

A Characterization of a Gaussian Process in Terms of Sufficient Estimators

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Abstract

In this paper we consider the estimation problem in a continuous time linear model. We establish that, under certain covariance structure of the process, if the best linear unbiased estimator for the expectation of the process is sufficient then the process involved has a Gaussian distribution. In particular, this implies that, under some conditions, the linear sufficiency and ordinary sufficiency properties are equivalent if and only if the distribution of the process is Gaussian.

Key words: sufficiency, linear sufficiency, Gaussian process

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1 Introduction

The present paper investigates the properties of linear sufficiency and ordinary sufficiency in a continuous time linear model. The main goal is to give a characterization of a Gaussian process by the above-mentioned properties.

In the last twenty years, several authors have introduced, developed, and characterized the concepts of linear sufficiency and linear completeness in a classical linear model, that is, a linear model in discrete time. These concepts coincide with the ordinary concepts when the normal distribution is imposed. For all these results see, Baksalary and Kala (1981), Drygas (1983) and Müller (1987), among others. The authors of the present paper have extended these concepts to a continuous time linear model. We have proved the equivalence between the linear concepts and ordinary concepts when the model is a Gaussian process (see Ibarrola and Pérez-Palomares (2003) and (2004)). On the other hand, and independently of the above results, several authors (using different hypotheses) have proved in a classical linear model that if the least square estimator is sufficient then the model is a Gaussian model, see Kelker and Matthes (1970), Eberl (1983) and Bischoff et al. (1987). The most general result concerning this characterization has been given by Bischoff (2000), where the covariance matrix of the model is different from the identity matrix. The objective of this paper is to give a characterization of a Gaussian process imposing the sufficiency on the BLUE estimator of the expectation of the process. This characterization (given under determined covariance structure) implies that the linear sufficiency and the ordinary sufficiency are equivalent only when a Gaussian distribution is assumed.

From now on, let $(Z_t, t \in [0, T])$, $T > 0$, be a stochastic process with distribution P_0 in $(\mathbb{R}^{[0, T]}, \mathcal{F}_T)$ where \mathcal{F}_T is the σ -algebra generated by $Z_t, t \in [0, T]$. Let E_0 be the mathematical expectation with respect to P_0 . Suppose that $E_0[Z_t] = 0, t \in [0, T]$ and $E_0[Z_s Z_t] = B(s, t), s, t \in [0, T]$ is a known continuous function in $[0, T] \times [0, T]$. For each $\theta \in \mathbb{R}^p$, we denote by P_θ the distribution of the process $(X_t, t \in [0, T])$ which is defined as $X_t = A(t)\theta + Z_t, t \in [0, T]$, where $A(t)'$ is a vector in \mathbb{R}^p , with known continuous components in $[0, T]$. Let μ be the normal distribution on $(\mathbb{R}^p, \mathcal{B}(\mathbb{R}^p))$ with zero mean and covariance matrix I . \bar{P} denotes the measure defined on $(\mathbb{R}^{[0, T]}, \mathcal{F}_T)$ as

$$\bar{P}(A) = \int_{\mathbb{R}^p} P_\theta(A) d\mu(\theta), \quad A \in \mathcal{F}_T.$$

The mathematical expectation with respect to \bar{P} will be denoted by \bar{E} and with respect to P_θ by E_θ . Thus, the process $(X_t, t \in [0, T])$ is an element of $L^2(\mathbb{R}^{[0, T]}, \mathcal{F}_T, \bar{P})$ with

$$\bar{E}[X_t] = 0, \quad \bar{E}[X_s X_t] = B(s, t) + A(s)A(t)', \quad s, t \in [0, T]. \quad (1)$$

We denote by $\bar{\mathcal{L}}(X_t, t \in [0, T])$ the closure on $L^2(\mathbb{R}^{[0, T]}, \mathcal{F}_T, \bar{P})$ of the set of finite linear combinations of type $\sum_{i=1}^n c_i X_{t_i}, c_i \in \mathbb{R}, t_i \in [0, T]$. $\bar{\mathcal{L}}(X_t, t \in [0, T])$ is a Hilbert space with the inner product $\langle Y, Z \rangle = \bar{E}[YZ]$. We are interested in estimators constructed by the observed paths of the process $(X_t, t \in [0, T])$ in a linear way, that is, we are interested in estimators belonging to the class $\bar{\mathcal{L}}(X_t, t \in [0, T])$. We refer to this class of estimators as linear estimators. With this framework, we can essentially use the same concepts that in a discrete time context with some technical differences. Next, we give the concepts we shall use throughout the paper. From now on, the terms minimum variance, unbiased and uncorrelated estimators are all referred to the measure P_θ ,

$\theta \in \mathbb{R}^p$. When the measure \bar{P} is involved, it will be mentioned explicitly.

