Singularly perturbed reaction-diffusion problems with discontinuities in the initial and/or the boundary data

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November 26, 2019

Abstract: Numerical approximations to the solutions of three different problem classes of singularly perturbed parabolic reaction-diffusion problems, each with a discontinuity in the boundary-initial data, are generated. For each problem class, an analytical function associated with the discontinuity in the data, is identified. Parameter-uniform numerical approximations to the difference between the analytical function and the solution of the singularly perturbed problem are generated using piecewise-uniform Shishkin meshes. Numerical results are given to illustrate all the theoretical error bounds established in the paper.

1 Introduction

To establish theoretical error bounds for a numerical method, one requires the solution of the continuous problem to be sufficiently regular, in order that certain partial derivatives of the solution are bounded on the closed domain. For parabolic problems, this often requires the assumption of sufficient compatibility conditions between the initial and boundary data, which can be viewed as solely a theoretical [3] constraint. However, accuracy can be lost in the numerical approximations if insufficient compatibility is imposed [4, 7], especially if a higher order numerical method is utilized. Moreover,

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certain mathematical models (e.g., Biot's consolidation theory of porous media [1, 2]) consider mathematical models with discontinuities between the initial and boundary data deliberately built into the problem formulation. In this paper, we consider the effect of discontinuous boundary/initial data on the accuracy of numerical approximations, in the context of singularly perturbed parabolic problems.

The solutions of singularly perturbed problems, with smooth data, typically exhibit boundary layers, whose widths depend on the singular perturbation parameter. Additional interior layers can appear when the coefficients of the differential operator are discontinuous or if the inhomogenous term contains a point source [5, 8, 15, 17]. In all of these problem classes, parameter-uniform numerical methods [6] have been constructed, by using *a priori* information about both the location and character of all boundary/interior layers that are present in the solution and using this analytical information to design an appropriate piecewise-uniform Shishkin mesh [6] for the problem.

In the main, the boundary and initial data for the problem need to be sufficiently smooth, in order to prevent further classical singularities appearing in the solution. As the smoothness of the boundary and initial data reduces, then the order of convergence can also reduce [18, 19, 21, 22]. In the case of sufficiently smooth data satisfying second-order compatibility conditions at the corners of the space-time domain (see Appendix 1), the typical error bound [14] in the L_∞ norm for singularly perturbed parabolic problems of reaction-diffusion type, is of the form

$$\|\bar{U} - u\| \leq C(N^{-1} \ln N)^2 + CM^{-1},$$

when one uses a tensor product of an appropriate piecewise-uniform Shishkin mesh in space (with N elements) and a uniform mesh (with M elements) in time, to generate a global approximation $\bar{U}$ to the continuous solution u. If there is only zero-order compatibility conditions assumed and a possible jump in the first time derivative of the boundary data, then the same numerical method retains parameter-uniform convergence, albeit with some minor reduction in the order [18] of convergence in space. If there is a discontinuity in the first space derivative of the initial condition, the order of convergence can drop to $O(N^{-1} + M^{-0.5})$ [19], [20, §14.2]. Nevertheless, the numerical method (based on an appropriate piecewise-uniform mesh) retains parameter-uniform convergence of some positive order.

However, for the singularly perturbed heat equation

$$-\varepsilon u_{xx} + u_t = 0, \;\; (x,t) \in (0,1) \times (0,T],$$

if the initial condition $u(x,0)$ is discontinuous [9] or if there is an incompatibility between the initial and boundary conditions $u(0^+,0) \neq u(0,0^+)$ then rectangular meshes do not produce parameter-uniform numerical methods [11, 12]. Note that if one incorporates a co-ordinate system aligned to the similarity transformation $\theta = x/(2\sqrt{\varepsilon t})$, then one can design a piecewise-uniform mesh [16] in this transformed co-ordinate system, to generate parameter-uniform numerical approximations. However, we will not consider the use of such transformed co-ordinate systems here.

In this paper, we introduce a mixed analytical/numerical method, which is based on the ideas in [7]. This method first identifies explicitly the main singular component $s(x,t)$ associated with the singularity and uses a piecewise-uniform Shishkin mesh to generate a parameter-uniform numerical approximation to the difference, $u - s$, between the exact solution u and the main singular component s. In this way, parameter-uniform numerical approximations are created for singularly perturbed problems with discontinuous initial conditions, problems with incompatible initial/boundary conditions and problems with discontinuous boundary conditions.

Below we examine singularly perturbed problems of the form

$$u_t - \varepsilon u_{xx} + b(t)u = f(x,t), \quad (x,t) \in Q := (0,1) \times (0,T]; \quad b(t) \geq 0,\ t \geq 0,$$

where the boundary/initial data will have a discontinuity at some point on the boundary $\bar{Q} \setminus Q$. Note that the coefficient $b(t)$ is assumed to be independent of the space variable x. This assumption permits us present relatively simple proofs for all of the pointwise bounds on the derivatives of the components of the continuous solutions, presented below. In [10], a related problem class was considered, which involved the differential equation

$$\varepsilon(u_t - u_{xx}) + b(x,t)u = f, (x,t) \in (0,1) \times (0,1]; \quad b(x,t) > 0;$$

with incompatible boundary-initial data. Note that the coefficient $b(x,t)$ can vary in space. However, the corresponding proofs (establishing bounds on the derivatives of the layer components) are significantly longer and contain much more technical detail to what is required for the three problem classes considered in the current paper. Nevertheless, in our numerical results section, we present test examples where the coefficient $b(x,t)$ does vary in space and we see that (from a computational perspective) the assumption $b(t)$ appears not necessary, in practice. In summary, the assumption $b(t)$ allows us present theoretical error bounds for three problem classes in a single publication. In this way, we see the minor modifications in the overall approach, when dealing with singularly perturbed problems with discontinuous boundary and initial data. In addition, initial layers appeared in the problem class studied in [10], which required the use of a Shishkin mesh in time. In the current paper, a uniform mesh in time suffices, as the time derivative of the continuous solution has a coefficient of order one in this paper.

The paper is structured as follows: In §2 the asymptotic behaviour of the solution u of the three classes problems is analysed. For each class of problems the singular component s is identified and the behaviour of $u-s$ is revealed by using an appropriate decomposition into regular and layer components. In §3 a finite difference scheme is proposed to approximate $u-s$ for each class of problems. Each scheme uses the backward Euler method in time and standard central differences in space defined on appropriately constructed meshes of Shishkin type. Error estimates in the maximum norm are established, which yield global parameter-uniform convergence for the methods. In §4 some numerical results for the three classes of problems are given and they indicate that our error estimates are sharp. The paper is completed with two technical appendices.

Notation. Throughout the paper, C denotes a generic constant that is independent of the singular perturbation parameter ε and of all discretization parameters. The L_∞ norm on the domain D will be denoted by $\|\cdot\|_D$ and the subscript is omitted if the domain is $\bar{Q}$.

2 Three classes of problem

Before we define the three problem classes to be examined in this paper, we define a set of singular functions which are associated with the singularities that are generated by discontinuous boundary/initial data in singularly perturbed problems.

The singular function $s : (-\infty, \infty) \times [0, \infty) \to (-1, 1)$ is defined as

$$s(x,t) := e^{-b(0)t} \operatorname{erf}\left(\frac{x}{2\sqrt{\varepsilon t}}\right), \text{ where } \operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_{r=0}^{z} e^{-r^2}\, dr. \qquad (1)$$

This function satisfies the constant coefficient quarter plane homogeneous problem

$$s_t - \varepsilon s_{xx} + b(0)s = 0, \quad (x,t) \in (0,\infty) \times (0,\infty),$$
$$s(0,t) = 0, \quad t > 0; \quad s(x,0) = 1, \quad x > 0.$$

Observe that $s \notin C^0([0,\infty) \times [0,\infty))$. Define the associated set of functions

$$s_n(x,t) := t^n s(x,t), \quad n \geq 0; \quad cs_n(x,t) := t^n(1 - s(x,t)).$$

Then $s_n, cs_n \in C^{n-1+\gamma}([0,\infty)\times[0,\infty)),\ n \geq 1.$ [1] Note further that

$$(s_n)_t - \varepsilon(s_n)_{xx} + b(0)(s_n) = n(s_{n-1})$$
$$\text{and}\quad (cs_n)_t - \varepsilon(cs_n)_{xx} + b(0)(cs_n) = n(cs_{n-1}) + b(0)t^n.$$

Hence, for all $n \geq 1$, we have the recurrence relationship

$$Ls_n(x,t) = ns_{n-1}(x,t) + (b(t)-b(0))s_n(x,t), \tag{2}$$

where $Lz := z_t - \varepsilon z_{xx} + b(t)z$.

