

# Quadrature Rules for $L^1$ -Weighted Norms of Orthogonal Polynomials

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**Abstract.** In this paper, we obtain  $L^1$ -weighted norms of classical orthogonal polynomials (Hermite, Laguerre and Jacobi polynomials) in terms of the zeros of these orthogonal polynomials; these expressions are usually known as quadrature rules. In particular, these new formulae are useful to calculate directly some positive defined integrals as several examples show.

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## 1. Introduction

A unified approach to classical orthogonal polynomials (Laguerre, Hermite and Jacobi polynomials) is via Rodrigues' formula, i.e.,

$$Q_{n,\omega}(t) = \frac{1}{\omega(t)\mu_n} \frac{d^n}{dt^n}(\omega Q^n)(t), \quad t \in (a, b), \quad (1.1)$$

where  $\omega$  is a weight in the range of definition  $(a, b)$ ,  $Q$  is a polynomial and  $\mu_n$  is a constant depending of  $n \geq 0$ . The following table shows how to obtain Laguerre, Hermite and Jacobi polynomials ( $L_n^{(\alpha)}$ ,  $H_n$ , and  $P_n^{(\alpha,\beta)}$ , respectively) taking different values of  $\omega$ ,  $Q$  and  $\mu_n$ :

Orthogonal polynomial, $Q_{n,\omega}$	$\mu_n$	$\omega$	$Q$	$(a, b)$
Laguerre polynomial, $L_n^{(\alpha)}$	$n!$	$t^\alpha e^{-t}$	$t$	$(0, \infty)$
Hermite polynomial, $H_n$	$(-1)^n$	$e^{-t^2}$	$1$	$(-\infty, \infty)$
Jacobi polynomial, $P_n^{(\alpha,\beta)}$	$(-1)^n 2^n n!$	$(1-t)^\alpha (1+t)^\beta$	$(1-t^2)$	$(-1, 1)$

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The zeros of the orthogonal polynomials  $Q_{n,\omega}$  associated to the distribution  $\omega(t)dt$  on the interval  $[a, b]$  are real and distinct and are located in the interior of the interval  $[a, b]$ , see [14, Theorem 3.3.1]. Note that  $\mathcal{Z}(Q_{n,\omega}) = \mathcal{Z}(\omega Q_{n,\omega})$  on the interval  $(a, b)$  where  $\mathcal{Z}(f)$  is the set the zeros of the function  $f$ .

The well-known Gauss–Jacobi quadrature rule states that

$$\int_a^b p(t)\omega(t)dt = \sum_{j=1}^n \lambda_j p(t_j), \quad t_j \in \mathcal{Z}(Q_{n,\omega}),$$

where  $p$  is an arbitrary polynomial of degree  $2n - 1$  and parameters  $(\lambda_j)_{1 \leq j \leq n}$  are known as Christoffel numbers. The distribution  $\omega(t)dt$  and the integer  $n$  uniquely determine these numbers  $(\lambda_j)_j$ , see for example [14, Theorem 3.4.1; Chapter XV]. It is difficult to state the origin of this theorem but Jacobi must have been aware of it in 1826 [10].

There exists a great number of papers and monographies about location of zeros of orthogonal polynomials and different types of quadrature rules: details of Gauss–Jacobi quadrature rule may be found, for example, in [8, 13] and [14, Chapter XV]. Szegő polynomials and Szegő quadrature formula on the unit circle are studied for the Fejér kernel in [12]. Connections with orthogonal polynomials on the line and Padé approximants are also obtained in [12]. In [9], a number of formulae are derived for the numerical evaluation of singular integrals in the interval  $(-1, 1)$ . These formulae are based on Gauss–Legendre quadrature rule. Later in [4], authors propose to approximate the Hilbert transform of smooth functions using the zeros of Hermite polynomials. In the nice paper [7], various concepts of orthogonality on the real line are reviewed in connection with quadrature rules. Finally, Gaussian and other positive quadrature rules are investigated to deduce some conditions about the existence of prescribed abscissa in [3].

In this paper, we prove a formula similar to Gauss–Jacobi quadrature rule (also named as Gaussian quadrature rule) to obtain  $L^1$ -weighted norm of classical orthogonal polynomials. This kind of result has not been considered before in the literature. In [5, 6], the error of the Gaussian quadrature rule is estimated in an  $L^1$ -weighted norm, only in the Jacobi setting.

A first approach to our problem is the following theorem. Again, the set of zeros of orthogonal polynomials plays an important role.

**Theorem 1.1.** *For  $n \geq 1$ , functions  $Q_{n,\omega}$  satisfy*

$$\|Q_{n,\omega}\|_1 := \int_a^b |Q_{n,\omega}(t)|\omega(t)dt = 2 \frac{\mu_{n-1}}{|\mu_n|} \sum_{j=1}^n (-1)^{j+1} \omega(t_j) Q(t_j) Q_{n-1,\omega}(t_j),$$

where  $t_j \in \mathcal{Z}(Q_{n,\omega})$  and  $a < t_1 < \dots < t_n < b$ .

*Proof.* We call  $t_0 = a$  and  $t_{n+1} = b$ . From Rodrigues’ formula, we obtain that

$$\begin{aligned} \int_a^b |Q_{n,\omega}(t)|\omega(t)dt &= \sum_{j=0}^n \int_{t_j}^{t_{j+1}} \left| \frac{1}{\mu_n} \frac{d^n}{dt^n} (\omega Q^n)(t) \right| dt = \sum_{j=0}^n \frac{(-1)^j}{|\mu_n|} \int_{t_j}^{t_{j+1}} \frac{d^n}{dt^n} (\omega Q^n)(t) dt \\ &= \sum_{j=0}^n \frac{(-1)^j}{|\mu_n|} \frac{d^{n-1}}{dt^{n-1}} (\omega Q^n)(t) \Big|_{t_j}^{t_{j+1}} = 2 \sum_{j=1}^n \frac{(-1)^{j+1}}{|\mu_n|} \frac{d^{n-1}}{dt^{n-1}} (\omega Q^n)(t_j) \end{aligned}$$

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where we have used that  $|\frac{1}{\mu_n} \frac{d^n}{dt^n}(\omega Q^n)| (t) = (-1)^j \frac{1}{|\mu_n|} \frac{d^n}{dt^n}(\omega Q^n)(t)$  for  $t_j < t < t_{j+1}$ , and the function  $\omega Q^n$  and its derivatives of order less than  $n$  vanish at the endpoints  $a$  and  $b$  in the three cases (Laguerre, Hermite, and Jacobi polynomials considered in the Introduction). Now, we apply the formula (1.1) to get that

$$\int_a^b |Q_{n,\omega}(t)|\omega(t)dt = 2 \frac{\mu_{n-1}}{|\mu_n|} \sum_{j=1}^n (-1)^{j+1} \omega(t_j) Q(t_j) Q_{n-1,\omega}(t_j),$$

and we conclude the result. □

In fact, this result may be improved using some recurrence relations; we show that

$$\int_a^b |Q_{n,\omega}(t)|\omega(t)dt = 2c_{n,\omega} \sum_{j=1}^n (-1)^{j+1} \omega(t_j) Q_{n-1,\omega}(t_j),$$

in Corollaries 2.4, 3.4 and 4.7 (where  $c_{n,\omega}$  is a parameter which depends on  $\omega$  and  $n$ ).