An estimable linear combination is a linear combination of θ which can be unbiasedly estimated by elements of $\bar{\mathcal{L}}(X_t, t \in [0, T])$. We say that a linear estimator is the BLUE for an estimable linear combination of θ if it is of minimum variance among all linear estimators unbiased for the linear combination. Let K be a compact subset of \mathbb{R} . We consider a family $(\theta_r, r \in K)$ of elements in $\bar{\mathcal{L}}(X_t, t \in [0, T])$. If K is not a finite set then we shall suppose that $(\theta_r, r \in K)$ is continuous in square mean sense. We denote by $\bar{\mathcal{L}}(\theta_r, r \in K)$ the closure in $L^2(\mathbb{R}^{[0, T]}, \mathcal{F}_T, \bar{P})$ of the linear combinations of $(\theta_r, r \in K)$. Then $(\theta_r, r \in K)$ is linearly sufficient if the BLUE of each estimable linear combination belongs to $\bar{\mathcal{L}}(\theta_r, r \in K)$. We say that $(\theta_r, r \in K)$ is linearly complete if for each $G \in \bar{\mathcal{L}}(\theta_r, r \in K)$ such that $E_\theta[G] = 0$, $\theta \in \mathbb{R}^p$, we have $G = 0$, a.s. Note that we are giving a light extension of the linear estimator concept with respect to that given in Ibarrola and Pérez-Palomares (2003) and (2004). In those papers, linear estimators of integral type have been considered.

Now, we present the construction of BLUE estimators for this model. Define, for each $j = 1, \dots, p$, the operator L_j as $L_j(\sum_{i=1}^n c_i X_{t_i}) = \sum_{i=1}^n c_i A^j(t_i)$, where A^j is the j -th component of A . These operators can be extended to $\bar{\mathcal{L}}(X_t, t \in [0, T])$ in the following way. For each $Y \in \bar{\mathcal{L}}(X_t, t \in [0, T])$, there exists a sequence $Y_n = \sum_{i=1}^n c_i X_{t_i}$ which converges to Y in $L^2(\mathbb{R}^{[0, T]}, \mathcal{F}_T, \bar{P})$. We can write $Y_n = \int_0^T V_n(dt) X_t$. From (1) it is easy to see that

$$\bar{E}[(Y_n - Y_m)^2] = E_0[(Y_n - Y_m)^2] + \sum_{j=1}^p \left(\int_0^T (V_n - V_m)(dt) A^j(t) \right)^2.$$

Since $(Y_n, n \geq 1)$ is a Cauchy sequence in $L^2(\mathbb{R}^{[0, T]}, \mathcal{F}_T, \bar{P})$ then $(Y_n, n \geq 1)$ and $\int_0^T V_n(dt) A(t)$ are Cauchy sequences, the first as an element in $L^2(\mathbb{R}^{[0, T]}, \mathcal{F}_T, P_0)$ and the second as a vector of real numbers. This implies that if

$(Y_n, n \geq 1)$ converges to Y in $L^2(\mathbb{R}^{[0,T]}, \mathcal{F}_T, \bar{P})$ then it converges to Y in $L^2(\mathbb{R}^{[0,T]}, \mathcal{F}_T, P_0)$ and $L_j(Y_n) = \int_0^T V_n(dt) A^j(t)$ is convergent. Thus, we can define $L_j(Y) = \lim_{n \rightarrow \infty} L_j(Y_n)$. Then, for each j , L_j is a continuous linear operator and, applying the Riesz representation theorem, we can assure the existence of an element $\hat{\theta}_j \in \bar{\mathcal{L}}(X_t, t \in [0, T])$ such that $L_j(Y) = \langle Y, \hat{\theta}_j \rangle$, $Y \in \bar{\mathcal{L}}(X_t, t \in [0, T])$. Defining $\hat{\theta} = (\hat{\theta}_1, \dots, \hat{\theta}_p)'$ and taking $Y = X_t$, we have

$$A(t) = \langle X_t, \hat{\theta}' \rangle = \bar{E}[X_t \hat{\theta}'], \quad t \in [0, T]. \quad (2)$$