2.1 Problem Class 1: incompatible boundary-initial data

Consider the singularly perturbed parabolic problem: Find $u : \bar{Q} \to \mathbb{R}$ with $Q := (0,1)\times(0,T]$, such that

$$Lu = u_t - \varepsilon u_{xx} + b(t)u = f(x,t),\ \text{in } Q; \quad b(t) \geq \beta \geq 0, \forall t \geq 0; \tag{3a}$$
$$u(0,t) = 0,\ u(1,t) = 0,\ t \geq 0, \quad u(x,0) = \phi(x),\ 0 < x < 1; \tag{3b}$$
$$\phi(0^+) \neq 0,\ \phi(1) = 0, \quad f, b \in C^{4+\gamma}(\bar{Q}), \quad \phi \in C^4(0,1); \tag{3c}$$
$$f(1,0) = -\varepsilon\phi''(1^-), \quad -\varepsilon\phi^{(iv)}(1^-) + b(0)\phi''(1^-) = (f_t + f_{xx})(1,0); \tag{3d}$$
$$f(0,0) = -\varepsilon\phi''(0^+). \tag{3e}$$

Since this problem is linear, there is no loss in generality in assuming homogeneous boundary conditions. Observe that there is a discontinuity in the data at the corner point $(0,0)$. The discontinuity in the data for this first problem class is the same discontinuity as that examined in [10]. The assumption (3e) on the data allows us present a simplified version of the numerical analysis. Without this assumption, we would require more of the analysis from [10]. By assuming the compatibility conditions (3d), we prevent any classical singularities appearing in the vicinity of the point $(1,0)$ (see Appendix 1).

In order to deduce the asymptotic behaviour of the solution of problem (3), it is decomposed into the sum

$$u = \phi(0^+)s(x,t) + y. \tag{4}$$

[1]The space $C^{n+\gamma}(\bar{Q})$ is the set of all functions, whose derivatives of order n are Hölder continuous of degree $\gamma > 0$. That is,

$$C^{n+\gamma}(\bar{Q}) := \left\{ z : \frac{\partial^{i+j} z}{\partial x^i \partial t^j} \in C^\gamma(\bar{Q}),\ 0 \leq i + 2j \leq n \right\}.$$

Note that $|\phi(0^+)|$ is the magnitude of the jump in the boundary/initial data, at $(0,0)$. The remainder y, defined by (4), satisfies the problem

$$Ly = F := f - (b(t) - b(0))\phi(0^+)s, \ (x,t) \in Q; \tag{5a}$$

$$y(0,t) = 0, \ y(1,t) = -\phi(0^+)s(1,t), \ t \geq 0; \tag{5b}$$

$$y(x,0) = \phi(x) - \phi(0^+), \ 0 < x < 1. \tag{5c}$$

Recall that $\phi(1) = 0$ and so $y(1^-,0) = y(1,0^+)$. Hence the boundary and initial data are continuous in the case of problem (5), $F \in C^{0+\gamma}(\bar{Q})$ and, using assumption (3e), we have that $y \in C^{2+\gamma}(\bar{Q})$. We further decompose the solution of (5) as follows:

$$y = v + w_L + w_R, \tag{6a}$$

where the regular component v satisfies the problem

$$L^*v^* = f^* - (b(t) - b(0))\phi(0^+), \ (x,t) \in Q_0^* := (-a, 1+a) \times (0,T], \tag{6b}$$

which is posed on an extended (in the spatial direction) domain Q_0^* [2] and a is an arbitrary positive parameter. The initial/boundary values for the regular component are determined by $v^* = v_0^* + \varepsilon v_1^*$, where these two subcomponents, in turn, satisfy

$$bv_0^* + (v_0^*)_t = f^* - (b(t) - b(0))\phi(0^+), \quad t > 0; \tag{6c}$$

$$v_0^*(x,0) = y^*(x,0), \quad x \in (-a, 1+a), \tag{6d}$$

$$L^*v_1^* = (v_0^*)_{xx}, \quad (x,t) \in Q_0^*, \quad v_1^* = 0, \ (x,t) \in \partial Q_0^*. \tag{6e}$$

Observe that the singular function s is not involved in the definition of the regular component. Moreover, observe that

$$v_0(0,t) = \int_{s=0}^{t} \big(f(0,s) - (b(s) - b(0)\phi(0^+)\big)e^{-(b(t)-b(s))}\, ds, \qquad t \geq 0.$$

The boundary layer components w_L, w_R satisfy the problems

$$L^*w_L^* = F^* - L^*v^*, \ (x,t) \in Q_1^* := (0, 1+a) \times (0,T], \tag{6f}$$

$$w_L^*(0,t) = -v^*(0,t), \ w_L^*(x,0) = 0, \ w_L^*(1+a,t) = 0; \tag{6g}$$

$$Lw_R = 0, \ (x,t) \in Q, \tag{6h}$$

$$w_R(1,t) = -(v^* + w_L^*)(1,t), \ w_R(x,0) = 0, \ w_R(0,t) = 0. \tag{6i}$$

[2] We use the notation $f^* : \bar{Q}^* \to \mathbb{R}$ to denote the extension of any function $f : \bar{Q} \to \mathbb{R}$ such that $f^*(x,t) \equiv f(x,t), \ (x,t) \in \bar{Q}$ and $\bar{Q} \subset \bar{Q}^*$.

By construction and by using the extended domains to avoid compatibility issues, $v, w_R \in C^{4+\gamma}(\bar{Q})$. Note that $L^*w_L^*(0,0) = 0$ and $v_t(0,0) = f(0,0)$, so the first order compatibility conditions are satisfied (see Appendix 1). Hence, $w_L \in C^{2+\gamma}(\bar{Q})$; but, in general, $w_L \notin C^{4+\gamma}(\bar{Q})$.

Theorem 1. *(Problem Class 1) For the regular component v we have, for all $0 \le i + 2j \le 4$ with $0 < \mu < 1$, the bounds*

$$\left\| \frac{\partial^{i+j} v}{\partial x^i \partial t^j} \right\| \le C(1 + \varepsilon^{1-i/2}), \tag{7a}$$

and for the boundary layer components, we have the bounds

$$\left| \frac{\partial^{i+j}}{\partial x^i \partial t^j} w_R(x,t) \right| \le C\varepsilon^{-i/2} e^{-\frac{(1-x)}{\sqrt{\varepsilon}}}, \quad 0 \le i + 2j \le 4; \quad (x,t) \in \bar{Q}; \tag{7b}$$

$$\left| \frac{\partial^{i+j}}{\partial x^i \partial t^j} w_L(x,t) \right| \le C\varepsilon^{-i/2} e^{-\frac{\mu}{2} \frac{x}{\sqrt{T\varepsilon}}}, \quad 0 \le i + 2j \le 2; \quad (x,t) \in \bar{Q}. \tag{7c}$$

In addition, for the higher derivatives, we have

$$\left| \frac{\partial^i}{\partial x^i} w_L(x,t) \right| \le \frac{C}{\varepsilon(\sqrt{\varepsilon t})^{i-2}} e^{-\frac{\mu}{2} \frac{x}{\sqrt{T\varepsilon}}}, \quad i = 3, 4;, \quad (x,t) \in Q; \tag{7d}$$

$$\left| \frac{\partial^2}{\partial t^2} w_L(x,t) \right| \le \frac{C}{t}, \quad (x,t) \in Q. \tag{7e}$$

Proof. We begin by establishing the bounds on the regular component. Note that v_0 is bounded independently of ε. Consider the problem (6e) transformed with the stretched variable $x/\sqrt{\varepsilon}$. Apply the *a priori* bounds [13] to establish bounds on the partial derivatives of v_1 in the stretched variables. Transforming back to the original variables, we deduce the bounds (7a). The bounds on w_R are obtained in the usual way [14].

