

HERMITE EXPANSIONS OF C_0 -GROUPS AND COSINE FUNCTIONS

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ABSTRACT. In this paper we introduce vector-valued Hermite expansions to approximate one-parameter operator families such as C_0 -groups and cosine functions. In both cases we estimate the rate of convergence of these Hermite expansions to the related family and compare with other known approximations. Finally we illustrate our results with particular examples of C_0 -groups and cosine functions and their Hermite expansions.

1. INTRODUCTION

Representations of functions through expansions of orthogonal polynomials such as Legendre, Hermite or Laguerre are well known in the classical analysis. They allow to approximate functions by series of orthogonal polynomials on different types of convergence: a pointwise way, uniformly in certain sets, or in Lebesgue norm. Two classical monographs where we can find this kind of results are [19, Chapter 4] and [25, Chapter IX]. In this paper we are concentrated on Hermite expansions.

For all $n \in \mathbb{N} \cup \{0\}$, classical Hermite polynomials are defined by Rodrigues' formula

$$H_n(t) := (-1)^n e^{t^2} \frac{d^n}{dt^n} (e^{-t^2})(t), \quad t \in \mathbb{R}.$$

We mention an interesting theorem which may be found in [19, Theorem 2, Sec. 4.15] and whose original statement was given in [26].

Theorem 1.1. *Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a differentiable function such that $\int_{-\infty}^{\infty} e^{-t^2} |f(t)|^2 dt < +\infty$.*

Then the series $\sum_{n=0}^{\infty} c_n(f) H_n(t)$, where

$$c_n(f) := \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} f(t) H_n(t) dt,$$

converges pointwise to $f(t)$.

For $\lambda \in \mathbb{C}$, we consider the exponential function e_λ where $e_\lambda(t) := e^{\lambda t}$ for $t \in \mathbb{R}$. Observe that the function e_λ is expressed in a Hermite expansion by

$$(1.1) \quad e^{\lambda t} = \sum_{n=0}^{\infty} \frac{1}{2^n n!} \lambda^n e^{\frac{\lambda^2}{4}} H_n(t), \quad t \in \mathbb{R},$$

for $\lambda \in \mathbb{C}$, see [19, Example 2, p.74].

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Now let $c(t) := \cos(\sqrt{a}t)$ with $a > 0$ for $t \in \mathbb{R}$. This function also verifies the hypothesis of Theorem 1.1, and we write

$$(1.2) \quad \cos(\sqrt{a}t) = \sum_{n=0}^{\infty} \frac{1}{2^{2n}(2n)!} (-a)^n e^{-\frac{a}{4}} H_{2n}(t), \quad t \in \mathbb{R}.$$

Different aspects of Hermite expansions have been studied in the literature: for example, the asymptotic behavior of coefficients in [8]; expansions of analytic functions in [24]; and estimations of coefficients and integrals related with Hermite expansions in [21].

Other approximation results involve Hermite functions defined by

$$\mathcal{H}_n(t) := \frac{1}{(2^n n! \pi^{\frac{1}{2}})^{\frac{1}{2}}} e^{-\frac{t^2}{2}} H_n(t), \quad t \in \mathbb{R}.$$

These functions form an orthonormal basis of the Hilbert space $L^2(\mathbb{R})$. Moreover, let f be in $L^p(\mathbb{R})$, $\frac{4}{3} < p < 4$, and $a_k(f) := \int_{-\infty}^{\infty} f(t) \mathcal{H}_k(t) dt$ for $k \in \mathbb{N} \cup \{0\}$. Then $\|S_n(f) - f\|_p \rightarrow 0$ as $n \rightarrow \infty$, with

$$S_n(f) := \sum_{k=0}^n a_k(f) \mathcal{H}_k, \quad n \in \mathbb{N} \cup \{0\},$$

see [5, Theorem 2].

Also these Hermite series allow to approximate tempered distributions. Let \mathcal{S} be the Schwartz class and \mathcal{S}' the space of tempered distribution on \mathbb{R} . Then the equality

$$T = \sum_{k=0}^{\infty} \langle T, \mathcal{H}_k \rangle \mathcal{H}_k$$

holds in the weak sense, see for example [23, pp. 143]. Moreover, Hermite expansions of Dirac distribution and the distribution principal value of $\frac{1}{x}$ may be found in [2, pp 191-193] and [11, Section 2]. Hermite expansions of products of tempered distributions are considered in detail in [11].

In other hand, a C_0 -group $(T(t))_{t \in \mathbb{R}}$ is a one parameter (strongly continuous) family of linear and bounded operators on a Banach space X , which verifies the exponential law, i.e., $T(t+s) = T(t)T(s)$ for all $t, s \in \mathbb{R}$, and $T(0) = I$. The (densely defined) operator A given by

$$Ax := \lim_{t \rightarrow 0} \frac{T(t)x - x}{t},$$

when the limit exists ($x \in D(A)$), is called the infinitesimal generator of C_0 -group; moreover, for $t \in \mathbb{R}$,

$$\frac{d}{dt} T(t)x = T(t)Ax, \quad x \in D(A),$$

see more details in monographs [3, 13]. This family of operators is interpreted as $(e^{tA})_{t \in \mathbb{R}}$, since it is the solution of the abstract Cauchy problem of first order. Likewise, it seems natural to consider identity (1.1) in the vector-valued version, and we have that

$$T(t)x = \sum_{n=0}^{\infty} \frac{1}{2^{2n} n!} A^n T^{(g)}\left(\frac{1}{4}\right)x H_n(t), \quad t \in \mathbb{R},$$

for $x \in D(A)$, Theorem 3.1 (ii), where $T^{(g)}(z)$ is a holomorphic C_0 -semigroup of angle $\frac{\pi}{2}$ whose generator is A^2 .

A cosine function $(C(t))_{t \in \mathbb{R}}$ is a one-parameter (strongly continuous) family of linear and bounded operators on a Banach space X associated to the second order abstract Cauchy problem. This family satisfies the cosine identity $C(t+s) + C(t-s) = 2C(t)C(s)$ for $s, t \in \mathbb{R}$, and $C(0) = I$. The (densely defined) operator A given by $Ax := C''(0)x$, for $x \in D(A)$, is the generator of the cosine function, see other details in [3, Section 3.14]. In Theorem 4.1 (ii), we show that

$$C(t)x = \sum_{n=0}^{\infty} \frac{1}{2^{2n}(2n)!} A^n T^{(c)}\left(\frac{1}{4}\right) x H_{2n}(t), \quad t \in \mathbb{R},$$

for $x \in D(A)$; in this case A also generates a holomorphic C_0 -semigroup of angle $\frac{\pi}{2}$, $(T^{(c)}(z))_{z \in \mathbb{C}^+}$. Note that this Hermite expansion extends the formula (1.2).

In the literature, there exist many different approximations of C_0 -semigroups, as Euler, Yosida, Dunford-Segal or subdiagonal Padé approximations, see [15] and references in [1]. However there are not some many approximations of C_0 -groups and cosine functions: stable rational approximations for exponential function are considered to treat hyperbolic problems, i.e., C_0 -groups in [9] and cosine functions in [16, Section 4]. In [4, Theorem 4.2], the author approximates the solution of fractional Cauchy problem of order $\alpha \geq 1$ and gives the rate of convergence; the case of cosine functions is included for $\alpha = 2$.

The paper is organized as follows. In the second section, we consider the functions $t \mapsto \frac{1}{2^{2n}n! \sqrt{\pi}} e^{-t^2} H_n(t)$ (for $n \in \mathbb{N} \cup \{0\}$) which have a key role in Theorem 1.1. They verify interesting properties, similar to Hermite polynomials (Proposition 2.3). We estimate their p -norm in Theorem 2.4 and obtain Hermite expansions for Dirichlet and Fejér kernels in Proposition 2.1. We also consider Hermite expansions in Lebesgue space $L^p(\mathbb{R})$ for $1 \leq p < \infty$. Finally, we give a vector-valued version of Theorem 1.1 on an abstract Banach space X .

Main theorems of this paper appear in the third and fourth sections. In Theorem 3.1 and Theorem 4.1, we express C_0 -groups and cosine functions through Hermite expansions. In Theorem 3.4 and Theorem 4.3, we give the rate of the Hermite expansions to the C_0 -group and cosine function respectively, which depends on the regularity of the initial data. We compare with the Laguerre expansions obtained for the case of C_0 -semigroups in [1, Theorem 5.2]. Two consequences of these Hermite expansions for C_0 -groups and cosine functions are the Hermite expansions for Féjer operators (Corollary 3.6 and Theorem 4.3 (iii)) and series representations for subordinated holomorphic semigroups (Theorem 3.1 (iii) and 4.1(iii)).