In this paper, we are interested to estimate and calculate the following  $L^1$ -weighted norms

$$\| \frac{t^i}{i!} Q_{n,\omega} \omega \|_1 = \int_a^b \frac{|t|^i}{i!} |Q_{n,\omega}(t)|\omega(t)dt, \quad n, i \in \mathbb{N} \cup \{0\}, \quad (1.2)$$

in the setting of classical orthogonal polynomials  $Q_{n,\omega}$ . These  $L^1$ -weighted norms are commonly used in applied and mathematical analysis and related to Sobolev norms (see Remark 2.5). Although a unified presentation might be considered (see Theorem 1.1 and compare Lemmata 2.2, 3.2 and 4.4), we dedicate different sections to results concerning about each family. The aim of this point of view is twofold: first, the situation of the number 0 with respect to the set  $\mathcal{Z}(Q_{n,\omega})$  is essential and different in each case; and second, it allows to handle easily constants and parameters involved in every case.

The main line of reasoning is to study a family of functions defined by  $q_{n,\omega} := \frac{1}{k_n} \omega Q_{n,\omega}$  where the constant  $k_n$  is given by the orthogonal condition,

$$\int_a^b Q_{n,\omega}(t) Q_{m,\omega}(t) \omega(t) dt = k_n \delta_{n,m}, \quad n, m \in \mathbb{N} \cup \{0\}, \quad (1.3)$$

and  $\delta_{n,m}$  is the Kronecker distribution. The exact value of  $k_n$  in each case is presented in the next table:

Orthogonal polynomial, $Q_{n,\omega}$	$k_n$
Laguerre polynomial, $L_n^{(\alpha)}$	$\frac{\Gamma(n+\alpha+1)}{n!}$
Hermite polynomial, $H_n$	$2^n n! \sqrt{\pi}$
Jacobi polynomial, $P_n^{(\alpha,\beta)}$	$\frac{2^{\alpha+\beta+1} \Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{(2n+\alpha+\beta+1) \Gamma(n+\alpha+\beta+1) n!}$

These functions  $q_{n,\omega}$  are fundamental in classical orthogonal expansions ([11, Chapter 4] and [14, Chapter IX]). Recently, the authors have treated them to introduce Laguerre expansions for  $C_0$ -semigroups in [1] and Hermite

expansions for  $C_0$ -groups and cosine function in [2]. In fact, to get sharp estimations of  $\|q_{n,\omega}\|_1$  is the motivating starting point of this paper: sharp estimations allow to assure convergence of vector-valued orthogonal expansions, see more details in [1, 2].

The paper is organized as follows. The second section deal with Laguerre polynomials, the third section with Hermite polynomials and the last one with Jacobi polynomials. Three recurrence relations, differential equations and other known relations for orthogonal polynomials are verified by functions  $q_{n,\omega}$ . We apply the Cauchy–Schwarz inequality to estimate (1.2) in Propositions 2.1, 3.1 and 4.2. Then, we integrate by parts to express  $\int_a^b \frac{t^i}{i!} q_{n,\omega}(t) dt$  by linear combinations of functions  $q_{k,\omega_k}$  (Lemmata 2.2, 3.2 and 4.4). A straightforward consequence of this identity is the equality

$$\int_a^b \frac{t^i}{i!} q_{n,\omega}(t) dt = 0, \quad 0 \leq i \leq n - 1,$$

which may be also shown from the orthogonal relation (1.3).

As consequences of previous results, Theorems 2.3, 3.3 and 4.5 are main results of this paper, where the exact value of (1.2) is obtained for  $0 \leq i \leq n - 1$ . For  $i = 0$  or  $i = n - 1$ , these theorems are improved in Corollaries 2.4, 3.4 and 4.7. These formulae provide a fast and efficient way to calculate some defined integrals, as Examples 2.6, 3.5 and 4.8 show, and may be of interest to general and specific public including mathematical software companies.

## 2. Laguerre Polynomials

Generalized Laguerre polynomials  $\{L_n^{(\alpha)}\}_{n \geq 0}$  ( $\alpha > -1$ ) are given by

$$L_n^{(\alpha)}(t) = \sum_{k=0}^n (-1)^k \binom{n + \alpha}{n - k} \frac{t^k}{k!}, \quad t \geq 0;$$

in particular  $L_0^{(\alpha)}(t) = 1$ ,  $L_1^{(\alpha)}(t) = -t + \alpha + 1$  and  $L_2^{(\alpha)}(t) = \frac{t^2}{2} - (\alpha + 2)t + \frac{(\alpha + 2)(\alpha + 1)}{2}$ . Polynomials  $\{L_n^{(\alpha)}\}_{n \geq 0}$  are solutions of second-order differential equation

$$ty'' + (\alpha + 1 - t)y' + ny = 0, \tag{2.1}$$

and satisfy the following recurrence relations

$$nL_n^{(\alpha)}(t) = (n + \alpha)L_{n-1}^{(\alpha)}(t) - tL_{n-1}^{(\alpha+1)}(t);$$

$$tL_n^{(\alpha+1)}(t) = (n + \alpha)L_{n-1}^{(\alpha)}(t) - (n - t)L_n^{(\alpha)}(t),$$

see for example [11, 14]. Note that  $L_n^\alpha(t) = L_n^{(\alpha+1)}(t) - L_{n-1}^{(\alpha+1)}(t)$  and we iterate to get that

$$L_n^{(\alpha)}(t) = \sum_{k=0}^n (-1)^k L_{n-k}^{(\alpha+1+k)}(t), \quad t \geq 0. \tag{2.2}$$

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Now, we consider the following Laguerre functions  $\{\ell_n^{(\alpha)}\}_{n \geq 0}$  defined by

$$\ell_n^{(\alpha)}(t) := \frac{n!}{\Gamma(n + \alpha + 1)} t^\alpha e^{-t} L_n^{(\alpha)}(t), \quad t \geq 0,$$

for  $\alpha \neq -1, -2, -3, \dots$ , and  $n \in \mathbb{N} \cup \{0\}$ . Recently, these functions have been studied in [1], and the following identity

$$\ell_n^{(\alpha)}(t) = \ell_{n-1}^{(\alpha)}(t) - \ell_{n-1}^{(\alpha+1)}(t), \quad t \geq 0, \quad (2.3)$$

holds, see [1, Proposition 2.3 (i)]. The family  $\{\ell_n^{(\alpha)}\}_{n \geq 0}$  is a total set in  $L^p(\mathbb{R}^+)$  for  $\alpha > -\frac{1}{p}$ , with  $1 \leq p < \infty$  [1, Theorem 3.1 (ii)]. Furthermore, the optimal estimate of  $\|\ell_n^{(\alpha)}\|_1$  has a key role on the study of vector-valued Laguerre expansions. In [1, Remark 2.10], we prove that

$$\frac{M_\alpha}{n^{\frac{\alpha+1}{2}}} \leq \|\ell_n^{(\alpha)}\|_1 \leq \frac{C_\alpha}{n^{\frac{\alpha}{2}}}, \quad n \in \mathbb{N},$$

for  $\alpha > -1$  and  $C_\alpha, M_\alpha > 0$ .