We now consider the component w_L. Observe that, with $cs_n := t^n(1-s)$, we have

$$L^*w_L^* = (b(t) - b(0) - tb'(0))\phi(0^+)cs_0 + b'(0)\phi(0^+)cs_1, \ (x,t) \in Q_1^*;$$

and $L^*w_L^* \in C^{0+\gamma}(\bar{Q}_1^*)$. Using $e^{-z^2} \le e^{0.25-z}$, it follows that

$$|cs_0(x,t)| \le Ce^{-\frac{x}{2\sqrt{\varepsilon T}}}, \qquad |cs_1(x,t)| \le Cte^{-\frac{x}{2\sqrt{\varepsilon T}}}, \ t \le T;$$

$$\left| \frac{\partial}{\partial t} cs_1(x,t) \right| \le Ce^{-\mu\sqrt{\frac{1}{2t\varepsilon}}x}, \qquad \mu < 1.$$

Hence, using a maximum principle, we can deduce that

$$|w_L(x,t)| \le Ce^{-\frac{x}{2\sqrt{\varepsilon T}}}e^{\theta t}, \quad \theta > \frac{1}{4T} - \beta, \quad (x,t) \in Q;$$

and, by applying the arguments from [14], we get that for $0 \le i + 2j \le 2$,

$$\left| \frac{\partial^{i+j}}{\partial x^i \partial t^j} w_L(x,t) \right| \le C\varepsilon^{-i/2} e^{-\frac{\mu x}{2\sqrt{T\varepsilon}}}, \quad (x,t) \in Q.$$

To obtain bounds on the higher derivatives of w_L, we introduce a further decomposition of this boundary layer function. Consider the continuous function

$$\begin{aligned} P(x,t) \quad &:= \quad B(t) - \int_{r=0}^{t} s_0(x,r)\; dr, \\ \text{where} \quad B(t) \quad &:= \quad \begin{cases} \frac{1-e^{-b(0)t}}{b(0)}, & \text{if } \; b(0) \neq 0, \\ t, & \text{if } \; b(0) = 0. \end{cases} \end{aligned}$$

This function has been constructed to satisfy the following problem

$$\begin{aligned} &P_t - \varepsilon P_{xx} + b(0)P = 0, \quad \text{in } Q, \\ &P(0,t) = B(t) \quad t \ge 0, \quad P(x,0) = 0, \quad 0 < x < 1. \end{aligned}$$

Note that

$$P(x,t) = B(t) - t + \int_{r=0}^{t} cs_0(x,r)\; dr.$$

We introduce the secondary expansion

$$w_L^*(x,t) = -v_t^*(0,0)P^*(x,t) + b'(0)\phi(0^+)\frac{1}{2}cs_2(x,t) + R^*(x,t);$$

and the remainder term R^*, defined over $\bar{Q}_1^*$, satisfies the problem

$$\begin{aligned} L^*R^* &= (b(t) - b(0) - b'(0)t)\phi(0^+)cs_0 + (b(t) - b(0))v_t^*(0,0)P^* \\ &\quad - (b(t) - b(0))b'(0)\phi(0^+)\frac{1}{2}cs_2, \\ R^*(x,0) &= 0, \; 0 < x < 1 + a; \quad R^*(1+a,t) = R^*(1+a,t), \; t \ge 0; \\ R^*(0,t) &= (tv_t^*(0,0) - v^*(0,t)) + v_t^*(0,0)(B(t) - t) - b'(0)\phi(0^+)\frac{t^2}{2}. \end{aligned}$$

Note that $v(0,0) = 0$ and $|R^*(0,t)| \le Ct^2$. Moreover, using properties of s_2 (see Appendix 2) one can check that $L^*R^* \in C^{2+\gamma}(\bar{Q}_1^*)$ and second level compatibility is satisfied at the point $(0,0)$. Hence, we have $R^* \in C^{4+\gamma}(\bar{Q}_1^*)$. Using this regularity, we can deduce the bounds

$$\left\| \frac{\partial^{i+j} R^*}{\partial x^i \partial t^j} \right\| \le C(1+\varepsilon^{-i/2}), \quad 0 \le i+2j \le 4.$$

From this, we obtain

$$\left| \frac{\partial^2}{\partial t^2} w_L(x,t) \right| \le Ct^{-1}, \quad (x,t) \in Q.$$

To obtain the desired bounds (involving the decaying exponential) on the third and fourth space derivatives of w_L, we form a problem for $R_2^* := \varepsilon R_{xx}^*$ by differentiating the differential equation satisfied by R^* twice to get

$$\begin{aligned} L^* R_2^* &= (b(0) + b'(0)t - b(t))\phi(0^+)\varepsilon(cs_0)_{xx} + (b(t) - b(0))v_t(0,0)\varepsilon P_{xx}^* \\ &\quad - (b(t) - b(0))b'(0)\phi(0^+)\varepsilon\frac{1}{2}(cs_2)_{xx}, \\ R_2^*(x,0) &= 0, R_2^*(1+a,t) = R_2^*(1+a,t); \quad R_2^*(0,t) = (g' + bg - h)(t); \end{aligned}$$

where $g(t) := R^*(0,t),\ h(t) := L^* R^*(0,t)$ are both smooth functions independent of ε. We can complete the proof (as in [14]) by noting that

$$\begin{aligned} |L^* R_2^*(x,t)| &\le Ce^{-\mu\frac{x}{\sqrt{4\varepsilon T}}}; \\ |(L^* R_2^*)_t(x,t)| + |\varepsilon(L^* R_2^*)_{xx}(x,t)| &\le Ce^{-\mu\frac{x}{\sqrt{4\varepsilon T}}}. \end{aligned}$$

□

2.2 Problem Class 2: discontinuous initial condition

Consider the singularly perturbed parabolic problem

$$Lu = f(x,t),\ (x,t) \text{ in } Q; \quad u(0,t) = 0,\ u(1,t) = 0,\ t \ge 0; \tag{8a}$$
$$u(x,0) = \phi(x),\ 0 < x < 1, \quad \phi(0) = \phi(1) = 0; \tag{8b}$$
$$\phi(d^-) \ne \phi(d^+), \quad 0 < d < 1; \tag{8c}$$
$$f(0,0) = -\varepsilon\phi''(0^+), -\varepsilon\phi^{(iv)}(0^+) + b(0)\phi''(0^+) = (f_t + f_{xx})(0,0); \tag{8d}$$
$$f(1,0) = -\varepsilon\phi''(1^-), -\varepsilon\phi^{(iv)}(1^-) + b(0)\phi''(1^-) = (f_t + f_{xx})(1,0); \tag{8e}$$
$$f, b \in C^{4+\gamma}(\bar{Q}), \quad \phi \in C^4((0,1) \setminus \{d\}), \tag{8f}$$
$$\phi'(d^-) = \phi'(d^+), \quad \phi''(d^-) = \phi''(d^+). \tag{8g}$$

The assumption of the compatibility conditions (8b), (8d) and (8e) ensures that no classical singularity appears near the corner points $(0,0),(1,0)$. However, observe that the initial function $\phi(x)$ is discontinuous at $x=d$. This will cause an interior layer to appear in the solution, near the point $(d,0)$. The assumption $\phi'(d^-)=\phi'(d^+)$ on the data prevents a drop in the order of convergence in our numerical approximations, as in the case of [20, §14.2].

Decompose the solution of (8) into the following sum

$$u(x,t)=\frac{[\phi](d)}{2}s(x-d,t)+y(x,t), \quad \text{where } [\phi](d):=\phi(d^+)-\phi(d^-). \tag{9}$$

By the definition (1) of the discontinuous function s, we have that

$$s(x-d,0)=\begin{cases}-1, & \text{for } x<d,\\ 0, & \text{for } x=d,\\ 1, & \text{for } x>d.\end{cases}$$

The component y is the solution of the problem

$$Ly=f-(b(t)-b(0))0.5[\phi](d)s(x-d,t),\ (x,t)\in Q, \tag{10a}$$
$$y(0,t)=-0.5[\phi](d)s(-d,t),\ y(1,t)=-0.5[\phi](d)s(1-d,t), \quad t\ge 0, \tag{10b}$$
$$y(x,0)=\phi(x)-0.5[\phi](d)s(x-d,0),\ x\ne d; \tag{10c}$$
$$y(d,0)=(\phi(d^+)+\phi(d^-))/2; \tag{10d}$$

and $y(x,0)$ is continuous for all $x\in[0,1]$. Moreover, due to (8g), we have $y(x,0)\in C^2(0,1)$. Using the maximum principle

$$\|y\|\le C.$$

The solution y is further decomposed into the sum

$$y=v+w_L+w_R+w_I,$$

where the components v and w_I are discontinuous functions and the components w_L and w_R are continuous functions. The regular component v is constructed to satisfy the problem

$$L^*v^*=f^*-(b(t)-b(0))0.5[\phi](d)e^{-b(0)t}s(x-d,0),\ (x,t)\in Q^*. \tag{11}$$

This problem is posed on the extended domain $Q^*:=(-a,1+a)\times(0,T], a>0$. The initial/boundary values for the regular component are determined by $v^* = v_0^* + \varepsilon v_1^*$, where the reduced solution v_0 satisfies the initial value problem

$$bv_0^* + (v_0^*)_t = f^* - (b(t) - b(0))0.5[\phi](d)e^{-b(0)t}s(x-d,0),\ t > 0; \tag{12a}$$
$$v_0^*(x,0) = y^*(x,0),\ x \in (-a, 1+a). \tag{12b}$$