It is immediate that

$$E_\theta[Y] = \bar{E}[Y \hat{\theta}'] \theta, \quad Y \in \bar{\mathcal{L}}(X_t, t \in [0, T]), \quad (3)$$

and, from (1),

$$\bar{E}[YZ] = E_0[YZ] + \bar{E}[Y \hat{\theta}'] \bar{E}[\hat{\theta} Z], \quad Y, Z \in \bar{\mathcal{L}}(X_t, t \in [0, T]). \quad (4)$$

For a linear estimator Y unbiased for 0 we have from (3) that $\bar{E}[Y \hat{\theta}'] = 0$ and using (4) we see that $E_0[Y \hat{\theta}'] = 0$. This means that $\hat{\theta}$ is uncorrelated with all linear estimators unbiased for 0, so $\hat{\theta}$ is the BLUE for its expectation. Finally, denoting by $\Sigma = \bar{E}[\hat{\theta} \hat{\theta}']$, we have that $A(t) \Sigma^{-1} \hat{\theta}$ is the BLUE for $A(t) \theta$, $t \in [0, T]$.

In the following section, we provide a Basu type result, which is the key to prove the subsequent results. In section 3, we characterize the distribution of the process we are considering and finally, in section 4, we will show an example.

2 A Basu type result

First, we define the set $S^\perp = \{Z \in \overline{\mathcal{L}}(X_t, t \in [0, T]) \text{ with } E_0[Z\hat{\theta}'] = 0\}$. Let $Z \in S^\perp$ and consider the estimator $\hat{\theta}_Z = \overline{E}[Z\hat{\theta}']\Sigma^{-}\hat{\theta}$ which is the BLUE for $(E_\theta[Z], \theta \in \mathbb{R}^p)$. Since $Z - \hat{\theta}_Z$ is unbiased for 0, $Cov(\hat{\theta}_Z, \hat{\theta}_Z - Z) = 0$. Then, $Var(\hat{\theta}_Z) = Cov(\hat{\theta}_Z, Z) = E_0[\hat{\theta}_Z Z] = 0$, due to $Z \in S^\perp$. Thus,

$$E_\theta[Z] = \overline{E}[Z\hat{\theta}']\Sigma^{-}\hat{\theta}, \text{ a.s. for } Z \in S^\perp, \quad (5)$$

in other words, the BLUE for $E_\theta[Z]$ is a deterministic estimator. Each element Z of S^\perp can be written as $Z = \hat{\theta}_Z + W$, that is, Z is the sum of a deterministic estimator plus an estimator W which has a distribution independent of θ .

In order to prove the main theorem of this section, we need a lemma which is similar to that given in Ibarrola and Pérez-Palomares (2004), so we omit the proof.

Lemma 1. $(\theta_r, r \in K)$ is linearly complete if and only if $\theta_r = g(r)\tilde{\theta}$, a.s. $r \in K$, where $\tilde{\theta} \in \overline{\mathcal{L}}(\theta_r, r \in K)$ and $g(r) = \overline{E}[\theta_r \hat{\theta}']$ (equivalently $E_\theta[\theta_r] = g(r)\theta$).

Now we are in a position to show the main result of this section.

Theorem 1. Let $(\theta_r, r \in K)$ be a family of linear estimators.

(i) If, for each $r \in K$ and $Z \in S^\perp$, θ_r and Z are uncorrelated, then $(\theta_r, r \in K)$ is linearly complete.

(ii) If $(\theta_r, r \in K)$ is linearly sufficient and linearly complete, then θ_r and Z are uncorrelated, for each $Z \in S^\perp$ and $r \in K$.

(iii) If $(\theta_r, r \in K)$ is sufficient and linearly complete, then $(\theta_r, r \in K)$ and Z

are independent, for each $Z \in S^\perp$.

Proof. (i) Let $Z \in \overline{\mathcal{L}}(\theta_r, r \in K)$ with $E_\theta[Z] = 0$, $\theta \in \mathbb{R}^p$, or equivalently $\overline{E}[Z\hat{\theta}'] = 0$. From (4) we have that $E_0[Z\hat{\theta}'] = 0$, so $Z \in S^\perp$. From hypothesis, Z and θ_r are uncorrelated, for all $r \in K$. Since $Z \in \overline{\mathcal{L}}(\theta_r, r \in K)$ it means that $Var(Z) = 0$ and therefore $Z = 0$, a.s. which shows that $(\theta_r, r \in K)$ is linearly complete.