Although, cosine functions and C_0 -semigroups differ really (see for example [7]), the nature of C_0 -groups and cosine functions are quite similar. In the case of Hermite expansions, analogous results hold (compare Theorem 3.1 and 4.1) and both approaches are compatible, see Remark 4.2 and subsection 5.4.

In the last section, we present some examples of C_0 -groups and cosine functions and their Hermite expansions: shift and multiplication groups, cosine functions on sequence spaces, and matrix approach to cosine functions and C_0 -groups. We also give some comments and open questions to motivate forthcoming papers in this topic.

Notation. Given $1 \leq p < \infty$, let $L^p(\mathbb{R})$ be the set of Lebesgue p -integrable functions, that is, f is a measurable function and

$$\|f\|_p := \left(\int_{-\infty}^{\infty} |f(t)|^p dt \right)^{\frac{1}{p}} < \infty;$$

for $p = 2$, remind that $L^2(\mathbb{R})$ is a Hilbert space with $\langle \cdot, \cdot \rangle$ the usual inner product, and $L^\infty(\mathbb{R})$ the set of essential bounded Lebesgue functions with the norm $\|f\|_\infty := \text{esssup}_{t \in \mathbb{R}} |f(t)|$. We call $C_0(\mathbb{R})$ the set of continuous functions defined in $(-\infty, \infty)$ such that $\lim_{t \rightarrow \pm\infty} f(t) = 0$, with the norm $\|\cdot\|_\infty$.

2. HERMITE FUNCTIONS AND HERMITE EXPANSIONS ON BANACH SPACES

Hermite polynomials are solutions of second order differential equation

$$(2.1) \quad y'' - 2ty' + 2ny = 0, \quad n \geq 0, \quad t \in \mathbb{R}.$$

Furthermore, they verify the following condition of orthogonality:

$$\frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} H_n(t) H_m(t) e^{-t^2} dt = \delta_{n,m}, \quad n, m \in \mathbb{N} \cup \{0\},$$

where $\delta_{n,m}$ is the Kronecker delta. These polynomials verify the following properties:

$$(2.2) \quad H_n(-t) = (-1)^n H_n(t), \quad n \in \mathbb{N} \cup \{0\},$$

$$H_{2m}(0) = (-1)^m \frac{(2m)!}{m!} \text{ and } H_{2m+1}(0) = 0,$$

$$(2.3) \quad H_{n+1}(t) = 2tH_n(t) - 2nH_{n-1}(t); \quad H_0(t) = 1; \quad H_1(t) = 2t;$$

for $t \in \mathbb{R}$.

From here on out, Muckenhoupt estimates for \mathcal{H}_n and H_n , which are well known in the classical theory of orthogonal polynomials, will be used several times: there exist constants $\gamma > 0$ and $C > 0$ (independent of n and t) such that

$$(2.4) \quad |\mathcal{H}_n(t)| \leq \begin{cases} C(N^{\frac{1}{3}} + |N - t^2|)^{-\frac{1}{4}}, & \text{for } t^2 \leq 2N; \\ C(e^{-\gamma t^2}), & \text{for } t^2 > 2N, \end{cases}$$

where $N = 2n + 1$ for all $n \geq 0$, see [22, Formula (2.4)]. Therefore, given $t \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that $t^2 \leq 2(2n_0 + 1)$, there exists $C > 0$ (independent of n_0 and t) such that

$$(2.5) \quad |H_n(t)| \leq C \left(\frac{e^{\frac{t^2}{2}} \sqrt{2^n n!}}{n^{\frac{1}{2}}} \right), \quad n \geq n_0,$$

where we have applied the first inequality of Muckenhoupt estimates (2.4).

Now we consider briefly the following functions. The Dirichlet kernel $(d_t)_{t \in \mathbb{R}}$ is defined by

$$d_t(s) := \frac{\sin(ts)}{\pi s}, \quad s \in \mathbb{R} \setminus \{0\},$$

and the Fejér kernel $(f_t)_{t \in \mathbb{R}}$, by

$$f_t(s) := \frac{1 - \cos(ts)}{\pi s^2}, \quad s \in \mathbb{R} \setminus \{0\}.$$

Note that $\|f_t\|_1 = |t|$ and $f_t(s) = \int_0^t d_u(s) du$, for $t, s \in \mathbb{R}$, see more details in [17]. Observe that Dirichlet and Fejér kernels verify the conditions of Theorem 1.1 and we obtain their Hermite expansions.

Proposition 2.1. *Fixed $s \in \mathbb{R}$, we have that:*

$$\begin{aligned} d_t(s) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} s^{2n-2} e^{-\frac{s^2}{4}}}{2^{2n-1} (2n-1)! \pi} H_{2n-1}(t), \quad t \in \mathbb{R}, \\ f_t(s) &= \frac{1 - e^{-\frac{s^2}{4}}}{\pi s^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} s^{2n-2} e^{-\frac{s^2}{4}}}{2^{2n} (2n)! \pi} H_{2n}(t), \quad t \in \mathbb{R}. \end{aligned}$$

Proof. As $d_t(s) = d_{-t}(s)$ for $t, s \in \mathbb{R}$, we have that $c_{2n}(d_{(\cdot)}(s)) = 0$ for $n \geq 0$ and

$$\begin{aligned} c_{2n-1}(d_{(\cdot)}(s)) &= \frac{(-1)^{2n-1}}{2^{2n-1} (2n-1)! \sqrt{\pi}} \int_{-\infty}^{\infty} \frac{d^{2n-1}}{dt^{2n-1}} (e^{-t^2}) \frac{\sin(ts)}{\pi s} dt \\ &= \frac{(-1)^{n-1}}{2^{2n-1} (2n-1)! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} s^{2n-2} \frac{\cos(ts)}{\pi} dt = \frac{(-1)^{n-1} s^{2n-2} e^{-\frac{s^2}{4}}}{2^{2n-1} (2n-1)! \pi}, \end{aligned}$$

for $n \geq 1$, and $s \in \mathbb{R}$. Using similar ideas, we show that $c_{2n-1}(f_{(\cdot)}(s)) = 0$ for $n \geq 1$,

$$\begin{aligned} c_0(f_{(\cdot)}(s)) &= \frac{1 - e^{-\frac{s^2}{4}}}{\pi s^2}, \\ c_{2n}(f_{(\cdot)}(s)) &= \frac{(-1)^{n-1} s^{2n-2} e^{-\frac{s^2}{4}}}{2^{2n} (2n)! \pi}, \end{aligned}$$

for $s \in \mathbb{R}$ and $n \geq 1$. □

Definition 2.2. *For $n \in \mathbb{N} \cup \{0\}$, we denote by h_n the family of functions defined by*

$$h_n(t) := \frac{1}{2^n n! \sqrt{\pi}} e^{-t^2} H_n(t), \quad t \in \mathbb{R}.$$

Note that

$$h_n(t) = \frac{1}{\sqrt{2^n n! \pi^{\frac{1}{2}}}} e^{-\frac{t^2}{2}} \mathcal{H}_n(t) = \frac{(-1)^n}{2^n n! \sqrt{\pi}} \frac{d^n}{dt^n} (e^{-t^2})(t), \quad t \in \mathbb{R},$$

with $n = 0, 1, 2, \dots$. As the Hermite polynomials, these functions verify recurrence relations and differential equations. We collect some of them in the following proposition which we will use later. We apply formulae (2.3) and (2.1) to show parts (i) and (ii). The complete proof is left to the reader.

Proposition 2.3. *The family of functions $\{h_n\}_{n \geq 0}$ satisfies:*

- (i) $2(n+1)h_{n+1}(t) = 2th_n(t) - h_{n-1}(t)$, for $t \in \mathbb{R}$ and $n \geq 1$.
- (ii) $h_n''(t) + 2th_n'(t) + 2(n+1)h_n(t) = 0$, for $t \in \mathbb{R}$ and $n \geq 0$.
- (iii) $h_n^{(k)} = (-1)^k 2^k (n+1) \dots (n+k) h_{n+k}$ for $n, k \geq 0$.

Theorem 2.4. *Take $1 \leq p < \infty$.*

- (i) *The set of functions $\{h_n\}_{n \geq 0} \subset L^p(\mathbb{R})$ and*

$$\|h_n\|_p \leq \frac{C_p}{\sqrt{2^n n!}}, \quad n \geq 1.$$

(ii) For $n \geq 1$,

$$\frac{n+1}{n} \max_{t \in \mathcal{Z}(H_n)} |h_{n+1}(t)| = \frac{1}{2n} \|h_{n-1}\|_\infty \leq \|h_n\|_1$$

where $\mathcal{Z}(H_n)$ is the set of zeros of H_n .