Observe that the reduced solution v_0^* is continuous, but in general

$$(v_0^*)_x(d^+,0) \neq (v_0^*)_x(d^-,0).$$

The first correction v_1 is defined as the multi-valued function

$$v_1^*(x,t) := \begin{cases} v_1^-, & \text{for } x \le d,\ t \ge 0, \\ v_1^+, & \text{for } x \ge d,\ t \ge 0, \end{cases}$$

and the two sides of this function are the solutions of

$$L^* v_1^- = (v_0^*)_{xx}, \quad -a < x \le d,\ t > 0;$$
$$v_1^-(-a,t) = 0,\ t \ge 0; \quad v_1^-(x,0) = 0,\ x \le d;$$

$$L^* v_1^+ = (v_0^*)_{xx}, \quad d \le x < 1+a,\ t > 0;$$
$$v_1^+(1+a,t) = 0,\ t \ge 0; \quad v_1^+(x,0) = 0,\ x \ge d;$$

where we use $Lv_1^-(d,t) := (v_0)_{xx}(d^-,t)$ and $Lv_1^+(d,t) := (v_0)_{xx}(d^+,t)$. Since the regular component v is multi-valued we now define the subdomains

$$Q^- := (0,d) \times (0,T] \quad \text{and} \quad Q^+ := (d,1) \times (0,T].$$

By using suitable extensions to these subdomains, we can have $v_1^\pm \in C^{4+\gamma}(\bar{Q}^\pm)$. For example,

$$L^*(v_1^-)^* = ((v_0)_{xx})^*, \quad (x,t) \in Q_*^- := (-a, d+a) \times (0,T];$$
$$(v_1^-)^*(-a,t) = (v_1^-)^*(d+a,t) = 0,\ t \ge 0,\ (v_1^-)^*(x,0) = 0,\ -a < x < d+a.$$

The boundary layer components w_L, w_R satisfy the problems

$$Lw_L = Lw_R = 0,\ (x,t) \in Q, \tag{13a}$$
$$w_L(0,t) = -v(0,t),\ w_L(x,0) = 0,\ w_L(1,t) = 0, \tag{13b}$$
$$w_R(0,t) = 0,\ w_R(x,0) = 0,\ w_R(1,t) = -v(1,t). \tag{13c}$$

Theorem 2. *(Problem Class 2) For the regular component v we have, for all $0 \le i + 2j \le 4$, the bounds*

$$\left\| \frac{\partial^{i+j} v^-}{\partial x^i \partial t^j} \right\|_{\bar{Q}^-}, \left\| \frac{\partial^{i+j} v^+}{\partial x^i \partial t^j} \right\|_{\bar{Q}^+} \le C(1 + \varepsilon^{1-i/2}). \tag{14a}$$

For the boundary layer components, for all $0 \le i + 2j \le 4$ and $(x,t) \in \bar{Q}$,

$$\left| \frac{\partial^{i+j} w_L}{\partial x^i \partial t^j}(x,t) \right| \le C\varepsilon^{-i/2} e^{-\frac{x}{\sqrt{\varepsilon}}}; \ \left| \frac{\partial^{i+j} w_R}{\partial x^i \partial t^j}(x,t) \right| \le C\varepsilon^{-i/2} e^{-\frac{(1-x)}{\sqrt{\varepsilon}}}. \tag{14b}$$

Proof. Adapt appropriately the argument from the proof of Theorem 1. □

Finally the multi-valued interior layer component

$$w_I(x,t) := \begin{cases} w_I^-, & \text{for } 0 \le x \le d, \ t \ge 0, \\ w_I^+, & \text{for } d \le x \le 1, \ t \ge 0; \end{cases}$$

is defined implicitly by the sum $y = v + w_L + w_R + w_I$. Hence, $\|w_I\| \le C$. Moreover, w_I satisfies the problem

$$Lw_I = R(x,t), \ (x,t) \in Q^- \cup Q^+, \tag{15a}$$

where

$$R(x,t) := (b(t) - b(0))0.5[\phi](d) \left(e^{-b(0)t} s(x-d,0) - s(x-d,t) \right), \tag{15b}$$

$$w_I(0,t) = 0, \ w_I(1,t) = 0, t \ge 0; \quad w_I(x,0) = 0, \ 0 \le x \le 1; \tag{15c}$$

$$[w_I](d,t) = -\varepsilon[v_1](d,t), \qquad [(w_I)_x](d,t) = -\varepsilon[(v_1)_x](d,t). \tag{15d}$$

Note that, $R \in C^{0+\gamma}(\bar{Q})$, $R(d,t) \ne 0$ for all $t > 0$ such that $b(t) \ne b(0)$. Moreover,

$$|R(x,t)| \le Cte^{-\frac{(x-d)^2}{4\varepsilon t}} \le Cte^{-\frac{|x-d|}{2\sqrt{\varepsilon T}}}, \ (x,t) \in \bar{Q}$$

and

$$\left| \frac{\partial}{\partial t} R(x,t) \right| \le Ce^{-\frac{\mu|x-d|}{2\sqrt{\varepsilon t}}}, \ \mu < 1, \ (x,t) \in Q.$$

Using a maximum principle, either side of $x = d$, we have for $0 \le i + 2j \le 2$

$$|w_I(x,t)| \le Ce^{-\frac{|x-d|}{2\sqrt{\varepsilon T}}}; \quad (x,t) \in \bar{Q}^- \cup \bar{Q}^+; \tag{16a}$$

$$\left| \frac{\partial^{i+j}}{\partial x^i \partial t^j} w_I(x,t) \right| \le C\varepsilon^{-i/2} e^{-\frac{\mu|x-d|}{2\sqrt{\varepsilon T}}}, \ \mu < 1, \quad (x,t) \in \bar{Q}^- \cup \bar{Q}^+. \tag{16b}$$

For the higher derivatives, we need to repeat the argument from the proof of Theorem 1, from the last section, to establish the additional bounds

$$\left|\frac{\partial^i}{\partial x^i} w_I(x,t)\right| \le \frac{C}{\varepsilon(\sqrt{\varepsilon t})^{i-2}} e^{-\frac{\mu}{2}\frac{|x-d|}{\sqrt{T\varepsilon}}}, \quad i=3,4, \quad (x,t)\in Q^-\cup Q^+; \tag{16c}$$

$$\left|\frac{\partial^2}{\partial t^2} w_I(x,t)\right| \le \frac{C}{t}, \quad (x,t)\in Q^-\cup Q^+. \tag{16d}$$

2.3 Problem Class 3: discontinuous boundary data

Consider the singularly perturbed parabolic problem

$$Lu = f(x,t) \text{ in } Q, \qquad u(1,t)=0,\ t\ge 0,\ u(x,0)=0,\ 0<x<1, \tag{17a}$$

and the boundary condition at $x=0$ is given by

$$u(0,t) = \begin{cases} \phi_1(t), & \text{if } 0\le t\le d, \\ \phi_2(t), & \text{if } d<t\le T, \end{cases} \quad \phi_1(d^-)\neq\phi_2(d^+),\ \phi_1'(d^-)=\phi_2'(d^+). \tag{17b}$$

Note that there is no loss in generality is assuming a homogenous initial condition. We assume that the following compatibility conditions are satisfied at $(0,0)$ and $(1,0)$:

$$\phi_1(0)=0,\ f(0,0)=\phi_1'(0^+),\ (f_t+f_{xx})(0,0)=\phi_1''(0^+)+b(0)\phi_1'(0^+), \tag{17c}$$

$$f(1,0)=(f_t+f_{xx})(1,0)=0, \tag{17d}$$

and also the following regularity conditions

$$f,b\in C^{4+\gamma}(\bar{Q}), \phi_1\in C^2(0,d), \phi_2\in C^2(d,T). \tag{17e}$$

The discontinuous boundary condition on the left, will cause a singularity to appear in the solution for $t\ge d$.

