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CESÀRO SUMS AND ALGEBRA HOMOMORPHISMS OF BOUNDED OPERATORS*

BY

LUCIANO ABADIAS

*Departamento de Matemáticas,
Instituto Universitario de Matemáticas y Aplicaciones,
Universidad de Zaragoza, 50009 Zaragoza, Spain.
e-mail: labadias@unizar.es*

AND

CARLOS LIZAMA

*Departamento de Matemática y Ciencia de la Computación,
Universidad de Santiago de Chile, Casilla 307-Correo 2, Santiago, Chile.
e-mail: carlos.lizama@usach.cl*

AND

PEDRO J. MIANA

*Departamento de Matemáticas,
Instituto Universitario de Matemáticas y Aplicaciones,
Universidad de Zaragoza, 50009 Zaragoza, Spain.
e-mail: pjmiana@unizar.es*

AND

M. PILAR VELASCO

*Centro Universitario de la Defensa,
Instituto Universitario de Matemáticas y Aplicaciones,
Instituto de Matemática Interdisciplinar, 50090 Zaragoza, Spain.
e-mail: velascom@unizar.es*

ABSTRACT

Let X be a complex Banach space. The connection between algebra homomorphisms defined on subalgebras of the Banach algebra $\ell^1(\mathbb{N}_0)$ and fractional versions of Cesàro sums of a linear operator $T \in \mathcal{B}(X)$ is established. In particular, we show that every (C,α)-bounded operator T induces an algebra homomorphism — and it is in fact characterized by such an algebra homomorphism. Our method is based on some sequence kernels, Weyl fractional difference calculus and convolution Banach algebras that are introduced and deeply examined. To illustrate our results, improvements to bounds for Abel means, new insights on the (C,α)-boundedness of the resolvent operator for temperated α-times integrated semigroups, and examples of bounded homomorphisms are given in the last section.

1. Introduction

Let X be a complex Banach space. Let T be an operator in the Banach algebra $\mathcal{B}(X)$ and denote by $\mathcal{T}$ the discrete semigroup given by $\mathcal{T}(n) := T^n$ for $n \in \mathbb{N}_0$. The Cesàro sum of order $\alpha > 0$ of T, $\{\Delta^{-\alpha}\mathcal{T}(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$, is defined by

$$\Delta^{-\alpha}\mathcal{T}(n)x = \sum_{j=0}^{n} k^{\alpha}(n-j)\mathcal{T}(j)x, \qquad x \in X, \quad n \in \mathbb{N}_0,$$

where

$$k^{\alpha}(n) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n+1)}, \qquad n \in \mathbb{N}_0,$$

is the Cesàro kernel. It is well-known that Cesàro sums are an important concept that appears in several contexts and ways in the literature. For instance, in Zygmund's book, it appeared in connection with summability of Fourier series [30, Chapter III, Section 3.11] and in [7] in relation to weighted norm inequalities for Jacobi polynomials and series. See also [20] and [25]. The starting point for our investigation is this definition of the fractional sum of the discrete semigroup

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$\mathcal{T}$. Certain fractional sums have been used in recent years to develop a theory of fractional differences with interesting applications to boundary value problems and concrete models coming from biological issues; see for example [5] and [19]. Note that this definition coincides or is connected with other fractional sums of the discrete semigroup $\mathcal{T}$ on the set $\mathbb{N}_0$; see [4, Section 1] or [6, Theorem 2.5].

Consider $\phi : \mathbb{N}_0 \to \mathbb{R}^+$ a positive weight, that is, $\phi(n+m) \leq C\phi(n)\phi(m)$ with $C > 0$, and the weighted Banach algebra ℓ^1_ϕ (endowed with their natural convolution product). Suppose $\frac{1}{\phi(\cdot)}\mathcal{T} \in \ell^\infty(\mathcal{B}(X))$. It is well-known and easy to show that the semigroup $\mathcal{T}$ induces an algebra homomorphism $\theta : \ell^1_\phi \to \mathcal{B}(X)$ defined by

$$\theta(f)x := \sum_{n=0}^{\infty} f(n)\mathcal{T}(n)x, \qquad f \in \ell^1_\phi, \quad x \in X.$$

Note that in the case that T is a power bounded operator, i.e., $\mathcal{T} \in \ell^\infty(\mathcal{B}(X))$, then $\theta : \ell^1 \to \mathcal{B}(X)$. Moreover, this homomorphism is a natural extension of the Z-transform. See Section 4 and [12] for more information on this concept.

In general, algebra homomorphisms are useful tools to treat different interesting aspects of operator theory: algebraic relations, sharp norm estimations, subordination operators, or ergodic behaviour (as Katznelson–Tzafriri type theorems, see [23]).

As mentioned before, it is remarkable that Cesàro sums have appeared in the literature some time ago but until now, their relation with the theory of fractional sums and their algebraic structure has not been noted. The first main purpose of this paper is to show how this connection provides new insight on properties and characterizations of Cesàro sums, notably concerning their interplay with algebra homomorphisms.

Cesàro sums are also a basic tool to define (C,α)-bounded operators, a natural extension of power-bounded operators. We recall that a bounded operator $T \in \mathcal{B}(X)$ is called (C,α)-bounded ($\alpha > 0$) if

$$\sup_n \| \frac{1}{k^{\alpha+1}(n)} \Delta^{-\alpha}\mathcal{T}(n) \| < \infty.$$

See [9, 27] for examples and properties of (C,α)-bounded operators. Note that if T is power bounded, then T is a (C,α)-bounded operator for every $\alpha > 0$. However, there are operators that do not satisfy the power-boundedness condition, but $\sup_{n\geq 1} \frac{1}{n}\|\Delta^{-1}\mathcal{T}(n)\| < \infty$, as the well-known Assani example shows

$$T = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix},$$

see [13, Section 4.7]; recently other examples have appeared in [9, 11, 27, 28, 29].

The following natural question then arises: (Q) Can T induce an algebra homomorphism from a proper subalgebra $\mathcal{A} \subset \ell^1$ to $\mathcal{B}(X)$?

The second purpose of this paper is to show that, surprisingly, the answer to (Q) is positive for every bounded operator such that their Cesàro sums are properly bounded (which includes (C,α)-bounded operators). More precisely, we construct appropriate subalgebras $\tau^\alpha(k^{\alpha+1}) \subset \ell^1$ and then we prove that the following assertions are equivalent:

(i) T is (C,α)-bounded operator.
(ii) There exists a bounded algebra homomorphism $\theta : \tau^\alpha(k^{\alpha+1}) \to \mathcal{B}(X)$ such that $\theta(e_1) = T$.

In the limit case $\alpha = 0$, the following assertions are equivalent:

(a) T is power bounded.
(b) There exists a bounded algebra homomorphism $\theta : \ell^1 \to \mathcal{B}(X)$ such that $\theta(e_1) = T$.
(c) For any $0 < \alpha < 1$, there exist a bounded algebra homomorphism $\theta_\alpha : \tau^\alpha(k^{\alpha+1}) \to \mathcal{B}(X)$ such that $\theta_\alpha(e_1) = T$ and $\sup_{0<\alpha<1} \|\theta_\alpha\| < \infty$.

This paper is organized as follows: In order to construct a suitable Banach algebra and the corresponding homomorphism, we introduce in Section 2 the notion of α-th fractional Weyl sum as follows:

$$W^{-\alpha} f(n) = \sum_{j=n}^{\infty} k^\alpha(j-n) f(j), \qquad n \in \mathbb{N}_0;$$

see Definition 2.2 below. We state their main algebraic properties in Proposition 2.4. Then, we introduce Banach algebras $\tau^\alpha(\phi)$ as the completion of the space of sequences $c_{0,0}$ under the norm $q_\phi(f) := \sum_{n=0}^{\infty} \phi(n)|W^\alpha f(n)|$ (Theorem 2.11). The weight sequences ϕ need to verify some summability conditions (Definition 2.8) to prove that the space $\tau^\alpha(\phi)$ is a Banach algebra. It is remarkable that such Banach algebras extend those defined for $\alpha \in \mathbb{N}_0$ and $\phi = k^{\alpha+1}$ in [17, Section 4]. Therefore, they are considered to study subalgebras of analytic functions on the unit disc contained in the Korenblyum and (analytic) Wiener algebra.

Section 3 contains an interesting characterization for the Cesàro sum of powers of a given (C,α)-bounded operator $T \in \mathcal{B}(X)$ solely in terms of a certain functional equation (Theorem 3.3). The obtained characterization corresponds to an extension of the well-known functional equation for the corresponding discrete semigroup $\mathcal{T}$, namely

$$T^nT^m = T^{n+m}, \quad n,m \in \mathbb{N}_0.$$

Theorem 3.5 gives a complete answer to question (Q) by defining a bounded algebra homomorphism $\theta : \tau^\alpha(\phi) \to \mathcal{B}(X)$ given explicitly by

$$\theta(f)x := \sum_{n=0}^{\infty} W^\alpha f(n)\Delta^{-\alpha}\mathcal{T}(n)x, \qquad f \in \tau^\alpha(\phi), \quad x \in X.$$

This homomorphism enjoys remarkable properties. The existence of bounded homomorphisms in these new Banach algebras completely characterizes the growth of Cesàro sums in Corollary 3.6; in particular, bounded homomorphisms from algebras $\tau^\alpha(k^{\alpha+1})$ characterize (C,α)-boundedness (Corollary 3.7). Such a connection seems to be new as well as the functional equation found in the beginning of this section.

The Z-transform technique may be traced back to De Moivre around the year 1730. In fact, De Moivre introduced the more general concept of "generating functions" to probability theory. It is interesting to compare the Z-transform (discrete case) to the Laplace transform (continuous case); see for example [12, Section 6.7]. In Section 4, we use the Widder space $C_W^\infty((\omega,\infty), X; \mathrm{m})$ where m is Borel measure on $\mathbb{R}_+$, introduced in [8], to give a new characterization of summable vector-valued sequences in terms of the Z-transform in Theorem 4.1. We complete the approach given in Section 3 involving the Z-transform and resolvent operators in Theorem 4.4.

Finally, in Section 5 we suggest several applications, counterexamples and final comments on this paper. A straightforward application is obtaining the Abel means by subordination to the Cesàro sums, as Theorem 5.1 shows. This point of view allows the improvement of some previous results given in [26]. Some results presented in this paper are inspired by similar ones obtained for α-times integrated semigroups; see [16]. In Section 5.2, we show a natural connection between both concepts. In Section 5.3, we present some counterexamples of algebra homomorphisms defined on certain Banach algebras which cannot be extended to some larger algebras. A future research line, the extension of the celebrated Katznelson–Tzafriri to (C,α)-bounded operators, is commented on in Section 5.4.

Notation. We denote by $\{e_n\}_{n\in\mathbb{N}_0}$ the set of canonical sequences given by $e_n(j)=\delta_{n,j}$ where $\delta_{n,j}$ is the known Kronecker delta, i.e., $\delta_{n,j}=1$ if $n=j$ and 0 otherwise. Let X be a Banach space and $\ell^p(X)$ the set of vector-valued sequences $f:\mathbb{N}_0\to X$ such that $\displaystyle\sum_{n=0}^{\infty}\|f(n)\|^p<\infty$, for $1\le p<\infty$; and $c_{0,0}(X)$ the set of vector-valued sequences with finite support. When $X=\mathbb{C}$ we write ℓ^p and $c_{0,0}$ respectively. It is well-known that ℓ^1 is a Banach algebra with the usual (commutative and associative) convolution product

$$(f*g)(n)=\sum_{j=0}^{n}f(n-j)g(j),\qquad n\in\mathbb{N}_0.$$

We write $f^{*n}=f*f^{*(n-1)}$ for $n\ge 2$, $f^{*1}=f$ and $f^{*0}=e_0$; in particular $e_n=e_1^{*n}$ for $n\in\mathbb{N}_0$. Consider $\phi:\mathbb{N}_0\to\mathbb{R}^+$ a positive sequence, and ℓ^1_ϕ is the Banach spaced formed by complex sequences $f:\mathbb{N}_0\to\mathbb{C}$ such that $\sum_{n\in\mathbb{N}_0}\phi(n)|f(n)|<\infty$.

Throughout the paper, we use the variable constant convention, in which C denotes a constant which may not be the same from line to line. The constant is frequently written with subindexes to emphasize some parameters.