**Proposition 2.1.** *For  $n, i \in \mathbb{N}$ , and  $\alpha > -(i + 1)$ , the inequality*

$$\int_0^\infty \frac{t^i}{i!} |\ell_n^{(\alpha)}(t)| dt \leq C_\alpha \frac{2^i i^{\frac{\alpha}{2} - \frac{1}{4}}}{n^{\frac{\alpha}{2}}},$$

holds with  $C_\alpha$  a constant which does not depend on  $n$  or  $i$ .

*Proof.* We apply the Cauchy–Schwarz inequality to get that

$$\begin{aligned} \int_0^\infty \frac{t^i}{i!} |\ell_n^{(\alpha)}(t)| dt &= \frac{n!}{i! \Gamma(\alpha + n + 1)} \int_0^\infty t^{i+\alpha} e^{-t} |L_n^{(\alpha)}(t)| dt \\ &\leq \frac{n!}{i! \Gamma(\alpha + n + 1)} \left( \int_0^\infty t^{2i+\alpha} e^{-t} dt \right)^{\frac{1}{2}} \left( \int_0^\infty t^\alpha e^{-t} |L_n^{(\alpha)}(t)|^2 dt \right)^{\frac{1}{2}} \\ &= \left( \frac{n! \Gamma(2i + \alpha + 1)}{(i!)^2 \Gamma(\alpha + n + 1)} \right)^{\frac{1}{2}}, \end{aligned}$$

where we have used that functions  $t \mapsto \left( \frac{n!}{\Gamma(\alpha + n + 1)} \right)^{\frac{1}{2}} t^{\frac{\alpha}{2}} e^{-\frac{t}{2}} L_n^{(\alpha)}(t)$  form a Hilbertian basis on  $L^2(\mathbb{R}^+)$ .

Since  $\lim_{n \rightarrow \infty} \frac{\Gamma(n + \alpha)}{\Gamma(n) n^\alpha} = 1$ , we deduce that

$$\frac{n! \Gamma(2i + \alpha + 1)}{(i!)^2 \Gamma(\alpha + n + 1)} \leq C_\alpha \frac{(2i)!(2i + 1)^\alpha}{n^\alpha (i!)^2} \leq C_\alpha \frac{2^{2i} i^{\alpha - \frac{1}{2}}}{n^\alpha},$$

where we have applied Stirling's formula and  $C_\alpha$  is a constant dependent on  $\alpha$  and independent of  $i$  and  $n$ . We conclude the result.  $\square$

**Lemma 2.2.** *For  $n \in \mathbb{N}$  and  $0 \leq i \leq n - 1$ , the Laguerre functions  $\ell_n^{(\alpha)}$  satisfy*

$$\int \frac{t^i}{i!} \ell_n^{(\alpha)}(t) dt = \sum_{k=0}^i \frac{(-1)^k}{(i-k)!} t^{i-k} \ell_{n-1-k}^{(\alpha+1+k)}(t).$$

*Proof.* We integrate by parts  $i$ -times to get that

$$\begin{aligned} \int_0^t \frac{t^i}{i!} \ell_n^{(\alpha)}(t) dt &= \frac{1}{\Gamma(\alpha + n + 1)} \int_0^t \frac{t^i}{i!} \frac{d^n}{dt^n} (t^{n+\alpha} e^{-t})(t) dt \\ &= \frac{1}{\Gamma(\alpha + n + 1)} \sum_{k=0}^i \frac{(-1)^k}{(i-k)!} t^{i-k} \frac{d^{n-k-1}}{dt^{n-k-1}} (t^{n+\alpha} e^{-t})(t) \\ &= \sum_{k=0}^i \frac{(-1)^k}{(i-k)!} t^{i-k} \ell_{n-1-k}^{(\alpha+1+k)}(t), \end{aligned}$$

and we conclude the result. □

**Theorem 2.3.** *Let  $n \in \mathbb{N} \cup \{0\}$ ,  $0 \leq i \leq n - 1$  and  $\alpha > -1$ . Then the Laguerre functions  $\ell_n^{(\alpha)}$  satisfy*

$$\int_0^\infty \frac{t^i}{i!} |\ell_n^{(\alpha)}(t)| dt = 2 \sum_{m=1}^n (-1)^{m+1} \sum_{k=0}^i \frac{(-1)^k}{(i-k)!} t_m^{i-k} \ell_{n-1-k}^{(\alpha+1+k)}(t_m), \quad (2.4)$$

with  $t_m \in \mathcal{Z}(L_n^{(\alpha)}) = \{t_1 < \dots < t_n\}$ .

*Proof.* We write by  $t_0 = 0$  and  $t_{n+1} = +\infty$ . Note that  $L_n^{(\alpha)}(0) = \binom{n+\alpha}{n} > 0$ ,  $|L_n^{(\alpha)}(t)| = (-1)^m L_n^{(\alpha)}(t)$  for  $t_m < t < t_{m+1}$  and then

$$\int_0^\infty \frac{t^i}{i!} |\ell_n^{(\alpha)}(t)| dt = \frac{1}{i!} \sum_{m=0}^n (-1)^m \int_{t_m}^{t_{m+1}} t^i \ell_n^{(\alpha)}(t) dt.$$

We apply Lemma 2.2 to deduce that

$$\begin{aligned} \int_0^\infty \frac{t^i}{i!} |\ell_n^{(\alpha)}(t)| dt &= \sum_{m=0}^n (-1)^m \sum_{k=0}^i \frac{(-1)^k}{(i-k)!} t^{i-k} \ell_{n-1-k}^{(\alpha+1+k)}(t) \Big|_{t_m}^{t_{m+1}} \\ &= 2 \sum_{m=1}^n (-1)^{m+1} \sum_{k=0}^i \frac{(-1)^k}{(i-k)!} t_m^{i-k} \ell_{n-1-k}^{(\alpha+1+k)}(t_m), \end{aligned}$$

and we have used that  $\lim_{t \rightarrow 0^+} t^{i-k} \ell_{n-1-k}^{(\alpha+1+k)}(t) = 0 = \lim_{t \rightarrow \infty} t^{i-k} \ell_{n-1-k}^{(\alpha+1+k)}(t)$ . □

**Corollary 2.4.** *For  $\alpha > -1$  and  $n \in \mathbb{N}$ , the Laguerre functions  $\ell_n^{(\alpha)}$  satisfy*

$$\begin{aligned} \|\ell_n^{(\alpha)}\|_1 &= 2 \sum_{m=1}^n (-1)^{m+1} \ell_{n-1}^{(\alpha)}(t_m), \\ \int_0^\infty t^{n-1} |\ell_n^{(\alpha)}(t)| dt &= \frac{2}{(\alpha + n)} \sum_{m=1}^n (-1)^{m+1} t_m^n \ell_{n-1}^{(\alpha)}(t_m), \end{aligned}$$

with  $t_m \in \mathcal{Z}(L_n^{(\alpha)}) = \{t_1 < \dots < t_n\}$ .