Decompose the solution of (17) into the sum

$$u = [\phi](d)H(t-d)cs(x,t-d)+y, \quad [\phi](d) := \phi_2(d^+)-\phi_1(d^-), \tag{18}$$

where $H(\cdot)$ is a unit step function defined by

$$H(x) := \begin{cases} 0, & \text{for } x<0, \\ 1, & \text{for } x\ge 0. \end{cases}$$

Note that $cs(x,0) = 0$. Observe that y is the solution of the parabolic problem

$$Ly = f + (b(d) - b(t))[\phi](d)H(t-d)cs(x,t-d) \text{ in } Q, \tag{19a}$$
$$y(1,t) = -[\phi](d)H(t-d)cs(1,t-d),\ t \geq 0, \tag{19b}$$
$$y(x,0) = \phi(x),\ 0 < x < 1, \tag{19c}$$

and

$$y(0,t) = u(0,t) - [\phi](d)H(t-d). \tag{19d}$$

As in previous sections, we decompose y into three subcomponents

$$y = v + w_R + w_L,$$

which are defined as the solutions of the following three parabolic problems. The regular component satisfies

$$L^* v^* = f^* \text{ in } Q^* := (-a, 1+a) \times (0,T], \tag{20a}$$
$$v^*(-a,t) = v^*(1+a,t) = 0,\ t \geq 0, \tag{20b}$$
$$v^*(x,0) = \phi^*(x),\ -a < x < 1+a; \tag{20c}$$

where $\phi^*(x)$ is a smooth extension of the initial condition (17a). The right boundary layer component satisfies

$$Lw_R = 0 \text{ in } Q, \quad w_R(x,0) = 0,\ 0 < x < 1, \tag{20d}$$
$$w_R(1,t) = y(1,t) - v(1,t),\ w_R(0,t) = 0,\ t \geq 0; \tag{20e}$$

and the left boundary layer component satisfies

$$Lw_L = Ly - f \text{ in } Q, \quad w_L(x,0) = 0,\ 0 < x < 1, \tag{20f}$$
$$w_L(1,t) = 0,\ w_L(0,t) = y(0,t) - v(0,t),\ t \geq 0. \tag{20g}$$

The regular component $v \in C^{4+\gamma}(\bar{Q})$ and since all time derivatives of $cs(1,t-d)$ are zero at $t = d$, we have that $w_R \in C^{4+\gamma}(\bar{Q})$. In addition, $w_L \in C^{2+\gamma}(\bar{Q})$. Hence, the character of the function y for Problem Class 3 is the same as for Problem Class 1. In other words, the bounds on the derivatives of the components of y given in Theorem 1 also apply in the case of Problem Class 3. However, the character of the singular component $u - y$ is different for the two problem classes.

3 Numerical Method

For all three problem classes we employ a classical finite difference operator (backward Euler in time and standard central differences in space) on an appropriate mesh (which will be piecewise-uniform in space and uniform in time). The piecewise-uniform Shishkin mesh for each of the three Problem Classes $n, (n = 1, 2, 3)$ will be denoted by $\bar{Q}_n^{N,M}$. The numerical method [3]:

$$L^{N,M}Y(x_i,t_j) = f(x_i,t_j), \qquad (x_i,t_j) \in Q_n^{N,M}, \tag{21a}$$

$$Y(x_i,t_j) = y(x_i,t_j), \qquad (x_i,t_j) \in \partial Q_n^{N,M}, \tag{21b}$$

$$\text{where} \quad L^{N,M}Y(x_i,t_j) := (-\varepsilon\delta_x^2 + b(t_j)I + D_t^-)Y(x_i,t_j). \tag{21c}$$

For Problem Class 1, the Shishkin mesh $\bar{Q}_1^{N,M}$ is defined via :

$$[0,1] = [0,\sigma] \cup [\sigma, 1-\sigma] \cup [1-\sigma, 1], \quad \sigma := \min\left\{\frac{1}{4}, \frac{4}{\mu}\sqrt{\varepsilon T}\ln N\right\}, \quad \mu < 1;$$

and $N/4, N/2, N/4$ grid points are uniformly distributed in each subinterval, respectively. For Problem Class 3, the Shishkin mesh is $\bar{Q}_3^{N,M} = \bar{Q}_1^{N,M}$.

For Problem Class 2, the Shishkin mesh is $\bar{Q}_2^{N,M}$, which is defined via

$$[0,1] = [0,\tau] \cup [\tau, d-\tau] \cup [d-\tau, d+\tau] \cup [d+\tau, 1-\tau] \cup [1-\tau, 1],$$

with

$$\tau := \min\left\{\frac{1}{8}, \frac{4}{\mu}\sqrt{\varepsilon T}\ln N\right\}, \quad \mu < 1$$

and $N/8, N/4, N/4, N/4, N/8$ grid points are uniformly distributed in each subinterval, respectively. Although it is required that $\mu < 1$ in the theoretical error analysis, in the numerical results section, we simply have taken $\mu = 1$.

Theorem 3. *Let Y be the solution of the finite difference scheme* (21) *and y the solution of the continuous problem. Then, the global approximation*

[3]The finite difference operators are defined by:

$$D_x^+U(x_i,t_j) := D_x^-U(x_{i+1},t_j); \quad D_x^-U(x_i,t_j) := \frac{U(x_i,t_j) - U(x_{i-1},t_j)}{h_i},$$

$$D_t^-U(x_i,t_j) := \frac{U(x_i,t_j) - U(x_i,t_{j-1})}{k_j}, \quad \delta_x^2U(x_i,t_j) := \frac{(D_x^+ - D_x^-)U(x_i,t_j)}{\hbar_i}$$

and the mesh steps are $h_i := x_i - x_{i-1}, \hbar_i = (h_{i+1} + h_i)/2, \quad k = k_j := t_j - t_{j-1}$.

$\bar{Y}$ on $\bar{Q}$ generated by the values of Y on $\bar{Q}_n^{N,M}$ and bilinear interpolation, satisfies

$$\|y - \bar{Y}\|_{\bar{Q}} \leq (CN^{-2}\ln^2 N + CM^{-1})\ln M, \tag{22}$$

for each of the three Problem Classes (3), (8) and (17).

Proof. For each of the three Problem Classes, the discrete solution Y is decomposed along the same lines as its continuous counterpart y.

Let us first consider Problem Classes 1 and 3. Using the bounds on the derivatives of the components in Theorem 1, truncation error bounds, discrete maximum principle and a suitable discrete barrier function and following the arguments in [14], we can establish the following bounds

$$\|v - V\|_{\bar{Q}^{N,M}}, \|w_R - W_R\|_{\bar{Q}^{N,M}} \leq CN^{-2}\ln^2 N + CM^{-1}.$$

It remains to bound the error due to the left boundary layer component. We introduce the following notation for this error and the associated truncation error

$$E_i^j := (w_L - W_L)(x_i, t_j) \quad \text{and} \quad \mathcal{T}_{i,j} := L^{N,M} E_i^j.$$

Note that

$$|\delta_x^2 w_L(x_i, t_j)| \leq C\left\|\frac{\partial^2 w_L}{\partial x^2}\right\|_{(x_{i-1},x_{i+1})\times\{t_j\}},$$
$$|D_t^- w_L(x_i, t_j)| \leq C\left\|\frac{\partial w_L}{\partial t}\right\|_{\{x_i\}\times(t_{j-1},t_j)}.$$

Hence, using the bounds (7c) on the derivatives of w_L, we have that outside the left layer

$$|\mathcal{T}_{i,j}| \leq CN^{-2}, \quad x_i \geq \sigma, t_j > 0.$$

Within the left layer, using the bounds (7d), (7e) on the higher derivatives of w_L, we have the truncation error bounds

$$\begin{aligned}
|\mathcal{T}_{i,1}| &\leq C\frac{(N^{-1}\ln N)^2}{t_1} + C\left\|\frac{\partial w_L}{\partial t}\right\|_{\{x_i\}\times(0,t_1)} \leq C\frac{(N^{-1}\ln N)^2}{t_1} + C, \quad x_i < \sigma \\
|\mathcal{T}_{i,j}| &\leq C\frac{(N^{-1}\ln N)^2}{t_j} + Ck\left\|\frac{\partial^2 w_L}{\partial t^2}\right\|_{\{x_i\}\times(t_{j-1},t_j)} \\
&\leq C\frac{(N^{-1}\ln N)^2}{t_j} + C\frac{k}{t_{j-1}}, \quad x_i < \sigma, t_j > t_1.
\end{aligned}$$

Hence, at all time levels, we have the truncation error bound

$$|\mathcal{T}_{i,j}| \leq C\frac{(N^{-1}\ln N)^2 + M^{-1}}{t_j}, \quad x_i < \sigma, t_j \geq t_1.$$

We now mimic the argument in [21] and note that at each time level,

$$-\varepsilon\delta_x^2 E_i^j + \left(b(x_i,t_j) + \frac{1}{k}\right) E_i^j = \mathcal{T}_{i,j} + \frac{1}{k}E_i^{j-1},\ t_j > 0.$$

From this we can deduce the error bound

$$\begin{aligned} |E_i^j| &\leq Ck\sum_{n=1}^{j}|\mathcal{T}_{i,n}| \leq C\left((N^{-1}\ln N)^2 + M^{-1}\right)\sum_{n=1}^{j}\frac{1}{n} \\ &\leq C\left((N^{-1}\ln N)^2 + M^{-1}\right)\left(1 + \int_{s=1}^{j}\frac{ds}{s}\right) \\ &\leq C((N^{-1}\ln N)^2 + M^{-1})\ln(1+j). \end{aligned} \tag{23}$$

In the case of Problem Class 2, we have an additional interior layer component w_I. The bounding of the error $\|w_I - W_I\|$ follows the same argument as above.