2. Weyl differences and convolution Banach algebras

In this section, we define certain spaces of sequences that correspond to an extension in two different directions of those considered in the recent paper [17, Definition 4.2]. We consider a positive order of regularity in Weyl differences (Definition 2.2) and different order of growth of Weyl differences (Definition 2.8). These spaces correspond to Banach subalgebras of the space ℓ^1 and are important to obtain a further characterization via homomorphisms for Cesàro sums in the next section.

We consider the usual difference operator $\Delta f(n)=f(n+1)-f(n)$, for $n\in\mathbb{N}_0$, its powers $\Delta^{k+1}=\Delta^k\Delta=\Delta\Delta^k$, for $k\in\mathbb{N}$, and we write $\Delta^0 f=f$ and $\Delta^1=\Delta$. It is easy to see that

$$\Delta^k f(n)=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}f(n+j),\qquad n\in\mathbb{N}_0$$

(see for example [12, (2.1.1)]), and then $\Delta^m : c_{0,0} \to c_{0,0}$ for $m \in \mathbb{N}_0$. In addition, for $\alpha > 0$, we consider the well-known scalar sequence $(k^\alpha(n))_{n=0}^\infty$ defined by

$$k^\alpha(n) := \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)} = \binom{n+\alpha-1}{\alpha-1}, \qquad n \in \mathbb{N}_0.$$

In Zygmund's classical monograph, the numbers $k^\alpha(n)$ are called Cesàro numbers of order α ([30, Vol. I, p.77]) and written $k^\alpha(n) = A_n^{\alpha-1}$. However, the notation as function k^α will facilitate the understanding of this paper. The kernels k^α may equivalently be defined by means of the generating function

$$\sum_{n=0}^{\infty} k^\alpha(n) z^n = \frac{1}{(1-z)^\alpha}, \quad |z| < 1, \quad \alpha > 0, \tag{2.1}$$

and satisfy the semigroup property, that is, $k^\alpha * k^\beta = k^{\alpha+\beta}$ for $\alpha, \beta > 0$. Furthermore, the following equality holds: for $\alpha > 0$,

$$k^\alpha(n) = \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1 + O\left(\frac{1}{n}\right)\right), \qquad n \in \mathbb{N} \tag{2.2}$$

([30, Vol. I, p.77 (1.18)]) and k^α is increasing (as a function of n) for $\alpha > 1$, decreasing for $1 > \alpha > 0$ and $k^1(n) = 1$ for $n \in \mathbb{N}$ ([30, Theorem III.1.17]). It is straightforward to check that $k^\alpha(n) \le k^\beta(n)$ for $\beta \ge \alpha > 0$ and $n \in \mathbb{N}_0$. The Gautschi inequality states that

$$x^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < (x+1)^{1-s}, \qquad x \ge 1, \quad 0 < s < 1 \tag{2.3}$$

([18]), which implies that

$$\frac{(n+1)^{\alpha-1}}{\Gamma(\alpha)} < k^\alpha(n) < \frac{n^{\alpha-1}}{\Gamma(\alpha)}, \qquad n \in \mathbb{N}, \quad 0 < \alpha < 1.$$

Note that when $\alpha = 0$ we have

$$k^0(n) := \lim_{\alpha \to 0^+} k^\alpha(n) = e_0(n), \qquad n \in \mathbb{N}_0.$$

Lemma 2.1: *For $\alpha > 0$, there exists $C_\alpha > 0$ such that*

$$k^\alpha(2n) \le C_\alpha k^\alpha(n), \qquad n \in \mathbb{N}_0.$$

In particular for $0 < \alpha < 1$, the following inequality holds:

$$k^{\alpha+1}(2n) < 2^\alpha k^{\alpha+1}(n)\left(1 + \frac{1-\alpha}{2(1+\alpha)}\right)^\alpha, \qquad n \in \mathbb{N}_0.$$

Proof. The proof of the first inequality is straightforward by the inequality (2.2). To show the second inequality, we use the known doubling equality for the Gamma function

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)=2^{1-2z}\sqrt{\pi}\Gamma(2z), \qquad \Re z>0,$$

to obtain that

$$\begin{aligned} k^{\alpha+1}(2n) &= \frac{\Gamma(\alpha+1+2n)}{\Gamma(\alpha+1)\Gamma(2n+1)} \\ &= 2^{\alpha}k^{\alpha+1}(n)\frac{\Gamma(\frac{\alpha}{2}+\frac{1}{2}+n)\Gamma(\frac{\alpha}{2}+1+n)}{\Gamma(\alpha+1+n)\Gamma(\frac{1}{2}+n)}, \qquad n\geq 1. \end{aligned}$$

We apply the Gautschi inequality (2.3) to get that

$$\begin{aligned} \frac{\Gamma(\frac{\alpha}{2}+\frac{1}{2}+n)}{\Gamma(\frac{1}{2}+n)} &< \left(\frac{\alpha}{2}+\frac{1}{2}+n\right)^{\frac{\alpha}{2}}, \\ \frac{\Gamma(\frac{\alpha}{2}+1+n)}{\Gamma(\alpha+1+n)} &< (\alpha+n)^{\frac{-\alpha}{2}}, \end{aligned}$$

for $0<\alpha<1$ and we conclude that

$$k^{\alpha+1}(2n)<2^{\alpha}k^{\alpha+1}(n)\left(1+\frac{1-\alpha}{2(\alpha+n)}\right)^{\frac{\alpha}{2}}\leq 2^{\alpha}k^{\alpha+1}(n)\left(1+\frac{1-\alpha}{2(1+\alpha)}\right)^{\alpha},$$

for $n\geq 1$ and $0<\alpha<1$. ∎

The Cesàro sum of order α of a sequence f is defined by

$$\Delta^{-\alpha}f(n):=(k^{\alpha}*f)(n)=\sum_{j=0}^{n}k^{\alpha}(n-j)f(j), \qquad n\in\mathbb{N}_0, \quad \alpha>0.$$

Again we prefer to follow the notation $\Delta^{-\alpha}f(n)$ instead of $S_n^{\alpha-1}$ used in [30]. Note that $\Delta^{-\alpha-\beta}f=k^{\beta}*(\Delta^{-\alpha}f)$ and then $\Delta^{-\alpha}\Delta^{-\beta}=\Delta^{-(\alpha+\beta)}=\Delta^{-\beta}\Delta^{-\alpha}$ for $\alpha,\beta>0$; for more details see again [30, Vol. I, pp. 76–77]. Note also that $\lim_{\alpha\to 0^+}\Delta^{-\alpha}f(n)=f(n)$ for all $n\in\mathbb{N}_0$ with $\alpha>0$.

We write $W=-\Delta$, $W^m=(-1)^m\Delta^m$ for $m\in\mathbb{N}$. The operator W has an inverse in $c_{0,0}$, $W^{-1}f(n)=\sum_{j=n}^{\infty}f(j)$ and its iterations are given by the sum

$$W^{-m}f(n)=\sum_{j=m}^{\infty}\frac{\Gamma(j-n+m)}{\Gamma(j-n+1)\Gamma(m)}f(j)=\sum_{j=n}^{\infty}k^{m}(j-n)f(j), \qquad n\in\mathbb{N}_0,$$

for each scalar-valued sequence f such that $\sum_{n=0}^{\infty} |f(n)|n^m < \infty$; see for example [17, p. 307]. These facts and the clear connection with the Weyl fractional calculus motivate the following definition:

Definition 2.2: Let $f : \mathbb{N}_0 \to X$ and $\alpha > 0$ be given. The Weyl sum of order α of f, $W^{-\alpha} f$, is defined by

$$W^{-\alpha} f(n) := \sum_{j=n}^{\infty} k^{\alpha}(j-n) f(j), \qquad n \in \mathbb{N}_0,$$

whenever the right-hand side makes sense. The Weyl difference of order α of f, $W^{\alpha} f$, is defined by

$$W^{\alpha} f(n) := W^m W^{-(m-\alpha)} f(n) = (-1)^m \Delta^m W^{-(m-\alpha)} f(n), \qquad n \in \mathbb{N}_0,$$

for $m = [\alpha] + 1$, whenever the right-hand side makes sense. In particular, $W^{\alpha} : c_{0,0} \to c_{0,0}$ for $\alpha \in \mathbb{R}$.

Remark 2.3: Note that the definition of W^{α} is dependent on $m = [\alpha] + 1$, but we can write

$$W^{\alpha} f(n) = W^m W^{-(m-\alpha)} f(n) = W^l W^{-(l-\alpha)} f(n), \qquad n \in \mathbb{N}_0,$$

for $l > m = [\alpha] + 1$ with $l \in \mathbb{N}$, whenever the right-hand side makes sense, since W^{-1} is the inverse operator of W and Proposition 2.3 (v) holds.

Observe that if $\alpha \in \mathbb{N}_0$, the Weyl difference of order α coincides with the definition given in [17, Section 4]. Some general properties are shown in the following proposition:

Proposition 2.4: *Let $f \in c_{0,0}(X)$. The following assertions hold:*

(i) *For $\alpha, \beta > 0$, $W^{-\alpha} W^{-\beta} f = W^{-(\alpha+\beta)} f = W^{-\beta} W^{-\alpha} f$.*
(ii) *For $\alpha > 0$ and $n \in \mathbb{N}_0$, we have $\lim_{\alpha \to 0^+} W^{-\alpha} f(n) = f(n)$.*
(iii) *For $\alpha > 0$, $W^{\alpha} W^{-\alpha} f = W^{-\alpha} W^{\alpha} f = f$.*
(iv) *For $\alpha > 0$ and $n \in \mathbb{N}_0$, we have $\lim_{\alpha \to 0^+} W^{\alpha} f(n) = f(n)$.*
(v) *For all $\alpha, \beta \in \mathbb{R}$ we have $W^{\alpha} W^{\beta} f = W^{\alpha+\beta} f = W^{\beta} W^{\alpha} f$.*

Proof. (i) It is clear using the Fubini theorem and the semigroup property $k^{\alpha+\beta} = k^{\alpha} * k^{\beta}$ for $\alpha, \beta > 0$.

(ii) It is sufficient to apply that f has finite support and $\lim_{\alpha\to 0^+} k^\alpha(j) = e_0(j)$ for $j \in \mathbb{N}_0$.

(iii) We write $m = [\alpha]+1$. Applying part (i), for $n \in \mathbb{N}_0$, we have that

$$W^\alpha W^{-\alpha} f(n) = W^m W^{-(m-\alpha)} W^{-\alpha} f(n) = W^m W^{-m} f(n) = f(n),$$

since W^{-m} is the inverse of W^m in $c_{0,0}(X)$, see [17, Section 4]. On the other hand,

$$\begin{aligned} W^{-\alpha} W^\alpha f(n) &= W^{-(\alpha+1-m)} W^{-(m-1)} W^m W^{-(m-\alpha)} f(n) \\ &= W^{-(\alpha+1-m)} W^1 W^{-(m-\alpha)} f(n) \\ &= W^{-(\alpha+1-m)} W^{-(m-\alpha)} f(n) - \sum_{j=n}^{\infty} k^{\alpha+1-m}(j-n) W^{-(m-\alpha)} f(j+1) \\ &= W^{-1} f(n) - \sum_{j=n+1}^{\infty} k^{\alpha+1-m}(j-n-1) W^{-(m-\alpha)} f(j) \\ &= W^{-1} f(n) - W^{-1} f(n+1) = f(n), \end{aligned}$$

where we use part (i).

(iv) It is sufficient to apply that f has finite support and $\lim_{\alpha\to 0^+} k^{1-\alpha}(j) = 1$ for $j \in \mathbb{N}_0$.

