Research Article

Accurate bidiagonal decomposition of collocation matrices of weighted φ-transformed systems †

E. Mainar,[1] J.M. Peña,[1] and B. Rubio[1]

[1] *Departamento de Matemática Aplicada/IUMA, Universidad de Zaragoza, Zaragoza, Spain*

Correspondence: *Corresponding author name, B. Rubio. Email: brubio@unizar.es

Summary

Given a system of functions, we introduce the concept of weighted φ-transformed system, which will include a very large class of useful representations in Statistics and Computer Aided Geometric Design. An accurate bidiagonal decomposition of the collocation matrices of these systems is obtained. This decomposition is used to present computational methods with high relative accuracy for solving algebraic problems with collocation matrices of weighted φ-transformed systems such as the computation of eigenvalues, singular values and the solution of some linear systems. Numerical examples illustrate the accuracy of the performed computations.

Keywords: Bidiagonal decompositions, Accurate computations, Rational basis.

1 Introduction

In this paper we present a very general procedure for generating, from an initial system and a positive function φ, new systems of functions useful for many applications. These systems, which we call weighted φ-transformed systems, arise with relevant probability distributions. They also include important rational bases (see [25], [14]) as well as systems belonging to spaces mixing algebraic, trigonometric and hyperbolic polynomials, which are useful in many applications of Approximation Theory and Computer Aided Geometric Design (CAGD). The weighted φ-transformed systems inherit from the initial system its accuracy when computing with its collocation matrices.

The accurate computation with structured classes of matrices is an important issue in Numerical Linear Algebra and it is receiving increasing attention in the recent years (cf. [6, 10, 21]). For this purpose, a parametrization adapted to the structure of the considered matrices is needed. Let us recall that an algorithm can be performed with high

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relative accuracy (HRA) if it does not include subtractions (except of the initial data), that is, if it only includes products, divisions, sums of numbers of the same sign and subtractions of the initial data (cf. [15]). Performing an algorithm with HRA is a very desirable goal because it implies that the relative errors of the computations are of the order of the machine precision, independently of the size of the condition number of the considered problem. Let us recall that a totally positive (TP) matrix has all its minors nonnegative. TP matrices arise in many applications (cf. [1]). It is known that, for some subclasses of TP matrices, many algebraic computations can be performed with HRA. For instance, the computation of their eigenvalues, singular values or the solutions of linear systems $Ax = b$ such that the components of b have alternating signs (see [9] and the references therein). The key tool for this purpose is provided by the algorithms of [15, 16] jointly with the use of a bidiagonal factorization of a nonsingular TP matrix, which can be obtained with HRA for some of those matrices. Up to now, this has been achieved with some relevant subclasses of TP matrices with applications to CAGD (cf. [7, 18, 20, 21]), to Finance (cf. [8]) or to Combinatorics (cf. [6]). In the case of CAGD, the importance of TP matrices comes from the fact that the normalized systems whose collocation matrices are TP provide shape preserving representations [3, 23]. In [18] we presented many important bases used in CAGD whose collocation matrices admit many computations with HRA.

In Section 3, we extend the analysis of [18] to the more general framework of this paper and assure that the algebraic computations mentioned above can be performed with HRA for the collocation matrices of weighted φ-transformed systems, assuming that the bidiagonal factorization of the corresponding collocation matrix of the initial system can be obtained with HRA and that the evaluation of φ does not requires subtractions up to initial data. Our numerical examples will illustrate that the solution of linear systems and the computation of eigenvalues and singular values can be solved accurately even when the above conditions do not hold. In particular, our results can be applied to perform interpolation with high precision.

The layout of the paper is as follows. Section 2 includes matrix notations and basic concepts. We recall the Neville elimination procedure, which allows us to introduce the bidiagonal factorization of a strictly totally positive matrix. Section 3 introduces the weighted φ-transformed systems and includes the results guaranteeing their nice computational properties. The bidiagonal factorization of the collocation matrices of the weighted φ-transformed systems is obtained. Section 4 includes many examples of weighted φ-transformed systems related to probabilistic distributions. Section 5 shows a class of rational spaces that can be generated by weighted φ-transformed systems. Curves generated by these weighted φ-transformed systems inherit geometric properties and algorithms of the traditional rational Bézier curves and so they can be considered as modeling tools in CAD/CAM systems. Finally, Section 6 includes numerical examples showing the accurate computation of eigenvalues and singular values and accurate solutions of linear systems associated to the collocation matrices of weighted φ-transformed systems.