(ii) If $(\theta_r, r \in K)$ is linearly sufficient and linearly complete, then it is the BLUE for its expectation and for each $r \in K$, $\theta_r = g(r)\Sigma^{-}\hat{\theta}$, a.s. with $g(r) = \overline{E}[\theta_r\hat{\theta}']$. Thus, $E_0[\theta_r Z] = g(r)\Sigma^{-}E_0[\hat{\theta}Z]$. Now, it is immediate that $E_0[\theta_r Z] = 0$, for $Z \in S^\perp$.

(iii) Since $(\theta_r, r \in K)$ is linearly complete, from Lemma 1, $\theta_r = g(r)\tilde{\theta}$, $r \in K$, a.s. with $\tilde{\theta} \in \overline{\mathcal{L}}(\theta_r, r \in K)$ and therefore $\tilde{\theta}$ is also linearly complete. It is clear that the σ -algebra generated by the process $(\theta_r, r \in K)$ and the σ -algebra generated by $\tilde{\theta}$ differ in null sets. Therefore, the sufficiency of $(\theta_r, r \in K)$ is translated to the $\tilde{\theta}$ estimator. Moreover, by the same argument, it suffices to show that $\tilde{\theta}$ and Z are independent for each $Z \in S^\perp$.

Thus, let $Z \in S^\perp$ of dimension m and let A and B be borel-sets in \mathbb{R}^p and \mathbb{R}^m , respectively. First of all, we will prove that $P_0(\tilde{\theta} \in A \mid Z \in B) = P_0(\tilde{\theta} \in A) P_0(Z \in B)$.

We have, by definition of P_θ , that

$$P_0(\tilde{\theta} \in A \mid Z \in B) = P_\theta(\tilde{\theta} - E_\theta[\tilde{\theta}] \in A \mid Z - E_\theta[Z] \in B).$$

By (5), there exists a matrix T such that $E_\theta[Z] = T\hat{\theta}$, P_θ - a.s. Thus,

$$P_0(\tilde{\theta} \in A \mid Z \in B) = P_\theta(\tilde{\theta} - E_\theta[\tilde{\theta}] \in A \mid Z - T\hat{\theta} \in B).$$

Since $\tilde{\theta}$ is a sufficient estimator, there exists a version of $P_\theta(Z - T\hat{\theta} \in B | \tilde{\theta})$ independent of θ . Let $h(\tilde{\theta})$ be this version. Then, we obtain

$$P_0(\tilde{\theta} \in A \mid Z \in B) = E_\theta[1_A(\tilde{\theta} - E_\theta[\tilde{\theta}])h(\tilde{\theta})] = E_0[1_A(\tilde{\theta})h(\tilde{\theta} + E_\theta[\tilde{\theta}])]. \quad (6)$$

On the other hand, since $\tilde{\theta}$ is linearly complete we have, from Lemma 1, that the support of $\tilde{\theta}$ is in the subspace $\{E_\theta[\tilde{\theta}], \theta \in \mathbb{R}^p\}$.

Now, we consider the measure ν in \mathbb{R}^p induced by $\tilde{\theta}$ with respect to P_0 , that is, $\nu(\cdot) = P_0(\tilde{\theta} \in \cdot)$. From (6) and the considerations above, we conclude that

$$P_0(\tilde{\theta} \in A \mid Z \in B) = \int_A h(u + z)\nu(du),$$

for all z in the subspace $\{E_\theta[\tilde{\theta}], \theta \in \mathbb{R}^p\}$, in particular, for all $z \in \mathbb{R}^p$ except in ν -null sets. Integrating this equality with respect to $\nu(dz)$ and applying Fubini's theorem, we assure that

$$P_0(\tilde{\theta} \in A \mid Z \in B) = \int_A \nu(du) \int_{\mathbb{R}^p} h(u + z)\nu(dz). \quad (7)$$

On the other hand,

$$P_0(Z \in B) = P_\theta(Z - T\hat{\theta} \in B) = E_\theta[h(\tilde{\theta})] = E_0[h(\tilde{\theta} + E_\theta[\tilde{\theta}])],$$

that is,

$$P_0(Z \in B) = \int_{\mathbb{R}^p} h(z + u)\nu(dz)$$

for all $u \in \mathbb{R}^p$ except in ν -null sets. Then, from this equality together with (7) we deduce that

$$P_0(\tilde{\theta} \in A \mid Z \in B) = \int_A \nu(du)P_0(Z \in B) = P_0(Z \in B)P_0(\tilde{\theta} \in A).$$

Finally, it is immediate that if $\tilde{\theta}$ and Z are independent under P_0 , then they are independent under P_θ .