(v) It is simple to check using the previous results. ∎

Example 2.5: (i) Let $\lambda \in \mathbb{C}\backslash\{0\}$ be given and define $p_\lambda(n) := \lambda^{-(n+1)}$ for $n \in \mathbb{N}_0$. An easy computation shows that the sequence p_λ is a pseudo-resolvent, that is, it satisfies the Hilbert equation

$$(\mu - \lambda)(p_\lambda * p_\mu)(n) = p_\lambda(n) - p_\mu(n), \quad n \in \mathbb{N}_0.$$

Moreover, the following identity holds:

$$p_\lambda * (\lambda e_0 - e_1) = e_0, \qquad \lambda \in \mathbb{C}\backslash\{0\}.$$

We claim that the functions p_λ are eigenfunctions for the operator W^α for $\alpha \in \mathbb{R}$ and $|\lambda| > 1$. In fact, we have, by (2.1), that

$$W^{-\alpha} p_\lambda(n) = \lambda^{-(n+1)} \sum_{j=0}^{\infty} k^\alpha(j) \lambda^{-j} = \frac{\lambda^\alpha}{(\lambda-1)^\alpha} p_\lambda(n), \qquad n \in \mathbb{N}_0.$$

By Proposition 2.4 (iii), we obtain that

$$W^\alpha p_\lambda = \frac{(\lambda-1)^\alpha}{\lambda^\alpha} p_\lambda, \qquad |\lambda| > 1,$$

proving the claim.

(ii) Let $\alpha \geq 0$ and $n \in \mathbb{N}_0$ be given. We define

$$h_n^\alpha(j) := \begin{cases} k^\alpha(n-j), & j \leq n, \\ 0, & j > n. \end{cases}$$

The functions h_n^α are denoted by $\Gamma_n^{\alpha-1}$ for $\alpha \in \mathbb{N}_0$ in [17, Section 4]. Note that $h_n^\alpha \in c_{0,0}$ for $n \in \mathbb{N}_0$. Moreover, $h_n^\alpha \in \operatorname{span}\{e_j \mid 0 \leq j \leq n\}$, $h_0^\alpha = e_0$, $h_1^\alpha = \alpha e_0 + e_1$, $h_n^0 := \lim_{\alpha \to 0^+} h_n^\alpha = e_n$, and

$$(2.4)\quad h_n^\alpha(j) = k^\alpha(n-j) = \sum_{l=0}^{n} k^\alpha(n-l) e_l(j) = \sum_{l=0}^{n} k^\alpha(n-l) e_1^{*l}(j), \quad 0 \leq j \leq n.$$

Then for all $\beta \geq 0$ it is easy to check that $W^{-\beta} h_n^\alpha = h_n^{\alpha+\beta}$, i.e.,

$$W^{-\beta} h_n^\alpha(j) = \sum_{i=j}^{\infty} k^\beta(i-j) h_n^\alpha(i) = h_n^{\alpha+\beta}(j), \qquad j \in \mathbb{N}_0.$$

Using Proposition 2.4 (iii), we obtain that

$$W^\beta h_n^\alpha(j) = h_n^{\alpha-\beta}(j), \qquad j \in \mathbb{N}_0,$$

for $0 \leq \beta \leq \alpha$ and $n \in \mathbb{N}_0$.

The following remark shows an interesting duality between the operator $\Delta^{-\alpha}$ and $W^{-\alpha}$. Similar results may be found in [1, Section 4] and [2, Theorem 4.1 and 4.4].

Remark 2.6: Let $f, g \in c_{0,0}$. We consider the usual duality product $\langle\, , \rangle$ given by

$$\langle f, g \rangle := \sum_{n=0}^{\infty} f(n) g(n).$$

By the Fubini theorem, we get that $\langle W^{-\alpha} f, g \rangle = \langle f, \Delta^{-\alpha} g \rangle$ and consequently

$$\langle f, g \rangle = \langle W^\alpha f, \Delta^{-\alpha} g \rangle = \langle \Delta^{-\alpha} f, W^\alpha g \rangle.$$

Note that these last three equalities also hold for the usual inner product in $c_{0,0}$.

The next lemma includes an equality which is an important tool for further developments in this paper. The proof runs parallel to the proof of the integer case given in [17, Lemma 4.4] and, therefore, we do not include it here.

Lemma 2.7: *Let $f, g \in c_{0,0}$ and $\alpha \geq 0$ be given. Then*

$$\begin{aligned} W^\alpha(f*g)(n) &= \sum_{j=0}^{n} W^\alpha g(j) \sum_{p=n-j}^{n} k^\alpha(p-n+j) W^\alpha f(p) \\ &\quad - \sum_{j=n+1}^{\infty} W^\alpha g(j) \sum_{p=n+1}^{\infty} k^\alpha(p-n+j) W^\alpha f(p). \end{aligned}$$

The following definition is inspired by [16, Definition 1.3]:

Definition 2.8: Let $\alpha > 0$ be given. We say that a positive sequence ϕ belongs to the class $\omega_{\alpha,loc}$ if there is a constant $c_\phi > 0$ such that

$$\left(\sum_{n=0}^{j} + \sum_{n=p+1}^{j+p}\right) k^\alpha(n)\phi(j+p-n) \leq c_\phi \phi(j)\phi(p), \qquad 1 \leq j \leq p. \tag{2.5}$$

We denote by ω_α the set of nondecreasing sequences $\phi \in \omega_{\alpha,loc}$ which are of exponential type and satisfy $\inf_{n\geq 0} (k^{\alpha+1}(n))^{-1}\phi(n) > 0$.

Examples of sequences in the set ω_α are the following ones:

(i) Any nondecreasing sequence ϕ satisfying $\max(k^{\alpha+1}(n), \phi(2n)) \leq M\phi(n)$ for some $M > 0$ and for each $n \geq 0$ (in particular, $\phi(n) = n^\beta(1+n^\mu)$ with $\beta + \mu \geq \alpha$ and $\beta, \mu \geq 0$ and $\phi(n) = k^\gamma(n)$ with $\gamma \geq \alpha + 1$).

(ii) $\phi(n) = k^{\alpha+1}(n)\rho(n)$, where ρ is a positive weight.

(iii) $\phi(n) = k^{\nu+1}(n)e^{\lambda n}$ for all $\nu, \lambda > 0$ and $n \in \mathbb{N}$.

By the property $k^\alpha(n) \sim \dfrac{n^{\alpha-1}}{\Gamma(\alpha)}$ (see formula (2.2)), equivalent examples may be given in terms of $n^{\alpha-1}$. The particular case $\phi(n) = k^{\alpha+1}(n)$ will play a fundamental role in this paper. Observe that in this case we obtain explicitly the value of the constant c_ϕ in (2.5).

Lemma 2.9: *For $0 < \alpha < 1$, the following inequality holds:*

$$\begin{aligned} &\left(\sum_{n=0}^{j} + \sum_{n=p+1}^{j+p}\right) k^\alpha(n) k^{\alpha+1}(j+p-n) \\ &\qquad \leq \left(2^{\alpha+1}\left(1 + \frac{1-\alpha}{2(1+\alpha)}\right)^\alpha - 1\right) k^{\alpha+1}(j) k^{\alpha+1}(p), \quad 1 \leq j \leq p. \end{aligned}$$

Proof. For $1 \le j \le p$ and $\alpha > 0$, we have that

$$\sum_{n=0}^{j} k^{\alpha}(n)k^{\alpha+1}(j+p-n) \le k^{\alpha+1}(j+p)\sum_{n=0}^{j} k^{\alpha}(n) = k^{\alpha+1}(j+p)k^{\alpha+1}(j),$$
$$\sum_{n=p+1}^{j+p} k^{\alpha}(n)k^{\alpha+1}(j+p-n) \le k^{\alpha+1}(j-1)\sum_{n=p+1}^{j+p} k^{\alpha}(n)$$
$$\le k^{\alpha+1}(j)\left(k^{\alpha+1}(j+p) - k^{\alpha+1}(p)\right).$$

As $k^{\alpha+1}$ is an increasing sequence, we have $k^{\alpha+1}(j+p) \le k^{\alpha+1}(2p)$ for $j \le p$ and we apply Lemma 2.1 to conclude the proof. ∎

Proposition 2.10: *Let $0 < \alpha \le \beta$ and $\phi \in \omega_{\alpha,loc}$ be given. The following properties hold:*

(i) $\omega_{\beta,loc} \subset \omega_{\alpha,loc}$ *and* $\omega_\beta \subset \omega_\alpha$.
(ii) $(k^\alpha * \phi)(2n) \le c_\phi \phi^2(n)$ *for all* $n \in \mathbb{N}$.
(iii) $k^\alpha(n) \le c_\phi\phi(n) \le a^n$ *for all* $n \in \mathbb{N}$ *and some* $a > 0$.
(iv) $k^{2\alpha}(2n) \le c\phi^2(n)$ *for all* $n \in \mathbb{N}_0$ *and some* $c > 0$.
(v) $\phi(n+1) \le C\phi(n)$ *for some* $C > 0$ *independent of* $n \in \mathbb{N}$.
(vi) $k^\beta \in \omega_{\alpha,loc}$ *if and only if* $\beta \ge \alpha + 1$.

Proof. (i) Since $k^\beta(n) \ge k^\alpha(n)$ for all $n \in \mathbb{N}_0$, then $\omega_{\beta,loc} \subset \omega_{\alpha,loc}$ and $\omega_\beta \subset \omega_\alpha$ for $\beta \ge \alpha > 0$.

(ii) It is enough to take $j = p$ in (2.5) in order to obtain the inequality.

(iii) By part (ii), we have that

$$k^\alpha(n)\phi(n) \le (k^\alpha * \phi)(2n) \le c_\phi\phi^2(n), \qquad n \in \mathbb{N},$$

and we get the first inequality. For $n \in \mathbb{N}$, we apply the inequality (2.5) $n-1$ times to obtain that

$$c_\phi\phi(n) = c_\phi k^\alpha(0)\phi(n-1+1) \le c_\phi^2\phi(1)\phi(n-1) \le (c_\phi\phi(1))^n.$$

(iv) We combine parts (ii), (iii) and the semigroup property of kernels k^α to conclude that

$$c_\phi\phi^2(n) \ge (k^\alpha * \phi)(2n) \ge c'(k^\alpha * k^\alpha)(2n) = c'k^{2\alpha}(2n), \qquad n \in \mathbb{N}_0,$$

for some $c' > 0$.

(v) Take $j=1$ and $p=n\in\mathbb{N}$ in (2.5) to get

$$\phi(n+1)=k^\alpha(0)\phi(n+1)\le\sum_{m=0}^{1}k^\alpha(m)\phi(n+1-m)\le c_\phi\phi(1)\phi(n),\qquad n\in\mathbb{N}.$$

(vi) If $k^\beta\in\omega_{\alpha,loc}$ then we can apply (2.2) and part (ii) to get

$$(k^\alpha*k^\beta)(2n)=k^{\alpha+\beta}(2n)\sim 2^{\alpha+\beta-1}\frac{n^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\le c\frac{n^{2(\beta-1)}}{\Gamma^2(\beta)},\qquad n\in\mathbb{N}.$$

We conclude that $\beta\ge\alpha+1$. Note that $k^{\alpha+1}\in\omega_{\alpha,loc}$ and then $k^\beta\in\omega_{\alpha,loc}$ for $\beta\ge\alpha+1$. By application of part (i) we conclude the proof. ∎

For $\alpha\ge0$, and $\phi\in\omega_{\alpha,loc}$, we define the application $q_\phi: c_{0,0}\to[0,\infty)$ given by

$$q_\phi(f):=\sum_{n=0}^{\infty}\phi(n)|W^\alpha f(n)|,\qquad f\in c_{0,0}.$$

Note that for $\alpha=0$ the above application corresponds to the usual norm in ℓ^1_ϕ. In the case of $\phi=k^{\alpha+1}$, we write q_α instead of $q_{k^{\alpha+1}}$ and $q_0=\|\ \ \|_1$ for $\alpha\ge0$. By (2.2), the norm q_α is equivalent to the norm $\widetilde{q_\alpha}$ given by

$$\widetilde{q_\alpha}(f):=|f(0)|+\sum_{n=1}^{\infty}n^\alpha|W^\alpha f(n)|.$$

The last formula was considered for the case $\alpha\in\mathbb{N}_0$ in [17, Definition 4.2].

Part of the following result extends [17, Theorem 4.5]. Their proof is similar to those given in [16, Proposition 1.4]. We include it in the following for the sake of completeness.

THEOREM 2.11: *Let $\alpha>0$ and $\phi\in\omega_{\alpha,loc}$ be given. The application q_ϕ defines a norm in $c_{0,0}$ which satisfies*

$$q_\phi(f*g)\le C_\phi\, q_\phi(f)\, q_\phi(g),\qquad f,g\in c_{0,0},$$

where the constant $C_\phi>0$ is independent of f and g.