*Proof.* To obtain the first equality, take  $i = 0$  in the Eq. (2.4) and we use that  $\ell_{n-1}^{(\alpha+1)}(t_m) = \ell_{n-1}^{(\alpha)}(t_m)$  for  $t_m \in \mathcal{Z}(L_n^{(\alpha)})$  by equality (2.3). Taking  $i = n - 1$  in (2.4), we get that

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$$\begin{aligned}
 \int_0^\infty \frac{t^{n-1}}{(n-1)!} |\ell_n^{(\alpha)}(t)| dt &= 2 \sum_{m=1}^n (-1)^{m+1} \sum_{k=0}^{n-1} \frac{(-1)^k}{(n-1-k)!} t_m^{n-1-k} \ell_{n-1-k}^{(\alpha+1+k)}(t_m) \\
 &= \frac{2}{\Gamma(\alpha+n+1)} \sum_{m=1}^n (-1)^{m+1} t_m^{\alpha+n} e^{-t_m} \sum_{k=0}^{n-1} (-1)^k L_{n-1-k}^{(\alpha+1+k)}(t_m) \\
 &= \frac{2}{(n-1)!(\alpha+n)} \sum_{m=1}^n (-1)^{m+1} t_m^n \ell_{n-1}^{(\alpha)}(t_m),
 \end{aligned}$$

where we have applied the equality (2.2). □

*Remark 2.5.* Note that the first equality in Corollary 2.4 improves the inequality

$$\|\ell_n^{(\alpha)}\|_1 \geq \max_{t \in \mathcal{Z}(L_n^{(\alpha)})} |\ell_{n-1}^{(\alpha)}(t)|, \quad n \geq 1,$$

shown in [1, Theorem 2.4 (iv)]. On other hand, the equality

$$\frac{d^k}{dt^k} \ell_n^{(\alpha)}(t) = \ell_{n+k}^{(\alpha-k)}(t), \quad t \geq 0,$$

holds for  $k \geq 1$  [1, Proposition 2.3 (vi)] and then, we obtain the following Sobolev norms

$$\int_0^\infty \left| \frac{d^k}{dt^k} \ell_n^{(\alpha)}(t) \right| dt = \int_0^\infty |\ell_{n+k}^{(\alpha-k)}(t)| dt = 2 \sum_{m=1}^{n+k} (-1)^{m+1} \ell_{n+k-1}^{(\alpha-k)}(t_m),$$

for  $\alpha > k - 1$  and  $t_m \in \mathcal{Z}(L_{n+k}^{(\alpha-k)})$ .

*Example 2.6.* We consider  $\ell_2^{(0)}(t) = \frac{1}{2}e^{-t}(t^2 - 4t + 2)$ . By Corollary 2.4, we conclude that

$$\begin{aligned}
 \int_0^\infty \frac{1}{2} e^{-t} |t^2 - 4t + 2| dt &= 2e^{-2} \left( e^{\sqrt{2}}(\sqrt{2} - 1) + e^{-\sqrt{2}}(1 + \sqrt{2}) \right); \\
 \int_0^\infty \frac{1}{2} e^{-t} t |t^2 - 4t + 2| dt &= 2e^{-2} \left( e^{\sqrt{2}}(5\sqrt{2} - 7) + e^{-\sqrt{2}}(5\sqrt{2} + 7) \right).
 \end{aligned}$$

Now, we take  $\ell_1^{(2)}(t) = \frac{1}{6}t^2(3-t)e^{-t}$  to check that  $\frac{1}{6} \int_0^\infty t^2 |3-t| e^{-t} dt = 9e^{-3}$ .

Finally, we take  $\ell_2^{(1)}(t) = \frac{1}{6}t(t^2 - 6t + 6)e^{-t}$  to get that

$$\begin{aligned}
 \frac{1}{6} \int_0^\infty t |t^2 - 6t + 6| e^{-t} dt &= e^{-3+\sqrt{3}}(4\sqrt{3} - 6) + e^{-3-\sqrt{3}}(4\sqrt{3} + 6), \\
 \frac{1}{6} \int_0^\infty t^2 |t^2 - 6t + 6| e^{-t} dt &= 2e^{-3} \left( e^{\sqrt{3}}(14\sqrt{3} - 24) + e^{-\sqrt{3}}(14\sqrt{3} + 24) \right).
 \end{aligned}$$

### 3. Hermite Polynomials

Hermite polynomials are polynomial solutions of second-order differential equation

$$y'' - 2ty' + 2ny = 0. \tag{3.1}$$

First, Hermite polynomials are the following ones:

$$H_0(t) = 1; \quad H_1(t) = 2t; \quad H_2(t) = 4t^2 - 2; \quad H_3(t) = 4t(2t^2 - 3).$$

In the following, we consider a family of Hermite functions in  $\mathbb{R}$  defined by

$$h_n(t) := \frac{1}{2^n n! \sqrt{\pi}} e^{-t^2} H_n(t), \quad t \in \mathbb{R},$$

for  $n \in \mathbb{N} \cup \{0\}$ . They have been studied in detail in [2, Section 2] and, in particular, the following identity

$$h_n^{(k)} = (-1)^k 2^k (n+1) \cdots (n+k) h_{n+k}, \tag{3.2}$$

is proved in [2, Proposition 2.3 (iii)]. The family  $\{h_n\}_{n \geq 0}$  is a total set in  $L^p(\mathbb{R})$ , with  $1 \leq p < \infty$ , and the optimal estimate of  $\|h_n\|_1$  has a great importance on the study of vector-valued Hermite expansions, see more details in [2]. By standard techniques, the known bound

$$\|h_n\|_1 \leq \frac{1}{\sqrt{n! 2^n}}, \quad n \in \mathbb{N},$$

is shown, see for example [2, Remark 2.5]. In the next proposition, we consider  $L^1$ -weighted norms.

**Proposition 3.1.** *For  $n, i \in \mathbb{N}$ , the Hermite functions  $h_n$  satisfy*

$$\int_{-\infty}^{\infty} \frac{|t|^i}{i!} |h_n(t)| dt \leq \frac{1}{\sqrt{2^n n! i! \sqrt{\pi}}}.$$

*Proof.* We apply the Cauchy–Schwarz inequality to obtain that

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{|t|^i}{i!} |h_n(t)| dt &\leq \frac{2}{i! \sqrt{2^n n! \sqrt{\pi}}} \left( \int_0^{\infty} t^{2i} e^{-t^2} dt \right)^{\frac{1}{2}} \left( \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{-t^2} |H_n(t)|^2}{2^n n! \sqrt{\pi}} dt \right)^{\frac{1}{2}} \\ &= \frac{2}{i! \sqrt{2^n n! \sqrt{\pi}}} \left( \frac{\Gamma(i + \frac{1}{2})}{2} \right)^{\frac{1}{2}} \left( \frac{1}{2} \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{2^n n! i! \sqrt{\pi}}}, \end{aligned}$$

where we have used that functions  $t \mapsto \frac{e^{-\frac{t^2}{2}} H_n(t)}{\sqrt{2^n n! \sqrt{\pi}}}$  (for  $n \geq 0$ ) form a Hilbertian basis on  $L^2(\mathbb{R})$ . □

The proof of the next lemma runs parallel to the proof of Lemma 2.2 and we do not include it here.