Proof. (i) Note that $|h_n|$ is an even function by (2.2). Then we divide the integral in two parts to apply the Muckenhoupt estimates (2.4). For $N = 2n + 1$, and $t^2 \leq 2N$, we have that

$$\int_0^{\sqrt{2N}} |h_n(t)|^p dt \leq \frac{C_p}{(2^n n! \pi^{\frac{1}{2}})^{\frac{p}{2}}} \int_0^{\sqrt{2N}} e^{-\frac{pt^2}{2}} (N^{\frac{1}{3}} + |N - t^2|)^{-\frac{p}{4}} dt \leq \frac{C_p}{(2^n n!)^{\frac{p}{2}} n^{\frac{p}{12}}}.$$

On the other hand, we have that

$$\int_{\sqrt{2N}}^\infty |h_n(t)|^p dt \leq \frac{C_{p,\gamma}}{(2^n n! \pi^{\frac{1}{2}})^{\frac{p}{2}}} \int_{\sqrt{2N}}^\infty e^{-\frac{pt^2}{2}} e^{-\gamma pt^2} dt \leq \frac{C_p}{(2^n n!)^{\frac{p}{2}}},$$

and we obtain the inequality for $1 \leq p < \infty$.

(ii) For $n \geq 1$, note that

$$|h_{n-1}(t)| \leq \int_t^\infty |h'_{n-1}(s)| ds \leq 2n \|h_n\|_1, \quad t \in \mathbb{R},$$

where we have applied Proposition 2.3 (iii) to get the second inequality. To show the first equality, due to $\lim_{t \rightarrow \pm\infty} h_{n-1}(t) = 0$, we obtain that

$$\begin{aligned} \|h_{n-1}\|_\infty &= \max_{\{t \in \mathbb{R} \mid (h_{n-1})'(t)=0\}} |h_{n-1}(t)| = \max_{\{t \in \mathbb{R}^+ \mid h_n(t)=0\}} |h_{n-1}(t)| \\ &= 2(n+1) \max_{\{t \in \mathbb{R}^+ \mid h_n(t)=0\}} |h_{n+1}(t)| = 2(n+1) \max_{\{t \in \mathbb{R}^+ \mid H_n(t)=0\}} |h_{n+1}(t)|, \end{aligned}$$

where we have used Proposition 2.3 (iii) and (i) to get the result. \square

Remark 2.5. For $p = 1$, we apply the Cauchy-Schwartz inequality and $\{\mathcal{H}_n\}_n$ is an orthonormal basis on $L^2(\mathbb{R})$ to get a direct proof:

$$\|h_n\|_1 = \frac{1}{\sqrt{2^n n! \pi^{\frac{1}{2}}}} \int_{-\infty}^\infty e^{-\frac{t^2}{2}} |\mathcal{H}_n(t)| dt \leq \frac{1}{\sqrt{2^n n! \pi^{\frac{1}{2}}}} \left(\int_{-\infty}^\infty e^{-t^2} dt \right)^{\frac{1}{2}} = \frac{1}{\sqrt{2^n n!}}.$$

for $n \geq 0$.

Now we consider the expansions of certain functions Hermite polynomials in spaces $L^p(\mathbb{R})$. This result will be used to prove that the span $\{h_n \mid n \geq 0\}$ is dense for all $1 \leq p < \infty$. Observe the set span $\{t^n e^{-t^2}\}_{n \geq 0}$ is dense in $L^1(\mathbb{R})$ ([25, Theorem 5.7.2]) and in $L^2(\mathbb{R})$ ([25, Theorem 5.7.1]).

Theorem 2.6. Take $1 \leq p < \infty$.

(i) Let $\lambda \in \mathbb{R}$, and $\eta_\lambda(t) := e^{-t^2} e^{\lambda t}$ for $t \in \mathbb{R}$. Then we have that

$$\eta_\lambda = \sqrt{\pi} e^{\frac{\lambda^2}{4}} \sum_{n=0}^\infty \lambda^n h_n \text{ in } L^p(\mathbb{R}).$$

(ii) The set span $\{h_n \mid n \geq 0\}$ is dense in $L^p(\mathbb{R})$.

Proof. (i) By (1.1), we get that $e^{-t^2} e^{\lambda t} = \sqrt{\pi} e^{\frac{\lambda^2}{4}} \sum_{n=0}^{\infty} \lambda^n h_n(t)$ pointwise for $\lambda \in \mathbb{R}$. This convergence is in $L^p(\mathbb{R})$ since

$$\left\| \sum_{n=0}^{\infty} \lambda^n h_n \right\|_p \leq C_p \sum_{n=0}^{\infty} \frac{|\lambda|^n}{\sqrt{2^n n!}} \leq C_p \sum_{n=0}^{\infty} \frac{(|\lambda| \sqrt{e})^n}{\sqrt{2^n n^{\frac{n}{2} + \frac{1}{4}}}} < \infty, \quad \lambda \in \mathbb{R},$$

where we have applied Theorem 2.4 (i) and Stirling's formula.

(ii) Using the part (i), it is enough to see that $\text{span}\{e^{-t^2} e^{\lambda t} \mid \lambda \in \mathbb{R}\}$ is dense in $L^p(\mathbb{R})$ to get the result. First, we use the decomposition $L^p(\mathbb{R}) = L^p(\mathbb{R}^-) \oplus L^p(\mathbb{R}^+)$, i.e., $g = g_- + g_+$, with $g_-(t) := g(t)$ ($t \in \mathbb{R}^-$) and $g_+(t) := g(t)$ ($t \in \mathbb{R}^+$). Then we shall prove that $\text{span}\{e^{-t^2} e^{\lambda t} \mid \lambda < 0\}$ is dense in $L^p(\mathbb{R}^+)$ and $\text{span}\{e^{-t^2} e^{\lambda t} \mid \lambda > 0\}$ is dense in $L^p(\mathbb{R}^-)$ to conclude the result. To do this, we apply the Hahn-Banach Theorem. Let $f \in L^q(\mathbb{R}^+)$ with $\frac{1}{p} + \frac{1}{q} = 1$ such that

$$\int_0^{\infty} f(t) e^{-t^2} e^{\lambda t} dt = 0, \quad \lambda < 0.$$

By Hölder inequality, $f e^{-t^2} \in L^1(\mathbb{R}^+)$ and then $0 = \mathcal{L}(f e^{-t^2})(-\lambda)$ for $\lambda < 0$. Since the Laplace transform is injective in $L^1(\mathbb{R}^+)$, we conclude that $f = 0$. The other case is similar. \square

We need to remind the differentiability of vector-valued functions to give a vector-valued version of classical Theorem 1.1. Let $(X, \|\cdot\|)$ be a Banach space, and $f : \mathbb{R} \rightarrow X$ a vector-valued function. We say that f is differentiable at t if exists

$$\lim_{h \rightarrow 0} \frac{1}{h} (f(t+h) - f(t))$$

on X , see more details, for example in [3, Chapter 1]. Using standard ideas on dual theory, we prove the following theorem. The proof runs parallel to [1, Theorem 3.5].

Theorem 2.7. *Let X be a Banach space and $f : \mathbb{R} \rightarrow X$ a differentiable function such that the integral $\int_{-\infty}^{\infty} e^{-t^2} \|f(t)\|^2 dt$ is finite, then the series $\sum_{n=0}^{\infty} c_n(f) H_n(t)$, with*

$$c_n(f) = \int_{-\infty}^{\infty} h_n(t) f(t) dt,$$

converges pointwise to $f(t)$.

3. HERMITE EXPANSIONS FOR C_0 -GROUPS

In this section, we approximate C_0 -groups by their Hermite expansions. A different approach, using stable rational approximations, is posed in [9]. The rate of convergence depends on the smoothness of initial data in both cases, compare [9, Theorem 3] and Theorem 3.4 (ii). We also give a new serie representation for holomorphic semigroups (Theorem 3.1 (iii)) and the Hermite expansion for Féjer operators (Corollary 3.6), both families of operators are subordinated to the initial C_0 -group.