We denote by $\tau^\alpha(\phi)$ the Banach algebra obtained as the completion of $c_{0,0}$ in the norm q_ϕ. In the case that $\phi\in\omega_\alpha$, then

(i) *the operator Δ is linear and bounded on $\tau^\alpha(\phi)$, in other words $\Delta\in\mathcal{B}(\tau^\alpha(\phi))$,*
(ii) *$\tau^\alpha(\phi)\hookrightarrow\tau^\alpha(k^{\alpha+1})\hookrightarrow\ell^1$, and $\lim_{\alpha\to0^+}q_\alpha(f)=\|f\|_1$, for $f\in c_{0,0}$,*
(iii) *for $0<\alpha<\beta$, $\tau^\beta(k^{\beta+1})\hookrightarrow\tau^\alpha(k^{\alpha+1})$,*

(iv) *for* $0<\alpha<1$,

$$q_\alpha(f*g)\le\left(2^{\alpha+1}\left(1+\frac{1-\alpha}{2(1+\alpha)}\right)^\alpha-1\right)q_\alpha(f)\,q_\alpha(g),\qquad f,g\in\tau^\alpha(k^{\alpha+1}).$$

Proof. It is clear that q_α is a norm in $c_{0,0}$. Now, applying Lemma 2.7 we have

$$\begin{aligned}
&q_\phi(f*g)\\
&\le\Big(\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{p=n-j}^{n}+\sum_{n=0}^{\infty}\sum_{j=n+1}^{\infty}\sum_{p=n+1}^{\infty}\Big)\phi(n)k^\alpha(p-n+j)|W^\alpha g(j)||W^\alpha f(p)|\\
&=\Big(\sum_{j=0}^{\infty}\sum_{n=j}^{\infty}\sum_{p=n-j}^{n}+\sum_{j=1}^{\infty}\sum_{n=0}^{j-1}\sum_{p=n+1}^{\infty}\Big)\phi(n)k^\alpha(p-n+j)|W^\alpha g(j)||W^\alpha f(p)|\\
&=\Big(\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\sum_{n=\max(j,p)}^{p+j}+\sum_{j=1}^{\infty}\sum_{p=1}^{\infty}\sum_{n=0}^{\min(j,p)-1}\Big)\phi(n)k^\alpha(p-n+j)|W^\alpha g(j)||W^\alpha f(p)|\\
&\le\phi(0)|W^\alpha g(0)||W^\alpha f(0)|+c_\phi\sum_{j=1}^{\infty}\sum_{p=1}^{\infty}\phi(j)\phi(p)|W^\alpha g(j)||W^\alpha f(p)|\\
&\le C_\phi\,q_\phi(f)\,q_\phi(g)
\end{aligned}$$

where we use Fubini's Theorem twice and the inequality (2.5) to show the first inequality.

Now let $\phi\in\omega_\alpha$ be given.

(i) It is clear that Δ is a linear operator. Moreover,

$$\begin{aligned}
q_\phi(\Delta(f))=\sum_{n=0}^{\infty}\phi(n)|W^\alpha f(n)-W^\alpha f(n+1)|&\le q_\phi(f)+\sum_{n=1}^{\infty}\phi(n-1)|W^\alpha f(n)|\\
&\le 2q_\phi(f),
\end{aligned}$$

for $f\in\tau^\alpha(\phi)$.

(ii) It is clear that $\tau^\alpha(\phi)\hookrightarrow\tau^\alpha(k^{\alpha+1})\hookrightarrow\ell^1$. Moreover, by the Monotone Convergence Theorem and Proposition 2.4 (ii), we have

$$\lim_{\alpha\to0^+}q_\alpha(f)=\lim_{\alpha\to0^+}\sum_{n=0}^{\infty}k^{\alpha+1}(n)|W^\alpha f(n)|=\sum_{n=0}^{\infty}|f(n)|=\|f\|_1,\qquad f\in c_{0,0}.$$

(iii) Let $f \in c_{0,0}$, and $0 < \alpha < \beta$ be given. Then

$$q_\alpha(f) = \sum_{n=0}^{\infty} k^{\alpha+1}(n)|W^\alpha f(n)| = \sum_{n=0}^{\infty} k^{\alpha+1}(n)\Big|\sum_{j=n}^{\infty} k^{\beta-\alpha}(j-n)W^\beta f(j)\Big|$$

$$\leq \sum_{j=0}^{\infty} |W^\beta f(j)| \sum_{n=0}^{j} k^{\beta-\alpha}(j-n)k^{\alpha+1}(n) = \sum_{j=0}^{\infty} k^{\beta+1}(j)|W^\beta f(j)| = q_\beta(f),$$

where we have applied Proposition 2.4 (v) and the semigroup property of k^α.

(iv) This inequality follows from Lemma (2.8). ∎

Example 2.12: Note that $\{h_n^\alpha\}_{n\in\mathbb{N}_0} \subset \tau^\alpha(\phi)$ where $\phi \in \omega_{\alpha,loc}$. By Example 2.5 (ii) we have $q_\phi(h_n^\alpha) = \phi(n)$ for all $n \in \mathbb{N}_0$. Then the series $\sum_{n=0}^{\infty} W^\alpha f(n)h_n^\alpha$ converges on $\tau^\alpha(\phi)$ for every $f \in \tau^\alpha(\phi)$. By Proposition 2.10 (iii)

$$|f(m)| \leq \sum_{n=m}^{\infty} k^\alpha(n-m)|W^\alpha(f)(n)| \leq c_\phi \sum_{n=m}^{\infty} \phi(n)|W^\alpha(f)(n)| \leq c_\phi q_\phi(f), \qquad m \in \mathbb{N}_0,$$

whenever k^α or ϕ is a non-decreasing function, i.e., for $\alpha \geq 1$ or $\phi \in \omega_\alpha$. We conclude that $f = \sum_{n=0}^{\infty} W^\alpha f(n)h_n^\alpha$ on $\tau^\alpha(\phi)$.

Let $\phi \in \omega_\alpha$ be such that $\phi(n) \leq Ca^n$ for some $a > 1$. Then $p_\lambda \in \tau^\alpha(\phi)$ for $|\lambda| > a$, where the sequences p_λ are defined in Example 2.5 (i), and

$$q_\phi(p_\lambda) \leq C\frac{|\lambda-1|^\alpha}{|\lambda|^\alpha(|\lambda|-a)}, \qquad |\lambda| > a.$$

In the particular case $\phi = k^\gamma$, we have $p_\lambda \in \tau^\alpha(k^\gamma)$ for $|\lambda| > 1$ and, for $\gamma \geq \alpha+1$, we obtain

$$q_{k^\gamma}(p_\lambda) = \frac{|\lambda-1|^\alpha|\lambda|^{\gamma-\alpha-1}}{(|\lambda|-1)^\gamma}, \qquad |\lambda| > 1, \tag{2.6}$$

where we have applied Example 2.5 (i) and the formula (2.1).

3. Cesàro sums and algebra homomorphisms

In this section and the following, we display our main results. The algebra structure of Cesàro sums are presented in several ways: A functional equation (Theorem 3.3), an algebra homomorphism (Theorem 3.5) and a characterization by means of pseudo-resolvents (Theorem 4.4). Note that this approach in fact characterizes the growth of Cesàro sums, as Corollary 3.6 and Corollary 3.7 for (C,α)-bounded operators show. We recall the following definition:

Definition 3.1: Given a bounded operator $T \in \mathcal{B}(X)$, the Cesàro sum of order $\alpha > 0$ of T, $\{\Delta^{-\alpha}\mathcal{T}(n)\}_{n\geq 0} \subset \mathcal{B}(X)$, is defined by

$$\Delta^{-\alpha}\mathcal{T}(n)x := (k^\alpha * \mathcal{T})(n)x = \sum_{j=0}^{n} k^\alpha(n-j)T^j x, \qquad x \in X, \quad n \in \mathbb{N}_0.$$

Note that we keep the notation $\mathcal{T}(n) = T^n$ for $n \in \mathbb{N}_0$.

Example 3.2: The canonical example for a Cesàro sum of order α in Banach algebras $\tau^\alpha(\phi)$ (in particular in ℓ^1) is the family $\{h_n^\alpha\}_{n\in\mathbb{N}_0}$ given in Example 2.5(ii). Note that $\{h_n^\alpha\}_{n\in\mathbb{N}_0} \subset \tau^\alpha(\phi)$ with $\phi \in \omega_{\alpha,loc}$, see Example 2.12. If we denote $\mathcal{E}(n) = e_1^{*n}$, then by equation (2.4) we get $h_n^\alpha = \Delta^{-\alpha}\mathcal{E}(n)$ for $n \in \mathbb{N}_0$.

The following theorem characterizes sequences of operators which are Cesàro sums of some order $\alpha > 0$ for a fixed operator T.

Theorem 3.3: *Let $\alpha > 0$ and $T, \{T_n\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$. The following assertions are equivalent:*

(i) *$T_n = \Delta^{-\alpha}\mathcal{T}(n)$ for $n \in \mathbb{N}_0$.*
(ii) *$T_1 = T + \alpha I$ and the following functional equation holds:*

$$T_nT_m = \sum_{u=m}^{n+m} k^\alpha(n+m-u)T_u - \sum_{u=0}^{n-1} k^\alpha(n+m-u)T_u \qquad n \geq 1,\, m \in \mathbb{N}_0. \tag{3.1}$$

Proof. Assume (i). We prove the identity (3.1). Indeed, for $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$, we have

$$\begin{aligned} T_nT_m &= \sum_{j=0}^{n}\sum_{i=0}^{m} k^\alpha(n-j)k^\alpha(m-i)T^{j+i} = \sum_{j=0}^{n}\sum_{u=j}^{m+j} k^\alpha(n-j)k^\alpha(m+j-u)T^u \\ &= \sum_{j=0}^{n}\sum_{u=0}^{m+j} k^\alpha(n-j)k^\alpha(m+j-u)T^u - \sum_{j=1}^{n}\sum_{u=0}^{j-1} k^\alpha(n-j)k^\alpha(m+j-u)T^u \\ &= \sum_{j=0}^{n} k^\alpha(n-j)T_{m+j} - \sum_{j=1}^{n}\sum_{u=0}^{j-1} k^\alpha(n-j)k^\alpha(m+j-u)T^u. \end{aligned}$$

Observe that

$$\begin{aligned}\sum_{j=1}^{n}\sum_{u=0}^{j-1}k^\alpha(n-j)k^\alpha(m+j-u)T^u &= \sum_{u=0}^{n-1}\sum_{j=u+1}^{n}k^\alpha(n-j)k^\alpha(m+j-u)T^u\\ &= \sum_{u=0}^{n-1}\sum_{l=u}^{n-1}k^\alpha(l-u)k^\alpha(m+n-l)T^u\\ &= \sum_{l=0}^{n-1}k^\alpha(m+n-l)\sum_{u=0}^{l}k^\alpha(l-u)T^u\\ &= \sum_{l=0}^{n-1}k^\alpha(m+n-l)T_l,\end{aligned}$$

and the equality (3.1) follows. This proves the claim. Conversely, assume (ii). Define

$$S_n := \sum_{j=0}^{n}k^\alpha(n-j)T^j, \quad n\in\mathbb{N}_0.$$

It is clear that $S_0 = T_0 = I$ (the equality $T_0 = I$ is easily deduced from the hypothesis) and $S_1 = T+\alpha I = T_1$. Inductively, we suppose that $S_n = T_n$. Then using that S_n satisfies (3.1), we have

$$S_{n+1}+k^\alpha(1)S_n - k^\alpha(n+1)I = S_nS_1 = T_nT_1 = T_{n+1}+k^\alpha(1)S_n-k^\alpha(n+1)I.$$

Then we conclude that $T_{n+1} = S_{n+1}$, and consequently $T_n = \Delta^{-\alpha}\mathcal{T}(n)$ for all $n\in\mathbb{N}_0$. ∎

Remark 3.4: If $\{T_n\}_{n\in\mathbb{N}_0}\subset\mathcal{B}(X)$ is a sequence of bounded operators which satisfies the equality (3.1), then the operator defined by $T := T_1-\alpha I$ is called the generator of $\{T_n\}_{n\in\mathbb{N}_0}$. By Theorem 3.3, $T_n = \Delta^{-\alpha}\mathcal{T}(n)$ where $\mathcal{T}(n) = T^n$ for $n\in\mathbb{N}_0$. In particular, note that $\{h_n^\alpha\}_{n\in\mathbb{N}_0}$ satisfies (3.1) in $\tau^\alpha(\phi)$ (see Example 3.2), and the generator is the element e_1.

