

Parameter-uniform numerical methods for singularly perturbed parabolic problems with incompatible boundary-initial data

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Abstract

Numerical approximations to the solution of a linear singularly perturbed parabolic reaction-diffusion problem with incompatible boundary-initial data are generated. The method involves combining the computational solution of a classical finite difference operator on a tensor product of two piecewise-uniform Shishkin meshes with an analytical function that captures the local nature of the incompatibility. A proof is given to show almost first order parameter-uniform convergence of these numerical/analytical approximations. Numerical results are given to illustrate the theoretical error bounds.

Keywords: Singularly perturbed problems, Incompatible boundary-initial data, Shishkin mesh

1 Introduction

We examine singularly perturbed parabolic problems in one space dimension, with an incompatibility between the initial condition and a boundary condition. These problems arise in mathematical models in fluid dynamics [9] and, in particular, models for flow in porous media [3]. The solutions of these problems typically exhibit boundary layers, initial layers and

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initial-boundary layers. In this paper we are interested in constructing a parameter-uniform numerical algorithm [2] for this class of singularly perturbed problems.

Within the literature on singularly perturbed problems, there are two common approaches to designing a parameter-uniform method: fitted operator (see e.g. [16]) or fitted mesh methods [2]. Numerical methods generate finite dimensional approximations U^N (where N is the number of mesh elements used in each coordinate direction) to the continuous solution u at the selected nodal points within the continuous domain \bar{Q} . A global approximation \bar{U}^N can also be created, using a user chosen choice of interpolating basis functions. In this paper, we shall simply employ bilinear basis functions. Parameter-uniform numerical methods [2] satisfy a theoretical error bound of the form:

$$\|u - \bar{U}^N\|_{\bar{Q}} \leq CN^{-p}, \quad p > 0;$$

where $\|\cdot\|_{\bar{Q}}$ is the L_∞ norm on the closed domain \bar{Q} , C is a generic constant, which depends on the problem data but is independent of N and the singular perturbation ε . We emphasize that this error bound estimates the pointwise error at all points in the domain \bar{Q} of the continuous solution. Parameter-uniform convergence at the nodes is a necessary, but not a sufficient, condition for parameter-uniform global convergence. If a numerical method is parameter-uniform at the nodes, then the distribution of the mesh points and the selected form of interpolation will determine whether the method is globally parameter-uniform or not.

For some classes of singularly perturbed problems with boundary layers, fitted operator methods on a uniform mesh exist which satisfy a parameter-uniform error bound at the nodes, but these fitted operator methods are not globally parameter-uniformly convergent [2], when some form of polynomial interpolation is employed. Moreover, a nodally parameter-uniform fitted operator method cannot be constructed for a class of singularly perturbed heat equations, if one only uses a uniform mesh [2, 14]. However, parameter-uniform numerical methods, using an appropriate Shishkin mesh have been designed for a wide class of singularly perturbed problems [12].

To establish pointwise parameter uniform error bounds on numerical approximations to the solutions of singularly perturbed parabolic problems, most publications assume second level compatibility conditions and sufficient regularity of the data so that the solution is in $C^{4+\gamma}(\bar{Q})$ ¹ in the closed

¹As in [8], we define the space $C^{0+\gamma}(D)$, where $D \subset \mathbf{R}^2$ is an open set, as the set of all functions that are Hölder continuous of degree $\gamma \in (0, 1)$ with respect to the metric $\|\cdot\|$,

domain \bar{Q} . Interested readers are referred, for example, to [19]. In the case of singularly perturbed parabolic problems in one space dimension and using appropriate fitted meshes, these compatibility constraints can be relaxed to zero order, without an adverse effect on the rate of uniform convergence [18]. Hence, parameter-uniform numerical methods exist when the boundary and initial data are simply assumed to be continuous.

However, there are difficulties with constructing a fitted mesh method for parabolic problems with an incompatibility between the initial and a boundary condition; or for a problem with a discontinuity in a boundary or the initial condition [10, 11]. Hemker and Shishkin [11] constructed a fitted operator method on a uniform mesh, which is nodally parameter-uniform for a singularly perturbed heat equation with a discontinuity in the initial condition; but the method is not globally parameter-uniform, using bilinear interpolation. An extension of this fitted operator method to a fitted operator method on a fitted piecewise-uniform mesh was constructed in [5], but this again failed to be parameter-uniform globally, using bilinear interpolation. The interpolation failed to produce an accurate global approximation in a neighbourhood of the point, where the initial condition and a boundary condition were incompatible.

Another approach to dealing with a problem having discontinuous data is to replace the problem with a regularized problem with continuous data [6], but these approximations are only accurate approximations to u outside a neighbourhood of the point $(0, 0)$. In other words, this approach will not generate parameter-uniform global approximations to the original problem with an incompatibility between the boundary and initial data.

In this paper, we examine an alternative approach to dealing with this problem class, which uses an idea examined numerically in [4] in the non-singularly perturbed case (set $\varepsilon = 1$). Given a differential operator L , the solution u of the continuous problem

$$Lu = f, \text{ in } Q, \quad u = g, \text{ on } \bar{Q} \setminus Q =: \partial Q, \quad \text{where } g \notin C^0(\partial Q);$$

where for all $\mathbf{p}_i = (x_i, t_i) \in \mathbf{R}^2, i = 1, 2; \|\mathbf{p}_1 - \mathbf{p}_2\|^2 = (x_1 - x_2)^2 + |t_1 - t_2|$. For f to be in $C^{0+\gamma}(D)$ the following semi-norm needs to be finite

$$[f]_{0+\gamma, D} = \sup_{\mathbf{p}_1 \neq \mathbf{p}_2, \mathbf{p}_1, \mathbf{p}_2 \in D} \frac{|f(\mathbf{p}_1) - f(\mathbf{p}_2)|}{\|\mathbf{p}_1 - \mathbf{p}_2\|^\gamma}.$$

The space $C^{n+\gamma}(D)$ is defined by

$$C^{n+\gamma}(D) = \left\{ z : \frac{\partial^{i+j} z}{\partial x^i \partial t^j} \in C^{0+\gamma}(D), 0 \leq i + 2j \leq n \right\},$$

and $\|\cdot\|_{n+\gamma}, [\cdot]_{n+\gamma}$ are the associated norms and semi-norms.

is written as the sum of two components $u = s + y$. The function s matches the incompatibility in the solution u and the other term y satisfies the singularly perturbed problem

$$Ly = f - Ls, \text{ in } Q, \quad y = g - s, \text{ on } \bar{Q} \setminus Q, \quad \text{where } g - s \in C^0(\partial Q).$$

In this paper, we design a parameter-uniform numerical method for this secondary problem, which generates a global approximation \bar{Y} to y . In this way, we can generate parameter-uniform numerical approximations $s + \bar{Y}$ to the solution u of a singularly perturbed problem with an incompatibility between the initial condition and a boundary condition. Note that here we restrict the discussion to problems in one space dimension. Extensions of the method to two space dimensions are not obvious [1] and require further investigation.

In the numerical analysis section of the paper, we are required to establish a convergence result for the numerical approximations to a boundary-initial layer component (w_{IB}), which has a classical weak singularity at the point $(x, t) = (0, 0)$. This component is not in $C^{2+\gamma}$ on the closed domain and so a further decomposition of w_{IB} is considered. The *a priori* pointwise bounds on the derivatives of these components of w_{IB} can be viewed as the main result of this paper. The now standard stability and consistency argument for piecewise-uniform Shishkin meshes is modified here in order to deal with the lack of regularity in this particular component w_{IB} . Further details for the more standard aspects of the proof are available in [7].

The rest of the paper is structured as follows. In the next section, we define the problem class to be examined, we decompose the continuous solution into various components and we derive parameter-explicit bounds on the derivatives of each of these components. In Section 3, we construct the numerical method and we establish a parameter-uniform bound on the error. In Section 4, we present the results of some numerical experiments with a representative test problem. For the sake of completeness, we write out the compatibility conditions of levels zero, one and two in the first appendix. In a second appendix, we present some properties of fundamental solutions of a singularly perturbed heat equation. Finally, in the last appendix, we collect some technical details used in the proof of Theorem 3.

Notation. Throughout the paper, C denotes a generic constant that is independent of the singular perturbation parameter ε and of all discretization parameters. The L_∞ norm on the domain D shall be denoted by $\|\cdot\|_D$ and the subscript is omitted if the domain is \bar{Q} .