3 Sufficiency and Gaussian Processes

In this section we will see that the sufficiency of $\hat{\theta}$ (with additional hypotheses) leads to the Gaussian character of the process $(X_t, t \in [0, T])$. For this purpose, we consider the eigenvalues λ_k and the eigenfunctions $e_k(t)$, $t \in [0, T]$, of the covariance function $B(t, s)$, which verify that

$$B(t, s) = \sum_{k=1}^{\infty} \lambda_k e_k(t) e_k(s), \quad t, s \in [0, T],$$

where

$$\lambda_k > 0, \quad \lambda_k e_k(t) = \int_0^T B(t, s) e_k(s) ds, \quad t \in [0, T]$$

and $\int_0^T e_k(t) e_j(t) dt = \delta_{kj}$, with $\delta_{kj} = 0$ if $k \neq j$ and $\delta_{kj} = 1$ if $k = j$.

Then, we can write the following Karhunen-Loève expansion, in square mean sense, of the process $(X_t, t \in [0, T])$,

$$X_t = A(t)\theta + \sum_{k=1}^{\infty} \sqrt{\lambda_k} e_k(t) Z_k,$$

where $Z_k = \frac{1}{\sqrt{\lambda_k}} \int_0^T e_k(t) Z_t dt$. The goal is to prove that if $\hat{\theta}$ is a sufficient estimator, then some of the variables Z_k have a Gaussian distribution. If we prove the Gaussian character of Z_k under P_0 it is automatically proved for P_θ , $\theta \in \mathbb{R}^p$, so we consider the above expansion of the process with respect to P_0 . Moreover, if we consider the inner product $\langle Y, Z \rangle_0 = E_0[YZ]$ in $\overline{\mathcal{L}}(X_t, t \in [0, T])$, then we can write the expansion of the process as

$$X_t = \sum_{k=1}^{\infty} \langle Z_t, Z_k \rangle_0 Z_k,$$

where the convergence is in square mean sense with respect to P_0 . Thus, we have

$$\hat{\theta} = \sum_{k=1}^{\infty} \langle \hat{\theta}, Z_k \rangle_0 Z_k.$$

We define the following index sets,

$$T_{S^\perp} = \{k \in \mathbb{N} : Z_k \in S^\perp\} = \{k \in \mathbb{N} : \langle \hat{\theta}, Z_k \rangle_0 = 0\}$$

and

$$T_S = \{k \in \mathbb{N} : Z_k = \hat{\theta}'c, P_0 - a.s. \text{ for some } c \in \mathbb{R}^p\}.$$

Thus, $\hat{\theta}$ can be written as

$$\hat{\theta} = \sum_{k \in T_S} \langle \hat{\theta}, Z_k \rangle_0 Z_k + \sum_{k \in (T_S \cup T_{S^\perp})^c} \langle \hat{\theta}, Z_k \rangle_0 Z_k. \quad (8)$$

Lemma 2. For each $k_0 \in (T_S \cup T_{S^\perp})^c$, we can find a $Z \in S^\perp$ and a finite subset $C \subseteq (T_S \cup T_{S^\perp})^c$ with $k_0 \in C$ such that

$$Z = \sum_{k \in C} \langle Z, Z_k \rangle_0 Z_k,$$

where $\langle Z, Z_{k_0} \rangle_0 \neq 0$.