The following theorem is one of the main results of this paper.

THEOREM 3.5: *Let $\alpha>0$ and $T\in\mathcal{B}(X)$ be such that $\|\Delta^{-\alpha}\mathcal{T}(n)\|\le C\phi(n)$ for $n\in\mathbb{N}_0$ with $\phi\in\omega_{\alpha,loc}$ and $C>0$. Then there exists a bounded algebra homomorphism $\theta:\tau^\alpha(\phi)\to\mathcal{B}(X)$ given by*

$$\theta(f)x := \sum_{n=0}^{\infty}W^\alpha f(n)\Delta^{-\alpha}\mathcal{T}(n)x, \qquad x\in X, \quad f\in\tau^\alpha(\phi).$$

Furthermore, the following properties hold:

(i) $\theta(h_n^\alpha) = \Delta^{-\alpha}\mathcal{T}(n)$ *for all* $n \in \mathbb{N}_0$. *In particular,* $\theta(e_0) = I$ *and* $\theta(e_1) = T$.

(ii) *For each* $f \in \tau^\alpha(\phi)$ *such that* $\Delta f \in \tau^\alpha(\phi)$ *we have*

$$T\theta(\Delta f)x = (I - T)\theta(f)x - f(0)x, \qquad x \in X.$$

(iii) *If*

$$\sup_{n\in\mathbb{N}_0} \frac{(k^{\beta-\alpha} * \phi)(n)}{\psi(n)} < \infty,$$

for some $0 < \alpha < \beta$ *and* $\psi \in \omega_{\beta,loc}$, *then* $\tau^\beta(\psi) \hookrightarrow \tau^\alpha(\phi)$ *and*

$$\theta(f)x = \sum_{n=0}^{\infty} W^\beta f(n)\Delta^{-\beta}\mathcal{T}(n)x, \quad x \in X, f \in \tau^\beta(\psi).$$

(iv) *If* $\|T\| \leq a$ *for some* $a > 0$, *then* $\theta(f)x = \sum_{n=0}^\infty f(n)T^n(x)$, *for* $f \in \tau^\alpha(\phi) \cap \ell^1_{a^n}$. *In particular,* $\theta(p_\lambda) = (\lambda - T)^{-1}$ *for each* $|\lambda| > a$.

Proof. Note that the map θ is well-defined, linear and continuous. Moreover, $\|\theta(f)x\| \leq Cq_\alpha(f)\|x\|$, for all $f \in \tau^\alpha(\phi)$ and $x \in X$. To see that θ is an algebra homomorphism it is sufficient to prove that $\theta(f * g) = \theta(f)\theta(g)$ for $f, g \in c_{0,0}$. Indeed, by Lemma 2.7, we get that

$$\begin{aligned}\theta(f * g)x &= \sum_{n=0}^{\infty} W^\alpha(f * g)(n)\Delta^{-\alpha}\mathcal{T}(n)x \\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{n} W^\alpha g(j) \sum_{p=n-j}^{n} k^\alpha(p-n+j)W^\alpha f(p)\Delta^{-\alpha}\mathcal{T}(n)x \\ &\quad - \sum_{n=0}^{\infty}\sum_{j=n+1}^{\infty} W^\alpha g(j) \sum_{p=n+1}^{\infty} k^\alpha(p-n+j)W^\alpha f(p)\Delta^{-\alpha}\mathcal{T}(n)x.\end{aligned}$$

We apply the Fubini theorem to get that

$$\begin{aligned}\theta(f*g)x=&\sum_{j=0}^{\infty}W^{\alpha}g(j)\sum_{p=0}^{j}W^{\alpha}f(p)\sum_{n=j}^{p+j}k^{\alpha}(p-n+j)\Delta^{-\alpha}\mathcal{T}(n)x\\&+\sum_{j=0}^{\infty}W^{\alpha}g(j)\sum_{p=j+1}^{\infty}W^{\alpha}f(p)\sum_{n=p}^{p+j}k^{\alpha}(p-n+j)\Delta^{-\alpha}\mathcal{T}(n)x\\&-\sum_{j=1}^{\infty}W^{\alpha}g(j)\sum_{p=1}^{j}W^{\alpha}f(p)\sum_{n=0}^{p-1}k^{\alpha}(p-n+j)\Delta^{-\alpha}\mathcal{T}(n)x\\&-\sum_{j=1}^{\infty}W^{\alpha}g(j)\sum_{p=j+1}^{\infty}W^{\alpha}f(p)\sum_{n=0}^{j-1}k^{\alpha}(p-n+j)\Delta^{-\alpha}\mathcal{T}(n)x.\end{aligned}$$

Therefore

$$\begin{aligned}\theta(f*g)x=&\sum_{j=1}^{\infty}W^{\alpha}g(j)\sum_{p=1}^{j}W^{\alpha}f(p)\Bigg(\sum_{n=j}^{p+j}-\sum_{n=0}^{p-1}\Bigg)k^{\alpha}(p-n+j)\Delta^{-\alpha}\mathcal{T}(n)x\\&+W^{\alpha}g(0)W^{\alpha}f(0)x\\&+\sum_{j=0}^{\infty}W^{\alpha}g(j)\sum_{p=j+1}^{\infty}W^{\alpha}f(p)\Bigg(\sum_{n=p}^{p+j}-\sum_{n=0}^{j-1}\Bigg)k^{\alpha}(p-n+j)\Delta^{-\alpha}\mathcal{T}(n)x\\=&\sum_{j=1}^{\infty}W^{\alpha}g(j)\sum_{p=1}^{j}W^{\alpha}f(p)\Delta^{-\alpha}\mathcal{T}(p)\Delta^{-\alpha}\mathcal{T}(j)x+W^{\alpha}g(0)W^{\alpha}f(0)x\\&+\sum_{j=0}^{\infty}W^{\alpha}g(j)\sum_{p=j+1}^{\infty}W^{\alpha}f(p)\Delta^{-\alpha}\mathcal{T}(p)\Delta^{-\alpha}\mathcal{T}(j)x\\=&\,\theta(f)\theta(g)x,\end{aligned}$$

where we have used the identity (3.1). This proves the claim. We now verify that the properties (i)-(iv) hold.

(i) Note that $W^{\alpha}h_n^{\alpha}=e_n$ (see Example 2.5 (ii)), and then $\theta(h_n^{\alpha})=\Delta^{-\alpha}\mathcal{T}(n)$ for $n\in\mathbb{N}_0$. As $e_0=h_0$ and $e_1=h_1^{\alpha}-\alpha h_0^{\alpha}$, it is clear that $\theta(e_0)=I$ and $\theta(e_1)=T$.

(ii) Let $f \in \tau^\alpha(\phi)$ be such that $\Delta f \in \tau^\alpha(\phi)$ and $x \in X$. We have that

$$\begin{aligned} T\theta(\Delta f)x &= T\left(\sum_{n=0}^{\infty} W^\alpha f(n+1)\Delta^{-\alpha}\mathcal{T}(n)x - \sum_{n=0}^{\infty} W^\alpha f(n)\Delta^{-\alpha}\mathcal{T}(n)x\right) \\ &= \sum_{n=0}^{\infty} W^\alpha f(n+1)\left(\Delta^{-\alpha}\mathcal{T}(n+1)x - k^\alpha(n+1)x\right) \\ &\quad - T\sum_{n=0}^{\infty} W^\alpha f(n)\Delta^{-\alpha}\mathcal{T}(n)x \\ &= (I-T)\theta(f)x - W^\alpha f(0)\Delta^{-\alpha}\mathcal{T}(0)x - \sum_{n=0}^{\infty} W^\alpha f(n+1)k^\alpha(n+1)x \\ &= (I-T)\theta(f)x - \sum_{n=0}^{\infty} W^\alpha f(n)k^\alpha(n)x \\ &= (I-T)\theta(f)x - f(0)x, \end{aligned}$$

where we have applied that $T\Delta^{-\alpha}\mathcal{T}(n) = \Delta^{-\alpha}\mathcal{T}(n+1) - k^\alpha(n+1)$ and $\sum_{n=0}^{\infty} W^\alpha f(n)k^\alpha(n) = f(0)$ for $f \in \tau(\phi)$.

(iii) Suppose that

$$\sup_{n\in\mathbb{N}_0} \frac{(k^{\beta-\alpha} * \phi)(n)}{\psi(n)} < \infty$$

for some $0 < \alpha < \beta$ and $\psi \in \omega_{\beta,loc}$. Then it is straightforward to check that $\tau^\beta(\psi) \hookrightarrow \tau^\alpha(\phi)$ and

$$\sum_{n=0}^{\infty} W^\alpha f(n)\Delta^{-\alpha}\mathcal{T}(n)x = \sum_{n=0}^{\infty} W^\beta f(n)\Delta^{-\beta}\mathcal{T}(n)x, \qquad f \in \tau^\beta(\psi),\ x \in X,$$

where we have applied Proposition 2.4 (v) and Remark 2.6.

(iv) Let $a > 0$ be such that $\|T\| \le a$. Then $\sigma(T) \subset \{z \in \mathbb{C} \mid |z| \le a\}$. For $f \in \tau^\alpha(\phi) \cap \ell^1_{a^n}$, we apply Remark 2.6 to get

$$\theta(f)x = \sum_{n=0}^{\infty} f(n)T^n(x), \qquad x \in X.$$

In particular $p_\lambda \in \tau^\alpha(\phi) \cap \ell(a^n)$ for $|\lambda| > a$ and

$$\theta(p_\lambda)x = \frac{1}{\lambda}\sum_{n=0}^{\infty} \frac{T^n}{\lambda^n}x = (\lambda - T)^{-1}x \qquad \text{for } x \in X. \qquad \blacksquare$$

Corollary 3.6: *Let $\alpha > 0$, $\phi \in \omega_\alpha$ and $\theta : \tau^\alpha(\phi) \to \mathcal{B}(X)$ be an algebra homomorphism. Then there exists $T \in \mathcal{B}(X)$ such that*

$$\theta(f)x = \sum_{n=0}^{\infty} W^\alpha f(n)\Delta^{-\alpha}\mathcal{T}(n)x, \qquad f \in \tau^\alpha(\phi), \quad x \in X;$$

in particular, $\theta(h_n^\alpha) = \Delta^{-\alpha}\mathcal{T}(n)$ for $n \in \mathbb{N}_0$ and $\theta(p_\lambda) = (\lambda - T)^{-1}$ for $|\lambda| > \|T\|$.

