qquad \gamma = f(\lambda) - e^{-\lambda} \int_0^\infty \frac{e^{-\theta}}{\lambda + \theta} \, d\theta, \quad \lambda > 0.$$

Proof. We first show identity (3.1). By (1.1), we see that

$$(3.2) \qquad \gamma = -\log \lambda - \int_0^\lambda e^{-x} \log \frac{x}{\lambda} \, dx - \int_\lambda^\infty e^{-x} \log \frac{x}{\lambda} \, dx.$$

By (2.2) and Lemma 2.1, the first integral in (3.2) can be written as

$$(3.3) \qquad \lambda \int_0^1 e^{-\lambda\theta} (-\log \theta) \, d\theta = \lambda \mathbb{E} e^{-\lambda UV} = \mathbb{E} H(N_\lambda).$$

Using integration by parts, the second integral in (3.2) equals to

$$-e^{-\lambda} \int_0^\infty e^{-\theta} \log\left(1 + \frac{\theta}{\lambda}\right) d\theta = -e^{-\lambda} \int_0^\infty \frac{e^{-\theta}}{\lambda + \theta} \, d\theta.$$

This, in conjunction with (3.2) and (3.3), shows (3.1).

On the other hand, differentiating under the expectation sign in the third expression in (2.7), we obtain from (1.4)

$$(3.4) \qquad f'(\lambda) = \mathbb{E} e^{-\lambda U} - \frac{1}{\lambda} = -\frac{e^{-\lambda}}{\lambda} = -\mathbb{E} e^{-(\lambda + (\lambda-1)T)},$$

where the last equality follows from (2.1). Differentiating again under the expectation sign in (3.4), we have for any $n = 1, 2, \dots$

$$\begin{aligned}
f^{(n)}(\lambda) &= (-1)^n \mathbb{E}(T+1)^{n-1} e^{-(\lambda + (\lambda-1)T)} \\
&= (-1)^n e^{-\lambda} \int_0^\infty (\theta + 1)^{n-1} e^{-\lambda\theta} \, d\theta \\
&= \frac{(-1)^n e^{-\lambda}}{\lambda} \int_0^\infty \left(\frac{x}{\lambda} + 1\right)^{n-1} e^{-x} \, dx \\
&= \frac{(-1)^n e^{-\lambda}}{\lambda} \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{1}{\lambda^{n-1-k}} \int_0^\infty x^{n-1-k} e^{-x} \, dx \\
&= (-1)^n \frac{(n-1)!}{\lambda^n} P(N_\lambda \le n-1),
\end{aligned}$$

(3.5)

where the last equality follows from (1.5). Finally, it follows from (3.1) that $f(\lambda) > 0$. This, together with (3.5), shows the complete monotonicity of f and completes the proof. $\qquad \square$

Looking at Theorem 3.1, it suffices to approximate $\mathbb{E} H(N_\lambda)$ by a finite sum in order to compute γ. This is done in two different ways in the following two corollaries.

For any $\lambda > 0$, denote by

$$(3.6) \qquad m(\lambda) = \inf\left\{ m \in \mathbb{N} \cap [\lambda, \infty) : \frac{\lambda^{m+1}}{(m+1)!} \le e^{-\lambda} \right\}.$$

This definition makes sense, because the sequence $(\lambda^m/m!)_{m \geq \lambda}$ is decreasing.

Corollary 3.2. *For any $\lambda > 0$, we have*

$$\left| \gamma + \sum_{k=1}^{m(\lambda)} \frac{(-\lambda)^k}{k!k} + \log \lambda \right| \leq \frac{2e^{-\lambda}}{\lambda}.$$

Proof. By (2.8), we have for any $m \in \mathbb{N}$

$$\left| \mathbb{E}e^{-\lambda UV} - \sum_{k=1}^{m} \frac{(-\lambda)^{k-1}}{k!\, k} \right| = \left| \mathbb{E}e^{-\lambda UV} - \sum_{j=0}^{m-1} \frac{(-\lambda)^j \mathbb{E}(UV)^j}{j!} \right|$$

$$\leq \frac{\lambda^m}{m!}\mathbb{E}(UV)^m = \frac{\lambda^m}{(m+1)!\,(m+1)},$$

thus implying, by virtue of Lemma 2.1, that

$$(3.7) \qquad \left| \mathbb{E}H(N_\lambda) + \sum_{k=1}^{m} \frac{(-\lambda)^k}{k!k} \right| \leq \frac{\lambda^{m+1}}{(m+1)!(m+1)}, \quad m \in \mathbb{N}.$$

The conclusion readily follows from Theorem 3.1, (3.6), and (3.7). □

For any $\lambda > 0$, denote by

$$(3.8) \qquad n(\lambda) = \inf \left\{ m \in \mathbb{N} \cap [\lambda/4, \infty) : \ \frac{(\lambda/2)^{2m+2}}{(2m+2)!} \leq e^{-\lambda/2} \right\}.$$

As in (3.6), this definition makes sense.