**Lemma 3.2.** *Take  $n \in \mathbb{N}$  and  $0 \leq i \leq n-1$ . Then the following identity holds:*

$$\int \frac{t^i}{i!} h_n(t) dt = -\frac{1}{n!} \sum_{k=0}^i \frac{(n-1-k)!}{2^{k+1}(i-k)!} t^{i-k} h_{n-1-k}(t).$$

**Theorem 3.3.** *Let be  $n \in \mathbb{N}$ ,  $0 \leq i \leq n-1$  and  $\mathcal{Z}(H_n) = \{t_1 < \dots < t_n\}$ .*

(i) *If  $i$  is even, then the Hermite functions  $h_n$  satisfy*

$$\int_{-\infty}^{\infty} \frac{t^i}{i!} |h_n(t)| dt = \frac{1}{n!} \sum_{m=1}^n (-1)^{m+n} \sum_{k=0}^i \frac{(n-1-k)!}{2^k (i-k)!} t_m^{i-k} h_{n-1-k}(t_m).$$



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(ii) If  $i$  is odd and  $n$  even, then they satisfy

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{|t|^i}{i!} |h_n(t)| dt &= \frac{1}{n!} \sum_{m=1}^{\frac{n}{2}} (-1)^{m+1} \sum_{k=0}^i \frac{(n-1-k)!}{2^k(i-k)!} t_m^{i-k} h_{n-1-k}(t_m) \\ &\quad + \frac{(-1)^{\frac{n}{2}} (n-1-i)!}{n! 2^i} h_{n-1-i}(0) \\ &\quad + \frac{1}{n!} \sum_{m=\frac{n}{2}+1}^n (-1)^m \sum_{k=0}^i \frac{(n-1-k)!}{2^k(i-k)!} t_m^{i-k} h_{n-1-k}(t_m); \end{aligned}$$

and in the case that  $n$  is odd,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{|t|^i}{i!} |h_n(t)| dt &= \frac{1}{n!} \sum_{m=1}^{\frac{n-1}{2}} (-1)^m \sum_{k=0}^i \frac{(n-1-k)!}{2^k(i-k)!} t_m^{i-k} h_{n-1-k}(t_m) \\ &\quad + \frac{1}{n!} \sum_{m=\frac{n+3}{2}}^n (-1)^{m+1} \sum_{k=0}^i \frac{(n-1-k)!}{2^k(i-k)!} t_m^{i-k} h_{n-1-k}(t_m). \end{aligned}$$

*Proof.* (i) We write  $t_0 = -\infty$  and  $t_{n+1} = +\infty$  to get that

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{t^i}{i!} |h_n(t)| dt &= \frac{1}{i!} \sum_{m=0}^n (-1)^{m+n} \int_{t_m}^{t_{m+1}} t^i h_n(t) dt \\ &= \frac{1}{n!} \sum_{m=0}^n (-1)^{m+n+1} \sum_{k=0}^i \frac{(n-1-k)!}{2^{k+1}(i-k)!} t^{i-k} h_{n-1-k}(t) \Big|_{t_m}^{t_{m+1}} \\ &= \frac{1}{n!} \sum_{m=1}^n (-1)^{m+n} \sum_{k=0}^i \frac{(n-1-k)!}{2^k(i-k)!} t_m^{i-k} h_{n-1-k}(t_m), \end{aligned}$$

where we have applied the Lemma 3.2 and  $\lim_{t \rightarrow \pm\infty} t^{i-k} h_{n-1-k}(t) = 0$ .

(ii) Since  $n$  is even, then we prove that

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{|t|^i}{i!} |h_n(t)| dt &= \frac{1}{i!} \sum_{m=0}^{\frac{n}{2}-1} (-1)^{m+1} \int_{t_m}^{t_{m+1}} t^i h_n(t) dt + \frac{(-1)^{\frac{n}{2}-1}}{i} \int_{t_{\frac{n}{2}}}^0 t^i h_n(t) dt \\ &\quad + \frac{(-1)^{\frac{n}{2}}}{i!} \int_0^{t_{\frac{n}{2}+1}} t^i h_n(t) dt + \frac{1}{i!} \sum_{m=\frac{n}{2}+1}^n (-1)^m \int_{t_m}^{t_{m+1}} t^i h_n(t) dt. \end{aligned}$$

By Lemma 3.2, we deduce that

$$\begin{aligned} \int_0^{t_{\frac{n}{2}+1}} t^i h_n(t) dt &= -\frac{1}{n!} \sum_{k=0}^i \frac{i!(n-1-k)!}{2^{k+1}(i-k)!} t_{\frac{n}{2}+1}^{i-k} h_{n-1-k}(t_{\frac{n}{2}+1}) \\ &\quad + \frac{i!(n-1-i)!}{2^{i+1}n!} h_{n-1-i}(0), \end{aligned}$$

and then

$$\begin{aligned} & \frac{(-1)^{\frac{n}{2}}}{i!} \int_0^{t^{\frac{n}{2}+1}} t^i h_n(t) dt + \frac{1}{i!} \sum_{m=\frac{n}{2}+1}^n (-1)^m \int_{t_m}^{t_{m+1}} t^i h_n(t) dt \\ &= (-1)^{\frac{n}{2}} \frac{(n-1-i)!}{2^{i+1} n!} h_{n-1-i}(0) \\ & \quad + \frac{1}{n!} \sum_{m=\frac{n}{2}+1}^n (-1)^m \sum_{k=0}^i \frac{(n-1-k)!}{2^k (i-k)!} t_m^{i-k} h_{n-1-k}(t_m). \end{aligned}$$

Analogously, we get the identities for the first two summands and we conclude the result.

Finally, we consider the case that  $n$  is odd; in this case  $t_{\frac{n+1}{2}} = 0$  and we get that

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{|t|^i}{i!} |h_n(t)| dt &= \frac{1}{i!} \sum_{m=0}^{\frac{n-1}{2}} (-1)^m \int_{t_m}^{t_{m+1}} t^i h_n(t) dt + \frac{1}{i!} \sum_{m=\frac{n+1}{2}}^n (-1)^{m+1} \int_{t_m}^{t_{m+1}} t^i h_n(t) dt \\ &= \frac{1}{n!} \sum_{m=1}^{\frac{n-1}{2}} (-1)^m \sum_{k=0}^i \frac{(n-1-k)!}{2^k (i-k)!} t_m^{i-k} h_{n-1-k}(t_m) \\ & \quad + (-1)^{\frac{n+1}{2}} \frac{(n-1-i)!}{2^{i+1} n!} h_{n-1-i}(0) \\ & \quad + (-1)^{\frac{n-1}{2}} \frac{(n-1-i)!}{2^{i+1} n!} h_{n-1-i}(0) \\ & \quad + \frac{1}{n!} \sum_{m=\frac{n+3}{2}}^n (-1)^{m+1} \sum_{k=0}^i \frac{(n-1-k)!}{2^k (i-k)!} t_m^{i-k} h_{n-1-k}(t_m) \\ &= \frac{1}{n!} \sum_{m=1}^{\frac{n-1}{2}} (-1)^m \sum_{k=0}^i \frac{(n-1-k)!}{2^k (i-k)!} t_m^{i-k} h_{n-1-k}(t_m) \\ & \quad + \frac{1}{n!} \sum_{m=\frac{n+3}{2}}^n (-1)^{m+1} \sum_{k=0}^i \frac{(n-1-k)!}{2^k (i-k)!} t_m^{i-k} h_{n-1-k}(t_m), \end{aligned}$$