Proof. Take $T := \theta(e_1)$. Note that $e_1 = h_1^\alpha - \alpha h_0^\alpha$ (see Example 2.5 (ii)), and $h_n^\alpha = \Delta^{-\alpha}\mathcal{E}(n)$ for $n \in \mathbb{N}_0$ where $\mathcal{E}(n) = e_1^{*n}$ (see Example 3.2). By Example 2.12, $f = \sum_{j=0}^{\infty} W^\alpha f(n) h_n^\alpha$ for $f \in \tau^\alpha(\phi)$. We apply the continuity of θ to get

$$\theta(h_n^\alpha)x = \sum_{j=0}^{n} k^\alpha(n-j)\,(\theta(e_1))^j\, x = \Delta^{-\alpha}\mathcal{T}(n)x$$

and hence

$$\theta(f)x = \sum_{n=0}^{\infty} W^\alpha f(n)\theta(h_n^\alpha)x = \sum_{n=0}^{\infty} W^\alpha f(n)\Delta^{-\alpha}\mathcal{T}(n)x,$$

for $x \in X$. By Theorem 3.5 (iv), we conclude the proof. ∎

By Theorem 3.5 and Corollary 3.6, we obtain the following characterizations of (C,α)-bounded and power-bounded operators:

Corollary 3.7: *Let $T \in \mathcal{B}(X)$ and $\alpha > 0$ be given. The following assertions are equivalent:*

(i) *T is a (C,α)-bounded operator.*
(ii) *There exists a bounded algebra homomorphism $\theta : \tau^\alpha(k^{\alpha+1}) \to \mathcal{B}(X)$ such that $\theta(e_1) = T$.*

In the limit case, the following assertions are equivalent:

(a) *T is power bounded.*
(b) *There exists a bounded algebra homomorphism $\theta : \ell^1 \to \mathcal{B}(X)$ such that $\theta(e_1) = T$.*
(c) *For any $0 < \alpha < 1$, there exist bounded algebra homomorphisms θ_α : $\tau^\alpha(k^{\alpha+1}) \to \mathcal{B}(X)$ such that $\theta_\alpha(e_1) = T$ and $\sup_{0<\alpha<1} \|\theta_\alpha\| < \infty$.*

Proof. Due to the previous results, we only have to check that (c) implies (b). Indeed, since the map θ_α is an algebra homomorphism, then $\theta_\alpha(e_n) = T^n$, $\theta_\alpha(f)$ is well defined for $f \in c_{0,0}$ and is independent of α. Let $C > 0$ be such that $\sup_{0<\alpha<1}\|\theta_\alpha\| < C$. We define $\theta(f) := \theta_\alpha(f)$ for $f \in c_{0,0}$ and some given $\alpha \in (0,1)$. Then $\|\theta(f)\| = \|\theta_\alpha(f)\| \le C\, q_\alpha(f)$ for $f \in c_{0,0}$. By Theorem 2.11 (ii), we get that $\|\theta(f)\| \le C\|f\|_1$, for $f \in c_{0,0}$. The result now follows by an argument of density. ∎

4. The Z-transform and resolvent operators

Let $f : \mathbb{N}_0 \to X$ be a scalar sequence on a Banach space X. We recall that the Z-transform of a given sequence $f : \mathbb{N}_0 \to X$ is defined by

$$\tilde{f}(z) = \sum_{n=0}^{\infty} f(n) z^{-n}, \tag{4.1}$$

for all z such that this series converges. The set of numbers z in the complex plane for which the series (4.1) converges is called the region of convergence of $\tilde{f}$. The uniqueness of the inverse Z-transform may be established as follows: suppose that there are two sequences f, and g with the same Z-transform, that is,

$$\sum_{n=0}^{\infty} f(n) z^{-n} = \sum_{n=0}^{\infty} g(n) z^{-n}, \qquad |z| > R.$$

It follows from Laurent's theorem that $f(n) = g(n)$ for $n \in \mathbb{N}_0$.

Let $\phi : \mathbb{N}_0 \to (0,\infty)$ be a sequence such that $\phi(n) \le C a^n$ for some $C > 0$ and $a > 0$. To follow the notation given in [8], we write $\omega = \log(a)$ where ω is a bound for the counting measure supported on $\mathbb{N}_0$, i.e., $\epsilon_\lambda \in \ell^1_\phi$ for $\lambda > \omega$ where $\epsilon_\lambda(n) := e^{-\lambda n}$ and $n \in \mathbb{N}_0$. Let $C^\infty((\omega,\infty), X)$ be the space of X-valued functions on (ω,∞) infinitely differentiable in the norm topology of X. For $r \in C^\infty((\omega,\infty), X)$, set

$$\|r\|_{W,\phi,\omega} := \sup\left\{ \frac{\|r^{(k)}(\lambda)\|}{\|\beta_{k,\lambda}\|_{1,\phi}} \;\middle|\; k \in \mathbb{N}_0, \lambda > \omega \right\},$$

where $\beta_{k,\lambda}(n) = n^k e^{-\lambda n}$ for $n \in \mathbb{N}_0$ and $\lambda > \omega$.

The Widder space $C_W^\infty((\omega,\infty), X; \phi)$ is defined by

$$C_W^\infty((\omega,\infty), X; \phi) = \{ r \in C^\infty((\omega,\infty), X) \mid \|r\|_{W,\phi,\omega} < \infty \}.$$

Endowed with the norm $\|\cdot\|_{W,\phi,\omega}$, the space $C_W^\infty((\omega,\infty), X; \phi)$ is a Banach space, see more details in [8, Section 1]. A direct consequence of [8, Theorem 1.2] is the following result.

Theorem 4.1: *Let $\phi : \mathbb{N}_0 \to (0,\infty)$ be a sequence such that $\phi(n) \le Ca^n$ for some $C > 0$ and $a > 0$. For a vector-valued sequence $f : \mathbb{N}_0 \to X$ the following assertions are equivalent.*

(i) $\sup_{n\in\mathbb{N}_0} \dfrac{\|f(n)\|}{\phi(n)} < \infty$.
(ii) *There exists $\theta : \ell^1_\phi \to X$ such that $\theta(\lambda p_\lambda) = \tilde{f}(\lambda)$ for $\lambda > a$.*
(iii) $\tilde{f} \circ \exp \in C_W^\infty((\log(a),\infty), X; \phi)$.

Proof. (i) $\Longrightarrow$ (ii): The mapping defined by $\theta(g) := \sum_{n=0}^\infty g(n)f(n)$ where $g \in \ell^1_\phi$ satisfies the required condition. (ii) $\Longrightarrow$ (i): We define $h(n) := \theta(e_n)$ for $n \in \mathbb{N}_0$. It is clear that $\sup_{n\in\mathbb{N}_0} \dfrac{\|h(n)\|}{\phi(n)} < \infty$, and

$$\tilde{f}(\lambda) = \theta(\lambda p_\lambda) = \sum_{n\in\mathbb{N}_0} \theta(e_n)\lambda^n = \tilde{h}(\lambda), \qquad |\lambda| > a,$$

from which we conclude that $h(n) = f(n)$ for all $n \in \mathbb{N}_0$. This proves (i). Now we prove that (ii) $\Longrightarrow$ (iii). Due to [8, Theorem 1.2], we have

$$\theta(\epsilon_\mu) = \theta(\exp(\mu) p_{\exp(\mu)}) = (\tilde{f} \circ \exp)(\mu), \qquad \mu > \log(a),$$

and (iii) is proved. (iii) $\Longrightarrow$ (ii) Suppose that $\tilde{f} \circ \exp \in C_W^\infty((\log(a),\infty), X; \phi)$. Again by [8, Theorem 1.2], there exists a bounded homomorphism $\theta : \ell^1_\phi \to X$ such that $\theta(\epsilon_\mu) = (\tilde{f} \circ \exp)(\mu)$ for $\mu > \log(a)$. Since $\epsilon_\mu(n) = e^{-\mu n} = e^\mu p_{e^\mu}(n)$, we conclude that $\theta(\lambda p_\lambda) = \tilde{f}(\lambda)$ for $\lambda > a$. ∎

Remark 4.2: Note that Theorem 4.1 is closely connected to [8, Theorem 4.2], where the representability of functions of the Widder space $C_W^\infty((\omega,\infty), X; \mathrm{m})$ through functions of $L^\infty(\mathbb{R}_+, X; \mathrm{m})$ is proved under the assumption that the Banach space X has the Radon–Nikodym property (RNP). The RNP is a well-known property in the theory of Banach spaces. This property is also true for closed subspaces (hereditary property) and is enjoyed by any reflexive space, any separable dual space, and any $\ell^1(\Gamma)$ space, where Γ is a set. See definitions and more details in [3, Section 1.2].

In the well-known scalar version, namely $X = \mathbb{C}$, the following Z-transforms are obtained directly:

$$\begin{aligned} \widetilde{e_n}(z) &= z^{-n}, \qquad z \neq 0, \quad n \in \mathbb{N}_0; \\ \widetilde{k^\alpha}(z) &= \frac{z^\alpha}{(z-1)^\alpha}, \qquad |z| > 1; \\ \widetilde{p_\lambda}(z) &= \frac{z}{z\lambda - 1}, \qquad |z| > \frac{1}{|\lambda|}, \ \lambda \in \mathbb{C}\backslash\{0\}, \\ \widetilde{h_n^\alpha}(z) &= \sum_{j=0}^{n} k^\alpha(n-j) z^{-j}, \qquad z \neq 0. \end{aligned}$$

It is also well-known that

$$\widetilde{(f * g)}(z) = \widetilde{f}(z)\widetilde{g}(z), \tag{4.2}$$

for all z such that $\widetilde{f}(z)$ and $\widetilde{g}(z)$ exist. For properties on the Z-transform we refer, for instance, to the book [12, Chapter 6]. In particular, given $\alpha > 0$ and $f : \mathbb{N}_0 \to X$ such that $\tilde{f}(z)$ exists for $|z| > R$, then

$$\widetilde{(\Delta^{-\alpha} f)}(z) = \frac{z^\alpha}{(z-1)^\alpha} \widetilde{f}(z), \qquad |z| > \max\{R, 1\}.$$

We denote by ${}_n f(m) := f(n+m)$ for all $m, n \in \mathbb{N}_0$. The next technical lemma for the Z-transform will be used in the forthcoming Theorem 4.4. We observe that similar results hold for the Laplace transform, see for example [24, Proposition 4.1].

Lemma 4.3: *Let X be a Banach space, $f : \mathbb{N}_0 \to \mathbb{C}$ a scalar-valued sequence and $S : \mathbb{N}_0 \to \mathcal{B}(X)$ an operator-valued sequence. Then*

$$\frac{1}{\mu - \lambda} \widetilde{f}(\mu) \Big(\mu \widetilde{S}(\lambda) x - \lambda \widetilde{S}(\mu) x \Big) = \sum_{n=0}^{\infty} \lambda^{-n} \sum_{m=0}^{\infty} \mu^{-m} (f * {}_n S)(m) x, \quad x \in X,$$

$$\frac{1}{\mu - \lambda} \Big(\mu \widetilde{f}(\lambda) - \lambda \widetilde{f}(\mu) \Big) \widetilde{S}(\mu) x = \sum_{n=0}^{\infty} \lambda^{-n} \sum_{m=0}^{\infty} \mu^{-m} ({}_n f * S)(m) x, \quad x \in X,$$

for all $|\lambda| > |\mu|$ sufficiently large.

Proof. To show the first identity, note that

$$\widetilde{{}_n S}(\mu) x = \sum_{m=0}^{\infty} \mu^{-m} S(m+n) x = \mu^n \sum_{j=n}^{\infty} \mu^{-j} S(j) x = \mu^n \Big(\widetilde{S}(\mu) x - \sum_{j=0}^{n-1} \mu^{-j} S(j) x \Big),$$

for $x \in X$ and $n \geq 1$. By (4.2) we get

$$\begin{aligned}\sum_{n=0}^{\infty}\lambda^{-n}\sum_{m=0}^{\infty}\mu^{-m}(f*{}_nS)(m)x &= \widetilde{f}(\mu)\sum_{n=0}^{\infty}\lambda^{-n}\widetilde{{}_nS}(\mu)x \\ &= \widetilde{f}(\mu)\Big(\widetilde{S}(\mu)x+\sum_{n=1}^{\infty}\lambda^{-n}\widetilde{{}_nS}(\mu)x\Big) \\ &= \widetilde{f}(\mu)\widetilde{S}(\mu)x\sum_{n=0}^{\infty}\Big(\frac{\mu}{\lambda}\Big)^n-\widetilde{f}(\mu)\sum_{n=1}^{\infty}\Big(\frac{\mu}{\lambda}\Big)^n\sum_{j=0}^{n-1}\mu^{-j}S(j)x.\end{aligned}$$

Finally, from the identities

$$\sum_{n=1}^{\infty}\Big(\frac{\mu}{\lambda}\Big)^n\sum_{j=0}^{n-1}\mu^{-j}S(j)x=\sum_{j=0}^{\infty}\mu^{-j}S(j)x\sum_{n=j+1}^{\infty}\Big(\frac{\mu}{\lambda}\Big)^n=\frac{\mu}{\lambda-\mu}\widetilde{S}(\lambda)x,$$

we conclude that

$$\sum_{n=0}^{\infty}\lambda^{-n}\sum_{m=0}^{\infty}\mu^{-m}(f*{}_nS)(m)x=\frac{1}{\lambda-\mu}\widetilde{f}(\mu)\Big(\lambda\widetilde{S}(\mu)x-\mu\widetilde{S}(\lambda)x\Big),$$

for all $|\lambda|>|\mu|$ sufficiently large and $x\in X$. The second identity in the Lemma can be proved similarly. ∎

Theorem 4.4: *Let $\alpha \geq 0$, $\phi \in \omega_\alpha$, $a>1$ be given and let X be a Banach space. Suppose that $\{T_n\}_{n\in\mathbb{N}_0}\subset\mathcal{B}(X)$ is such that $T_0=I$ and satisfies $\|T_n\|\leq C\phi(n)\leq C'a^n$ for all $n\in\mathbb{N}_0$ with $C,C'>0$. The following statements are equivalent:*

(i) *The operator-valued sequence $\{T_n\}_{n\in\mathbb{N}_0}$ satisfies the equation* (3.1).
(ii) *There exists a bounded algebra homomorphism $\theta:\tau^\alpha(\phi)\to\mathcal{B}(X)$ such that $\theta(h_n^\alpha)=T_n$ for $n\in\mathbb{N}_0$.*
(iii) *The family $\{R(\lambda)\}_{|\lambda|>a}$ defined by*

$$R(\lambda)x:=\frac{(\lambda-1)^\alpha}{\lambda^{\alpha+1}}\sum_{n=0}^{\infty}\lambda^{-n}T_n(x),\qquad |\lambda|>a,\ x\in X,$$

is a pseudo-resolvent.