First of all, we give some basic results from C_0 -group theory. Given $(A, D(A))$ a closed operator on a Banach space X , the resolvent operator $\lambda \rightarrow (\lambda - A)^{-1}$ is analytic in the resolvent

set, and

$$\frac{d^n}{d\lambda^n}(\lambda - A)^{-1}x = (-1)^n n! (\lambda - A)^{-n-1}x \text{ for all } n \in \mathbb{N}, \quad x \in X,$$

see [13, p.240]. It is known that A generates a C_0 -group, $(T(t))_{t \in \mathbb{R}}$, if and only if $\pm A$ generates a C_0 -semigroup. Every C_0 -group is exponentially bound, i.e., there exist $M, \omega \geq 0$ such that $\|T(t)\| \leq M e^{\omega|t|}$ for all $t \in \mathbb{R}$. For $\alpha > 0$ and $\Re(\lambda) > \omega$, the fractional powers of the resolvent operator is defined by

$$(\lambda \mp A)^{-\alpha}x := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\lambda t} T(\pm t)x dt, \quad x \in X,$$

see [18, Proposition 11.1]. Note that the operator $(A^2, D(A^2))$ generates a holomorphic C_0 -semigroup of angle $\frac{\pi}{2}$,

$$T^{(g)}(z)x := \frac{1}{\sqrt{4\pi z}} \int_{-\infty}^\infty e^{-\frac{t^2}{4z}} T(t)x dt, \quad \Re(z) > 0,$$

for $x \in X$, see [3, Corollary 3.7.15]. We say that a C_0 -group is uniformly bounded if $\|T(t)\| \leq M$ for all $t \in \mathbb{R}$. For further details see, for example, monographies [3, 13].

Theorem 3.1. *Let $(T(t))_{t \in \mathbb{R}}$ be a C_0 -group on a Banach space X with infinitesimal generator $(A, D(A))$.*

(i) *For $n \in \mathbb{N} \cup \{0\}$, we get that*

$$\int_{-\infty}^\infty h_n(t) T(t)x dt = \frac{1}{2^n n!} A^n T^{(g)}\left(\frac{1}{4}\right)x, \quad x \in X.$$

In the case that $\sup_{t \in \mathbb{R}} \|T(t)\| < \infty$, we have that

$$\|A^n T^{(g)}\left(\frac{1}{4}\right)x\| \leq M \sqrt{2^n n!} \|x\|, \quad n \in \mathbb{N}, \quad x \in X,$$

for some $M > 0$.

(ii) *For $x \in D(A)$, the following equality holds:*

$$T(t)x = \sum_{n=0}^\infty \frac{1}{2^n n!} A^n T^{(g)}\left(\frac{1}{4}\right)x H_n(t), \quad t \in \mathbb{R}.$$

(iii) *In the case that $\sup_{t \in \mathbb{R}} \|T(t)\| < \infty$, we have that*

$$T^{(g)}(z)x = \sum_{n=0}^\infty \frac{1}{2^{2n} n!} A^{2n} T^{(g)}\left(\frac{1}{4}\right)x (4z - 1)^n, \quad \left|z - \frac{1}{4}\right| < \frac{1}{4},$$

for $x \in X$.

Proof. (i) For $n \in \mathbb{N} \cup \{0\}$, we obtain

$$\int_{-\infty}^\infty h_n(t) T(t)x dt = \frac{(-1)^n}{2^n n! \sqrt{\pi}} \int_{-\infty}^\infty \frac{d^n}{dt^n} (e^{-t^2}) T(t)x dt = \frac{A^n}{2^n n! \sqrt{\pi}} \int_{-\infty}^\infty e^{-t^2} T(t)x dt,$$

for $x \in X$, where we have integrated by parts. Then the result is obtained. In the case that $\sup_{t \in \mathbb{R}} \|T(t)\| \leq M$, we apply Theorem 2.4 (i) to get the bound.

(ii) Note that $\int_{-\infty}^{\infty} e^{-t^2} \|T(t)x\|^2 dt < \infty$, for $x \in X$, the function $T(\cdot)x : \mathbb{R} \rightarrow X$ is differentiable at every point and $\frac{d}{dt}T(t)x = T(t)Ax$, for $t \in \mathbb{R}$ and $x \in D(A)$. Then, we apply Theorem 2.7 to get

$$\|T(t)x - \sum_{n=0}^m \frac{1}{2^n n!} A^n T^{(g)}\left(\frac{1}{4}\right)x H_n(t)\| \rightarrow 0, \quad x \in D(A),$$

as $m \rightarrow \infty$, for all $t \in \mathbb{R}$.

(iii) By the part (i) and Stirling's formula, we have that

$$\sum_{n=0}^{\infty} \frac{\|A^{2n}T^{(g)}(\frac{1}{4})x\|}{2^{2n}n!} |4z-1|^n \leq M\|x\| \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} |4z-1|^n \leq M' \sum_{n=0}^{\infty} \frac{|4z-1|^n}{n^{\frac{1}{4}}} < \infty,$$

for $x \in X$ and $|z - \frac{1}{4}| < \frac{1}{4}$. Then we define the analytic family of operators

$$F(z)x := \sum_{n=0}^{\infty} \frac{A^{2n}T^{(g)}(\frac{1}{4})x}{2^{2n}n!} (4z-1)^n, \quad |z - \frac{1}{4}| < \frac{1}{4}, \quad x \in X.$$

Note that for $x \in D(A^2)$, we have that

$$F'(z)x = \sum_{n=1}^{\infty} \frac{A^{2n}T^{(g)}(\frac{1}{4})x}{2^{2n}n!} 4n(4z-1)^{n-1} = \sum_{n=0}^{\infty} \frac{A^{2n}T^{(g)}(\frac{1}{4})(A^2(x))}{2^{2n}n!} (4z-1)^{n-1} = F(z)A^2(x),$$

where we conclude that $F(z)x = T^{(g)}(z)x$ for $x \in X$ and $|z - \frac{1}{4}| < \frac{1}{4}$. \square

Remark 3.2. Now we compare Hermite expansions of C_0 -groups and Laguerre expansions of C_0 -semigroups studied in [1]. Let $(T(t))_{t \in \mathbb{R}}$ be a uniformly bounded C_0 -group with generator $(A, D(A))$. Since $(T(t))_{t \geq 0}$ and $(T(-t))_{t \geq 0}$ are C_0 -semigroups whose generators are A and $-A$, we get that

$$T(t)x = \begin{cases} \sum_{n=0}^{\infty} (-A)^n (1-A)^{-n-\alpha-1} x L_n^{(\alpha)}(t), & \text{for } t > 0, \\ \sum_{n=0}^{\infty} A^n (1+A)^{-n-\alpha-1} x L_n^{(\alpha)}(-t), & \text{for } t < 0, \end{cases}$$

for $x \in D(A)$, $\alpha > -1$ where $L_n^{(\alpha)}$ is the Laguerre polynomial, see [1, Theorem 4.1.(iii)]. Since

$$H_{2n}(t) = (-1)^n 2^{2n} n! L_n^{-\frac{1}{2}}(t^2) \text{ and } H_{2n+1}(t) = (-1)^n 2^{2n+1} n! t L_n^{(\frac{1}{2})}(t^2), \quad t \in \mathbb{R},$$

see for example [19, Formula 4.19.5], we also express the C_0 -group $(T(t))_{t \in \mathbb{R}}$ by

$$T(t)x = \begin{cases} \sum_{n=0}^{\infty} \frac{1}{2^{2n}n!} A^n (1-A)^{-n-\frac{1}{2}} x H_{2n}(\sqrt{t}), & \text{for } t > 0, \\ \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}n!\sqrt{-t}} A^n (1+A)^{-n-\frac{3}{2}} x H_{2n+1}(\sqrt{-t}), & \text{for } t < 0, \end{cases}$$

for $x \in D(A)$. In any case, it is more natural to express a C_0 -group in terms of its Hermite expansion due to the order of convergence is sharper, see Remark 3.5.

As we have commented, it is important to know the rate of approximation of truncated Hermite expansion to the C_0 -group. This theory has a key role in many areas of maths, as in PDE's, harmonic analysis or numerical analysis. Before to calculate the order of convergence in Theorem 3.4, we prove the following technical lemma.

Lemma 3.3. *Let $(T(t))_{t \in \mathbb{R}}$ be a uniformly bounded C_0 -group in a Banach space X with infinitesimal generator $(A, D(A))$ and p a positive integer number. Then*

$$(3.1) \quad \left\| \int_{-\infty}^{\infty} h_n(t) T(t) x dt \right\| \leq \frac{C \sqrt{(n-p)!}}{2^{\frac{n+p}{2}} n!} \|A^p x\|,$$

for $x \in D(A^p)$, $n \geq p$ and C a positive constant.