2 Continuous problem

Consider the singularly perturbed parabolic problem: Find $u : \bar{Q} \rightarrow \mathbb{R}$ with $Q := (0, 1) \times (0, T]$, such that

$$Lu := \varepsilon(u_t - u_{xx}) + b(x, t)u = f(x, t), \quad (x, t) \in Q; \quad (1a)$$

$$u(0, t) = g_L(t), \quad u(1, t) = g_R(t) \quad t \geq 0, \quad u(x, 0) = \phi(x), \quad 0 < x < 1; \quad (1b)$$

$$\phi(0^+) \neq g_L(0), \quad \phi(1^-) = g_R(0), \quad (1c)$$

$$b_x(0, 0) = 0 \quad \text{and} \quad b(x, t) > \beta > 0, \quad (x, t) \in Q; \quad (1d)$$

$$f, b \in C^{4+\gamma}(\bar{Q}), \quad g_L, g_R \in C^2[0, T], \quad \phi \in C^4(0, 1). \quad (1e)$$

Observe that the solution of this problem is discontinuous at the corner $(0, 0)$ of the domain \bar{Q} . We define the related constant coefficient differential operator

$$L_0 z := \varepsilon(z_t - z_{xx}) + b(0, 0)z, \quad (2)$$

so that (by (1d))

$$|(L - L_0)z(x, t)| \leq C(x^2 + t)|z(x, t)|.$$

It is important to point out that the coefficient $b(x, t)$ can depend on both the space and time variables. In the special case where this coefficient only depends on time, then the singularity associated with the incompatibility at $(0, 0)$ can be found analytically. However, our final theoretical error bound in Theorem 4 requires $b_x(0, 0) = 0$. This restriction on the data appears to be an artifact of our method of proof, as in our numerical experiments there is no obvious difference in the performance of the method when this restriction is ignored.

We also assume the compatibility conditions at the point $(1, 0)$

$$\varepsilon(g'_R(0) - \phi_{xx}(1^-)) + b(1, 0)g_R(0) = f(1, 0); \quad (3)$$

$$\begin{aligned} &\varepsilon(g''_R(0) - \phi_{xxxx}(1^-)) + b(1, 0)(g'_R(0) + \phi_{xx}(1^-)) \\ &+ b_t(1, 0)g_R(0) + 2b_x(1, 0)\phi_x(1^-) + b_{xx}(1, 0)\phi(1^-) = (f_t + f_{xx})(1, 0); \end{aligned} \quad (4)$$

Here we simply assume these additional compatibility conditions in order to concentrate on the issues near $(0, 0)$, associated with the lack of corresponding compatibility conditions being assumed at $(0, 0)$. Thus, in this section, the solution u is decomposed into a sum of terms, some associated with the layers in the solution and some terms (denoted below by $A_0 z_0(x, t) + A_1 z_1(x, t) + A_2 z_2(x, t)$) associated with the lack of compatibility

being assumed at $(0, 0)$; while, if we did not assume (3) and (4), then additional terms of the form $A_1^R z_1(1-x, t) + A_2^R z_2(1-x, t)$ would be included in the expansion of the continuous solution; and the influence of these additional terms on the numerical analysis, could be tracked in the exact same way as the terms $A_1 z_1(x, t) + A_2 z_2(x, t)$ are handled in the error analysis below. Hence, it is solely for the sake of clarity of exposition in this section of the paper, that we assume the compatibility conditions (3) and (4). In other words, the numerical method presented below will satisfy the same error bound, established in Theorem 4, even when the data does not satisfy the constraints (3) and (4).

Decompose the solution of (1) into the sum

$$u = A_0 e^{-\frac{b(0,0)t}{\varepsilon}} \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) + y, \quad A_0 := g_L(0) - \phi(0^+), \quad (5)$$

where $\operatorname{erfc}(z)$ is the complementary error function

$$\operatorname{erfc}(z) := \frac{2}{\sqrt{\pi}} \int_{s=z}^{\infty} e^{-s^2} ds.$$

Note that the function

$$z_0(x, t) := e^{-\frac{b(0,0)t}{\varepsilon}} \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right)$$

is the first of a family of functions defined as the solutions of the constant coefficient homogeneous quarter plane problems, where for all $n = 0, 1, 2, \dots$

$$L_0 z_n = 0, \quad x, t > 0, \quad z_n(0, t) = t^n e^{-\frac{b(0,0)t}{\varepsilon}}, \quad t > 0, \quad z_n(x, 0) = 0, \quad x > 0. \quad (6)$$

In Appendix A.2, we explicitly write out several derivatives of these functions and we discuss the regularity of these functions.

In this section, we establish *a priori* bounds on the derivatives of the continuous function $y := u - A_0 z_0$, which satisfies the problem

$$Ly = f(x, t) - A_0(L - L_0)z_0(x, t), \quad \text{in } Q; \quad (7a)$$

$$y(0, t) = g_L(t) - A_0 e^{-\frac{b(0,0)t}{\varepsilon}}, \quad y(1, t) = g_R(t) - A_0 z_0(1, t), \quad t \geq 0; \quad (7b)$$

$$y(x, 0) = \phi(x), \quad 0 < x < 1. \quad (7c)$$

We introduce extended domains, where various subcomponents of the solution y will be defined: For arbitrary positive constants p, q, r ,

$$\bar{Q}^* := [-p, 1+q] \times [-r, T]; \quad \bar{Q}_S^* := [0, 1] \times [-r, T]; \quad \bar{Q}_B^* := [-p, 1+q] \times [0, T].$$

To avoid excessive notation, we shall denote smooth extensions of any function g to some larger domain by g^* (such that $g^*|_{\bar{Q}} \equiv g$), even though these extensions will be taken over different domains. A discussion of this technical technique of extending the problem definition to a larger domain Q^* may be found, for example, in [16, Chapter 12] or [17].

The solution of (7) can be decomposed into a sum of a regular component v and several layer components w (with a subscript to identify the location of the layer) defined as follows:

$$y = v + w_L + w_R + w_I + w_{IB}; \quad (8a)$$

where the regular component v satisfies the problem

$$L^*v^* = f^*, \text{ in } Q^*, \quad v^* = v^*, \text{ on } \partial Q^* \quad (8b)$$

and $L^*v^*(x, y) \equiv Lv(x, y)$, $\forall(x, y) \in Q$. The boundary/initial values for the regular component are determined from the reduced solution v_0 and a correction v_1 . We write $v^* = v_0^* + \varepsilon v_1^*$, where the reduced solution v_0 and the correction v_1 are defined via

$$v_0^* = \left(\frac{f}{b}\right)^*, \text{ in } Q^*; \quad (8c)$$

$$L^*v_1^* = (v_0)_{xx}^* - (v_0)_t^*, \quad \text{in } Q^*, \quad v_1^* = 0, \quad \text{on } \partial Q^*. \quad (8d)$$

The boundary layer components w_L, w_R satisfy the homogeneous problems

$$L^*w_R^* = L^*w_L^* = 0, \quad \text{in } Q_S^*; \quad (8e)$$

$$w_L^*(0, t) = (y - v^*)^*(0, t), \quad w_L^*(x, -r) = 0, \quad w_L^*(1, t) = 0, \quad \text{on } \partial Q_S^*; \quad (8f)$$

$$w_R^*(0, t) = 0, \quad w_R^*(x, -r) = 0, \quad w_R^*(1, t) = (y - v^*)^*(1, t), \quad \text{on } \partial Q_S^*. \quad (8g)$$

The initial layer function w_I satisfies the problem

$$L^*w_I^* = 0 \quad \text{in } Q_B^*; \quad (8h)$$

$$w_I^*(-p, t) = 0, \quad w_I^*(x, 0) = (y - v^*)^*(x, 0), \quad w_I^*(1 + q, t) = 0, \quad \text{on } \partial Q_B^*. \quad (8i)$$

Having defined the problems over the extended domains, to avoid compatibility issues, the components v, w_L, w_R, w_I are all in $C^{4+\gamma}(\bar{Q})$.

Finally, the initial-boundary layer component w_{IB} satisfies the problem

$$Lw_{IB} = -A_0(L - L_0)z_0(x, t), \quad (x, t) \in Q; \quad (8j)$$

$$w_{IB}(0, t) = -w_I^*(0, t), \quad w_{IB}(1, t) = -w_I^*(1, t), \quad t \geq 0; \quad (8k)$$

$$w_{IB}(x, 0) = -w_L^*(x, 0) - w_R^*(x, 0), \quad 0 < x < 1. \quad (8l)$$

The regularity of this key component w_{IB} is discussed below in Theorem 3.

Theorem 1. For all $0 \leq i + 2j \leq 4$, we have the following bounds. For the regular component $v \in C^{4+\gamma}(\bar{Q})$,

$$\left\| \frac{\partial^{i+j} v}{\partial x^i \partial t^j} \right\| \leq C \left(1 + \varepsilon^{1-(i/2+j)} \right); \quad (9a)$$

and, for all points $(x, t) \in Q$, the boundary layer components $w_L, w_R \in C^{4+\gamma}(\bar{Q})$ satisfy

$$|w_L(x, t)| \leq C e^{-\sqrt{\frac{\beta}{\varepsilon}} x}; \quad |w_R(x, t)| \leq C e^{-\sqrt{\frac{\beta}{\varepsilon}}(1-x)}; \quad (9b)$$

$$\left| \frac{\partial^{i+j}}{\partial x^i \partial t^j} w_L(x, t) \right| \leq C \varepsilon^{-(i/2+j)} e^{-\sqrt{\frac{\beta}{\varepsilon}} x}, \quad (9c)$$

$$\left| \frac{\partial^{i+j}}{\partial x^i \partial t^j} w_R(x, t) \right| \leq C \varepsilon^{-(i/2+j)} e^{-\sqrt{\frac{\beta}{\varepsilon}}(1-x)}. \quad (9d)$$

In addition, for $1 \leq j \leq 2$, the time derivatives satisfy

$$\max \left\{ \left\| \frac{\partial^j w_L}{\partial t^j} \right\|, \left\| \frac{\partial^j w_R}{\partial t^j} \right\| \right\} \leq C \varepsilon^{1-j}. \quad (9e)$$