Proof. Let $k_0 \in (T_S \cup T_{S^\perp})^c$. Since k_0 does not belong to T_S , there exists a $Z \in S^\perp$ such that $\langle Z, Z_{k_0} \rangle_0 \neq 0$. Since $Z \in S^\perp$, $\langle Z, \hat{\theta} \rangle_0 = 0$ and from (8)

$$\sum_{k \in (T_S \cup T_{S^\perp})^c} \langle \hat{\theta}, Z_k \rangle_0 \langle Z_k, Z \rangle_0 = 0.$$

Each $\langle \hat{\theta}, Z_k \rangle_0$ is a vector of \mathbb{R}^p , so the last equality implies that there exists a finite subset $C \subseteq (T_S \cup T_{S^\perp})^c$, with $k_0 \in C$, such that,

$$\sum_{k \in C} \langle \hat{\theta}, Z_k \rangle_0 \langle Z_k, Z \rangle_0 = 0.$$

Thus, it is immediate that

$$Z^* = \sum_{k \in C} \langle Z_k, Z \rangle_0 Z_k$$

belongs to S^\perp and $\langle Z^*, Z_{k_0} \rangle_0 = \langle Z, Z_{k_0} \rangle_0 \neq 0$, which proves the lemma.

Now we enunciate the main theorem of this section.

Theorem 2. Suppose that the random variables $(Z_k, k \in (T_S \cup T_{S^\perp})^c)$ are jointly independent. Then, we have that if $\hat{\theta}$ is a sufficient estimator then, $(Z_k, k \in (T_S \cup T_{S^\perp})^c)$ are Gaussian random variables.

Proof. First, for each $k \in T_S$, there exists a vector c_k such that $Z_k = \hat{\theta}' c_k$, P_0 -a.s. Using (8) we can find a matrix P such that

$$P\hat{\theta} = \sum_{k \in (T_S \cup T_{S^\perp})^c} \langle \hat{\theta}, Z_k \rangle_0 Z_k,$$

in fact, the matrix $P = (I - \sum_{k \in T_S} \langle \hat{\theta}, Z_k \rangle_0 c_k' c_k, c_k \in \mathbb{R}^p)$.

We fix a $k_0 \in (T_S \cup T_{S^\perp})^c$. From Lemma 2 there exist a $Z \in S^\perp$ and a finite subset $C \subseteq (T_S \cup T_{S^\perp})^c$ such that

$$Z = \sum_{k \in C} c_k Z_k,$$

with $c_{k_0} \neq 0$. We rewrite $P\hat{\theta}$ as

$$P\hat{\theta} = \sum_{k \in C} \langle \hat{\theta}, Z_k \rangle_0 Z_k + Y,$$

where Y is the limit in L^2 of a linear combination of $Z_k, k \in (T_S \cup T_{S^\perp})^c \setminus C$. Hence the random variables $((Z_j, j \in C), Y)$ are mutually independent. From the hypothesis, $\hat{\theta}$ is a sufficient estimator, therefore by Theorem 1(iii), $\hat{\theta}$ and Z are independent random variables. Thus, we have that the linear combination $\sum_{k \in C} \langle \hat{\theta}, Z_k \rangle_0 Z_k + Y$ is independent of $\sum_{k \in C} c_k Z_k$ and the random variable Z_{k_0} is present in both linear combinations. The Darmois-Skitovich theorem leads to that Z_{k_0} is a Gaussian random variable. Since k_0 is arbitrary, we have proved Theorem 2.

Corollary 1. Suppose that $T_S = T_{S^\perp} = \emptyset$ and $Z_k, k \geq 1$ are mutually inde-

pendent random variables. Then,

- (i) If $\hat{\theta}$ is a sufficient estimator, then $(X_t, t \in [0, T])$ is a Gaussian process.
- (ii) The linear sufficiency is equivalent to the ordinary sufficiency if and only if $(X_t, t \in [0, T])$ is a Gaussian process.

Proof. (i) is immediate from Theorem 2. (ii) In Ibarrola and Pérez-Palomares (2003) the “if part” is proved when linear estimators of integral type are considered. For estimators in the class $\overline{\mathcal{L}}(X_t, t \in [0, T])$ the result remains valid with some little variations in the proof. For the “only if” part, it suffices to take into account that $\hat{\theta}$ is a linearly sufficient estimator. The corollary is proved.

Corollary 2. Suppose that $T_S = \emptyset$ and $(Z_k, k \in (T_S \cup T_{S^\perp})^c)$ are jointly independent random variables. If $\hat{\theta}$ is a sufficient estimator then every estimator BLUE has a normal distribution.

Proof. First, from (8) and Theorem 2, $\hat{\theta}$ has a normal distribution. On the other hand, every estimator BLUE is unique and is therefore a linear combination of the $\hat{\theta}$ estimator, showing that it has a normal distribution.