Proof. We define by $B(x) := \frac{1}{2^n n!} A^n T(g) \left(\frac{1}{4}\right) x = \int_{-\infty}^{\infty} h_n(t) T(t) x dt$, for $x \in X$, see Theorem 3.1 (i). For $x \in D(A^p)$, we apply Proposition 2.3 (iii) and integrate by parts to get that

$$\begin{aligned} B(x) &= \frac{(-1)^p}{2^p n \dots (n-p+1)} \int_{-\infty}^{\infty} \frac{d^p}{dt^p} (h_{n-p})(t) T(t) x dt \\ &= \frac{1}{2^p n \dots (n-p+1)} \int_{-\infty}^{\infty} h_{n-p}(t) A^p T(t) x dt, \end{aligned}$$

where we have used that $\lim_{t \rightarrow \pm\infty} \frac{d^{p-j}}{dt^{p-j}} (h_{n-p})(t) = 0$, for $j = 1, 2, \dots, p$. By Theorem 2.4 (i) for $p = 1$, we conclude that

$$\|B(x)\| \leq \frac{1}{2^p n \dots (n-p+1)} \|h_{n-p}\|_1 \sup_{t \in \mathbb{R}} \|A^p T(t) x\| \leq \frac{C \sqrt{(n-p)!}}{2^{\frac{n+p}{2}} n!} \|A^p x\|$$

for $x \in D(A^p)$. □

We define the m -th partial sum of the Hermite expansion,

$$T_m(t)x := \sum_{n=0}^m \frac{1}{2^n n!} A^n T(g) \left(\frac{1}{4}\right) x H_n(t), \quad x \in X, \quad t \in \mathbb{R},$$

for $m \geq 0$, and we are interested to estimate the rate of convergence of the m -th partial sum $(T_m(t))_{t \in \mathbb{R}}$ to the C_0 -group $(T(t))_{t \in \mathbb{R}}$.

Theorem 3.4. *Let $(T(t))_{t \in \mathbb{R}}$ be a uniformly bounded C_0 -group on a Banach space X with infinitesimal generator $(A, D(A))$. Then for each $t \in \mathbb{R}$ there is a $m_0 \in \mathbb{N}$ such that for $m \geq m_0$ and $2 \leq p \leq m+1$,*

$$\|T(t)x - T_m(t)x\| \leq \frac{C_{t,p}}{m^{\frac{p}{2} - \frac{11}{12}}} \|A^p x\|, \quad x \in D(A^p).$$

Proof. Fix $t \in \mathbb{R}$. By inequalities (2.5) and (3.1), there is a $m_0 \in \mathbb{N}$ such that for $m \geq m_0$,

$$\begin{aligned} \|T(t)x - T_m(t)x\| &\leq \sum_{n=m+1}^{\infty} \left\| \int_{-\infty}^{\infty} h_n(s) T(s) x ds \right\| \|H_n(t)\| \leq \sum_{n=m+1}^{\infty} \frac{C_t}{2^{\frac{p}{2}} n^{\frac{1}{12}}} \sqrt{\frac{(n-p)!}{n!}} \|A^p x\| \\ &\leq \sum_{n=m+1}^{\infty} \frac{C_t}{2^{\frac{p}{2}} n^{\frac{p}{2} + \frac{1}{12}}} \|A^p x\| \leq \frac{C_{t,p}}{m^{\frac{p}{2} - \frac{11}{12}}} \|A^p x\|, \end{aligned}$$

for $x \in D(A^p)$. □

Remark 3.5. Note that for $t > 0$, we consider the m -th partial sum of Laguerre expansion $S_{m,\alpha}(t)x = \sum_{n=0}^m (-A)^n (1-A)^{-n-\alpha-1} x L_n^{(\alpha)}(t)$, ($x \in X$) introduced in [1], see also Remark 3.2.

By [1, Theorem 5.2], we have that

$$\|T(t)x - S_{m,\alpha}(t)x\| \leq \frac{C_{t,p}}{m^{\frac{p}{2}-1}} \|A^p x\|, \quad x \in D(A^p),$$

for $t > 0$ and therefore Hermite approximation is sharper than Laguerre approximation, compare with Theorem 3.4.

Finally, let $T \equiv (T(t))_{t \in \mathbb{R}}$ be a uniformly bounded C_0 -group on a Banach space X and we consider the family of operators

$$\begin{aligned} \mathcal{D}_N^T(t)x &:= \int_{-N}^N d_t(s) T(s)x \, ds, \\ \mathcal{F}^T(t)x &:= \int_{-\infty}^{\infty} f_t(s) T(s)x \, ds, \end{aligned}$$

for $x \in X$ and $N \geq 1$ (functions d_t and f_t are considered in Section 1). The uni-parametric family $(\mathcal{F}^T(t))_{t \in \mathbb{R}}$ is a Fejér family and is studied in detail in [14, Section 5]. In the case that X is a UMD Banach space then there exists $\lim_{N \rightarrow +\infty} \mathcal{D}_N^T(t)x$ for $x \in X$ and defined bounded operators, called $(\mathcal{D}^T(t))_{t \in \mathbb{R}}$ a Dirichlet family, see [14, Section 4].

Corollary 3.6. *Let X be a Banach space and $T \equiv (T(t))_{t \in \mathbb{R}}$ a uniformly bounded C_0 -group on X . For $x \in X$ and $t \in \mathbb{R}$, we have that*

$$(3.2) \quad \mathcal{F}^T(t)x = \int_{-\infty}^{\infty} \frac{1 - e^{-\frac{s^2}{4}}}{\pi s^2} T(s)x \, ds + \sum_{n=1}^{\infty} \left(\int_{-\infty}^{\infty} \frac{(-1)^{n-1} s^{2n-2} e^{-\frac{s^2}{4}}}{2^{2n} (2n)! \pi} T(s)x \, ds \right) H_{2n}(t).$$

Proof. We use Proposition 2.1 to express the Féjér kernel in its Hermite expansion. We need to commute the integral and the serie. Note that the vector-valued coefficients in the expansion (3.2) are bounded by $\|c_{2n}(f_{(\cdot)})\|_1$, where $c_{2n}(f_{(\cdot)})$ is the coefficient of Hermite expansion of Féjér kernel given in Proposition 2.1. Then

$$\|c_{2n}(f_{(\cdot)})\|_1 = \frac{1}{2^{2n-1} (2n)! \pi} \int_0^{\infty} s^{2(n-1)} e^{-\frac{s^2}{4}} \, ds = \frac{(2n-3)!!}{2^{n-1} (2n)! \sqrt{\pi}} = \frac{1}{(2n-1) 2^{2n-1} n! \sqrt{\pi}}.$$

Then we have that

$$\|c_{2n}(f_{(\cdot)})\|_1 |H_{2n}(t)| \leq \frac{C_t}{n^{\frac{4}{3}}},$$

where we have used Muckenhoupt estimates (2.5) and Stirling's formula. Therefore the serie is absolutely convergent and we obtain the result. \square

Remark 3.7. In the case that X is a UMD Banach space, $\mathcal{D}^T(t)x = \frac{d}{dt} \mathcal{F}^T(t)x$ holds for $x \in X$. However we can not conclude that

$$\mathcal{D}^T(t)x = \sum_{n=1}^{\infty} \left(\int_{-\infty}^{\infty} \frac{(-1)^{n-1} s^{2n-2} e^{-\frac{s^2}{4}}}{2^{2n-1} (2n-1)! \pi} T(s)x \, ds \right) H_{2n-1}(t), \quad x \in X,$$

for $t \in \mathbb{R}$ due to

$$\|c_{2n-1}(d_{(\cdot)})\|_1 |H_{2n-1}(t)| \leq \frac{C_t}{n^{\frac{5}{6}}},$$

and the series does not converge.

4. HERMITE EXPANSIONS FOR COSINE FUNCTIONS

In this section we consider operator families related to the second order Cauchy problem, the cosine and sine functions. We obtain their Hermite expansions (Theorem 4.1 (ii)) and a new representation serie for subordinated holomorphic semigroup (Theorem 4.1 (iii)). We present that Hermite expansions for C_0 -group and Hermite expansions for cosine functions are consistent, see Remark 4.2. We also give the rate of convergence of the truncated Hermite expansion to the cosine function, Theorem 4.3 (ii).