Proof. To deduce the bounds (9a), use the stretched variables $\zeta = x/\sqrt{\varepsilon}, \eta = t/\varepsilon$ to transform problem (8d) to a classical problem, apply the *a priori* bounds [13, Theorem 5.2, p. 320] on the derivatives of the solution and then transform back to the original variables (x, t) . A maximum principle, the assumption $b(x, t) > \beta$ and the argument from [15, Theorem 4] yield the bounds on the boundary layer components w_L, w_R (9b)-(9d). See [7] for further details on completing the proof for (9e). \square

Theorem 2. For the initial layer component $w_I \in C^{4+\gamma}(\bar{Q})$ and

$$|w_I(x, t)| \leq C e^{-\frac{\beta t}{\varepsilon}}, \quad (x, t) \in Q; \quad (10a)$$

$$\left| \frac{\partial^{i+j}}{\partial x^i \partial t^j} w_I(x, t) \right| \leq C \varepsilon^{-i/2} \varepsilon^{-j} e^{-\frac{\beta t}{\varepsilon}}, \quad 0 \leq i + 2j \leq 4; \quad (10b)$$

$$\left| \frac{\partial^i}{\partial x^i} w_I(x, t) \right| \leq C (1 + \varepsilon^{1-i/2}) e^{-\frac{\beta t}{\varepsilon}}, \quad i = 1, 2, 3, 4. \quad (10c)$$

Proof. The argument follows the same lines as the proof of the previous theorem. See [7] for details. \square

We consider now the initial-boundary layer component w_{IB} defined in (8j)-(8l). As y is continuous and the components v, w_L, w_R, w_I are smooth, zero-order compatibility conditions (for w_{IB}) are satisfied. We further decompose the initial-boundary layer term via

$$w_{IB}(x, t) = A_1 z_1(x, t) + w_C(x, t),$$

where the constant A_1 is specified in (36) and the function z_1 is defined in (6). Note that $z_1 \notin C^{2+\gamma}(\bar{Q})$ and, moreover, $y \notin C^{2+\gamma}(\bar{Q})$.

Theorem 3. *The initial-boundary layer component $w_C \in C^{2+\gamma}(\bar{Q})$. For all $(x, t) \in \bar{Q}$*

$$|w_C(x, t)| \leq C e^{-\frac{\beta t}{\varepsilon}}; \quad (11a)$$

and

$$\left| \frac{\partial^2 w_C}{\partial x^2}(x, t) \right| + \left| \frac{\partial w_C}{\partial t}(x, t) \right| \leq C + C\varepsilon^{-1} \left(e^{-\sqrt{\frac{\beta}{\varepsilon}}x} + e^{-\sqrt{\frac{\beta}{\varepsilon}}(1-x)} + e^{-\frac{\beta t}{\varepsilon}} \right) \quad (11b)$$

$$\left| \frac{\partial^4 w_C}{\partial x^4}(x, t) \right| + \left| \frac{\partial^2 w_C}{\partial t^2}(x, t) \right| \leq C\varepsilon^{-1} + C\varepsilon^{-2} \left(e^{-\sqrt{\frac{\beta}{\varepsilon}}x} + e^{-\sqrt{\frac{\beta}{\varepsilon}}(1-x)} + e^{-\frac{\beta t}{\varepsilon}} \right) \quad (11c)$$

Proof. From Appendix A.3, the initial-boundary layer component w_C can be written in the form

$$w_C(x, t) = (A_2 z_2 + A_0 \Psi + R_C)(x, t),$$

where the function z_2 is defined in (6) and the other terms $A_2, \Psi(x, t)$ are defined in (35), (34) and (37) in Appendix A.3. From this construction

$$\begin{aligned} R_C &= w_C - A_2 z_2 - A_0 \Psi = w_{IB} - A_1 z_1 - A_2 z_2 - A_0 \Psi \\ &= (u - A_0 z_0 - A_1 z_1 - A_2 z_2 - A_0 \Psi) - (v + w_L + w_R + w_I) \\ &= y_2 - (v + w_L + w_R + w_I). \end{aligned}$$

In Appendix A.3, it is established that $y_2 \in C^{4+\gamma}(\bar{Q})$, which implies that $R_C \in C^{4+\gamma}(\bar{Q})$.

The remainder $R_C(x, t)$ satisfies the problem

$$\begin{aligned} LR_C &= Ly_2 - f \\ R_C(x, 0) &= -(w_L^* + w_R^*)(x, 0), \quad 0 < x < 1, \\ R_C(0, t) &= -(w_I^* + A_1 z_1 + A_2 z_2 + A_0 \Psi)(0, t), \quad t > 0, \\ R_C(1, t) &= -(w_I^* + A_1 z_1 + A_2 z_2 + A_0 \Psi)(1, t), \quad t > 0. \end{aligned}$$

Although the functions $z_2, \Psi \notin C^{4+\gamma}(\bar{Q})$, we still have the necessary bounds on the higher derivatives of these functions. See Appendices A.2 and A.3 for details. It remains to bound the derivatives of R_C .

Using the bounds in (38) we have that

$$\left| \frac{\partial^{i+j}}{\partial x^i \partial t^j} (LR_C(x, t)) \right| \leq C \varepsilon^{-j} e^{-\frac{\beta t}{\varepsilon}}, \quad 0 \leq i + 2j \leq 2;$$

and, from the previous two theorems and the fact that $R_C \in C^{4+\gamma}(\bar{Q})$, we deduce that

$$\begin{aligned} \left| \frac{\partial^j}{\partial t^j} R_C(0, t) \right| + \left| \frac{\partial^j}{\partial t^j} R_C(1, t) \right| &\leq C \varepsilon^{-j} e^{-\frac{\beta t}{\varepsilon}}; \quad 0 \leq j \leq 2; \\ \left| \frac{\partial^i}{\partial x^i} R_C(x, 0) \right| &\leq C \varepsilon^{-i/2} \left(e^{-\sqrt{\frac{\beta}{\varepsilon}} x} + e^{-\sqrt{\frac{\beta}{\varepsilon}}(1-x)} \right); \quad 0 \leq i \leq 4. \end{aligned}$$

From the maximum principle we then have that

$$|R_C(x, t)| \leq C e^{-\frac{\beta t}{\varepsilon}}.$$

Using the stretched variables $x/\sqrt{\varepsilon}, t/\varepsilon$ and the argument from the proofs of the previous theorems, we can deduce that

$$\left| \frac{\partial^{i+j}}{\partial x^i \partial t^j} R_C(x, t) \right| \leq C \varepsilon^{-j} e^{-\frac{\beta t}{\varepsilon}}, \quad 0 \leq i + 2j \leq 4.$$

This completes the proof. \square

3 Numerical Method

To accurately capture the layers in both space and time, we use a tensor product of two piecewise-uniform Shishkin meshes [2] $\bar{Q}^{N,M} := \omega_x^N \times \omega_t^M$. The space Shishkin mesh $\omega_x^N := \{x_i\}_{i=0}^N$ is fitted to the two boundary layers by splitting the space domain as follows:

$$[0, \sigma] \cup [\sigma, 1 - \sigma] \cup [1 - \sigma, 1]. \quad (12a)$$

The N space mesh points are distributed in the ratio $N/4 : N/2 : N/4$ across these three subintervals. The transition point σ (in space) is taken to be

$$\sigma := \min \left\{ 0.25, 2\sqrt{\frac{\varepsilon}{\beta}} \ln N \right\}. \quad (12b)$$

The Shishkin mesh $\omega_t^M := \{t_j\}_{j=0}^M$ splits the time domain into two subintervals $[0, \tau] \cup [\tau, 1]$ and the mesh points in time are distributed equally between these two subintervals. The transition point τ (in time) is taken to be

$$\tau := \min \left\{ 0.5, \frac{\varepsilon}{\beta} \ln M \right\}. \quad (12c)$$

We confine our attention to the case where $\sigma < 0.25$ and $\tau < 0.5$. For the other case, where $\sigma = 0.25$ or $\tau = 0.5$, a classical argument can be applied. We denote by $Q^{N,M} := \bar{Q}^{N,M} \cap Q$ and $\partial Q^{N,M} := \bar{Q}^{N,M} \setminus Q^{N,M}$.

We use a classical finite difference operator on this mesh to produce the following discrete problem: Find Y such that

$$L^{N,M}Y(x_i, t_j) = \left(f - A_0(b - b(0, 0))z_0 \right)(x_i, t_j), \quad (x_i, t_j) \in Q^{N,M}, \quad (13a)$$

$$Y(x_i, t_j) = y(x_i, t_j), \quad (x_i, t_j) \in \partial Q^{N,M}, \quad (13b)$$

$$\text{where } L^{N,M}Y(x_i, t_j) := (\varepsilon D_t^- - \varepsilon \delta_x^2 + b(x_i, t_j)I)Y(x_i, t_j). \quad (13c)$$

The finite difference operators are defined by

$$\begin{aligned} D_x^+ Y(x_i, t_j) &:= D_x^- Y(x_{i+1}, t_j), & D_x^- Y(x_i, t_j) &:= \frac{Y(x_i, t_j) - Y(x_{i-1}, t_j)}{h_i}, \\ D_t^- Y(x_i, t_j) &:= \frac{Y(x_i, t_j) - Y(x_i, t_{j-1})}{k_j}, & \delta_x^2 Y(x_i, t_j) &:= \frac{(D_x^+ - D_x^-)Y(x_i, t_j)}{\bar{h}_i}, \end{aligned}$$

and the mesh steps are $h_i := x_i - x_{i-1}$, $\bar{h}_i = (h_{i+1} + h_i)/2$, $k_j := t_j - t_{j-1}$.