In these cases the generator of $\{T_n\}_{n\in\mathbb{N}_0}$, defined by $T:=T_1-\alpha I$ (see Remark 3.4), satisfies that $T_n=\Delta^{-\alpha}\mathcal{T}(n)$ for $n\in\mathbb{N}_0$, $\theta(e_1)=T$, $\{\lambda\in\mathbb{C}\mid|\lambda|>a\}\subset\rho(T)$ and

$$R(\lambda)=(\lambda-T)^{-1},\qquad |\lambda|>a.$$

Proof. The proof (i)⇒(ii) is a direct consequence of Theorem 3.3 and Theorem 3.5. To show that (ii)⇒(iii), we use Corollary 3.6. Finally we prove (iii)⇒(i). It is clear that

$$R(\lambda)=\frac{\widetilde{\mathfrak{T}}(\lambda)}{\lambda\widetilde{k^\alpha}(\lambda)},\qquad |\lambda|>a,$$

where $\mathfrak{T}=\{T_n\}_{n\in\mathbb{N}_0}$ and $\widetilde{\mathfrak{T}}$ is given by (4.1). Since $\{R(\lambda)\}_{|\lambda|>a}$ is a pseudo-resolvent, then

$$(\mu-\lambda)\frac{\widetilde{\mathfrak{T}}(\lambda)\widetilde{\mathfrak{T}}(\mu)}{\lambda\widetilde{k^\alpha}(\lambda)\mu\widetilde{k^\alpha}(\mu)}=\frac{\widetilde{\mathfrak{T}}(\lambda)}{\lambda\widetilde{k^\alpha}(\lambda)}-\frac{\widetilde{\mathfrak{T}}(\mu)}{\mu\widetilde{k^\alpha}(\mu)},\qquad |\lambda|,|\mu|>a,\quad \mu\neq\lambda,$$

so

$$\widetilde{\mathfrak{T}}(\lambda)\widetilde{\mathfrak{T}}(\mu)=\frac{1}{\mu-\lambda}\Big(\mu\widetilde{k^\alpha}(\mu)\widetilde{\mathfrak{T}}(\lambda)-\lambda\widetilde{k^\alpha}(\lambda)\widetilde{\mathfrak{T}}(\mu)\Big),\qquad |\lambda|,|\mu|>a,\quad \mu\neq\lambda.$$

On the other hand, note that the condition (3.1) can be rewritten as

$$\begin{aligned}(k^\alpha * {}_n\mathfrak{T})(m)-({}_nk^\alpha * \mathfrak{T})(m)+k^\alpha(n)T_m&\\ =\sum_{u=n}^{n+m}k^\alpha(n+m-u)T_u-\sum_{u=0}^{m-1}k^\alpha(n+m-u)T_u,&\end{aligned}$$

for $m\geq 1$ and $n\geq 0$. We apply Lemma 4.3 in order to obtain, after a simple algebraic manipulation, that

$$\begin{aligned}\sum_{n=0}^{\infty}\lambda^{-n}\sum_{m=0}^{\infty}\mu^{-m}((k^\alpha * {}_n\mathfrak{T})(m)-({}_nk^\alpha * \mathfrak{T})(m)+k^\alpha(n)T_m)&\\ =\frac{\mu\widetilde{k^\alpha}(\mu)\widetilde{\mathfrak{T}}(\lambda)-\lambda\widetilde{k^\alpha}(\lambda)\widetilde{\mathfrak{T}}(\mu)}{\mu-\lambda},&\end{aligned}$$

for $|\lambda|,|\mu|>a$, and $\mu\neq\lambda$. Then we conclude that $\{T_n\}_{n\in\mathbb{N}_0}$ satisfies (3.1), as consequence of the injectivity of the double Z-transform. Finally, by Corollary 3.6

$$R(\lambda)=\theta(p_\lambda)=(\lambda-T)^{-1},\qquad |\lambda|>a,$$

and we finish the proof. ∎

5. Applications, examples and final comments

In this last section, we present applications of, comments on, examples and counterexamples to the results presented in this paper.

5.1. Bounds for Abel means. Given $T \in \mathcal{B}(X)$ and $0 \le r < 1$ we recall that the Abel mean of order r to the operator T, denoted by $A_r(T)$, is defined by

$$A_r(T)x := (1-r)\sum_{n=0}^{\infty} r^n T^n(x), \qquad x \in X,$$

whenever this series converges, see for example [26]. Denoting $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ the spectral radius of T, we have for $0 < r < \frac{1}{r(T)}$ that $\frac{1}{r} \in \rho(T)$ and

$$A_r(T) = \frac{(1-r)}{r}\left(\frac{1}{r} - T\right)^{-1}, \qquad 0 < r < \min\left\{1, \frac{1}{r(T)}\right\}.$$

The next theorem improves [26, Proposition 2.1 (i)], given there for $\alpha \in \{0,1\}$.

Theorem 5.1: *Take $\alpha \ge 0$ and $T \in \mathcal{B}(X)$. Then*

$$A_r(T)x = (1-r)^{\alpha+1}\sum_{n=0}^{\infty} r^n \Delta^{-\alpha}\mathcal{T}(n)x, \qquad 0 \le r < \min\left\{1, \frac{1}{r(T)}\right\}.$$

In the case that $\|\Delta^{-\alpha}\mathcal{T}(n)\| \le Ck^{\gamma+1}(n)$ for $n \ge 1$ and $\gamma \ge \alpha$ then we have

$$\|A_r(T)\| \le C(1-r)^{-(\gamma-\alpha)}, \qquad 0 \le r < 1.$$

In particular, if T is a (C,α)-bounded operator then $\sup_{0\le r<1}\|A_r(T)\| < \infty$.

Proof. Let $\alpha \ge 0$ be given and $p_{\frac{1}{r}}(n) = r^{n+1}$ for $0 < r < 1$. By Remark 2.6, we have that

$$\begin{aligned} A_r(T)x &= (1-r)\sum_{n=0}^{\infty} r^n T^n(x) = \frac{1-r}{r}\sum_{n=0}^{\infty} W^\alpha p_{\frac{1}{r}}(n)\Delta^{-\alpha}\mathcal{T}(n)x \\ &= \frac{(1-r)^{\alpha+1}}{r}\sum_{n=0}^{\infty} p_{\frac{1}{r}}(n)\Delta^{-\alpha}\mathcal{T}(n)x = (1-r)^{\alpha+1}\sum_{n=0}^{\infty} r^n\Delta^{-\alpha}\mathcal{T}(n)x, \end{aligned}$$

where we have used Example 2.5 (i) for $0 < r < \min\{1, \frac{1}{r(T)}\}$. For $r = 0$ it is obvious.

In the case that $\|\Delta^{-\alpha}\mathcal{T}(n)\| \le Ck^{\gamma+1}(n)$ for $n \ge 1$ and $\gamma \ge \alpha$, there exists a bounded algebra homomorphism $\theta : \tau^\alpha(k^{\gamma+1}) \to \mathcal{B}(X)$ such that

$$\theta(f) = \sum_{n=0}^{\infty} W^\alpha f(n)\Delta^{-\alpha}\mathcal{T}(n), \quad x \in X,\ f \in \tau^\alpha(k^{\gamma+1});$$

see Theorem 3.5. Note that $p_{\frac{1}{r}} \in \tau^\alpha(k^{\gamma+1})$ and $A_r(T) = \frac{1-r}{r}\theta(p_{\frac{1}{r}})$, for $0 < r < 1$. By formula (2.6), we obtain that

$$\|A_r(T)\| \leq C\frac{1-r}{r} q_{k^{\gamma+1}}(p_{\frac{1}{r}}) = C\frac{1-r}{r}\frac{r}{(1-r)^{\gamma+1-\alpha}} = \frac{C}{(1-r)^{\gamma-\alpha}}, \quad 0 < r < 1,$$

and we conclude the proof. ∎

Remark 5.2: If we consider $\|T^n\| \leq Cn^\gamma$, with $\gamma \geq 0$, and using the estimate $n^\gamma \leq \Gamma(\gamma+1)k^{\gamma+1}(n)$ which follows easily from (2.3), then we get that

$$\|A_r(T)\| \leq C\Gamma(\gamma+1)(1-r)^{-\gamma},$$

which improves the bound of [26, Proposition 2.1 (i) (2.3)]. One can use similar arguments to improve the bound of [26, Proposition 2.1 (i) (2.4)].

Remark 5.3: An inverse result exists on Banach lattices (see [26, Corollary 3.2]), which proves that for any $\alpha > -1$ and a positive bounded operator T, $\{(1-r)^\alpha A_r(T),\ 0 \leq r < 1\}$ is bounded if and only if $\|\Delta^{-1}\mathcal{T}(n)\| \leq C(n+1)^\alpha$, $n \in \mathbb{N}_0$. In particular, T is Abel-mean bounded if and only if it is $(C,1)$-bounded. Note that there are examples of positive $(C,1)$-bounded operators in Banach lattices which are not power bounded, see the remarks following [26, Corollary 3.2].

5.2. α-Times integrated semigroups and Cesàro sums. Now, let A be a closed linear operator on X, $\alpha > 0$ and $\{S_\alpha(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ an α-times integrated semigroup generated by A, that is, $S_\alpha(0) = 0$, the map $[0,\infty) \to X$, $r \mapsto S_\alpha(r)x$ is strongly continuous and

$$S_\alpha(t)S_\alpha(s)x = \frac{1}{\Gamma(\alpha)}\left(\int_t^{t+s}(t+s-r)^{\alpha-1}S_\alpha(r)xdr - \int_0^s (t+s-r)^{\alpha-1}S_\alpha(r)xdr\right),$$

$x \in X$, for $t, s > 0$; for $\alpha = 0$, $\{S_0(t)\}_{t\geq 0}$ is a usual C_0-semigroup, $S_0(0) = I$ and $S_0(t+s) = S_0(t)S_0(s)$ for $t, s > 0$. In the case that $\{S_\alpha(t)\}_{t\geq 0}$ is a non-degenerate family and $\|S_\alpha(t)\| \leq Ce^{\omega t}$ for $C > 0$, $\omega \in \mathbb{R}$, then there exists a closed operator, $(A, D(A))$, called the generator of $\{S_\alpha(t)\}_{t\geq 0}$, such that

$$(5.1) \qquad (\lambda - A)^{-1}x = \lambda^\alpha \int_0^\infty e^{-\lambda t}S_\alpha(t)xdt, \qquad \Re\lambda > \omega, \qquad x \in X.$$

Moreover, the following integral equality holds

$$A\int_0^t S_\alpha(s)xds = S_\alpha(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}x, \qquad t>0, \quad x\in X. \tag{5.2}$$

For more details see [22].