2 Basic notations and auxiliary results

A matrix is *totally positive* (TP) if all its minors are nonnegative and *strictly totally positive* (STP) if they are positive (see [1]). Let us now recall some basic matrix notations and results on Neville elimination. Our notation follows the notation used in [11, 12]. Given $n \in \mathbb{N}$ and $k \in \{1,\dots,n\}$, let $Q_{k,n}$ be the set of increasing sequences of k positive integers less than or equal to n. If $\alpha,\beta \in Q_{k,n}$, we denote by $A[\alpha|\beta]$ the $k\times k$ submatrix of A containing rows of places α and columns of places β. Neville elimination (see [11, 12]), is a procedure to make zeros in a column of a matrix by adding to a given row an appropriate multiple of the previous one. For a given nonsingular matrix $A=(a_{i,j})_{1\le i,j\le n}$, this elimination procedure consists of at most $n-1$ successive major steps, resulting in the sequence of matrices:

$$A^{(1)} := A \to A^{(2)} \to \cdots \to A^{(n)} = U.$$

For $1 \le k \le n-1$, $A^{(k+1)} = (a_{i,j}^{(k+1)})_{1\le i,j\le n}$ is obtained from $A^{(k)} = (a_{i,j}^{(k)})_{1\le i,j\le n}$ by defining

$$\begin{cases} a_{i,k}^{(k+1)} := 0, & i = k+1,\dots,n, \\ a_{i,j}^{(k+1)} := a_{i,j}^{(k)} - \frac{a_{i,k}^{(k)}}{a_{i-1,k}^{(k)}} a_{i-1,j}^{(k)} & \text{if } a_{i-1,k}^{(k)} \neq 0, \quad k+1 \le i,j \le n, \end{cases}$$

so that $A^{(k+1)}$ has zeros below its main diagonal in the k first columns. Finally, U is an upper triangular matrix. The element $p_{i,j} := a_{i,j}^{(j)}$, is called the (i,j) pivot of the Neville elimination of A for $1 \le j \le i \le n$. The pivots $p_{i,i}$ are called diagonal pivots. The Neville elimination can be performed without row exchanges if all the pivots are nonzero and, in this case, Lemma 2.6 of [11] implies that $p_{i,1} = a_{i,1}$, for $1 \le i \le n$, and

$$p_{i,j} = \frac{\det A[i-j+1,\dots,i|1,\dots,j]}{\det A[i-j+1,\dots,i-1|1,\dots,j-1]}, \quad 1 < j \le i \le n. \tag{1}$$

Furthermore, the (i,j) multiplier of the Neville elimination of A is

$$m_{i,j} := \frac{a_{i,j}^{(j)}}{a_{i-1,j}^{(j)}} = \frac{p_{i,j}}{p_{i-1,j}}, \quad 1 \le j < i \le n. \tag{2}$$

Neville elimination has been used to characterize TP and STP matrices (see [11, 12]). From Theorem 4.1 of [11] and p. 116 of [12] (see also Theorem 2.1 of [5]), a given matrix A is STP if and only if the Neville elimination of A and A^T can be performed without row exchanges, all the multipliers of the Neville elimination of A and A^T are positive and all the diagonal pivots of the Neville elimination of A are positive.

Bidiagonal factorizations have played a crucial role to derive for TP matrices algorithms with HRA (cf. [15]). According to the arguments of p.116 of [12], an STP matrix $A \in \mathbb{R}^{(n+1)\times(n+1)}$ can be factorized in the form

$$A = F_n F_{n-1} \cdots F_1 D G_1 \cdots G_{n-1} G_n, \tag{3}$$

where F_i and G_i are the lower and upper triangular bidiagonal matrices

$$F_i = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & m_{i+1,1} & 1 & & \\ & & & \ddots & \ddots & \\ & & & & m_{n+1,n+1-i} & 1 \end{pmatrix}, \quad G_i^T = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & \widehat{m}_{i+1,1} & 1 & & \\ & & & \ddots & \ddots & \\ & & & & \widehat{m}_{n+1,n+1-i} & 1 \end{pmatrix}, \tag{4}$$

and $D = \operatorname{diag}(p_{1,1}, \ldots, p_{n+1,n+1})$. The entries $m_{i,j}$ and $\widehat{m}_{i,j}$ are the multipliers of the Neville elimination of A and A^T, respectively, and the diagonal entries $p_{i,i}$ are the diagonal pivots of the Neville elimination of A.