One can extend this nodal error bound to a global error bound by applying the argument in [6, pp. 56-57] and using the modification in [10] to manage the initial singularity. □

4 Numerical Results

The orders of convergence of the finite difference scheme (21) are estimated using the two-mesh principle [6]. We denote by $Y^{N,M}$ and $Y^{2N,2M}$ the computed solutions with (21) on the Shishkin meshes $Q_n^{N,M}$ and $Q_n^{2N,2M}$, respectively. These solutions are used to computed the maximum two-mesh global differences

$$D_\varepsilon^{N,M} := \|\bar{Y}^{N,M} - \bar{Y}^{2N,2M}\|_{Q_n^{N,M}\cup Q_n^{2N,2M}}$$

where $\bar{Y}^{N,M}$ and $\bar{Y}^{2N,2M}$ denote the bilinear interpolation of the discrete solutions $Y^{N,M}$ and $Y^{2N,2M}$ on the mesh $Q_n^{N,M}\cup Q_n^{2N,2M}$. Then, the orders of global convergence $P_\varepsilon^{N,M}$ are estimated in a standard way [6]

$$P_\varepsilon^{N,M} := \log_2\left(\frac{D_\varepsilon^{N,M}}{D_\varepsilon^{2N,2M}}\right).$$

The uniform two-mesh global differences $D^{N,M}$ and their corresponding uniform orders of global convergence $P^{N,M}$ are calculated by

$$D^{N,M} := \max_{\varepsilon \in S} D_\varepsilon^{N,M}, \quad P^{N,M} := \log_2 \left(\frac{D^{N,M}}{D^{2N,2M}} \right),$$

where $S = \{2^0, 2^{-1}, \ldots, 2^{-30}\}$.

In order that the temporal discretization error dominates the spatial discretization error, in all the tables of this paper, except in Table 2, we have taken $N = 2^4 \times M$.

4.1 Problem Class 1

We present numerical results for two examples from this first class of problems. In the first example the coefficient of the reaction term depends only on the temporal variable; while in the second example, it depends on the spatial variable. The numerical results computed with the analytical/numerical method of this paper suggest that the method is uniformly and globally convergent in both cases.

Example 1. *Consider problem (3), with the data given by*

$$b(x,t) = 1 + t, \;\; f(x,t) = 4x(1-x)t + t^2, \;\; \phi(x) = (1-x)^2. \qquad (24)$$

The maximum two-mesh global differences associated with the component y and the orders of convergence are given in Table 1. Observe that the numerical results show that the method is first-order globally parameter-uniformly convergent. In Table 1 we take $N = 16M$, which means that the largest spatial mesh step is significantly smaller than the uniform mesh step in time; so the temporal convergence rate dominates. In Table 2, we give the uniform two-mesh global differences taking $N = M$ and the computed orders of convergence illustrate that the method is almost second order convergent; in this case the spatial discretization errors dominate the temporal discretization errors. The numerical results in Tables 1 and 2 are in agreement with our error estimates in Theorem 3.

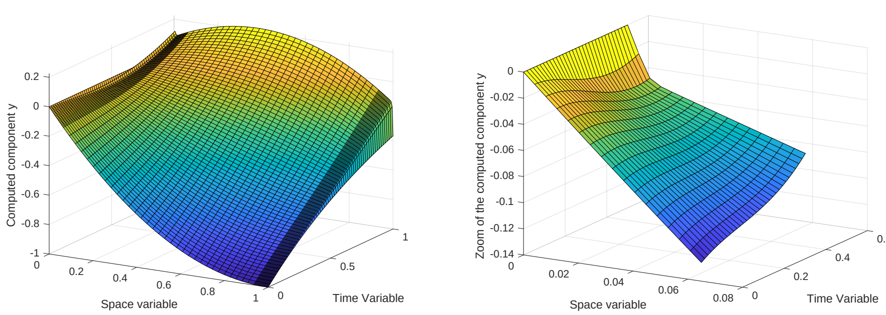
We display in Figure 1 the numerical approximation to the function y defined in (5), which exhibits only boundary layers. The numerical solution to problem (3)-(24) is displayed in Figure 2, which exhibits both boundary layers and the singularity caused by the incompatibility between the initial and boundary conditions.

Example 2. *Consider problem (3), with the data given by*

$$b(x,t) = 1 + 10x, \quad f(x,t) = 4x(1-x)t + t^2, \quad \phi(x) = (1-x)^2. \qquad (25)$$

Table 1: Example 1 from Problem Class 1: Maximum two-mesh global differences and orders of convergence for the function y in (5) using a piecewise uniform Shishkin mesh

	N=256,M=16	N=512,M=32	N=1024,M=64	N=2048,M=128	N=4096,M=256
$\varepsilon = 2^{0}$	**1.295E-02** 0.890	**6.990E-03** 0.938	**3.650E-03** 0.965	**1.870E-03** 0.984	**9.453E-04**
$\varepsilon = 2^{-2}$	3.789E-03 0.936	1.980E-03 0.966	1.013E-03 0.983	5.128E-04 0.991	2.580E-04
$\varepsilon = 2^{-4}$	2.214E-03 1.035	1.081E-03 1.017	5.339E-04 1.009	2.653E-04 1.005	1.322E-04
$\varepsilon = 2^{-6}$	4.216E-03 1.017	2.083E-03 1.008	1.035E-03 1.004	5.161E-04 1.002	2.577E-04
$\varepsilon = 2^{-8}$	4.971E-03 1.017	2.456E-03 1.009	1.220E-03 1.004	6.084E-04 1.002	3.037E-04
$\varepsilon = 2^{-10}$	5.269E-03 1.020	2.598E-03 1.010	1.290E-03 1.005	6.428E-04 1.003	3.208E-04
$\varepsilon = 2^{-12}$	6.402E-03 1.263	2.668E-03 1.015	1.321E-03 1.008	6.569E-04 1.004	3.276E-04
$\varepsilon = 2^{-14}$	1.092E-02 1.445	4.011E-03 1.580	1.342E-03 1.012	6.655E-04 1.006	3.313E-04
$\varepsilon = 2^{-16}$	1.092E-02 1.445	4.013E-03 1.575	1.347E-03 1.011	6.685E-04 1.007	3.326E-04
. . .	. . .	. . .	. . .	. . .	. . .
$\varepsilon = 2^{-30}$	1.093E-02 1.445	4.014E-03 1.571	1.352E-03 1.011	6.707E-04 1.007	3.337E-04
$D^{N,M}$ $P^{N,M}$	1.295E-02 0.890	6.990E-03 0.938	3.650E-03 0.965	1.870E-03 0.984	9.453E-04



(a) Entire domain $\bar{Q}$

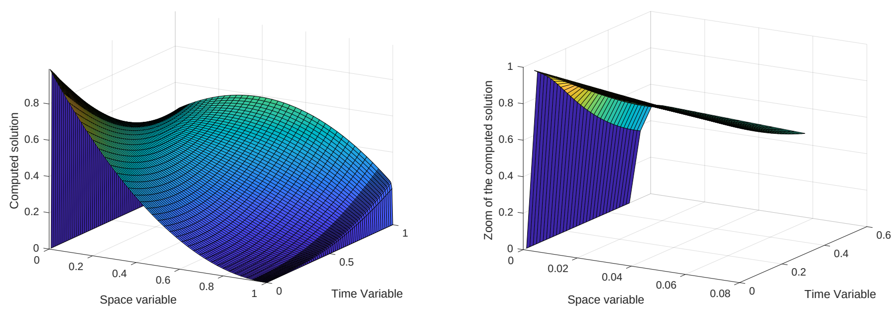
(b) A zoom in on the corner $(0, 0)$

Figure 1: Example 1 from Problem Class 1: The numerical approximation to y with $\varepsilon = 2^{-16}$ and $N = M = 64$

The maximum two-mesh global differences associated with the component y and the orders of convergence are given in Table 3. Observe that the numerical results indicate that the method is globally parameter-uniformly convergent. Comparing these orders of convergence with those in Table 1,

Table 2: Example 1 from Problem Class 1: Uniform two-mesh global differences and orders of convergence for the function y in (5) using a piecewise uniform Shishkin mesh with $N = M$

	N=M=64	N=M=128	N=M=256	N=M=512	N=M=1024
$D^{N,M}$	4.972E-02	2.548E-02	1.117E-02	3.983E-03	1.330E-03
$P^{N,M}$	0.964	1.189	1.488	1.583	



(a) Entire domain $\bar{Q}$

(b) A zoom in on the corner $(0, 0)$

Figure 2: Example 1 from Problem Class 1: The approximation $s + Y$ to the solution u with $\varepsilon = 2^{-16}$ and $N = M = 64$

we see that the theoretical assumption of b being independent of the space variable appears to be not necessary in order to observe parameter-uniform convergence.