We prove below in Theorem 4 that the scheme (13) is uniformly convergent using a truncation error argument. It is well known that the scheme (13) satisfies a discrete maximum principle and it is used to derive error estimates from appropriate truncation error estimates. We recall that the discrete maximum principle establishes that if Z is a grid function that satisfies

$$L^{N,M}Z \geq 0 \text{ on } Q^{N,M} \text{ and } Z \geq 0 \text{ on } \partial Q^{N,M}, \text{ then } Z \geq 0 \text{ on } \bar{Q}^{N,M}.$$

We now describe how the truncation error estimates are deduced. Away from the transition points, the mesh is uniform and a classical truncation error argument yields the bound

$$|L^{N,M}(Y - y)(x_i, t_j)| \leq Ch_i^2 \varepsilon \|y_{xxxx}\| + Ck_j \varepsilon \|y_{tt}\|, \quad x_i \neq \sigma, 1 - \sigma.$$

In addition, the discrete solution can be decomposed along the same lines as the continuous solution. That is,

$$Y = V + W_L + W_R + W_{IB} + W_I,$$

where these discrete functions are defined by

$$L^{N,M}V = Lv(x_i, t_j), \quad L^{N,M}W_{L,R,IB,I}(x_i, t_j) = Lw_{L,R,IB,I}(x_i, t_j)$$

and on the boundary

$$V(x_i, t_j) = v(x_i, t_j), \quad W_{L,R,IB,I}(x_i, t_j) = w_{L,R,IB,I}(x_i, t_j), \quad (x_i, t_j) \in \partial Q^{N,M}.$$

Theorem 4. *Let Y be the solution of the finite difference scheme (13) and y the solution of the problem (7). Then, the following nodal error estimates are satisfied*

$$\|y - Y\|_{\bar{Q}^{N,M}} \leq C (N^{-2}(\max\{\ln^2 N, \ln M\}) + M^{-1} \ln^2 M). \quad (14)$$

Proof. From the estimates (9a), the truncation error for the regular component is bounded by:

$$\begin{aligned} |L^{N,M}(V - v)(x_i, t_j)| &\leq Ch_i^2 \varepsilon \|v_{xxxx}\| + Ck_j \varepsilon \|v_{tt}\| \\ &\leq C(N^{-1} \ln N)^2 + CM^{-1} \ln M, \quad x_i \neq \sigma, 1 - \sigma; \end{aligned} \quad (15a)$$

$$|L^{N,M}(V - v)(x_i, t_j)| \leq C\sqrt{\varepsilon} N^{-1} \ln N + CM^{-1} \ln M, \quad x_i = \sigma, 1 - \sigma. \quad (15b)$$

By employing a suitable barrier function, described in [15, pp. 183-184], we can deduce from the estimates (15) and the discrete maximum principle that

$$\|v - V\|_{\bar{Q}^{N,M}} \leq C(N^{-1} \ln N)^2 + CM^{-1} \ln M. \quad (16)$$

Using the exponential bounds on the derivatives of the boundary layer components w_L and w_R given in Theorem 1 and the definition of the space Shishkin mesh ω_x^N , we bound its truncation errors (see [15, Theorem 6] for details) as follows:

$$|L^{N,M}(W_L - w_L)(x_i, t_j)| \leq C(N^{-1} \ln N)^2 + CM^{-1} \ln M, \quad (17a)$$

$$|L^{N,M}(W_R - w_R)(x_i, t_j)| \leq C(N^{-1} \ln N)^2 + CM^{-1} \ln M, \quad (17b)$$

for all $(x_i, t_j) \in Q^{N,M}$. The discrete maximum principle yields the bound

$$\|W_L - w_L\|_{\bar{Q}^{N,M}} \leq C(N^{-1} \ln N)^2 + CM^{-1} \ln M, \quad (18a)$$

$$\|W_R - w_R\|_{\bar{Q}^{N,M}} \leq C(N^{-1} \ln N)^2 + CM^{-1} \ln M. \quad (18b)$$

In the case of the initial layer function, note that

$$e^{-b(0,0)k_j/\varepsilon} \leq \left(1 + \frac{b(0,0)k_j}{\varepsilon}\right)^{-1}$$

and, using a discrete barrier function $B(t_j)$, we deduce that

$$|W_I(x_i, t_j)| \leq \prod_{m=1}^j \left(1 + \frac{\beta k_m}{\varepsilon}\right)^{-1} =: B(t_j),$$

as

$$\varepsilon D_t^- B(t_j) + b(x_i, t_j)B(t_j) = (b(x_i, t_j) - \beta)B(t_j) \geq 0.$$

This barrier function $B(t_j)$ and the estimates (10) are used to deduce bounds for truncation error associated with the component w_I . First, outside the initial layer, where $t_j \geq \tau$, we have

$$|W_I(x_i, t_j) - w_I(x_i, t_j)| \leq |W_I(x_i, t_j)| + |w_I(x_i, t_j)| \leq CM^{-1}, \quad (19)$$

and within the initial layer, where $t_j < \tau$,

$$\begin{aligned} |L^{N,M}(W_I - w_I)(x_i, t_j)| \\ \leq C(N^{-1} \ln N)^2 + CM^{-1} \ln M, \quad x_i \neq \sigma, 1 - \sigma, \end{aligned} \quad (20a)$$

$$\begin{aligned} |L^{N,M}(W_I - w_I)(x_i, t_j)| \\ \leq C\sqrt{\varepsilon}N^{-1} \ln N + CM^{-1} \ln M, \quad x_i = \sigma, 1 - \sigma. \end{aligned} \quad (20b)$$

As in the case of the continuous initial-boundary layer component w_{IB} , we introduce the secondary decomposition

$$W_{IB} = W_C + A_1 Z_1,$$

where the components W_C, Z_1 are defined as the solutions of

$$\begin{aligned} L_0^{N,M} W_C &:= (\varepsilon D_t^- - \varepsilon \delta_x^2 + b(0, 0))W_C = 0, \quad \text{on } Q^{N,M}, \quad W_C = w_C \text{ on } \partial Q^{N,M}; \\ L_0^{N,M} Z_1 &= 0, \quad \text{on } Q^{N,M}, \quad Z_1 = z_1 \text{ on } \partial Q^{N,M}. \end{aligned}$$

Note that, using a discrete maximum principle,

$$|W_C(x_i, t_j)| \leq C \prod_{m=1}^j \left(1 + \frac{\beta k_m}{\varepsilon}\right)^{-1}, \quad |Z_1(x_i, t_j)| \leq C\varepsilon \prod_{m=1}^j \left(1 + \frac{\beta k_m}{\varepsilon}\right)^{-1}.$$

The error $W_{IB} - w_{IB}$ is decomposed into the sum

$$W_{IB} - w_{IB} = W_C - w_C + A_1(Z_1 - z_1), \quad \varepsilon|A_1| \leq C.$$

From the earlier exponential bounds on each of the four individual terms W_C, w_C, Z_1, z_1 we establish that for $t_j \geq \tau$

$$\begin{aligned} |(W_{IB} - w_{IB})(x_i, t_j)| &\leq |W_C(x_i, t_j)| + |w_C(x_i, t_j)| \\ &\quad + A_1(|Z_1(x_i, t_j)| + |z_1(x_i, t_j)|) \\ &\leq CM^{-1}. \end{aligned} \quad (21)$$

Within the initial layer, from Theorem 3, we have that for $t_j < \tau$,

$$\begin{aligned} |L^{N,M}(W_C - w_C)(x_i, t_j)| \\ \leq C(N^{-1} \ln N)^2 + CM^{-1} \ln M, \quad x_i \notin [\sigma, 1 - \sigma], \end{aligned} \quad (22a)$$

$$\begin{aligned} |L^{N,M}(W_C - w_C)(x_i, t_j)| &\leq C\sqrt{\varepsilon}(N^{-1} \ln N) + C\frac{H}{\sqrt{\varepsilon}}e^{-\frac{\beta(\sigma+H)}{\sqrt{\varepsilon}}} + CM^{-1} \ln M \\ &\leq C\sqrt{\varepsilon}(N^{-1} \ln N) + C(N^{-1} \ln N)^2 + CM^{-1} \ln M, \quad x_i = \sigma, 1 - \sigma, \end{aligned} \quad (22b)$$

$$\begin{aligned} |L^{N,M}(W_C - w_C)(x_i, t_j)| \\ \leq C(N^{-1} \ln N)^2 + C\frac{H^2}{\varepsilon}e^{-\frac{\beta(\sigma+H)}{\sqrt{\varepsilon}}} + CM^{-1} \ln M \\ \leq C(N^{-1} \ln N)^2 + CM^{-1} \ln M, \quad x_i \in (\sigma, 1 - \sigma), \end{aligned} \quad (22c)$$

where H is the mesh width in the coarse region, i.e., $H = 2(1 - 2\sigma)/N = O(N^{-1})$.