and we conclude the result. □

**Corollary 3.4.** *For  $n \in \mathbb{N}$ , the Hermite functions  $h_n$  satisfy*

$$\begin{aligned} \|h_n\|_1 &= \frac{1}{n} \sum_{m=1}^n (-1)^{m+n} h_{n-1}(t_m), \\ \int_{-\infty}^{\infty} \frac{t^{2n}}{(2n)!} |h_{2n+1}(t)| dt &= \frac{1}{(2n+1)!} \sum_{m=1}^{2n+1} (-1)^{m+1} \sum_{k=0}^{2n} \frac{1}{2^k} t_m^{2n-k} h_{2n-k}(t_m); \\ \int_{-\infty}^{\infty} \frac{|t|^{2n-1}}{(2n-1)!} |h_{2n}(t)| dt &= \frac{1}{(2n)!} \sum_{m=1}^n (-1)^{m+1} \sum_{k=0}^{2n-1} \frac{1}{2^k} t_m^{2n-1-k} h_{2n-1-k}(t_m) \\ & \quad + \frac{(-1)^n}{(2n)! 2^{2n-1} \sqrt{\pi}} + \frac{1}{(2n)!} \sum_{m=n+1}^{2n} (-1)^m \sum_{k=0}^{2n-1} \frac{1}{2^k} t_m^{2n-1-k} h_{2n-1-k}(t_m); \end{aligned}$$

with  $t_m \in \mathcal{Z}(H_n) = \{t_1 < \dots < t_n\}$ .

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*Example 3.5.* Let us consider functions  $h_1$ ,  $h_2$  and  $h_3$  defined by

$$h_1(t) = \frac{1}{\sqrt{\pi}} t e^{-t^2}, \quad h_2(t) = \frac{1}{4\sqrt{\pi}} e^{-t^2} (2t^2 - 1), \quad h_3(t) = \frac{1}{12\sqrt{\pi}} e^{-t^2} t(2t^2 - 3).$$

Then, we apply Theorem 3.3 and Corollary 3.4 to get that

$$\begin{aligned} 4\sqrt{\pi} \int_{-\infty}^{\infty} |h_1(t)| dt &= \int_{-\infty}^{\infty} |2t^2 - 1| e^{-t^2} dt = 2\sqrt{2} e^{-\frac{1}{2}}; \\ 4\sqrt{\pi} \int_{-\infty}^{\infty} |t h_2(t)| dt &= \int_{-\infty}^{\infty} |t(2t^2 - 1)| e^{-t^2} dt = 4e^{-\frac{1}{2}} - 1; \\ 12\sqrt{\pi} \int_{-\infty}^{\infty} |h_3(t)| dt &= \int_{-\infty}^{\infty} |t(2t^2 - 3)| e^{-t^2} dt = 1 + 4e^{-\frac{3}{2}}; \\ 12\sqrt{\pi} \int_{-\infty}^{\infty} |t h_3(t)| dt &= \int_{-\infty}^{\infty} |t^2(2t^2 - 3)| e^{-t^2} dt = 3\sqrt{6} e^{-\frac{3}{2}}; \\ 12\sqrt{\pi} \int_{-\infty}^{\infty} |t^2 h_3(t)| dt &= \int_{-\infty}^{\infty} |t^3(2t^2 - 3)| e^{-t^2} dt = 2 \left( 7e^{-\frac{3}{2}} - \frac{1}{2} \right). \end{aligned}$$

## 4. Jacobi Polynomials

Jacobi polynomials  $P_n^{(\alpha, \beta)}$  where

$$P_n^{(\alpha, \beta)}(t) := \sum_{j=0}^n \frac{1}{2^n} \binom{n+\alpha}{n-j} \binom{n+\beta}{j} (t-1)^j (t+1)^{n-j}, \quad t \in \mathbb{R},$$

for  $n \in \mathbb{N}$  and  $\alpha, \beta \in \mathbb{R}$ , are polynomials solutions of second-order differential equation

$$(1-t^2)y''(t) + (\beta - \alpha - (\alpha + \beta + 2)t)y'(t) + n(n + \alpha + \beta + 1)y(t) = 0. \quad (4.1)$$

Note that  $P_n^{(\alpha, \beta)}(1) = \binom{n+\alpha}{n}$ .

Other interesting identities are the following ones,

$$P_n^{(\alpha, \beta)}(t) = (-1)^n P_n^{(\beta, \alpha)}(-t),$$

$$\frac{d}{dt} P_n^{(\alpha, \beta)}(t) = \frac{n + \alpha + \beta + 1}{2} P_{n-1}^{(\alpha+1, \beta+1)}(t), \quad t \in \mathbb{R}.$$

First Jacobi polynomials are  $P_0^{(\alpha, \beta)}(t) = 1$ ; and  $P_1^{(\alpha, \beta)}(t) = \frac{1}{2}(\alpha + \beta + 2)t + \frac{1}{2}(\alpha - \beta)$ . For  $\alpha = \beta = 0$ , polynomials  $P_n^{(0,0)}$  are the known as Legendre polynomials, see for example [14, Chapter 4].

In the following, we define Jacobi functions  $p_n^{(\alpha, \beta)}$  by

$$\begin{aligned} p_n^{(\alpha, \beta)}(t) &:= \frac{(2n + \alpha + \beta + 1)\Gamma(n + \alpha + \beta + 1)n!}{2^{\alpha+\beta+1}\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)} (1-t)^\alpha (1+t)^\beta P_n^{(\alpha, \beta)}(t) \\ &= \frac{(-1)^n (2n + \alpha + \beta + 1)\Gamma(n + \alpha + \beta + 1)}{2^{n+\alpha+\beta+1}\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)} \frac{d^n}{dt^n} ((1-t)^{n+\alpha} (1+t)^{n+\beta})(t), \end{aligned}$$

for  $n \in \mathbb{N} \cup \{0\}$  and  $t \in (-1, 1)$ . The following lemma contains some results for Jacobi functions  $p_n^{(\alpha, \beta)}$  which are similar to some equalities for Jacobi polynomials  $P_n^{(\alpha, \beta)}$ .

**Lemma 4.1.** For  $n \in \mathbb{N}$  and  $\alpha, \beta \in \mathbb{R}$ , the following equalities hold.

(i)

$$\left(p_n^{(\alpha, \beta)}\right)^{(k)} = \frac{(-1)^k}{2^k} \frac{\Gamma(n + \alpha + \beta + 1)}{\Gamma(n + \alpha + \beta + 1 - k)} p_{n+k}^{(\alpha-k, \beta-k)}, \quad k \in \mathbb{N}.$$

(ii)

$$\begin{aligned} p_{n-1}^{(\alpha+1, \beta+1)}(t) &= \frac{(n + \alpha + \beta)(n + \alpha + \beta + 1)}{(2n + \alpha + \beta)(2n + \alpha + \beta - 1)} p_{n-1}^{(\alpha, \beta)}(t) \\ &+ \frac{(n + \alpha + \beta + 1)(\alpha - \beta)}{(2n + \alpha + \beta)(2n + \alpha + \beta + 2)} p_n^{(\alpha, \beta)}(t) \\ &- \frac{(n + \alpha + 1)(n + \beta + 1)}{(2n + \alpha + \beta + 2)((2n + \alpha + \beta + 3))} p_{n+1}^{(\alpha, \beta)}(t). \end{aligned}$$