4.2 Problem Class 2

Example 3. *Consider problem (8) with the data given by*

$$b(x,t) = 1 + 10xt, \quad f(x,t) = 4x(1-x)t + t^2, \tag{26a}$$

$$\phi(x) = \begin{cases} -1 + (2x-1)^2, & \text{if} \quad 0 \le x \le 0.5, \\ 1 - (2x-1)^2, & \text{if} \quad 0.5 < x \le 1. \end{cases} \tag{26b}$$

Observe that in this example the coefficient b depends on the temporal and spatial variables and, moreover, $\phi'(0.5^+) = \phi'(0.5^-)$ but $\phi''(0.5^+) \neq \phi''(0.5^-)$. The schemes considered here to approximate the solution of this example are defined on the Shishkin mesh $\bar{Q}_2^{N,M}$.

If the singularity is not stripped off and Example 3 is simply solved with backward Euler method and standard central finite differences on the

Table 3: Example 2 from Problem Class 1: Maximum two-mesh global differences and orders of convergence for the function y in (5) using a piecewise uniform Shishkin mesh

	N=256,M=16	N=512,M=32	N=1024,M=64	N=2048,M=128	N=4096,M=256
$\varepsilon = 2^{0}$	4.837E-03 0.181	4.267E-03 0.878	2.321E-03 1.000	1.160E-03 0.995	5.823E-04
$\varepsilon = 2^{-2}$	9.114E-03 0.966	4.665E-03 0.994	2.341E-03 0.998	1.172E-03 0.999	5.863E-04
$\varepsilon = 2^{-4}$	1.086E-02 0.975	5.523E-03 0.987	**2.787E-03** 0.993	**1.400E-03** 0.997	**7.016E-04**
$\varepsilon = 2^{-5}$	**1.092E-02** 0.982	**5.531E-03** 0.990	2.784E-03 0.994	1.398E-03 0.997	7.006E-04
$\varepsilon = 2^{-6}$	1.068E-02 0.988	5.387E-03 0.990	2.712E-03 0.995	1.361E-03 0.998	6.814E-04
$\varepsilon = 2^{-8}$	1.047E-02 0.980	5.305E-03 0.990	2.672E-03 0.995	1.341E-03 0.997	6.717E-04
$\varepsilon = 2^{-10}$	1.056E-02 0.982	5.349E-03 0.990	2.693E-03 0.995	1.351E-03 0.997	6.769E-04
$\varepsilon = 2^{-12}$	1.059E-02 0.982	5.361E-03 0.990	2.698E-03 0.995	1.354E-03 0.997	6.782E-04
$\varepsilon = 2^{-14}$	1.060E-02 0.982	5.365E-03 0.991	2.700E-03 0.995	1.355E-03 0.997	6.786E-04
$\varepsilon = 2^{-16}$	1.061E-02 0.983	5.368E-03 0.991	2.701E-03 0.995	1.355E-03 0.997	6.787E-04
. . .	. . .	. . .	. . .	. . .	. . .
$\varepsilon = 2^{-30}$	1.062E-02 0.984	5.371E-03 0.991	2.702E-03 0.995	1.355E-03 0.998	6.788E-04
$D^{N,M}$ $P^{N,M}$	1.092E-02 0.982	5.531E-03 0.989	2.787E-03 0.993	1.400E-03 0.997	7.016E-04

Shishkin mesh $\bar{Q}_2^{N,M}$, the method is not globally convergent for any value of ε. This is illustrated in Table 4 where the uniform two-mesh global differences are given.

Table 4: Example 3 from Problem Class 2: Maximum two-mesh global differences and orders of convergence for u using a piecewise uniform Shishkin mesh, without separating off the singularity

	N=256,M=16	N=512,M=32	N=1024,M=64	N=2048,M=128	N=4096,M=256
$D^{N,M}$ $P^{N,M}$	6.698E-01 0.231	5.707E-01 0.193	4.992E-01 -0.001	4.994E-01 -0.001	4.996E-01

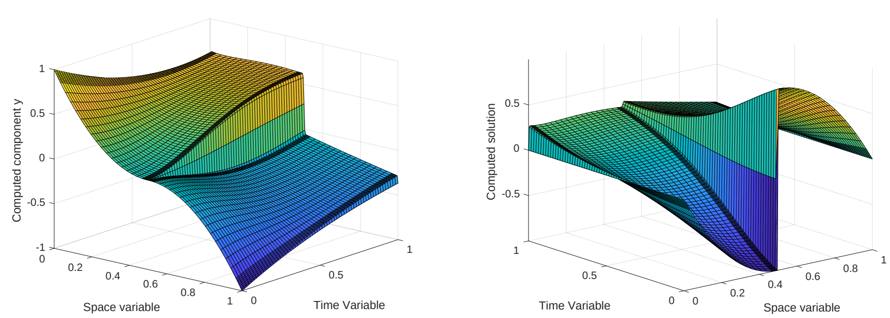
We show now the numerical results when the singularity is stripped off. The maximum two-mesh global differences associated with the component y and the orders of convergence are given in Table 5. Observe that the numerical results indicate that the method is globally and uniformly convergent.

Figure 3 displays both the numerical approximation to the function y de-

Table 5: Example 3 from Problem Class 2: Maximum two-mesh global differences and orders of convergence for the function y in (10) using a piecewise uniform Shishkin mesh

	N=256,M=16	N=512,M=32	N=1024,M=64	N=2048,M=128	N=4096,M=256
$\varepsilon = 2^{0}$	1.683E-02 0.978	8.549E-03 0.999	4.277E-03 1.003	**2.134E-03** 1.001	**1.066E-03**
$\varepsilon = 2^{-2}$	6.557E-03 1.045	3.177E-03 1.024	1.563E-03 1.014	7.741E-04 1.007	3.852E-04
$\varepsilon = 2^{-4}$	5.748E-03 0.942	2.992E-03 0.970	1.527E-03 0.985	7.717E-04 0.992	3.879E-04
$\varepsilon = 2^{-6}$	8.330E-03 0.939	4.346E-03 0.969	2.220E-03 0.984	1.122E-03 0.992	5.642E-04
$\varepsilon = 2^{-8}$	9.535E-03 0.937	4.981E-03 0.968	2.546E-03 0.984	1.287E-03 0.992	6.469E-04
$\varepsilon = 2^{-10}$	1.128E-02 1.005	5.618E-03 1.003	2.804E-03 1.001	1.401E-03 1.001	6.999E-04
$\varepsilon = 2^{-12}$	1.245E-02 1.003	6.212E-03 1.001	3.103E-03 1.001	1.551E-03 1.000	7.751E-04
$\varepsilon = 2^{-14}$	1.880E-02 1.519	6.559E-03 1.000	3.278E-03 1.000	1.639E-03 1.000	8.194E-04
$\varepsilon = 2^{-15}$	**3.134E-02** 1.505	1.104E-02 1.727	3.335E-03 1.000	1.667E-03 1.000	8.338E-04
$\varepsilon = 2^{-16}$	2.964E-02 1.228	**1.266E-02** 1.464	**4.588E-03** 1.442	1.689E-03 1.000	8.445E-04
. . .	. . .	. . .	. . .	. . .	. . .
$\varepsilon = 2^{-30}$	2.957E-02 1.226	1.264E-02 1.463	4.584E-03 1.388	1.752E-03 1.001	8.755E-04
$D^{N,M}$ $P^{N,M}$	3.134E-02 1.308	1.266E-02 1.464	4.588E-03 1.104	2.134E-03 1.001	1.066E-03

fined in (10) and the numerical solution to problem (8) and (26) is displayed in Figure 3. The presence of an interior layer is evident in both figures.



(a) The computed Y

(b) The approximation $s + Y$

Figure 3: Example 3 from Problem Class 2: The numerical approximation to y and the approximation $s + Y$ to the solution u, with $\varepsilon = 2^{-16}$ and $N = M = 64$

4.3 Problem Class 3

Example 4. *Consider the problem (17), with the data taken to be*

$$b(x,t) = 1 + x, \quad f(x,t) = 4(1+x)(1-x)t + t^2, \tag{27}$$

and

$$u(0,t) = \begin{cases} 0, & \text{if } \ 0 \le t \le 0.25, \\ 0.5, & \text{if } \ 0.25 < t \le 1. \end{cases}$$

Observe that in this example the function $b = b(x)$. The schemes considered here to approximate the solution are defined on the Shishkin mesh $\bar{Q}_3^{N,M}$.

Once again, we first confirm the need to use our analytical/numerical approach to approximate the Problem Class 3. If we use backward Euler method and standard central finite differences on the Shishkin mesh $\bar{Q}_3^{N,M}$ to approximate Example 4 without separating off the singularity, it is not globally convergent for any value of ε. By way of illustration, the uniform two-mesh global differences are given in Table 6.