4 An example

We shall illustrate the results given in the preceding sections by considering two type of covariance functions. First, we consider $B(s, t) = \min\{s, t\}$, $s, t \in [0, 1]$ and the function $A(t) = c't^a$, $c \in \mathbb{R}^p$, $a > 1/2$. When $0 < a \leq 1/2$, the BLUE estimator $\hat{\theta}$ has variance zero so we do not consider these trivial cases. Define the measure $F(u, 1] = c a u^{a-1}$, $0 \leq u < 1$, then it is easy to check that

$\int_0^1 B(t, s)F'(ds) = A(t)$ and that $\int_0^1 A'(s)F'(ds) = \frac{a^2}{2a-1}(c'c)$. Therefore

$$\int_0^1 (B(t, s) + A(t)A'(s))F'(ds) = A(t) + \frac{a^2}{2a-1}t^a c'c' = (1 + \frac{a^2}{2a-1}c'c)A(t).$$

This implies that the estimator $\hat{\theta}$ verifying (2) is

$$\begin{aligned}\hat{\theta} &= (1 + \frac{a^2}{2a-1}c'c)^{-1} \int_0^1 X_u F(du) \\ &= (1 + \frac{a^2}{2a-1}c'c)^{-1} ca(X_1 - (a-1)) \int_0^1 X_u u^{a-2} du.\end{aligned}$$

It is well known that the eigenvalues and the eigenfunctions of the covariance function B are $\lambda_k = (k - 1/2)^2 \pi^2$ and $e_k(t) = \sqrt{2} \sin \frac{\pi}{2}(2k-1)t$, $t \in [0, 1]$, $k = 1, 2, \dots$. Next, we are going to identify the subsets T_{S^\perp} and T_S (defined in Section 3). From definition, $k \in T_{S^\perp} \Leftrightarrow E_0[\hat{\theta}Z_k] = 0$ and it is immediate that $E_0[\hat{\theta}Z_k] = 0 \Leftrightarrow \int_0^1 t^a e_k(t)dt = \sqrt{2} \int_0^1 t^a \sin \frac{\pi}{2}(2k-1)t dt = 0$. On the other hand, applying a change of variable to the last integral and integrating by parts, we obtain

$$\int_0^1 t^a \sin \frac{\pi}{2}(2k-1)t dt = \frac{a}{(\frac{\pi}{2}(2k-1))^{a+1}} \int_0^{\frac{\pi}{2}(2k-1)} t^{a-1} \cos t dt.$$

Thus, $k \in T_{S^\perp} \Leftrightarrow \int_0^{\frac{\pi}{2}(2k-1)} t^{a-1} \cos t dt = 0$. We shall show that $T_{S^\perp} = \emptyset$. First, we will prove that $2k \notin T_{S^\perp}$, $\forall k$.

$$\begin{aligned}\int_0^{\frac{\pi}{2}(4k-1)} t^{a-1} \cos t dt &= \int_0^{\frac{\pi}{2}} t^{a-1} \cos t dt + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} t^{a-1} \cos t dt \\ &\quad + \sum_{j=1}^{k-1} \left(\int_{\frac{\pi}{2}(4j-1)}^{\frac{\pi}{2}(4j+1)} t^{a-1} \cos t dt + \int_{\frac{\pi}{2}(4j+1)}^{\frac{\pi}{2}(4j+3)} t^{a-1} \cos t dt \right).\end{aligned}$$

Applying a change of variable to the integrals $\int_{\frac{\pi}{2}(4j+1)}^{\frac{\pi}{2}(4j+3)} t^{a-1} \cos t dt$, we have

$$\begin{aligned} \int_0^{\frac{\pi}{2}(4k-1)} t^{a-1} \cos t dt &= \int_0^{\frac{\pi}{2}} t^{a-1} \cos t dt + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} t^{a-1} \cos t dt \\ &+ \sum_{j=1}^{k-1} \int_{\frac{\pi}{2}(4j-1)}^{\frac{\pi}{2}(4j+1)} (t^{a-1} - (t+\pi)^{a-1}) \cos t dt. \end{aligned}$$

Since $\cos t > 0$ if $t \in (\frac{\pi}{2}(4j-1), \frac{\pi}{2}(4j+1))$ and $t^{a-1} - (t+\pi)^{a-1}$ has not sign changes for each a fixed, we have $\int_0^{\frac{\pi}{2}(4k-1)} t^{a-1} \cos t dt \neq 0$.