(iii)

$$\begin{aligned} \frac{(n + \alpha + 1)(n + \beta + 1)}{(2n + \alpha + \beta + 3)} p_{n+1}^{(\alpha, \beta)}(t) &= -\frac{(2n + \alpha + \beta + 2)(n + \alpha + \beta)n}{(2n + \alpha + \beta)((2n + \alpha + \beta - 1))} p_{n-1}^{(\alpha, \beta)}(t) \\ &+ \frac{1}{2(2n + \alpha + \beta)} ((2n + \alpha + \beta + 2)(2n + \alpha + \beta)t + \alpha^2 - \beta^2) p_n^{(\alpha, \beta)}(t). \end{aligned}$$

(iv)

$$\begin{aligned} p_{n-1}^{(\alpha+1, \beta+1)}(t) &= \frac{(n + \alpha + \beta)(2n + \alpha + \beta + 1)}{(2n + \alpha + \beta)(2n + \alpha + \beta - 1)} p_{n-1}^{(\alpha, \beta)}(t) \\ &+ \frac{1}{2} \left( \frac{\alpha - \beta}{2n + \alpha + \beta} - t \right) p_n^{(\alpha, \beta)}(t). \end{aligned}$$

*Proof.* To show the first part, note that

$$\begin{aligned} \frac{d}{dt} p_n^{(\alpha, \beta)}(t) &= \frac{(-1)^n (2n + \alpha + \beta + 1) \Gamma(n + \alpha + \beta + 1)}{2^{n+\alpha+\beta+1} \Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)} \frac{d^{n+1}}{dt^{n+1}} ((1-t)^{n+\alpha} (1+t)^{n+\beta})(t) \\ &= -\frac{n + \alpha + \beta}{2} p_{n+1}^{(\alpha-1, \beta-1)}(t). \end{aligned}$$

We iterate this equality to get  $\left(p_n^{(\alpha, \beta)}\right)^{(k)}$ .

The part (ii) is straightforward consequence of a formula similar to Jacobi polynomials, [14, (4.5.5)]. The part (iii) is obtained from the recurrence formula for Jacobi polynomials, [14, (4.5.1)]. To finish, the part (iv) is obtained from part (ii) and (iii).  $\square$

**Proposition 4.2.** For  $n, i \in \mathbb{N}$ ,  $\alpha, \beta > -1$ , the Jacobi functions  $p_n^{(\alpha, \beta)}$  satisfy the inequality

$$\int_{-1}^1 |t|^i |p_n^{(\alpha, \beta)}(t)| dt \leq C_{\alpha, \beta} \sqrt{\frac{n}{i+1+\gamma}},$$

where  $C_{\alpha, \beta}$  is a independent constant of  $n$  and  $i$  and  $\gamma = \min(\alpha, \beta)$ .

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*Proof.* We denote by  $c_n^{(\alpha,\beta)} = \frac{(2n + \alpha + \beta + 1)\Gamma(n + \alpha + \beta + 1)n!}{2^{\alpha+\beta+1}\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)}$ . By the Cauchy–Schwarz inequality, we get that

$$\begin{aligned} \int_{-1}^1 |t|^i |p_n^{(\alpha,\beta)}(t)| dt &\leq c_n^{(\alpha,\beta)} \left( \int_{-1}^1 t^{2i} (1-t)^\alpha (1+t)^\beta dt \right)^{\frac{1}{2}} \\ &\quad \left( \int_{-1}^1 (1-t)^\alpha (1+t)^\beta |P_n^{(\alpha,\beta)}(t)|^2 dt \right)^{\frac{1}{2}} \\ &\leq \sqrt{c_n^{(\alpha,\beta)}} \left( C_\alpha \int_{-1}^0 t^{2i} (1+t)^\beta dt + C_\beta \int_0^1 t^{2i} (1-t)^\alpha dt \right)^{\frac{1}{2}} \\ &\leq C_{\alpha,\beta} \sqrt{c_n^{(\alpha,\beta)}} \left( \frac{(2i)!}{\Gamma(2i + \beta + 2)} + \frac{(2i)!}{\Gamma(2i + \alpha + 2)} \right)^{\frac{1}{2}} \end{aligned}$$

where we have used that functions  $t \mapsto \sqrt{c_n^{(\alpha,\beta)}} (1-t)^{\frac{\alpha}{2}} (1+t)^{\frac{\beta}{2}} P_n^{(\alpha,\beta)}(t)$  form a Hilbertian basis on  $L^2(-1, 1)$ .

Since  $\lim_{n \rightarrow \infty} \frac{\Gamma(n + \alpha)}{(n-1)! n^\alpha} = 1$ , we deduce that  $\frac{(2i)!}{\Gamma(2i + \beta + 2)} \leq C_\beta \frac{1}{i^{\beta+1}}$ , then

$$c_n^{(\alpha,\beta)} \leq C_{\alpha,\beta} \frac{n(n-1)! n^{\alpha+\beta+1} n!}{(n-1)!^2 n^{\alpha+1} n^{\beta+1}} \leq C_{\alpha,\beta} n$$

and we conclude the result.  $\square$

*Remark 4.3.* In [14, (7.34.1)], the equivalence  $\int_{-1}^1 |p_n^{(\alpha,\beta)}(t)| dt \sim \sqrt{n}$  when  $n \rightarrow \infty$  is stated.

The proof of next lemma is similar to the proof of Lemma 2.2 and we avoid it here.

**Lemma 4.4.** *For  $n \in \mathbb{N}$  and  $0 \leq i \leq n-1$ , the Jacobi functions  $p_n^{(\alpha,\beta)}$  satisfy*

$$\int \frac{t^i}{i!} p_n^{(\alpha,\beta)}(t) dt = - \sum_{k=0}^i \frac{2^{k+1} \Gamma(n + \alpha + \beta + 1)}{\Gamma(n + k + \alpha + \beta + 2)} \frac{t^{i-k}}{(i-k)!} p_{n-1-k}^{(\alpha+1+k, \beta+1+k)}(t).$$

**Theorem 4.5.** *Take  $n \in \mathbb{N}$ ,  $0 \leq i \leq n-1$ ;  $\alpha, \beta > -1$  and  $\mathcal{Z}(P_n^{(\alpha,\beta)}) = \{t_1 < \dots < t_{n_0} < t_{n_0+1} < t_n\}$  with  $0 \in [t_{n_0}, t_{n_0+1})$ .*

(i) *In the case that  $i$  is even, the Jacobi functions  $p_n^{(\alpha,\beta)}$  satisfy*

$$\begin{aligned} &\int_{-1}^1 \frac{|t|^i}{i!} |p_n^{(\alpha,\beta)}(t)| dt \\ &= \sum_{m=1}^n (-1)^{m+n} \sum_{k=0}^i \frac{2^{k+2} \Gamma(n + \alpha + \beta + 1)}{\Gamma(n + k + \alpha + \beta + 2)} \frac{t_m^{i-k}}{(i-k)!} p_{n-1-k}^{(\alpha+1+k, \beta+1+k)}(t_m). \end{aligned}$$

(ii) In the case that  $i$  is odd, they satisfy

$$\begin{aligned} & \int_{-1}^1 \frac{|t|^i}{i!} |p_n^{(\alpha,\beta)}(t)| dt \\ &= \sum_{m=1}^{n_0} (-1)^{m+n+1} \sum_{k=0}^i \frac{2^{k+2}\Gamma(n+\alpha+\beta+1)}{\Gamma(n+k+\alpha+\beta+2)} \frac{t_m^{i-k}}{(i-k)!} p_{n-1-k}^{(\alpha+1+k,\beta+1+k)}(t_m) \\ & \quad + (-1)^{n_0+n} \frac{2^{i+2}\Gamma(n+\alpha+\beta+1)}{\Gamma(n+i+\alpha+\beta+2)} p_{n-1-i}^{(\alpha+1+i,\beta+1+i)}(0) \\ & \quad + \sum_{m=n_0+1}^n (-1)^{m+n} \sum_{k=0}^i \frac{2^{k+2}\Gamma(n+\alpha+\beta+1)}{\Gamma(n+k+\alpha+\beta+2)} \frac{t_m^{i-k}}{(i-k)!} p_{n-1-k}^{(\alpha+1+k,\beta+1+k)}(t_m). \end{aligned}$$

*Proof.* (i) The proof is similar to the proof of Theorem 3.3 (i) due to  $P_n^{(\alpha,\beta)}(1) > 0$ ; in this case, we apply Lemma 4.4.