Let X be a Banach space and $C : \mathbb{R} \rightarrow \mathcal{B}(X)$ a cosine function, see definition in the Introduction or [3, Section 3.14]. All cosine functions are even, exponentially bounded, $\|C(t)\| \leq Me^{wt}$ with $M \geq 1$ and $w \geq 0$, and

$$\lambda(\lambda^2 - A)^{-1}x = \int_0^\infty e^{-\lambda t} C(t)x dt, \quad \Re(\lambda) > w, x \in X,$$

where A is the generator of $(C(t))_{t \in \mathbb{R}}$. The Euler approximation for cosine functions (for C_0 -semigroups, see for example [3, Corollary 3.3.6] and [15]),

$$C(t)x = \lim_{n \rightarrow \infty} (-1)^n \frac{1}{n!} \left(\frac{n}{t}\right)^{n+1} (\lambda(\lambda^2 - A)^{-1})^{(n)} \left(\frac{n}{t}\right) x, \quad x \in X, \quad t \in \mathbb{R},$$

is a consequence of the Post-Widder inversion formula for the Laplace transform (see for example [3, Theorem 1.7.7]). Other different approach is followed in [16, Section 4] where stable rational schemes are used to approximate cosine functions.

Observe that A generates a holomorphic C_0 -semigroup of angle $\frac{\pi}{2}$,

$$T^{(c)}(z)x = \frac{1}{\sqrt{\pi z}} \int_0^\infty e^{-\frac{t^2}{4z}} C(t)x dt, \quad \Re(z) > 0,$$

for $x \in X$, see for example [3, Corolary 3.14.17]. There are other operators associated to the second order Cauchy problem. The sine function $S : \mathbb{R} \rightarrow \mathcal{B}(X)$ associated with C is defined by

$$S(t)x := \int_0^t C(s)x ds \quad t \in \mathbb{R}, x \in X.$$

Observe that S is an odd function; see more details in [3, Section 3.14].

Theorem 4.1. *Let $(C(t))_{t \in \mathbb{R}}$ be a cosine function on a Banach space X with infinitesimal generator $(A, D(A))$.*

(i) *For $n \in \mathbb{N} \cup \{0\}$, we get that*

$$\int_{-\infty}^\infty h_{2n+1}(t)C(t)x dt = \int_{-\infty}^\infty h_{2n}(t)S(t)x dt = 0, \quad x \in X,$$

$$\int_{-\infty}^\infty h_{2n}(t)C(t)x dt = \frac{1}{2^{2n}(2n)!} A^n T^{(c)}\left(\frac{1}{4}\right)x, \quad x \in X,$$

and

$$\int_{-\infty}^\infty h_{2n+1}(t)S(t)x dt = \frac{1}{2^{2n+1}(2n+1)!} A^n T^{(c)}\left(\frac{1}{4}\right)x, \quad x \in X.$$

In the case that $\sup_{t \in \mathbb{R}} \|C(t)\| < \infty$, we have that

$$\|A^n T^{(c)}\left(\frac{1}{4}\right)x\| \leq M 2^n \sqrt{(2n)!} \|x\|, \quad n \in \mathbb{N}, \quad x \in X.$$

for some $M > 0$.

(ii) For $x \in D(A)$, the following Hermite expansions hold:

$$C(t)x = \sum_{n=0}^{\infty} \frac{1}{2^{2n}(2n)!} A^n T^{(c)}\left(\frac{1}{4}\right)x H_{2n}(t), \quad t \in \mathbb{R},$$

and

$$S(t)x = \sum_{n=0}^{\infty} \frac{1}{2^{2n+1}(2n+1)!} A^n T^{(c)}\left(\frac{1}{4}\right)x H_{2n+1}(t), \quad t \in \mathbb{R}.$$

(iii) In the case that $\sup_{t \in \mathbb{R}} \|C(t)\| < \infty$, the following serie representation holds:

$$T^{(c)}(z)x = \sum_{n=0}^{\infty} \frac{1}{2^{2n}n!} A^n T^{(c)}\left(\frac{1}{4}\right)x (4z-1)^n, \quad \left|z - \frac{1}{4}\right| < \frac{1}{4},$$

for $x \in X$.

Proof. (i) Since h_{2n+1} is an odd function and h_{2n} is even, first identities are verified for $n \in \mathbb{N} \cup \{0\}$. On the other hand,

$$\int_{-\infty}^{\infty} h_{2n}(t)C(t)x dt = \frac{1}{2^{2n}(2n)!\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{d^{2n}}{dt^{2n}}(e^{-t^2})C(t)x dt = 2 \frac{A^n}{2^{2n}(2n)!\sqrt{\pi}} \int_0^{\infty} e^{-t^2} C(t)x dt,$$

for $x \in X$, and

$$\begin{aligned} \int_{-\infty}^{\infty} h_{2n+1}(t)S(t)x dt &= \frac{1}{2^{2n+1}(2n+1)!\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{d^{2n}}{dt^{2n}}(e^{-t^2})C(t)x dt \\ &= 2 \frac{A^n}{2^{2n+1}(2n+1)!\sqrt{\pi}} \int_0^{\infty} e^{-t^2} C(t)x dt. \end{aligned}$$

(ii) Note that $\int_{-\infty}^{\infty} e^{-t^2} \|C(t)x\|^2 dt < \infty$ for $x \in X$; the function $C(\cdot)x : \mathbb{R} \rightarrow X$ is twice differentiable at every point and $\frac{d^2}{dt^2} C(t)x = C(t)Ax$ for $x \in D(A)$, see [3, Proposition 3.14.5 a)]. Then we apply Theorem 2.7 and we have that

$$\|C(t)x - \sum_{n=0}^m \frac{1}{2^{2n}(2n)!} A^n T^{(c)}\left(\frac{1}{4}\right)x H_{2n}(t)\| \rightarrow 0$$

as $m \rightarrow \infty$, for all $t \in \mathbb{R}$. . By the fundamental theorem of calculus, $S(\cdot)x : \mathbb{R} \rightarrow X$ is also differentiable at every point, and the rest is a simple check.

(iii) The proof runs parallel to the proof of Theorem 3.1(iii). \square

Remark 4.2. It is well-know that if A is the generator of a C_0 -group $(T(t))_{t \in \mathbb{R}}$ on a Banach space X , then A^2 generates a cosine function in X given by

$$C(t) = \frac{T(t) + T(-t)}{2}, \quad t \in \mathbb{R},$$

see for example [3, Example 3.14.15]. We apply Theorem 3.1 (ii) to get that

$$C(t)x = \sum_{n=0}^{\infty} \frac{1}{2^{2n}(2n)!} A^{2n} T^{(c)}\left(\frac{1}{4}\right) x H_{2n}(t), \quad t \in \mathbb{R},$$

for $x \in D(A)$ (note that $T^{(c)}(\frac{1}{4}) = T^{(g)}(\frac{1}{4})$). In this particular case, we improve Theorem 4.1 (ii) due to the equality holds for $x \in D(A)$ (larger than the set $D(A^2)$).

Conversely if A generates a uniformly bounded cosine function $(C(t))_{t \in \mathbb{R}}$ in a UMD Banach space X , then $i(-A)^{\frac{1}{2}}$ generates a C_0 -group, $(\tilde{T}(t))_{t \in \mathbb{R}}$, where

$$\tilde{T}(t) := C(t) + i(-A)^{\frac{1}{2}} S(t), \quad t \in \mathbb{R},$$

see for example [3, Theorem 3.16.7] and [12, Corollary 2.6]. By Theorem 4.1 (ii), we have that

$$\begin{aligned} \tilde{T}(t)x &= \sum_{n=0}^{\infty} \frac{1}{2^{2n}(2n)!} A^n T^{(c)}\left(\frac{1}{4}\right) x H_{2n}(t) + i(-A)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{1}{2^{2n+1}(2n+1)!} A^n T^{(c)}\left(\frac{1}{4}\right) x H_{2n+1}(t) \\ &= \sum_{n=0}^{\infty} \frac{1}{2^n n!} (i(-A)^{\frac{1}{2}})^n \tilde{T}^{(g)}\left(\frac{1}{4}\right) x H_n(t) \end{aligned}$$

for $x \in D(A)$, $t \in \mathbb{R}$ (again $T^{(c)}(\frac{1}{4}) = \tilde{T}^{(g)}(\frac{1}{4})$). Note that in this case Theorem 3.1 (ii) is an extension of Theorem 4.1 (ii) due to $D(A) \subset D((-A)^{\frac{1}{2}})$.

Following the same type of arguments as in the previous section, we give the order of convergence of truncated Hermite expansions $(C_m(t))_{t \in \mathbb{R}}$ where

$$C_m(t)x := \sum_{n=0}^m \frac{1}{2^{2n}(2n)!} A^n T\left(\frac{1}{4}\right) x H_{2n}(t), \quad t \in \mathbb{R}, \quad x \in X, \quad m \geq 0,$$

to the cosine function $(C(t))_{t \in \mathbb{R}}$. We also give the Hermite expansion of subordinated Féjer families, $(\mathcal{F}^C(t))_{t \in \mathbb{R}}$, where

$$\mathcal{F}^C(t)x := 2 \int_0^{\infty} f_t(s) C(s)x ds, \quad x \in X,$$

see [14, Corollary 5.5].