3 Weighted φ-transformed systems

Let us first introduce a key concept of this paper. Let $(u_0, \ldots, u_n)$ be a system of functions defined on $I = [a, b]$, $\varphi : [a, b] \to \mathbb{R}$ a positive function and $d_0, \ldots, d_n$ positive real values. The corresponding *weighted φ-transformed* system from $(u_0, \ldots, u_n)$ is the system $(\widetilde{u}_0, \ldots, \widetilde{u}_n)$ of functions defined by

$$\widetilde{u}_i(t) := d_i \varphi(t) u_i(t), \quad t \in [a, b], \quad i = 0, \ldots, n. \tag{5}$$

Let us suppose that $(u_0, \ldots, u_n)$ is a system of functions defined on $I = [a, b]$ and $a < t_1 < \cdots < t_{n+1} < b$ is a sequence of nodes such that the corresponding collocation matrix

$$A := \big(u_{j-1}(t_i)\big)_{1 \le i,j \le n+1} \tag{6}$$

is STP. Let

$$A = F_n F_{n-1} \cdots F_1 D G_1 \cdots G_{n-1} G_n \tag{7}$$

be the bidiagonal factorization (3) such that F_i and G_i are the lower and upper triangular bidiagonal matrices of the form (4) and D is a diagonal matrix.

The following result proves that the collocation matrix of the corresponding weighted φ-transformed system $(\widetilde{u}_0, \ldots, \widetilde{u}_n)$ at nodes $a < t_1 < \cdots < t_{n+1} < b$

$$\widetilde{A} := \big(\widetilde{u}_{j-1}(t_i)\big)_{1\le i,j\le n+1} = \big(d_{j-1}\varphi(t_i)u_{j-1}(t_i)\big)_{1\le i,j\le n+1} \tag{8}$$

is also STP and obtains its bidiagonal factorization (3) from the factorization (7) of the collocation matrix A given in (6).

Theorem 1. *The collocation matrix* (8) *is STP and it can be factorized as*

$$\widetilde{A} = \widetilde{F}_n \widetilde{F}_{n-1} \cdots \widetilde{F}_1 \widetilde{D} \widetilde{G}_1 \cdots \widetilde{G}_{n-1} \widetilde{G}_n, \tag{9}$$

where $\widetilde{\mathrm{F}}_\mathrm{i}$ *and* $\widetilde{\mathrm{G}}_\mathrm{i}$ *are the lower and upper bidiagonal matrices of the form*

$$\widetilde{F}_i = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & \mathrm{r_{i+1,1}} & 1 & & \\ & & & \ddots & \ddots & \\ & & & & \mathrm{r_{n+1,n+1-i}} & 1 \end{pmatrix}, \quad \widetilde{G}_i^T = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & \widehat{\mathrm{r}}_{\mathrm{i+1,1}} & 1 & & \\ & & & \ddots & \ddots & \\ & & & & \widehat{\mathrm{r}}_{\mathrm{n+1,n+1-i}} & 1 \end{pmatrix} \tag{10}$$

and $\widetilde{\mathrm{D}} = \mathrm{diag}\,(\mathrm{q_{1,1}}, \ldots, \mathrm{q_{n+1,n+1}})$. *The entries* $\mathrm{r_{i,j}}, \widehat{\mathrm{r}}_{\mathrm{i,j}}$ *and* $\mathrm{q_{i,i}}$ *are given by*

$$\mathrm{r_{i,j}} = \frac{\varphi(\mathrm{t_i})}{\varphi(\mathrm{t_{i-1}})}\mathrm{m_{i,j}}, \quad \widehat{\mathrm{r}}_{\mathrm{i,j}} = \frac{\mathrm{d_{i-1}}}{\mathrm{d_{i-2}}}\widehat{\mathrm{m}}_{\mathrm{i,j}}, \quad 1 \le \mathrm{j} < \mathrm{i} \le \mathrm{n}+1,$$

$$\mathrm{q_{i,i}} = \mathrm{d_{i-1}}\varphi(\mathrm{t_i})\mathrm{p_{i,i}}, \quad 1 \le \mathrm{i} \le \mathrm{n}+1, \tag{11}$$

where $\mathrm{m_{i,j}}$, $\widehat{\mathrm{m}}_{\mathrm{i,j}}$ *and* $\mathrm{p_{i,i}}$ *are the entries of the matrices of the bidiagonal factorization* (7) *of the collocation matrix* A *defined in* (6).