(ii) Suppose that  $n$  is even. Again, we denote by  $t_0 = -1$  and  $t_{n+1} = 1$ . Then, we show that

$$\begin{aligned} \int_{-1}^1 \frac{|t|^i}{i!} |p_n^{(\alpha,\beta)}(t)| dt &= \frac{1}{i!} \sum_{m=0}^{n_0-1} (-1)^{m+1} \int_{t_m}^{t_{m+1}} t^i p_n^{(\alpha,\beta)}(t) dt + \frac{(-1)^{n_0+1}}{i} \int_{t_{n_0}}^0 t^i p_n^{(\alpha,\beta)}(t) dt \\ & \quad + \frac{(-1)^{n_0}}{i!} \int_0^{t_{n_0+1}} t^i p_n^{(\alpha,\beta)}(t) dt + \frac{1}{i!} \sum_{m=n_0+1}^n (-1)^m \int_{t_m}^{t_{m+1}} t^i p_n^{(\alpha,\beta)}(t) dt. \end{aligned}$$

By Lemma 4.4, we deduce that

$$\begin{aligned} \int_0^{t_{n_0+1}} \frac{t^i}{i!} p_n^{(\alpha,\beta)}(t) dt &= - \sum_{k=0}^i \frac{2^{k+1}\Gamma(n+\alpha+\beta+1)}{\Gamma(n+k+\alpha+\beta+2)} \frac{t_{n_0+1}^{i-k}}{(i-k)!} p_{n-1-k}^{(\alpha+1+k,\beta+1+k)}(t_{n_0+1}) \\ & \quad + \frac{2^{i+1}\Gamma(n+\alpha+\beta+1)}{\Gamma(n+i+\alpha+\beta+2)} p_{n-1-i}^{(\alpha+1+i,\beta+1+i)}(0) \end{aligned}$$

and then

$$\begin{aligned} & \frac{(-1)^{n_0}}{i!} \int_0^{t_{n_0+1}} t^i p_n^{(\alpha,\beta)}(t) dt + \frac{1}{i!} \sum_{m=n_0+1}^n (-1)^m \int_{t_m}^{t_{m+1}} t^i p_n^{(\alpha,\beta)}(t) dt \\ &= (-1)^{n_0} \frac{2^{i+1}\Gamma(n+\alpha+\beta+1)}{\Gamma(n+i+\alpha+\beta+2)} p_{n-1-i}^{(\alpha+1+i,\beta+1+i)}(0) \\ & \quad + \sum_{m=n_0+1}^n (-1)^m \sum_{k=0}^i \frac{2^{k+2}\Gamma(n+\alpha+\beta+1)}{\Gamma(n+k+\alpha+\beta+2)} \frac{t_m^{i-k}}{(i-k)!} p_{n-1-k}^{(\alpha+1+k,\beta+1+k)}(t_m). \end{aligned}$$

Analogously, we get the identities for the first two summands and we conclude the result.

Finally, the case when  $n$  is odd is similar to the previous one. □

*Remark 4.6.* In the case that  $\alpha = \beta$ , then  $p_{2n}^{(\alpha,\alpha)}$  is a even function and  $p_{2n-1}^{(\alpha,\alpha)}$  is odd. In this last case,  $0 \in \mathcal{Z}(P_{2n-1}^{(\alpha,\alpha)})$  and  $n_0 = n - 1$ .

**Corollary 4.7.** For  $\alpha, \beta > -1$  and  $n \in \mathbb{N}$ , the Jacobi functions  $p_n^{(\alpha, \beta)}$  satisfy

$$\|p_n^{(\alpha, \beta)}\|_1 = \frac{4}{(n + \alpha + \beta + 1)} \frac{(n + \alpha + \beta)}{(2n + \alpha + \beta)} \frac{(2n + \alpha + \beta + 1)}{(2n + \alpha + \beta - 1)} \sum_{m=1}^n (-1)^{m+n} p_{n-1}^{(\alpha, \beta)}(t_m),$$

with  $t_j \in \mathcal{Z}(P_n^{(\alpha, \beta)}) = \{t_1 < \dots < t_n\}$ .

*Proof.* By Theorem 4.5, we get that

$$\begin{aligned} \int_{-1}^1 |p_n^{(\alpha, \beta)}(t)| dt &= \sum_{m=1}^n (-1)^{m+n} \frac{2^2 \Gamma(n + \alpha + \beta + 1)}{\Gamma(n + \alpha + \beta + 2)} p_{n-1}^{(\alpha+1, \beta+1)}(t_m) \\ &= \frac{4}{(n + \alpha + \beta + 1)} \frac{(n + \alpha + \beta)}{(2n + \alpha + \beta)} \frac{(2n + \alpha + \beta + 1)}{(2n + \alpha + \beta - 1)} \sum_{m=1}^n (-1)^{m+n} p_{n-1}^{(\alpha, \beta)}(t_m) \end{aligned}$$

where we have applied Lemma 4.1 (iv). □

*Example 4.8.* For  $p_2^{(0,0)}(t) = \frac{5}{4}(3t^2 - 1)$ , we conclude that

$$\int_{-1}^1 \frac{5}{4} |3t^2 - 1| dt = \frac{10\sqrt{3}}{9}; \quad \int_{-1}^1 \frac{5}{4} |t(3t^2 - 1)| dt = \frac{25}{24}.$$

In the case that  $p_3^{(0,0)}(t) = \frac{7}{4}(5t^3 - 3t)$ , we get that

$$\int_{-1}^1 \frac{7}{4} |5t^3 - 3t| dt = \frac{91}{40}; \quad \int_{-1}^1 \frac{7}{4} |t(5t^3 - 3t)| dt = \frac{42}{25} \sqrt{\frac{3}{5}}.$$

Now, we consider  $p_2^{(1,0)}(t) = \frac{3}{4}(-5t^3 + 3t^2 + 3t - 1)$  and we obtain that

$$\begin{aligned} \int_{-1}^1 \frac{3}{4} |-5t^3 + 3t^2 + 3t - 1| dt &= \frac{72}{125} \sqrt{6}; \\ \int_{-1}^1 \frac{3}{4} |t(-5t^3 + 3t^2 + 3t - 1)| dt &= \frac{18921}{25000}. \end{aligned}$$

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