THEOREM 5.4: *Suppose that $\{S_\alpha(t)\}_{t\geq 0}$ is an α-times integrated semigroup generated by $(A, D(A))$ such that $\|S_\alpha(t)\| \leq Ce^{\omega t}$ with $0\leq\omega<1$. Then $1\in\rho(A)$ and for $R := (1-A)^{-1}$, $\mathcal{R}(n) = R^n$ we have*

$$\begin{aligned}\Delta^{-\alpha}\mathcal{R}(n)x &= (I-A)\int_0^\infty \frac{e^{-t}t^n}{n!}S_\alpha(t)xdt, \qquad n\in\mathbb{N}_0,\\ &= \int_0^\infty \frac{e^{-t}t^{n-1}}{(n-1)!}S_\alpha(t)xdt + k^{\alpha+1}(n)x - k^{\alpha+1}(n-1)x, \qquad n\geq 1, \quad x\in X.\end{aligned}$$

In particular, if $\{S_\alpha(t)\}_{t\geq 0}$ has temperated growth, i.e. $\|S_\alpha(t)\|\leq Ct^\alpha$ for $t>0$, then $(I-A)^{-1}$ is a (C,α)-bounded operator.

Proof. Let $\lambda\in\rho(A)$ be given. We have

$$\frac{(-1)^n}{n!}\frac{d^n}{d\lambda^n}(\lambda^{-\alpha}(\lambda-A)^{-1}) = \sum_{j=0}^n \frac{k^\alpha(n-j)}{\lambda^{\alpha+n-j}}(\lambda-A)^{-j-1}.$$

On the other hand, for $\Re\lambda>\omega$, we apply formula (5.1) to get that

$$\frac{(-1)^n}{n!}\frac{d^n}{d\lambda^n}(\lambda^{-\alpha}(\lambda-A)^{-1})x = \int_0^\infty \frac{t^n}{n!}e^{-\lambda t}S_\alpha(t)x\,dt, \qquad x\in X.$$

Finally, we set $\lambda=1$ to conclude the first equality. Now for $n\geq 1$, we have that

$$\begin{aligned}\Delta^{-\alpha}\mathcal{R}(n)x &= \int_0^\infty \frac{e^{-t}t^n}{n!}S_\alpha(t)xdt + A\int_0^\infty \frac{e^{-t}t^{n-1}}{(n-1)!}\left(1-\frac{t}{n}\right)\int_0^t S_\alpha(s)xdsdt\\ &= \int_0^\infty \frac{e^{-t}t^{n-1}}{(n-1)!}S_\alpha(t)xdt + k^{\alpha+1}(n)x - k^{\alpha+1}(n-1)x, \qquad x\in X,\end{aligned}$$

where we applied the equality (5.2).

In the case that $\|S_\alpha(t)\|\leq Ct^\alpha$, we use the second equality and that the sequence $k^{\alpha+1}$ is increasing to conclude that

$$\sup_{n\in\mathbb{N}_0}\frac{\|\Delta^{-\alpha}\mathcal{R}(n)\|}{k^{\alpha+1}(n)} < \infty$$

and $(I-A)^{-1}$ is a (C,α)-bounded operator. ∎

Classical examples of generators of temperated α-times integrated semigroups are differential operators A such that their symbol $\hat{A}$ is of the form $\hat{A} = ia$, where a is a real elliptic homogeneous polynomial on $\mathbb{R}^n$ or $a \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is a real homogeneous function on $\mathbb{R}^n$ such that if $a(t) = 0$ then $t = 0$; see [21, Theorem 4.2], and other different examples in [21, Section 6].

Remark 5.5: In the case of uniformly bounded C_0-semigroups, i.e. $\{T(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ such that $\sup_{t>0} \|T(t)\| < \infty$, the resolvent $(1-A)^{-1}$ is power-bounded due to

$$(1-A)^{-n}x = \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-t} T(t) x dt, \qquad x \in X.$$

Note that Theorem 5.4 includes a natural extension of this fact: the resolvent $(1-A)^{-1}$ is a (C,α)-bounded operator when A generates a temperated α-times integrated semigroup.

We may also consider the homomorphism θ defined in Theorem 3.5, and in this case

$$\theta(\Delta f)x = -A\theta(f)x - (I-A)f(0)x, \qquad f \in \tau^\alpha(k^{\alpha+1}), \quad x \in D(A),$$

when A generates a temperated α-times integrated semigroup. This equality shows that if we know the generator A, we can transfer properties between f and Δf for sequences in $\tau^\alpha(k^{\alpha+1})$.

5.3. Counterexamples of bounded homomorphisms.

Example 5.6: In [10, Section 2] there is an example of a positive, Cesàro bounded but not power bounded operator T on the space ℓ^1. As the author comments in [9, Section 4. Examples], one has $\|T^n\|_1 \leq Kn/\ln(n)$ where K is the uniform bound of the Cesàro averages of T. In this example T is also a contraction in ℓ^∞. In [13, Section (VI)], it is proved that $\sup_{n\geq 0} \|T^n\|_p \geq (2^k)^{\frac{1}{p}}$ for any $k \geq 1$ and $1 \leq p < \infty$. We conclude that T is not power bounded in ℓ^p $(1 \leq p < \infty)$ and T is Cesàro bounded in ℓ^p $(1 \leq p \leq \infty)$. By Corollary 3.7, there exists a bounded homomorphism $\theta : \tau^1(k^2) \to \mathcal{B}(\ell^p)$ such that $\theta(e_1) = T$ which extends to $\theta : \ell^1 \to \mathcal{B}(\ell^p)$ if and only if $p = \infty$.

Example 5.7: In [28], a simple matrix construction, which unifies different approaches to the Ritt condition and ergodicity of matrix semigroups, is studied in detail. Consider the Banach space $\mathfrak{X} := X \oplus X$ with norm

$$\|x_1 \oplus x_2\|_{X\oplus X} := \sqrt{\|x_1\|^2 + \|x_2\|^2}, \qquad x_1 \oplus x_2 \in \mathfrak{X}.$$

Let the bounded linear operator $\mathfrak{T}$ on $\mathfrak{X}$ be defined by the operator matrix

$$\mathfrak{T} := \begin{pmatrix} T & T-I \\ 0 & T \end{pmatrix}$$

where $T \in \mathcal{B}(X)$. In [28, Lemma 2.1], some connected properties between T and $\mathfrak{T}$ are given. Now we consider $X = \ell^2$ and the backward shift operator $T \in \mathcal{L}(\ell^2)$ defined by

$$T(x)(n) := x(n+1), \qquad x \in \ell^2, n \in \mathbb{N}_0.$$

By [28, Example 3.1], $\|\mathfrak{T}^n\| \geq 2n$ and $\mathfrak{T}$ is a $(C,1)$-bounded operator. We apply Corollary 3.7 to conclude that there exists an algebra homomorphism $\theta : \tau^1(k^2) \to \mathcal{B}(\mathfrak{X})$ such that $\theta(e_1) = \mathfrak{T}$ which does not extend continuously to ℓ^1. In [28, Remark 3.2], the growth $\|\mathfrak{T}^n\| \geq 2n$ is pointed at as the fastest possible for a Cesàro bounded operator.

Example 5.8: In [26, Proposition 4.3], the following example is given. For any γ with $0 < \gamma < 1$, there exists a positive linear operator T on an L_1-space such that

$$\sup_{n\geq 0} \left\| \frac{\Delta^{-\gamma}\mathcal{T}(n)}{k^{\gamma+1}(n)} \right\| = \infty, \quad \text{but} \quad \sup_{n\geq 0} \left\| \frac{\Delta^{-\beta}\mathcal{T}(n)}{k^{\beta+1}(n)} \right\| < \infty \quad \text{for all } \beta > \gamma.$$

By Corollary 3.7, we conclude that there exists a bounded algebra homomorphism θ such that $\theta : \tau^\beta(k^{\beta+1}) \to \mathcal{B}(X)$ for all $\beta > \gamma$, $\theta(e_1) = T$, and the homomorphism θ does not extend continuously to the algebra $\tau^\gamma(k^{\gamma+1})$ with $0 < \gamma < 1$.

Example 5.9: In [26, Proposition 4.4 (i)], the following operator is constructed: Let $dimX = \infty$. For any integer $j \geq 0$, there exists a bounded linear operator T on X such that

$$\sup_{n\geq 0} \left\| \frac{\Delta^{-(j+1)}\mathcal{T}(n)}{k^{j+2}(n)} \right\| < \infty, \quad \text{but} \quad \sup_{n\geq 0} \left\| \frac{\Delta^{-\gamma}\mathcal{T}(n)}{k^{\gamma+1}(n)} \right\| = \infty \quad \text{for } 0 \leq \gamma < j+1.$$

By Corollary 3.7, we conclude that there exists a bounded algebra homomorphism θ such that $\theta : \tau^{j+1}(k^{j+2}) \to \mathcal{B}(X)$, $\theta(e_1) = T$, and the homomorphism θ does not extend continuously to the algebra $\tau^\gamma(k^{\gamma+1})$ with $0 \leq \gamma < j+1$.

Example 5.10: In [26, Proposition 4.4 (ii)], the following operator is constructed. Let $dimX = \infty$. There exists a bounded linear operator T on X with $r(T) = 1$,

$\|T\| = 2$, and

$$\|A_r(T)\| \le 1 - r, \quad 0 < r < 1; \quad \text{and} \quad \sup_{n\ge 0} \left\| \frac{\Delta^{-j}\mathcal{T}(n)}{k^{j+1}(n)} \right\| = \infty, \quad \text{for } j \ge 1.$$

Since $k^j(n) \le k^{j+1}(n)$ for $n \ge 0$, we also conclude that

$$\left\| \frac{\Delta^{-j}\mathcal{T}(n)}{k^{j}(n)} \right\| = \infty \qquad \text{for } j \ge 1$$

and the converse of Theorem 5.1 does not hold for $\gamma < \alpha$.

5.4. Application to Katznelson–Tzafriri type theorems. Let $A(\mathbb{T})$ be the regular convolution Wiener algebra formed by all continuous periodic functions $f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{int}$, $t \in [-\pi, \pi]$, where $\{\hat{f}(n)\}_{n\in\mathbb{Z}}$ are the Fourier coefficients of f, that is

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt, \qquad n \in \mathbb{Z},$$

with the norm $\|f\|_{A(\mathbb{T})} := \sum_{n=-\infty}^{\infty} |\hat{f}(n)|$, and let $A_+(\mathbb{T})$ be the closed convolution subalgebra of $A(\mathbb{T})$ where the functions satisfy that $\hat{f}(n) = 0$ for $n < 0$. Note that both $A(\mathbb{T})$ and $\ell^1_{\mathbb{Z}}$, and $A_+(\mathbb{T})$ and ℓ^1 are isometrically isomorphic, where $\ell^1_{\mathbb{Z}}$ denotes the complex summable sequences indexed by $\mathbb{Z}$.

Katznelson and Tzafriri proved in 1986 the following well-known theorem: if $T \in \mathcal{B}(X)$ is power-bounded and $f \in A_+(\mathbb{T})$ is of spectral synthesis in $A(\mathbb{T})$ with respect to $\sigma(T) \cap \mathbb{T}$, then

$$\lim_{n\to\infty} \|T^n\theta(\hat{f})\| = 0;$$

see [23, Theorem 5]. Moreover, for $T \in \mathcal{B}(X)$ a power-bounded operator, one has $\lim_{n\to\infty} \|T^n - T^{n+1}\| = 0$ if and only if $\sigma(T) \cap \mathbb{T} \subseteq \{1\}$; see [23, Theorem 1].

The authors have obtained some similar results for (C, α)-bounded operators, which will appear in a forthcoming paper. We define $A^\alpha(\mathbb{T})$ to be a new regular Wiener algebra contained in $A(\mathbb{T})$, and $A^\alpha_+(\mathbb{T})$ a convolution closed subalgebra of $A^\alpha(\mathbb{T})$, which is isometrically isomorphic to $\tau^\alpha(k^{\alpha+1})$. The result states that if $\alpha > 0$, $T \in \mathcal{B}(X)$ is a (C, α)-bounded operator and $f \in A^\alpha_+(\mathbb{T})$ is of spectral synthesis in $A^\alpha(\mathbb{T})$ with respect to $\sigma(T) \cap \mathbb{T}$, then

$$\lim_{n\to\infty} \frac{1}{k^{\alpha+1}(n)} \|\Delta^{-\alpha}\mathcal{T}(n)\theta(\hat{f})\| = 0.$$

On the continuous case, Katznelson–Tzafriri theorems have been proved for C_0-semigroups and extended later for α-times integrated semigroups; see [14] and [15], respectively.

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