Proof. *Observe that*

$$\widetilde{A} = \mathrm{diag}\,(\varphi(t_1), \ldots, \varphi(t_{n+1}))\, A\, \mathrm{diag}\,(d_0, \ldots, d_n)\,.$$

So, taking into account the positivity of φ and the coefficients $d_0,\dots,d_n$, we deduce that $\operatorname{diag}(\varphi(t_1),\dots,\varphi(t_{n+1}))$ *and* $\operatorname{diag}(d_0,\dots,d_n)$ *are nonsingular and TP matrices. Since A is STP and, by Theorem 3.1 of* [1]*, the product of STP matrices by a nonsingular TP matrix is a STP matrix, we conclude that $\widetilde{A}$ is STP. In order to compute the pivots and the multipliers of the Neville elimination of $\widetilde{A}$ we need to obtain its minors with j initial consecutive columns and j consecutive rows starting with row $i-j+1$.*

Let $1\le j\le i\le n+1$. For any $1\le k\le j$, each entry of the k-th row of the matrix $\widetilde{A}[i-j+1,\dots,i|1,\dots,j]$ has as common factor $\varphi(t_{i-j+k})$ and each entry of the k-th column of the matrix $\widetilde{A}[i-j+1,\dots,i|1,\dots,j]$ has as common factor d_{k-1}. Therefore we can write

$$\widetilde{A}[i-j+1,\dots,i|1,\dots,j]=D_1A[i-j+1,\dots,i|1,\dots,j]D_2,$$

where $D_1:=\operatorname{diag}(\varphi(t_{i-j+1}),\dots,\varphi(t_i))$ *and* $D_2:=\operatorname{diag}(d_0,\dots,d_{j-1})$*. Using properties of determinants we can write*

$$\det\widetilde{A}[i-j+1,\dots,i|1,\dots,j]=\prod_{k=1}^{j}\left(d_{k-1}\varphi(t_{i-j+k})\right)\det A[i-j+1,\dots,i|1,\dots,j].$$

Let us denote by $\widetilde{p}_{i,j}$ the pivot obtained in the Neville elimination procedure of $\widetilde{A}$. Taking into account the previous formula and (1)*, we deduce that*

$$\widetilde{p}_{i,j}=\frac{\det\widetilde{A}[i-j+1,\dots,i|1,\dots,j]}{\det\widetilde{A}[i-j+1,\dots,i-1|1,\dots,j-1]}=d_{j-1}\varphi(t_i)p_{i,j},\tag{12}$$

where $p_{i,j}$ is the pivot obtained in the Neville elimination procedure of the matrix A. Observe that, for the particular case $i=j$, we have

$$q_{i,i}=d_{i-1}\varphi(t_i)p_{i,i},\quad 1\le i\le n+1.$$

Finally, from formulae (12) *and* (2)*,*

$$r_{i,j}=\frac{\widetilde{p}_{i,j}}{\widetilde{p}_{i-1,j}}=\frac{d_{j-1}\varphi(t_i)p_{i,j}}{d_{j-1}\varphi(t_{i-1})p_{i-1,j}}=\frac{\varphi(t_i)}{\varphi(t_{i-1})}m_{i,j},\quad 1\le j<i\le n+1.$$

Analogously, for any $1\le k\le j$, each entry of the k-th row of $\widetilde{A}^T$ has as common factor d_{k-1} and each entry of the k-th column of the matrix $\widetilde{A}^T$ has as common factor $\varphi(t_k)$. Then

$$\widetilde{A}^T[i-j+1,\dots,i|1,\dots,j]=\widehat{D}_1A^T[i-j+1,\dots,i|1,\dots,j]\widehat{D}_2,$$

where $\widehat{D}_1 := \operatorname{diag}(d_{i-j}, \dots, d_{i-1})$ *and* $\widehat{D}_2 := \operatorname{diag}(\varphi(t_1), \dots, \varphi(t_j))$. *Using properties of determinants we can write*

$$\det \widetilde{A}^T[i-j+1, \dots, i|1, \dots, j] = \prod_{k=1}^{j} d_{i-k}\varphi(t_k) \det A^T[i-j+1, \dots, i|1, \dots, j].$$