Corollary 3.3. *For any $\lambda > 0$, we have*

$$\left| \gamma - \sum_{k=0}^{n(\lambda)} \frac{2}{2k+1} P\left(N_{\lambda/2} > 2k\right) + \log \lambda \right| \leq \left(\lambda + \frac{1}{\lambda} \right) e^{-\lambda}.$$

Proof. Let $k \in \mathbb{N}$. Since the function $h_k(\theta) = P(N_\theta = k)$ increases in $\theta \in [0, k]$, we have from (2.3)

$$(3.9) \qquad P(N_\lambda > k) \leq \lambda P(N_\lambda = k), \quad 0 < \lambda \leq k.$$

Let $m \geq n(\lambda)$, as defined in (3.8). Since $2m \geq 2n(\lambda) \geq \lambda/2$, we have from (3.9)

$$\sum_{k=m+1}^{\infty} \frac{2}{2k+1} P\left(N_{\lambda/2} > 2k\right) \leq \frac{2}{2m+3} \sum_{k=m+1}^{\infty} P\left(N_{\lambda/2} > 2k\right)$$

$$\leq \frac{\lambda}{2m+3} \sum_{k=m+1}^{\infty} P\left(N_{\lambda/2} = 2k\right) \leq \frac{\lambda}{2m+3} P\left(N_{\lambda/2} > 2m+1\right)$$

$$\leq \frac{\lambda^2}{2(2m+3)} P\left(N_{\lambda/2} = 2m+1\right) \leq \lambda \frac{(\lambda/2)^{2m+2}}{(2m+2)!} e^{-\lambda/2} \leq \lambda e^{-\lambda},$$

the last inequality by definition (3.8). Hence, the result follows from Lemma 2.1 and Theorem 3.1. □

From a computational point of view, the terms in Corollary 3.3 are more involved than those in Corollary 3.2, since a precomputation of the tail probabilities $P(N_{\lambda/2} > 2k) = 1 - P(N_{\lambda/2} \leq 2k)$ is required. However, as far as the number of terms is concerned, Stirling's formula shows that $m(\lambda) \leq 3.6\lambda$, whereas

$n(\lambda) \leq 0.9\lambda$, for moderate or large values of λ. As a result, numerical computations show that the total CPU time used for computing in Corollary 3.3 is less than that in Corollary 3.2.

Finally, to obtain a rational approximation to Euler's constant at a geometric rate of convergence, we will use the following formula to compute the logarithm function (cf. [1])

$$(3.10) \qquad \log(1+a) = 2\sum_{j=0}^{\infty} \frac{1}{2j+1} \left(\frac{a}{a+2}\right)^{2j+1}, \quad a > -1.$$

Corollary 3.4. *Let $\lambda = 2^N$, for some $N \in \mathbb{N}$. In the setting of Corollary 3.2, we have*

$$(3.11) \qquad \left| \gamma + \sum_{k=1}^{m(\lambda)} \frac{(-\lambda)^k}{k!\,k} + 2N \sum_{k=0}^{\lambda/2} \frac{1}{(2k+1)3^{2k+1}} \right| \leq 3\frac{e^{-\lambda}}{\lambda}.$$

Proof. Choosing $a = 1$ in (3.10), we have

$$(3.12) \qquad \log\lambda = 2N\sum_{j=0}^{\infty} \frac{1}{(2j+1)3^{2j+1}}.$$

Observe that

$$(3.13) \qquad 2N\sum_{j=\lambda/2+1}^{\infty} \frac{1}{(2j+1)3^{2j+1}} \leq \frac{2N}{3\lambda} \sum_{j=\lambda/2+1}^{\infty} \frac{1}{3^{2j}} = \frac{N}{12}\left(\frac{e}{3}\right)^\lambda \frac{e^{-\lambda}}{\lambda} \leq \frac{e^{-\lambda}}{\lambda},$$

since $\lambda = 2^N$. Hence, the conclusion follows from Corollary 3.2, (3.12), and (3.13).
$\square$

Corollary 3.4 could be improved by choosing a rational number λ of the form

$$\lambda = \left(1 + \frac{1}{q}\right)^N, \quad q, N \in \mathbb{N}.$$

Applying (3.10) with $a = 1/q$, the second finite sum on the left-hand side in (3.11) could be replaced by

$$2N\sum_{k=0}^{M(\lambda)} \frac{1}{(2k+1)(1+2q)^{2k+1}},$$

with number of terms $M(\lambda) \leq \lambda/2$. However, the improvement would not be significant, in view of the fact that $m(\lambda) \geq \lambda$, as defined in (3.6).

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