Collecting the truncation error bounds (20) and (22) we have for $t_j < \tau$

$$\begin{aligned} |L^{N,M}[(W_I + W_C) - (w_I + w_C)](x_i, t_j)| \\ \leq C(N^{-1} \ln N)^2 + CM^{-1} \ln M, \quad x_i \neq \sigma, 1 - \sigma, \\ |L^{N,M}[(W_I + W_C) - (w_I + w_C)](x_i, t_j)| \\ \leq C\sqrt{\varepsilon}(N^{-1} \ln N) + CM^{-1} \ln M, \quad x_i = \sigma, 1 - \sigma, \end{aligned}$$

and $|[(W_I + W_C) - (w_I + w_C)](x_i, \tau)| \leq CM^{-1}$. Hence, the truncation error is first order only along the spatial transition lines $x_i = \sigma, 1 - \sigma$. By employing again a suitable barrier function, described in [15], we can deduce for $(x_i, t_j) \in [0, 1] \times [0, \tau]$

$$\left| [(W_I + w_C) - (w_I + w_C)](x_i, \tau) \right| \leq C(N^{-1} \ln N)^2 + CM^{-1} \ln M. \quad (23)$$

We employ a different argument to bound the error

$$E_i^j := A_1(Z_1 - z_1)(x_i, t_j), \quad \text{for } 1 \leq j \leq M/2.$$

From Appendix A.2, $\varepsilon|A_1| \leq C$ and below we bound the truncation error

$$\mathcal{T}_{i,j} := |(L_0^{N,M} - L_0)A_1 z_1(x_i, t_j)|$$

using the bounds on the derivatives of z_1 given in Appendix A.2. The truncation error at the first time level $t = t_1$ is thus bounded as follows

$$\mathcal{T}_{i,1} \leq C \left\| \frac{\partial^2 z_1}{\partial x^2} \right\|_{X_i^1} + C \left\| \frac{\partial z_1}{\partial t} \right\|_{T_i^1} \leq C$$

where $X_i^j := [x_{i-1}, x_{i+1}] \times \{t_j\}$, $T_i^j := \{x_i\} \times [t_{j-1}, t_j]$. For the other time levels $2 \leq j \leq M/2$, we have for $x_i \in (0, \sigma) \cup (1 - \sigma, 1)$

$$\mathcal{T}_{i,j} \leq C \left(h^2 \left\| \frac{\partial^4 z_1}{\partial x^4} \right\|_{X_i^j} + k \left\| \frac{\partial^2 z_1}{\partial t^2} \right\|_{T_i^j} \right) \leq C \left(h^2 \left(1 + \frac{\varepsilon}{t_j} \right) + \frac{k}{t_{j-1}} \right)$$

and for $x_i \in [\sigma, 1 - \sigma]$

$$\mathcal{T}_{i,j} \leq C \left(\left\| \frac{\partial^2 z_1}{\partial x^2} \right\|_{X_i^j} + k \left\| \frac{\partial^2 z_1}{\partial t^2} \right\|_{T_i^j} \right) \leq C \left(z_0(x_{i-1}, t_j) + \frac{k}{t_{j-1}} \right).$$

As in [18, p. 916] and also using $(x - 2t\sqrt{b(0,0)/\varepsilon})^2 \geq 0$, we have

$$\operatorname{erfc}(z) \leq \frac{e^{-z^2}}{z + \sqrt{z^2 + 4/\pi}} \leq C e^{-z^2}, \quad z \geq 0, \quad e^{-\frac{x^2}{4t}} \leq e^{\frac{b(0,0)t}{\varepsilon}} e^{-\sqrt{\frac{b(0,0)}{\varepsilon}}x},$$

which yield, for all $x_i \geq \sigma$,

$$z_0(x_{i-1}, t_j) \leq C e^{-\sqrt{\frac{b(0,0)}{\varepsilon}}x_{i-1}} \leq C e^{\sqrt{\frac{b(0,0)}{\varepsilon}}h} e^{-\sqrt{\frac{b(0,0)}{\varepsilon}}\sigma} \leq C N^{-2}.$$

Thus, for $x_i \in [\sigma, 1 - \sigma]$, we have the truncation error bound

$$\mathcal{T}_{i,j} \leq C \left(\frac{k}{t_{j-1}} + N^{-2} \right).$$

We again follow the argument in [18] and note that at each time level, $t = t_j$, $1 \leq j \leq M/2$,

$$-\varepsilon \delta_x^2 E_i^j + \left(b(x_i, t_j) + \frac{\varepsilon}{k} \right) E_i^j = \mathcal{T}_{i,j} + \frac{\varepsilon}{k} E_i^{j-1}.$$

From this we can deduce that

$$\begin{aligned}
|E_i^j| &\leq C \frac{k}{\varepsilon} \left(\mathcal{T}_{i,1} + \sum_{n=2}^j \mathcal{T}_{i,n} \right) \\
&\leq CM^{-1}(\ln M) + C \frac{k}{\varepsilon} MN^{-2} + C \left(\frac{k}{\varepsilon} + h^2 \right) \int_{s=1}^{M/2} \frac{ds}{s} \\
&\leq C(M^{-1} \ln M + N^{-2}) \ln M.
\end{aligned} \tag{24}$$

The error estimates (16), (18), (19), (21), (23) and (24) prove the nodal error bound (14) and the result follows. \square

One can extend the nodal error estimate (14) to a global error estimate by applying the argument in [2, pp. 56-57]. Note that, in general, we use a bound on the second space and time derivative (of each component) to establish this global error bound. However, in the case of the terms w_{IB}, w_C, z_1 , we use the alternative interpolation bound over each rectangle $Q_{i,j} := (x_i, x_{i+1}) \times (t_j, t_{j+1})$ of the form

$$\|(z - \bar{z})(x, t)\|_{Q_{i,j}} \leq C(t_{j+1} - t_j) \|z_t\|_{Q_{i,j}} + C(x_{i+1} - x_i)^2 \|z_{xx}\|_{Q_{i,j}}.$$

Moreover, in the case of the term involving z_1 , note that

$$A_1 |(z_1)_t(x, t)| \leq \frac{C}{\varepsilon} e^{-\beta t/\varepsilon}.$$

Corollary 1. *Let Y be the solution of the finite difference scheme (13) and y the solution of the problem (7). Then, the following global error estimates are satisfied*

$$\|y - \bar{Y}\| \leq C (N^{-2}(\max\{\ln^2 N, \ln M\}) + M^{-1} \ln^2 M), \tag{25}$$

where \bar{Y} denotes the bilinear interpolant of the discrete function Y from the values of the grid $\bar{Q}^{N,M}$ to the domain \bar{Q} .

Note that it is the presence of the fitted mesh (in both space and time) that yields global parameter-uniform convergence.

4 Numerical results

Consider the following sample problem from the problem class (1):

$$b(x, t) = 1 + x^2 + t, \quad f(x, t) = e^{-x}, \tag{26a}$$

$$\phi(x) = 1 - x, \quad g_L(t) = 0, \quad g_R(t) = -t^2, \tag{26b}$$

and the domain is $Q = (0, 1) \times (0, 1]$. Observe that $1 = \phi(0^+) \neq g_L(0) = 0$ and the compatibility conditions (3) and (4) are not satisfied at $x = 1, t = 0$. This problem is a minor variant of a problem considered on the half line $x > 0$ in [9, §2] to illustrate the interaction of initial and boundary layers.

The component y , which is defined in (7), is the solution of the problem

$$Ly = e^{-x} + (b(x, t) - b(0, 0))z_0(x, t) \quad (x, t) \in Q, \quad (27a)$$

$$y(0, t) = z_0(0, t), \quad y(1, t) = -t^2 + z_0(1, t), \quad t \geq 0, \quad (27b)$$

$$y(x, 0) = 1 - x, \quad 0 < x < 1, \quad (27c)$$

where we recall that $z_0(x, t) = e^{-b(0,0)t/\varepsilon} \operatorname{erfc}(0.5x/\sqrt{t})$ and $y \in C^0(\partial Q)$. Nevertheless, the component y for this example does not satisfy the first-order compatibility condition (29c) at the corner $(0, 0)$.

The exact solution of problem (27) is unknown. In Figure 1 the computed approximation U (generated from the finite difference scheme (13)) to the solution u of Example (26) is displayed. The solution surface reveals that u has initial and boundary layers. In all the figures of this section we consider the values of $\varepsilon = 2^{-12}$ and $N = M = 64$.