Table 6: Example 4 from Problem Class 3: Maximum two-mesh global differences and orders of convergence for u using a piecewise uniform Shishkin mesh, without separating off the singularity

	N=256,M=16	N=512,M=32	N=1024,M=64	N=2048,M=128	N=4096,M=256
$D^{N,M}$	2.500E-01	2.500E-01	2.500E-01	2.500E-01	2.500E-01
$P^{N,M}$	0.000	0.000	0.000	0.000	

We show now the numerical results when the singularity is stripped off. The maximum two-mesh global differences associated with the component y and the orders of convergence are given in Table 7. Observe that the numerical results indicate that the method is globally parameter-uniformly convergent. Figure 4 displays the numerical approximation to the function y defined in (19) and the approximation to the solution of problem (17) and (27). Thin boundary layer regions near $x = 0$ and $x = 1$ are visible in both plots, while large time derivatives near $t = 0.25$ are visible only in the plot of the approximation $s + Y$.

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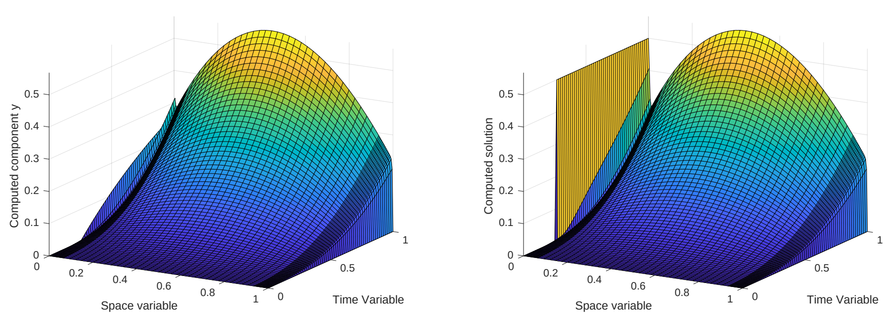
Table 7: Example 4 from Problem Class 3: Maximum two-mesh global differences and orders of convergence for the function y in (19) using a piecewise uniform Shishkin mesh

	N=256,M=16	N=512,M=32	N=1024,M=64	N=2048,M=128	N=4096,M=256
$\varepsilon = 2^{0}$	2.765E-03 0.903	1.478E-03 0.904	7.901E-04 0.962	4.057E-04 0.979	2.058E-04
$\varepsilon = 2^{-2}$	4.421E-03 0.975	2.250E-03 0.985	1.136E-03 0.992	5.714E-04 0.996	2.866E-04
$\varepsilon = 2^{-4}$	7.926E-03 1.020	3.909E-03 1.010	1.940E-03 1.005	9.664E-04 1.003	4.823E-04
$\varepsilon = 2^{-6}$	1.022E-02 1.009	5.079E-03 1.004	2.532E-03 1.002	1.264E-03 1.001	6.315E-04
$\varepsilon = 2^{-8}$	1.050E-02 1.010	5.214E-03 1.005	2.598E-03 1.003	1.297E-03 1.001	6.478E-04
$\varepsilon = 2^{-10}$	1.059E-02 1.010	5.260E-03 1.005	2.621E-03 1.003	1.308E-03 1.001	6.535E-04
$\varepsilon = 2^{-12}$	1.062E-02 1.010	5.275E-03 1.005	2.628E-03 1.003	1.312E-03 1.001	6.553E-04
$\varepsilon = 2^{-14}$	1.063E-02 1.010	5.278E-03 1.005	2.630E-03 1.003	1.313E-03 1.001	6.558E-04
$\varepsilon = 2^{-16}$	1.063E-02 1.009	5.279E-03 1.005	2.630E-03 1.003	1.313E-03 1.001	6.559E-04
⋮	⋮	⋮	⋮	⋮	⋮
$\varepsilon = 2^{-30}$	**1.063E-02** 1.010	**5.279E-03** 1.005	**2.631E-03** 1.003	**1.313E-03** 1.001	**6.559E-04**
$D^{N,M}$ $P^{N,M}$	1.063E-02 1.010	5.280E-03 1.005	2.631E-03 1.003	1.313E-03 1.001	6.559E-04



(a) The computed Y (b) The approximation $s + Y$

Figure 4: Example 4 from Problem Class 3: The numerical approximation to y and the approximation $s + Y$ to the solution u, with $\varepsilon = 2^{-16}$ and $N = M = 64$

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Appendix 1: Compatibility conditions

Below we place certain regularity and compatibility restrictions on the data of the problem

$$Lu := u_t - \varepsilon u_{xx} + b(x,t)u = f(x,t), \quad (x,t) \in Q, \tag{28a}$$
$$u(0,t) = g_L(t),\ u(1,t) = g_R(t)\ t \geq 0, \quad u(x,0) = \phi(x),\ 0 < x < 1, \tag{28b}$$

in order that the solution $u \in C^{4+\gamma}(\overline{Q})$. Compatibility conditions at the zero-order level correspond to:

$$\phi(0^+) = g_L(0) \quad \text{and} \quad \phi(1^-) = g_R(0). \tag{29a}$$

Assuming (29a), we can write $u = \Phi(x,t) + z,\ (x,t) \in Q$ where

$$\Phi(x,t) := \phi(x) + (1-x)(g_L(t) - g_L(0)) + x\left(g_R(t) - g_R(0)\right);$$

$Lz = f - L\Phi$; and $z(x,t) = 0,\ (x,t) \in \partial Q$. Note that

$$L\Phi = (1-x)g_L'(t) + xg_R'(t) - \varepsilon\phi''(x) + b\Phi.$$

From [13], if $b, f, L\Phi \in C^{0+\gamma}(\bar{Q})$ and the first-order compatibility conditions

$$(g_R'(0) - \varepsilon\phi''(1^-)) + b(1,0)\phi(1^-) = f(1,0), \tag{29b}$$
$$(g_L'(0) - \varepsilon\phi''(0^+)) + b(0,0)\phi(0^+) = f(0,0), \tag{29c}$$

are satisfied, then $u \in C^{2+\gamma}(\bar{Q})$. If $b, f, L\Phi \in C^{2+\gamma}(\bar{Q})$ and we further assume second-order compatibility (so that the mixed derivative z_{xxt} is well defined at $(0,0)$ and $(1,0)$), such that

$$(f - L\Phi)_t(0,0^+) + (f - L\Phi)_{xx}(0^+,0) = 0; \tag{29d}$$
$$(f - L\Phi)_t(1,0^+) + (f - L\Phi)_{xx}(1^-,0) = 0, \tag{29e}$$

then the solution of (28) satisfies $u \in C^{4+\gamma}(\bar{Q})$.

Appendix 2: Properties of $s_2(x,t)$

Recall that

$$s_2(x,t) := t^2 e^{-b(0)t}\,\mathrm{erf}\left(\frac{x}{2\sqrt{\varepsilon t}}\right).$$

Using the inequality $t^p e^{-t} \le C_{p,\mu} e^{-\mu t},\ t \in [0,\infty),\ \mu < 1, p > 0$; we have the following bounds, for all $(x,t) \in \bar{Q}$,

$$\|s_2\| \le C; \tag{30a}$$

$$\left|\frac{\partial s_2}{\partial t}(x,t)\right| + \varepsilon\left|\frac{\partial^2 s_2}{\partial x^2}(x,t)\right| \le C\frac{xt}{\sqrt{\varepsilon t}} e^{-\frac{x^2}{4\varepsilon t}} \le Cte^{-\mu\frac{x}{\sqrt{4\varepsilon T}}}; \tag{30b}$$

$$\begin{aligned}\varepsilon^2\left|\frac{\partial^4 s_2}{\partial x^4}(x,t)\right| &= \varepsilon^2 \frac{e^{-b(0)t}}{2}\frac{\sqrt{t}}{\varepsilon\sqrt{\varepsilon\pi}}\left|\frac{3x}{2\varepsilon t} - \frac{x^3}{4(\varepsilon t)^2}\right| e^{-\frac{x^2}{4\varepsilon t}} \\ &\le Ce^{-\mu\frac{x}{\sqrt{4\varepsilon T}}},\end{aligned} \tag{30c}$$

and

$$\begin{aligned}\frac{\partial^2 s_2}{\partial t^2}(x,t) = e^{-b(0)t}\frac{1}{4\sqrt{\pi}}&\left(\frac{3x}{\sqrt{\varepsilon t}} - \frac{x^3}{2\varepsilon t\sqrt{\varepsilon t}}\right)e^{-\frac{x^2}{4\varepsilon t}} \\ &+ \mathrm{erf}\left(\frac{x}{2\sqrt{\varepsilon t}}\right) 2\left(1 - 2b(0)t + \frac{(b(0)t)^2}{2}\right)e^{-b(0)t}.\end{aligned}$$

Hence,

$$\left|\frac{\partial^2 s_2}{\partial t^2}(x,t)\right| \le C, \quad x > 0. \tag{30d}$$

The second order time derivative is bounded, but not continuous, on the closed domain.