Theorem 4.3. *Let $(C(t))_{t \in \mathbb{R}}$ be a uniformly bounded cosine function on a Banach space X with infinitesimal generator $(A, D(A))$.*

(i) *Let p be a positive integer. Then for $x \in D(A^p)$ and $n \geq p$, we get*

$$\left\| \int_{-\infty}^{\infty} h_{2n}(t) C(t)x dt \right\| \leq \frac{C \sqrt{(2n-2p)!}}{2^{n+p}(2n)!} \|A^p x\|,$$

with C a positive constant.

(ii) *Then for each $t \in \mathbb{R}$ there is a $m_0 \in \mathbb{N}$ such that for $m \geq m_0$, and $1 \leq p \leq m+1$, we have that*

$$\|C(t)x - C_m(t)x\| \leq \frac{C_{t,p}}{m^{p-\frac{11}{12}}} \|A^p x\|, \quad x \in D(A^p).$$

(iii) For $x \in X$ and $t \in \mathbb{R}$, the following equality holds:

$$\mathcal{F}^C(t)x = 2 \int_0^\infty \frac{1 - e^{-\frac{s^2}{4}}}{\pi s^2} C(s)x ds + 2 \sum_{n=1}^\infty \left(\int_0^\infty \frac{(-1)^{n-1} s^{2n-2} e^{-\frac{s^2}{4}}}{2^{2n}(2n)!\pi} C(s)x ds \right) H_{2n}(t).$$

Proof. (i) We write $B(x) := \frac{1}{2^{2n}(2n)!} A^n T(\frac{1}{4})x = \int_{-\infty}^\infty h_{2n}(t)C(t)x dt$, for $x \in X$, see Theorem 4.1 (i). We apply Proposition 2.3 (iii) and integrate by parts to obtain

$$B(x) = \frac{1}{2^{2p}2n \dots (2n - 2p + 1)} \int_{-\infty}^\infty h_{2n-2p}(t)A^p C(t)x dt, \quad x \in D(A^p).$$

By Theorem 2.4 (i), we get the following inequality

$$\|B(x)\| \leq \frac{1}{2^{2p}2n \dots (2n - 2p + 1)} \|h_{2n-2p}\|_1 \sup_{t \in \mathbb{R}} \|A^p C(t)x\| \leq \frac{C \sqrt{(2n - 2p)!}}{2^{n+p}(2n)!} \|A^p x\|,$$

for $x \in D(A^p)$.

(ii) Following the same steps as in Theorem 3.4 and using part (i), we take $t \in \mathbb{R}$, and $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ with $1 \leq p \leq m + 1$,

$$\begin{aligned} \|C(t)x - C_m(t)x\| &\leq \sum_{n=m+1}^\infty \left\| \int_{-\infty}^\infty h_{2n}(s)C(s)x ds \right\| \|H_{2n}(t)\| \\ &\leq \sum_{n=m+1}^\infty \frac{C_t}{2^p n^{\frac{1}{12}}} \sqrt{\frac{(2n - 2p)!}{2n!}} \|A^p x\| \leq \frac{C_{t,p}}{m^{p - \frac{1}{12}}} \|A^p x\|, \end{aligned}$$

for $x \in D(A^p)$.

(iii) The proof of this part is similar to the proof of Corollary 3.6. \square

5. EXAMPLES, REMARKS AND FINAL COMMENTS

In this last section we present some concrete examples of C_0 -groups and cosine functions and we apply our results to give their vector-valued Hermite expansions. We also comment some connections with well-known (scalar) Hermite expansions for tempered distributions. Finally we point out some open questions and interesting problems associated to other families of operators.

5.1. Shift group and cosine function. Let $L^p(\mathbb{R})$ with $1 \leq p < \infty$. The shift group (or translation group) in $L^p(\mathbb{R})$, $(T(t))_{t \in \mathbb{R}}$, defined by

$$T(t)f(x) := f(x - t) = \delta_t * f(x) \quad x \in \mathbb{R},$$

is an isometry C_0 -group (we denote by $(\delta_t)_{t \in \mathbb{R}}$ the usual Dirac delta distribution concentrated at t). The infinitesimal generator A is the usual derivation operator, $A = -\frac{d}{dx}$. Furthermore,

$$T^{(g)}(z)f = g_z * f,$$

where g_z is the Gaussian kernel, $g_z(r) = \frac{1}{\sqrt{4\pi z}} e^{-r^2/4z}$, for $\Re z > 0$ and $r \in \mathbb{R}$. By Theorem 3.1

(ii), we obtain the formula

$$\delta_t * f = \sum_{n=0}^\infty (h_n * f)H_n(t) = \sum_{n=0}^\infty \frac{(-1)^n}{2^n n!} (g_{\frac{1}{4}}^{(n)} * f)H_n(t), \quad f \in W^{(1),p}(\mathbb{R})$$

where $W^{(1),p}(\mathbb{R})$ is the Sobolev space defined by $W^{(1),p}(\mathbb{R}) = \{f \in L^p(\mathbb{R}) \mid f' \in L^p(\mathbb{R})\}$.

The distribution principal value of $\frac{1}{x}$ is a tempered distribution whose is expanded in terms of Hermite polynomials, see for example [2, pp 193], [11, Example 2.4]. In general, given $T \equiv (T(t))_{t \in \mathbb{R}}$, a uniformly bounded C_0 -group in a UMD space X , the Hilbert transform \mathcal{H}^T associated to T is defined by

$$\mathcal{H}^T(x) = \lim_{\varepsilon \rightarrow 0^+, N \rightarrow \infty} \frac{i}{\pi} \int_{\varepsilon < |t| < N} \frac{T(t)x}{t}, \quad x \in X,$$

see [20]; relations between Dirichlet and Féjer families and the Hilbert transform \mathcal{H}^T are studied in [14]. It would be interested to obtain the Hermite expansion of the bounded operator \mathcal{H}^T .

Now we define also in $L^p(\mathbb{R})$, the isometry cosine function $(C(t))_{t \in \mathbb{R}}$, by

$$C(t)f(x) := \frac{1}{2}(f(x-t) + f(x+t)) = \frac{1}{2}(\delta_t * f(x) + \delta_{-t} * f(x)) \quad x \in \mathbb{R}.$$

The generator is $A = \frac{d^2}{dx^2}$ and is clear that

$$C(t)f = \sum_{n=0}^{\infty} \frac{1}{2^{2n}(2n)!} (g_{\frac{1}{4}}^{(2n)} * f) H_{2n}(t), \quad f \in W^{(2),p}(\mathbb{R})$$

where $W^{(2),p}(\mathbb{R})$ is the Sobolev space defined by $W^{(2),p}(\mathbb{R}) = \{f \in L^p(\mathbb{R}) \mid f'' \in L^p(\mathbb{R})\}$.

5.2. Multiplication groups and cosine functions. Let (Ω, Σ, μ) be a σ -finite measure space, $1 \leq p < \infty$ and the Lebesgue Banach space $L^p(\Omega)$. We consider the unitary C_0 -group $(T(t))_{t \in \mathbb{R}}$ in $L^p(\Omega)$ defined by

$$T(t)f(s) = e^{itq(s)}f(s), \quad s \in \Omega,$$

where $q : \Omega \rightarrow \mathbb{R}$ is a Lebesgue measurable function. The infinitesimal generator is $A = iq$, $D(A) = \{f \in L^p(\Omega) \mid qf \in L^p(\Omega)\}$. These multiplication groups are treated deeply in [13, Chapter I, Section 4; Chapter II, Section 2.9]. Some known examples are the Fourier Transform of the Gaussian and Poisson semigroups, $q(s) = -s^2$ and $q(s) = -|s|$; recently some interesting examples for $q(s) = -\log(1 + s^2)$ and $q(s) = -\log(1 + |s|)$, are been studied in [10].

Note that

$$T^{(g)}\left(\frac{1}{4}\right)f(s) = \frac{1}{\sqrt{\pi}} \left(\int_{-\infty}^{\infty} e^{-x^2} e^{ixq(s)} dx \right) f(s) = e^{-\frac{q^2(s)}{4}} f(s), \quad s \in \Omega.$$

By Theorem 3.1 (ii), we obtain that

$$T(t)f(s) = \sum_{n=0}^{\infty} \frac{i^n q^n(s)}{2^n n!} e^{-\frac{q^2(s)}{4}} f(s) H_n(t), \quad f \in D(A).$$

For $q(s) = \lambda \in \mathbb{R}$, we get the formula (1.1).