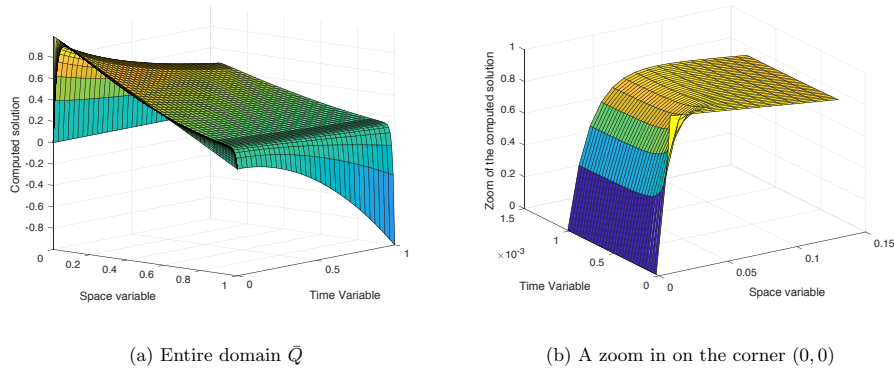


Figure 1: Example (26): The numerical approximation to the solution u with $\varepsilon = 2^{-12}$ and $N = M = 64$

The orders of convergence of the finite difference scheme (13) are estimated using the two-mesh principle [2]. We denote by $Y^{N,M}$ and $Y^{2N,2M}$ the computed solutions with (13) on the Shishkin meshes $Q^{N,M}$ and $Q^{2N,2M}$, respectively. These solutions are used to compute the maximum two-mesh

global differences

$$D_\varepsilon^{N,M} := \|\bar{Y}^{N,M} - \bar{Y}^{2N,2M}\|_{Q^{N,M} \cup Q^{2N,2M}}$$

where $\bar{Y}^{N,M}$ and $\bar{Y}^{2N,2M}$ denote the bilinear interpolation of the discrete solutions $Y^{N,M}$ and $Y^{2N,2M}$ on the mesh $Q^{N,M} \cup Q^{2N,2M}$. Then, the orders of global convergence $P_\varepsilon^{N,M}$ are estimated in a standard way [2]

$$P_\varepsilon^{N,M} := \log_2 \left(\frac{D_\varepsilon^{N,M}}{D_\varepsilon^{2N,2M}} \right).$$

The uniform two-mesh global differences $D^{N,M}$ and their corresponding uniform orders of global convergence $P^{N,M}$ are calculated by

$$D^{N,M} := \max_{\varepsilon \in S} D_\varepsilon^{N,M}, \quad P^{N,M} := \log_2 \left(\frac{D^{N,M}}{D^{2N,2M}} \right),$$

where $S = \{2^0, 2^{-1}, \dots, 2^{-30}\}$. The maximum two-mesh global differences $D_\varepsilon^{N,M}$ and the orders of global convergence $P_\varepsilon^{N,M}$ associated with the problem (27) are displayed in Table 1. The uniform two-mesh global differences $D^{N,M}$ and their orders of convergence $P^{N,M}$ are given in the last row of this table. These numerical results are in line with the error estimate (25) showing that the method is an almost first-order uniformly global convergent scheme.

Finally, we give some information about the distribution of the errors. In Figure 2 we display the approximate errors $|(Y^{64,64} - \bar{Y}^{1024,1024})(x_i, t_j)|$ with $(x_i, t_j) \in \bar{Q}^{64,64}$; and it is observed that the largest errors occur within the layers and, within the initial layer, the maximum errors occur at the earlier times.

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Table 1: Example (26): Maximum and uniform two-mesh global differences and orders of global convergence associated with the solution y of problem (27)

	N=64 M=16	N=128 M=32	N=256 M=64	N=512 M=128	N=1024 M=256	N=2048 M=512	N=4096 M=1024
$\epsilon = 2^0$	3.287E-03 0.851	1.822E-03 0.929	9.570E-04 0.968	4.893E-04 0.988	2.467E-04 0.996	1.237E-04 0.999	6.192E-05
$\epsilon = 2^{-1}$	5.854E-03 0.769	3.435E-03 0.885	1.860E-03 0.947	9.647E-04 0.978	4.899E-04 0.991	2.466E-04 0.996	1.236E-04
$\epsilon = 2^{-2}$	9.217E-03 0.594	6.107E-03 0.799	3.510E-03 0.905	1.875E-03 0.957	9.660E-04 0.980	4.896E-04 0.991	2.464E-04
$\epsilon = 2^{-3}$	1.266E-02 0.500	8.951E-03 0.518	6.250E-03 0.820	3.540E-03 0.915	1.877E-03 0.960	9.654E-04 0.981	4.892E-04
$\epsilon = 2^{-4}$	1.266E-02 0.466	9.162E-03 0.495	6.499E-03 0.631	4.197E-03 0.722	2.545E-03 0.781	1.481E-03 0.820	8.388E-04
$\epsilon = 2^{-5}$	2.200E-02 1.248	9.263E-03 0.506	6.521E-03 0.635	4.199E-03 0.723	2.544E-03 0.781	1.480E-03 0.820	8.384E-04
$\epsilon = 2^{-6}$	2.990E-02 1.408	1.127E-02 0.787	6.530E-03 0.637	4.199E-03 0.723	2.543E-03 0.781	1.480E-03 0.820	8.382E-04
$\epsilon = 2^{-7}$	3.123E-02 1.058	1.499E-02 1.194	6.551E-03 0.640	4.204E-03 0.725	2.543E-03 0.781	1.480E-03 0.820	8.382E-04
$\epsilon = 2^{-8}$	3.125E-02 1.001	1.561E-02 1.056	7.510E-03 0.832	4.218E-03 0.728	2.547E-03 0.783	1.480E-03 0.821	8.382E-04
$\epsilon = 2^{-9}$	4.082E-02 1.386	1.562E-02 1.001	7.807E-03 0.879	4.245E-03 0.735	2.550E-03 0.783	1.482E-03 0.821	8.388E-04
$\epsilon = 2^{-10}$	7.276E-02 1.720	2.209E-02 1.499	7.812E-03 0.862	4.299E-03 0.743	2.569E-03 0.790	1.486E-03 0.824	8.397E-04
$\epsilon = 2^{-11}$	7.349E-02 1.281	3.025E-02 1.432	1.121E-02 1.341	4.426E-03 0.778	2.582E-03 0.793	1.490E-03 0.825	8.410E-04
$\epsilon = 2^{-12}$	7.352E-02 1.281	3.025E-02 1.461	1.099E-02 1.330	4.372E-03 0.743	2.611E-03 0.803	1.497E-03 0.826	8.446E-04
$\epsilon = 2^{-13}$	7.354E-02 1.281	3.026E-02 1.461	1.099E-02 1.330	4.372E-03 0.743	2.611E-03 0.797	1.503E-03 0.835	8.428E-04

$\epsilon = 2^{-29}$	7.360E-02 1.282	3.027E-02 1.461	1.100E-02 1.331	4.372E-03 0.743	2.611E-03 0.797	1.503E-03 0.835	8.428E-04
$\epsilon = 2^{-30}$	7.360E-02 1.282	3.027E-02 1.461	1.100E-02 1.331	4.372E-03 0.743	2.611E-03 0.797	1.503E-03 0.835	8.428E-04
$D^{N,M}$	7.360E-02	3.027E-02	1.121E-02	4.426E-03	2.611E-03	1.503E-03	8.446E-04
$P^{N,M}$	1.282	1.433	1.341	0.761	0.797	0.832	

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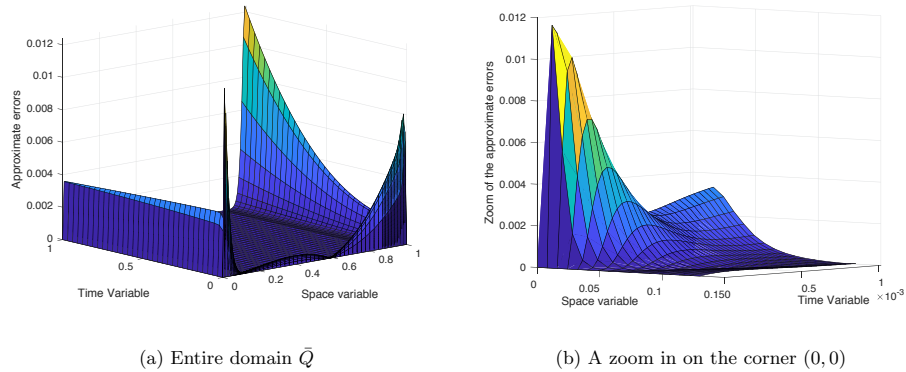


Figure 2: Example (26): Approximate errors $|Y^{64,64} - \bar{Y}^{1024,1024}|$ for problem (27) on the coarsest mesh $\bar{Q}^{64,64}$ with $\varepsilon = 2^{-12}$

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A Appendices

A.1 Compatibility conditions

In this appendix, we explicitly write down the compatibility conditions (see e.g. [8, 13]) of zero, first and second order associated with a singularly perturbed parabolic problem in one space dimension. Consider the following problem: Find $s(x, t)$ such that

$$Ls = \varepsilon(s_t - s_{xx}) + b(x, t)s = g(x, t), \quad (x, t) \in Q, \quad (28a)$$

$$s(0, t) = g_L(t), \quad s(1, t) = g_R(t) \quad t \geq 0, \quad s(x, 0) = \phi(x), \quad 0 < x < 1. \quad (28b)$$

Below we place certain regularity and compatibility restrictions on the data in order that the solution $s \in C^{n+\gamma}(\bar{Q})$, $n = 2, 4$.