In a similar way, it is proved that $(2k-1) \notin T_{S^\perp}$ and we can conclude that $T_{S^\perp} = \emptyset$. In order to characterize the subset T_S , note that if $k \in T_S$ then, from the definition of this set, $E_0[X_t Z_k] = E_0[X_t \hat{\theta}'] c^*$, $c^* \in \mathbb{R}^p$. But, on the other hand, we know that $E_0[X_t Z_k] = \sqrt{\lambda_k} e_k(t)$ and that $E_0[X_t \hat{\theta}'] = t^a c^{*'}$, for a vector $c^* \in \mathbb{R}^p$ (possibly different from the preceding one). Thus, we have that if $k \in T_S$, then $e_k(t) = t^a s_k$, $s_k \in \mathbb{R}$, which is a contradiction. So we can assure that $T_{S^\perp} = T_S = \emptyset$. If we suppose that $(Z_k, k \in \mathbb{N})$ are independent random variables, Corollary 1 let us to conclude that $\hat{\theta}$ is a sufficient estimator if and only if $(X_t, t \in [0, 1])$ is a Brownian motion.

The second example corresponds to the covariance function $B(s, t) = \min\{s, t\} - st$, $s, t \in [0, 1]$ which is associated to the Brownian bridge. Consider $A(t) = c'(t^a - t)$, $c \in \mathbb{R}^p$ and $a > 1/2$, $a \neq 1$. As before, the case $0 < a \leq 1/2$ is trivial. The construction of $\hat{\theta}$ is straightforward and its expression is

$$\hat{\theta} = (1 + \frac{(a-1)^2}{2a-1} c'c)^{-1} \int_0^1 X_u F(du),$$

with $F(u, 1] = cau^{a-1}$, $0 \leq u < 1$, the same measure as in the preceding example.

The eigenvalues and the eigenfunctions for this covariance function are $\lambda_k = (k\pi)^{-2}$ and $e_k(t) = \sqrt{2} \sin(k\pi t)$, $t \in [0, 1]$, $k = 1, 2, \dots$. We also have, $k \in T_{S^\perp} \Leftrightarrow \int_0^1 e_k(t)(t^a - t)dt = 0$. First, if we consider an even natural number, we

obtain from several integrations by parts

$$\int_0^1 (t^a - t) \sin(2k\pi t) dt = \frac{-a(a-1)}{(2k\pi)^2} \int_0^1 t^{a-2} \sin(2k\pi t) dt.$$

Then, $2k \in T_{S^\perp} \Leftrightarrow \int_0^1 t^{a-2} \sin(2k\pi t) dt = 0 \Leftrightarrow \int_0^{2k\pi} t^{a-2} \sin t dt = 0$. Now,

$$\begin{aligned} \int_0^{2k\pi} t^{a-2} \sin t dt &= \sum_{j=0}^{k-1} \left(\int_{2j\pi}^{(2j+1)\pi} t^{a-2} \sin t dt + \int_{(2j+1)\pi}^{(2j+2)\pi} t^{a-2} \sin t dt \right) \\ &= \sum_{j=0}^{k-1} \int_{2j\pi}^{(2j+1)\pi} (t^{a-2} - (t+\pi)^{a-2}) \sin t dt. \end{aligned}$$

As in the above example, we have that if $t \in (2j\pi, (2j+1)\pi)$ then $\sin t > 0$. On the other hand, if $1/2 < a < 2$, $t^{a-2} - (t+\pi)^{a-2} > 0$ but, if $a > 2$ then $t^{a-2} - (t+\pi)^{a-2} < 0$. So, we can conclude that if $a \neq 2$, $2k \notin T_{S^\perp}$. If $a = 2$, we have $2k \in T_{S^\perp}$, $\forall k$.

In a similar way we can prove that $(2k-1) \notin T_{S^\perp}$, $\forall k$ and, as before, it is immediate that $T_S = \emptyset$. Finally, we conclude that if $(Z_k, k = 1, 2, \dots)$ are independent random variables, then $\hat{\theta}$ is a sufficient estimator if and only if $(X_t, t \in [0, 1])$ is a Brownian bridge (with $A(t) = (t^a - t)c'$, $c \in \mathbb{R}^p$, $a > 1/2$, $a \neq \{1, 2\}$).

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