Now we suppose that $m : \Omega \rightarrow \mathbb{R}^-$ is a Lebesgue measurable function. We consider the cosine function $(C(t))_{t \in \mathbb{R}}$ in $L^p(\Omega)$ defined by

$$C(t)f(s) = \cos(t\sqrt{-m(s)})f(s), \quad s \in \Omega,$$

whose generator is $A = m$, $D(A) = \{f \in L^p(\Omega) \mid mf \in L^p(\Omega)\}$. Note that

$$T^{(c)}\left(\frac{1}{4}\right)f(s) = \frac{2}{\sqrt{\pi}} \left(\int_0^{\infty} e^{-t^2} \cos(t\sqrt{-m(s)}) dt \right) f(s) = e^{\frac{m(s)}{4}} f(s), \quad s \in \Omega,$$

where he have used [6, Formula 7.80, p.66]. By Theorem 4.1 (ii), we have that

$$C(t)f(s) = \sum_{n=0}^{\infty} \frac{m^n(s)}{2^{2n}(2n)!} e^{\frac{m(s)}{4}} f(s) H_{2n}(t), \quad f \in D(A),$$

for $t \in \mathbb{R}$ and $s \in \Omega$; in the particular case $m(s) = -a$, (with $a > 0$) we obtain the formula (1.2).

5.3. Cosine functions on sequence spaces. Let $X = c_0, \ell^p$ be spaces of all complex sequence $x = (x_k)_{k \in \mathbb{N}}$ convergent to 0 equipped with the usual norm $\|x\|_{\infty} := \max_{k \in \mathbb{N}} |x_k|$ for $X = c_0$; and

$$\|x\|_p < \infty \text{ where } \|x\|_p := \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}}, \text{ for } 1 \leq p < \infty.$$

For each $n \in \mathbb{N}$ let e_n be the element of X such that $(e_n)_k = \delta_{n,k}$ the Kronecker delta. Every $x \in X$ can be represented as the series $x = \sum_{k=1}^{\infty} x_k e_k$. The family of linear operators $(C(t))_{t \in \mathbb{R}}$ given by

$$C(t)x = \sum_{k=1}^{\infty} \cos(kt) x_k e_k, \quad x \in X,$$

is a strongly continuous cosine family of contractions on X , generated by $(A, D(A))$ where

$$Ax = \sum_{k=1}^{\infty} -k^2 x_k e_k, \quad x \in D(A),$$

and $D(A) = \{x \in X \mid (k^2 x_k)_{k=1}^{\infty} \in X\}$. For $X = c_0$, this example has been studied in [7, Example 2] in an approximation process.

Note that $T^{(c)}\left(\frac{1}{4}\right)x = \sum_{k=1}^{\infty} e^{-\frac{k^2}{4}} x_k e_k$, and

$$A^n T\left(\frac{1}{4}\right)x = (-1)^n \sum_{k=1}^{\infty} k^{2n} e^{-\frac{k^2}{4}} x_k e_k, \quad x \in X.$$

In this case, for $1 \leq p < \infty$, we have that

$$\|A^n T\left(\frac{1}{4}\right)\|_{\mathcal{B}(X)} = \sup_{k \geq 1} (k^{2n} e^{-\frac{k^2}{4}}) \cong \left(\frac{4n}{e}\right)^n \cong C 2^n \sqrt{2n!}, \quad n \geq 1,$$

and Theorem 4.1 (i) is sharp. Then by Theorem 4.1 (ii), we get

$$C(t)x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(2n)!} \left(\sum_{k=1}^{\infty} k^{2n} e^{-\frac{k^2}{4}} x_k e_k \right) H_{2n}(t), \quad x \in D(A),$$

which is a vector-valued version of the identity (1.2).

5.4. Matrix approach to cosine functions and C_0 -groups. ([3, Theorem 3.14.11]) Let A be an operator on a Banach space X . Then the following assertions are equivalent,

- (i) A generates a cosine function $(C(t))_{t \in \mathbb{R}}$ on X .

- (ii) There exists a Banach space V such that $D(A) \hookrightarrow V \hookrightarrow X$ and such that the operator \mathcal{B} , given by

$$\mathcal{B} \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ Ax \end{pmatrix},$$

with $D(\mathcal{B}) := D(A) \times V$, generates a C_0 -group $(\mathcal{J}(t))_{t \in \mathbb{R}}$ on $V \times X$ with the norm $\|(x, y)\|_{V \times X} = \|x\|_V + \|y\|_X$. In this case

$$\mathcal{J}(t) = \begin{pmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{pmatrix}, \quad t \in \mathbb{R},$$

where $S(t)x = \int_0^t C(s)x ds$, for $x \in X$ and $t \in \mathbb{R}$.

Applying Theorem 3.1 (ii), we expand C_0 -group $(\mathcal{J}(t))_{t \in \mathbb{R}}$ through Hermite polynomials: for $(x, y) \in D(\mathcal{B})$

$$\mathcal{J}(t)(x, y) = \sum_{n=0}^{\infty} \mathcal{C}_n(x, y) H_n(t), \quad t \in \mathbb{R},$$

where $\mathcal{C}_n = \frac{1}{2^n n!} \mathcal{B}^n \mathcal{J}^{(g)}\left(\frac{1}{4}\right)$ for $n \geq 0$. Note that

$$\mathcal{B}^{2n} = \begin{pmatrix} A^n & 0 \\ 0 & A^n \end{pmatrix}, \quad \mathcal{B}^{2n+1} = \begin{pmatrix} 0 & A^n \\ A^{n+1} & 0 \end{pmatrix}, \quad n \geq 0,$$

and

$$\mathcal{J}^{(g)}\left(\frac{1}{4}\right)(x, y) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \mathcal{J}(t)(x, y) dt = \begin{pmatrix} T^{(c)}\left(\frac{1}{4}\right) & 0 \\ 0 & T^{(c)}\left(\frac{1}{4}\right) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad (x, y) \in V \times X,$$

where we have used that

$$T^{(c)}\left(\frac{1}{4}\right) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} C(t) dt, \quad \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} AS(t) dt = 0.$$

Then

$$\mathcal{C}_{2n} = \frac{1}{2^{2n} (2n)!} \begin{pmatrix} A^n T^{(c)}\left(\frac{1}{4}\right) & 0 \\ 0 & A^n T^{(c)}\left(\frac{1}{4}\right) \end{pmatrix}, \quad \mathcal{C}_{2n+1} = \frac{1}{2^{2n+1} (2n+1)!} \begin{pmatrix} 0 & A^n T^{(c)}\left(\frac{1}{4}\right) \\ A^{n+1} T^{(c)}\left(\frac{1}{4}\right) & 0 \end{pmatrix},$$

for $n \geq 0$. In other hand, we apply Theorem 4.1(ii) to get that

$$\begin{aligned} \mathcal{J}(t)(x, y) &= \begin{pmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \sum_{n=0}^{\infty} \frac{1}{2^{2n} (2n)!} \begin{pmatrix} A^n T^{(c)}\left(\frac{1}{4}\right) & 0 \\ 0 & A^n T^{(c)}\left(\frac{1}{4}\right) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} H_{2n}(t) \\ &\quad + \sum_{n=0}^{\infty} \frac{1}{2^{2n+1} (2n+1)!} \begin{pmatrix} 0 & A^n T^{(c)}\left(\frac{1}{4}\right) \\ A^{n+1} T^{(c)}\left(\frac{1}{4}\right) & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} H_{2n+1}(t) \\ &= \sum_{n=0}^{\infty} \mathcal{C}_n(x, y) H_n(t) \end{aligned}$$

for $(x, y) \in D(\mathcal{B})$ and $t \in \mathbb{R}$.

In the case that we consider the Banach space $X \times X$ with the norm $\|(x, y)\|_{X \times X} = \|x\| + \|y\|$, the operator A generates a cosine function if and only if $(\mathcal{A}, D(\mathcal{A}))$ generates a once integrated semigroup \mathcal{S} on $X \times X$, where $D(\mathcal{A}) = D(A) \times X$,

$$\mathcal{A} := \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}, \quad \mathcal{S}(t) = \begin{pmatrix} S(t) & \int_0^t S(s) ds \\ C(t) - I & S(t) \end{pmatrix}, \quad t \in \mathbb{R},$$

see [3, Theorem 3.14.7] and [3, Definition 3.2.1]. Hermite expansions for n -times integrated groups have not considered in the literature and seems to be natural to develop this theory. In [16, Theorem 4.1], stable rational approximations for exponential functions are considered to approximate n -times integrated semigroups (and then cosine functions) for smooth initial data. Both approaches might be compared in a forthcoming paper.

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