Level zero-order compatibility conditions corresponds to:

$$\phi(0^+) = g_L(0) \quad \text{and} \quad \phi(1^-) = g_R(0). \quad (29a)$$

Assuming (29a), we can write

$$\begin{aligned} s(x, t) &= \Phi(x, t) + z(x, t), \quad (x, t) \in Q \quad \text{where} \\ \Phi(x, t) &:= \phi(x) + (1-x)(g_L(t) - g_L(0)) + x(g_R(t) - g_R(0)); \end{aligned}$$

$Lz = g - L\Phi$; and $z(x, t) = 0$, $(x, t) \in \partial Q$. Note that

$$L\Phi = \varepsilon \left((1-x)g'_L(t) + xg'_R(t) - \phi''(x) \right) + b(x, t)\Phi.$$

From [13], if $b, g, L\Phi \in C^{0+\gamma}(\bar{Q})$ and the first-order compatibility conditions

$$\varepsilon(g'_R(0) - \phi''(1^-)) + b(1, 0)\phi(1^-) = g(1, 0) \quad (29b)$$

$$\varepsilon(g'_L(0) - \phi''(0^+)) + b(0, 0)\phi(0^+) = g(0, 0) \quad (29c)$$

are satisfied, then $s \in C^{2+\gamma}(\bar{Q})$. If $b, g, L\Phi \in C^{2+\gamma}(\bar{Q})$ and we further assume second-order compatibility conditions such that

$$(g - L\Phi)_t(0, 0^+) + (g - L\Phi)_{xx}(0^+, 0) = 0; \quad (29d)$$

$$(g - L\Phi)_t(1, 0^+) + (g - L\Phi)_{xx}(1^-, 0) = 0; \quad (29e)$$

then the solution of (28) $s \in C^{4+\gamma}(\bar{Q})$. Note that the constraint (29d) corresponds to

$$\begin{aligned} (g_t + g_{xx})(0, 0) &= \varepsilon g''_L(0) + b(0, 0)g'_L(0) + b_t(0, 0)g_L(0) - \varepsilon \phi^{iv}(0^+) \\ &\quad + 2b_x(0, 0)\phi'(0^+) + b_{xx}(0, 0)\phi(0^+) + b(0, 0)\phi''(0^+). \end{aligned}$$

A.2 Regularity of the function z_n from (6)

Consider the solutions $z_n(x, t)$, $n \geq 0$ of the following problems:

$$\begin{aligned} \frac{\partial z_n}{\partial t} - \frac{\partial^2 z_n}{\partial x^2} + \frac{b(0, 0)}{\varepsilon} z_n &= 0, \quad x > 0, t > 0 \\ z_n(x, 0) &= 0, \quad x \geq 0; \quad z_n(0, t) = t^n e^{-\frac{b(0, 0)t}{\varepsilon}}, \quad t > 0. \end{aligned}$$

Note that

$$z_0(x, t) = \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) e^{-\frac{b(0, 0)t}{\varepsilon}}; \quad \varepsilon \left| \frac{\partial z_0}{\partial t}(x, t) \right| \leq C \frac{1}{t} (\varepsilon + t) e^{-\frac{\beta t}{\varepsilon}};$$

and

$$z_n = n \int_{s=0}^t z_{n-1}(x, s) e^{-\frac{b(0,0)(t-s)}{\varepsilon}} ds; \quad (z_n)_t + \frac{b(0,0)}{\varepsilon} z_n = n z_{n-1}, \quad n \geq 1. \quad (30)$$

Observe that $z_n = v_n e^{-b(0,0)t/\varepsilon}$ and the functions $\{v_n\}_{n=0}^4$ are explicitly given in [4, (5), p.538] and we also note that $(v_n)_{xx} = (v_n)_t = n v_{n-1}$. From a maximum principle, we have the following bounds:

$$|z_n(x, t)| \leq t^n e^{-\frac{b(0,0)t}{\varepsilon}} \leq C \varepsilon^n e^{-\frac{\beta t}{\varepsilon}}; \quad n = 0, 1, 2. \quad (31)$$

Let us list some of the derivatives of the fundamental function $z_0(x, t)$

$$\begin{aligned} \frac{\partial z_0}{\partial x} &= \frac{-1}{\sqrt{\pi t}} e^{-\frac{x^2}{4t}} e^{-\frac{b(0,0)t}{\varepsilon}} = O\left(\frac{1}{\sqrt{t}}\right) \\ \frac{\partial^2 z_0}{\partial x^2} &= \frac{\partial z_0}{\partial t} + \frac{b(0,0)}{\varepsilon} z_0 = \frac{x}{2t\sqrt{\pi t}} e^{-\frac{x^2}{4t}} e^{-\frac{b(0,0)t}{\varepsilon}} = O\left(\frac{1}{t}\right) \\ \frac{\partial^3 z_0}{\partial x^3} &= \frac{1}{2t\sqrt{\pi t}} \left(1 - \frac{x^2}{2t}\right) e^{-\frac{x^2}{4t}} e^{-\frac{b(0,0)t}{\varepsilon}} = O\left(\frac{1}{t\sqrt{t}}\right) \\ \frac{\partial^4 z_0}{\partial x^4} &= \frac{\partial^2 z_0}{\partial t^2} + 2 \frac{b(0,0)}{\varepsilon} \frac{\partial z_0}{\partial t} + \left(\frac{b(0,0)}{\varepsilon}\right)^2 z_0 \\ &= \frac{-x}{4t^2\sqrt{\pi t}} \left(3 - \frac{x^2}{2t}\right) e^{-\frac{x^2}{4t}} e^{-\frac{b(0,0)t}{\varepsilon}} = O\left(\frac{1}{t^2}\right) \\ \frac{\partial z_0}{\partial t} &= O\left(\frac{1}{\varepsilon}\right) z_0 + O\left(\frac{1}{t}\right), \quad \frac{\partial^2 z_0}{\partial t^2} = O\left(\frac{1}{\varepsilon^2}\right) z_0 + O\left(\frac{1}{\varepsilon t}\right) + O\left(\frac{1}{t^2}\right). \end{aligned}$$

Observe that the function

$$h(x, t) := \left(\frac{x}{\sqrt{t}}\right)^p e^{-\frac{x^2}{4t}}, \quad t > 0, \quad h(x, 0) := 0, \quad 0 \leq x \leq 1, \quad p \geq 0;$$

is bounded, but not continuous on \bar{Q} . From this and the explicit expressions for the derivatives of z_0 given above, we deduce² that

$$S_{i,j}(x, t) := x^i t^j z_0 \in C^{n-1+\gamma}(\bar{Q}), \quad n = i + 2j \geq 1.$$

When $i + 2j = 4$, the second derivative in time of these functions $S_{i,j}$ are all bounded on \bar{Q} , but are not continuous. Moreover, for $i + 2j \geq 4$, we have

$$\left| \frac{\partial^{n+m}}{\partial x^n \partial t^m} S_{i,j}(x, t) \right| \leq C \varepsilon^{-m} e^{-\frac{\beta t}{\varepsilon}}, \quad 0 \leq n + 2m \leq 4; \quad (32)$$

²For example,

$$t^2 \frac{\partial^2 z_0}{\partial x^2} \in C^{0+\gamma}(\bar{Q}) \quad \text{and} \quad x^4 \frac{\partial z_0}{\partial t} \in C^{0+\gamma}(\bar{Q}).$$

and for all integers $m, n \geq 1$

$$\begin{aligned} L_0(t^n z_0) &= \varepsilon n t^{n-1} z_0; \\ L_0(t^n x^m z_0) &= \varepsilon (x^m n t^{n-1} z_0 - m(m-1)x^{m-2}t^n z_0 - 2mx^{m-1}t^n (z_0)_x). \end{aligned}$$

Using the recurrence relation (30) and the above properties of z_0 we have that

$$\left| \frac{\partial^2 z_1}{\partial x^2}(x, t) \right| + \left| \frac{\partial z_1}{\partial t}(x, t) \right| \leq C e^{-\frac{\beta t}{\varepsilon}}; \quad (33a)$$

$$\left| \frac{\partial^3 z_1}{\partial x^3}(x, t) \right| \leq C \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}} e^{-\frac{b(0,0)t}{\varepsilon}}; \quad (33b)$$

$$\left| \frac{\partial^4 z_1}{\partial x^4}(x, t) \right| + \varepsilon \left| \frac{\partial^2 z_1}{\partial t^2}(x, t) \right| \leq C \frac{1}{t} (\varepsilon + t) e^{-\frac{\beta t}{\varepsilon}}; \quad (33c)$$

$$\left| \frac{\partial^4 z_2}{\partial x^4}(x, t) \right| + \left| \frac{\partial^2 z_2}{\partial t^2}(x, t) \right| \leq C e^{-\frac{\beta t}{\varepsilon}}. \quad (33d)$$

Note also that

$$\begin{aligned} \left| \frac{\partial^2 z_0}{\partial t^2}(x, t) \right| &\leq \frac{C}{(\varepsilon + t)^2} e^{-\frac{b(0,0)t}{\varepsilon}}; \quad \left| \frac{\partial^2 z_1}{\partial t^2}(x, t) \right| \leq \frac{C}{t} e^{-\frac{b(0,0)t}{\varepsilon}}; \\ \left| \frac{\partial^2 z_2}{\partial t^2}(x, t) \right| &\leq C e^{-\frac{b(0,0)t}{\varepsilon}}. \end{aligned}$$

A.3 Regularity of the function y_2 from Theorem 3

Recall that the solution u of problem (1) is discontinuous at the point $(0, 0)$ and by subtracting the discontinuous function $A_0 z_0$, we see that $y = u - A_0 z_0$ satisfies zero order compatibility at the point $(0, 0)$. Hence the solution y of problem (7) is a continuous function. By subtracted off appropriate multiples A_n of z_n (see (30)) from u we can satisfy up to the n^{th} order compatibility conditions at the point $(0, 0)$. Since $L_0 z_n = 0$, one can check that

$$L(u - A_0 z_0), L(u - A_0 z_0 - A_1 z_1) \in C^{0+\gamma}(\bar{Q})$$

and this implies that $(u - A_0 z_0) \in C^{2+\gamma}(\bar{Q})$. Thus $y \in C^0(\bar{Q})$ and $(y - A_1 z_1) \in C^{2+\gamma}(\bar{Q})$, but $(y - A_1 z_1 - A_2 z_2) \notin C^{4+\gamma}(\bar{Q})$. We now define the following two functions

$$y_1 := y - A_1 z_1 \quad \text{and} \quad y_2 := y_1 - A_2 z_2 - A_0 \Psi;$$

where

$$\begin{aligned} \Psi(x, t) &:= b_t(0, 0)\chi_{0,1} + \frac{b_{xx}(0, 0)}{2}\chi_{2,0} + b_{xt}(0, 0)\chi_{1,1} + \frac{b_{xxx}(0, 0)}{6}\chi_{3,0}; \quad (34) \\ L_0\Psi &= \left(b_t(0, 0)t + \frac{b_{xx}(0, 0)}{2}x^2 + b_{xt}(0, 0)xt + \frac{b_{xxx}(0, 0)}{6}x^3 \right) z_0, \end{aligned}$$

and the set of functions, $\{\chi_{i,j}\}$ are defined to be

$$L_0\chi_{i,j} = x^i t^j z_0(x, t); \quad i, j = 0, 1, 2, \dots$$

The first few functions in the set $\{\chi_{i,j}\}$ are explicitly given as

$$\varepsilon\chi_{1,0} = xt z_0 + t^2(z_0)_x \quad \text{and} \quad |\chi_{1,0}(0, t)| \leq C\sqrt{t}e^{-\frac{\beta t}{\varepsilon}}; \quad (35a)$$

$$\varepsilon\chi_{0,1} = \frac{t^2 z_0}{2} \quad \text{and} \quad |\chi_{0,1}(0, t)| \leq Ct e^{-\frac{\beta t}{\varepsilon}}; \quad (35b)$$

$$\varepsilon\chi_{2,0} = (x^2 t + t^2)z_0 + 2xt^2(z_0)_x + (4/3)t^3(z_0)_{xx}, \quad (35c)$$

$$\text{and } \varepsilon\chi_{2,0}(0, t) = t^2 e^{-\frac{b(0,0)t}{\varepsilon}};$$

$$\varepsilon\chi_{1,1} = \frac{3xt^2 z_0 + 2t^3(z_0)_x}{6} \quad \text{and} \quad |\chi_{1,1}(0, t)| \leq Ct\sqrt{t}e^{-\frac{\beta t}{\varepsilon}}, \quad (35d)$$

$$\varepsilon\chi_{3,0} = (x^3 t + 3xt^2)z_0 + (4t^3 + 3x^2 t^2)(z_0)_x + 4xt^3(z_0)_{xx} + 2t^4(z_0)_{xxx},$$

$$\text{and } |\chi_{3,0}(0, t)| \leq Ct\sqrt{t}e^{-\frac{\beta t}{\varepsilon}}.$$

We shall see below that the additional term $A_0\Psi$ has been included so that $y_2 \in C^{4+\gamma}(\bar{Q})$. The amplitude A_0 has been specified in (5). Below we specify the amplitudes A_1 and A_2 . Observe that the function y_1 satisfies

$$\begin{aligned} Ly_1 &= f - (b(x, t) - b(0, 0))(A_1 z_1 + A_0 z_0); \\ y_1(0, t) &= g_L(t) - (A_0 z_0 + A_1 z_1)(0, t); \quad y_1(1, t) = g_R(t) - (A_0 z_0 + A_1 z_1)(1, t); \\ y_1(x, 0) &= \phi(x). \end{aligned}$$

From (29c) in the first Appendix, first order compatibility is satisfied (for y_1) if A_1 is such that

$$f(0, 0) = \varepsilon(g'_L(0) - A_1 - \phi''(0^+)) + b(0, 0)(A_0 + \phi(0^+)). \quad (36)$$

In general, $A_1 = O(\varepsilon^{-1})$. Since $Ly_1 \in C^{0+\gamma}(\bar{Q})$, then $y_1 \in C^{2+\gamma}(\bar{Q})$.

Next we move onto the regularity of y_2 . Note first that, since $b_x(0, 0) = 0$,

$$(L - L_0)z_n = \left(b_t(0, 0)t + \frac{b_{xx}(0, 0)}{2}x^2 + b_{xt}(0, 0)xt + \frac{b_{xxx}(0, 0)}{6}x^3 \right) z_n + \text{H.O.T.}$$

(H.O.T. stands for high order terms). Hence,

$$\begin{aligned}
Ly_2 &= Ly_1 - (L - L_0)(A_2z_2 + A_0\Psi) - A_0L_0\Psi \\
&= f - (L - L_0)(A_1z_1 + A_0z_0) - A_0L_0\Psi - (b(x, t) - b(0, 0))(A_2z_2 + A_0\Psi) \\
&= f + O(t^2 + x^4 + x^2t)A_0z_0 - (L - L_0)(A_1z_1) - (b(x, t) - b(0, 0))(A_2z_2 + A_0\Psi) \\
&= f + O(t^2 + x^4 + x^2t)A_0z_0 + O(t + x^2)A_2z_2 + O(t + x^2)A_1z_1 \\
&\quad + \frac{A_0}{\varepsilon}O(t + x^2)(O(x^2t + t^2)z_0 + O(xt^2 + t^3)(z_0)_x + O(t^3)(z_0)_{xx} + O(t^4)(z_0)_{xxx})
\end{aligned}$$

and the boundary and initial conditions are

$$\begin{aligned}
y_2(0, t) &= g_L(t) - (A_0z_0 + A_1z_1 + A_2z_2 + A_0\Psi)(0, t) \\
&= g_L(t) - (A_0 + A_1t + A_2t^2) e^{-\frac{b(0,0)t}{\varepsilon}} - A_0\frac{t^2}{2\varepsilon}b_t(0, 0)e^{-\frac{b(0,0)t}{\varepsilon}} \\
&\quad - A_0\frac{t^2}{2\varepsilon} \left(b_{xx}(0, 0) \left(z_0 + \frac{4t}{3} \frac{\partial^2 z_0}{\partial x^2} \right) + \frac{2b_{xt}(0, 0)t}{3} \frac{\partial z_0}{\partial x} \right. \\
&\quad \left. + \frac{2b_{xxx}(0, 0)}{3} \left(2t \frac{\partial z_0}{\partial x} + t^2 \frac{\partial^3 z_0}{\partial x^3} \right) \right) (0, t); \\
y_2(1, t) &= y(1, t) - (A_1z_1 + A_2z_2 + A_0\Psi)(1, t); \quad y_2(x, 0) = \phi(x).
\end{aligned}$$

Note that $\lim_{t \rightarrow 0^+} y_2(0, t) = g_L(0) - A_0$ and

$$\begin{aligned}
\lim_{t \rightarrow 0^+} \frac{\partial y_2(0, t)}{\partial t} &= g'_L(0) - A_1 + A_0 \frac{b(0, 0)}{\varepsilon} = \frac{f(0, 0) - b(0, 0)\phi(0^+) + \varepsilon\phi''(0^+)}{\varepsilon}, \\
\lim_{t \rightarrow 0^+} \frac{\partial^2 y_2(0, t)}{\partial t^2} &= g''_L(0) + 2\varepsilon^{-1}A_1b(0, 0) - 2A_2 - 2\varepsilon^{-1}A_0(b_t(0, 0) + b_{xx}(0, 0)) \\
&\quad - A_0 \left(\frac{b(0, 0)}{\varepsilon} \right)^2.
\end{aligned}$$

First order compatibility is satisfied (for y_2) if A_1 is such that (36) is satisfied and the above construction has been designed in order that $Ly_2 \in C^{2+\gamma}(\bar{Q})$. Finally second order compatibility is satisfied (for y_2) if A_2 is such that

$$\begin{aligned}
&\varepsilon(g''_L(0) - \phi^{iv}(0^+)) + (A_1 + g'_L(0))b(0, 0) - 2\varepsilon A_2 - 2A_0(b_t + b_{xx})(0, 0) \\
&\quad + b_t(0, 0)(g_L(0) - A_0) + b_{xx}(0, 0)\phi(0^+) + b(0, 0)\phi''(0^+) \\
&= (f_t + f_{xx})(0, 0). \tag{37}
\end{aligned}$$

Note that $A_0 = O(1)$, $A_1 = O(\varepsilon^{-1})$ and $A_2 = O(\varepsilon^{-2})$. By this construction we have that $y_2 \in C^{4+\gamma}(\bar{Q})$. Moreover,

$$\left| \frac{\partial^{i+j}}{\partial x^i \partial t^j} (Ly_2(x, t)) \right| \leq C\varepsilon^{-j} e^{-\frac{\beta t}{\varepsilon}}, \quad 0 \leq i + 2j \leq 2. \tag{38}$$