Taking into account the previous formula, (1) *and* (2) *we deduce that*

$$\frac{\det \widetilde{A}^T[i-j+1, \dots, i|1, \dots, j]}{\det \widetilde{A}^T[i-j+1, \dots, i-1|1, \dots, j-1]} = d_{i-1}\varphi(t_j)p_{i,j}.$$

Finally, taking into account (2)*, we conclude*

$$\widehat{r}_{i,j} = \frac{d_{i-1}}{d_{i-2}}\widehat{m}_{i,j}, \quad 1 \le j < i \le n+1.$$

□

We say that a system of functions $(u_0, \dots, u_n)$ defined on $I = [a, b]$ is TP (respectively, STP) if all its collocation matrices (6) in I are TP (respectively, STP). As a consequence of the previous result we have that weighted φ-transformed systems inherit the property of being STP, as stated in the following result.

Corollary 1. *Let* $(\mathrm{u}_0, \dots, \mathrm{u}_\mathrm{n})$ *be a STP system of functions defined on* $\mathrm{I} = [\mathrm{a}, \mathrm{b}]$. *Then any weighted* φ*-transformed system* $(\widetilde{\mathrm{u}}_0, \dots, \widetilde{\mathrm{u}}_\mathrm{n})$ *given by (5) is STP.*

Remark 1. *Observe that, if the evaluation of* φ *can be performed by arithmetic operations and it does not require subtractions (except for the initial data), the entries of the bidiagonal factorization of Theorem 1 can be obtained from the bidiagonal factorization of (7) without performing subtractions. Therefore, if the bidiagonal factorization of (7) can be performed with HRA, then the bidiagonal factorization of Theorem 1 can be also performed with HRA. It is known that the bidiagonal factorization* (3) *of the collocation matrices associated to some important bases used in CAGD can be performed with HRA (see* [18]*). In consequence, the bidiagonal factorization of the collocation matrices of their corresponding weighted* φ*-transformed systems can also be performed with HRA and we can apply the algorithms presented in* [15] *and* [16] *to perform many algebraic computations with HRA. For instance, the computation of their eigenvalues, singular values or the solutions of some linear systems associated to these collocation matrices.*

4 Weighted φ-transformed probability distributions

A probability distribution is a mathematical function that provides the probabilities of occurrence of different possible results in an experiment. In this section we are going to see some interesting bases that can be defined from probability distributions and can be considered as weighted φ-transformed systems from other bases whose collocation matrices are STP and can be factorized as in (9).

The *binomial distribution* is frequently used to model the number of successes in a sample of size n. If the probability of success is $t \in [0,1]$, $n \in \mathbb{N}$ is the number of trials and $k \in \mathbb{N}$ is the number of successes, then the probability of getting exactly k successes in n trials is given by

$$P(k \text{ successes in } n \text{ trials}) = \binom{n}{k} t^k (1-t)^{n-k}, \quad k = 0,1,\ldots,n.$$

The binomial functions coincide with the Bernstein polynomials of degree n,

$$B_k^n(t) := \binom{n}{k} t^k (1-t)^{n-k}, \quad t \in [0,1], \quad k = 0,1,\ldots,n.$$

The collocation matrix of the Bernstein basis $(B_0^n, \ldots, B_n^n)$ for any sequence of parameters $0 < t_1 < \cdots < t_{n+1} < 1$ is STP and its corresponding bidiagonal factorization (3) - (4) is given by

$$\begin{aligned}
m_{i,j} &= \frac{(1-t_i)^{n-j+1}(1-t_{i-j})}{(1-t_{i-1})^{n-j+2}} \frac{\prod_{k=1}^{j-1}\left(t_i - t_{i-k}\right)}{\prod_{k=2}^{j}\left(t_{i-1} - t_{i-k}\right)}, \quad 1 \le j < i \le n+1, \\
\widehat{m}_{i,j} &= \frac{n-i+2}{i-1} \frac{t_j}{(1-t_j)}, \quad 1 \le j < i \le n+1, \\
p_{i,i} &= \binom{n}{i-1} \frac{(1-t_i)^{n-i+1}}{\prod_{k=1}^{i-1}(1-t_k)} \prod_{k=1}^{i-1}\left(t_i - t_k\right), \quad 1 \le i \le n+1
\end{aligned} \tag{13}$$

(see [19] or Theorem 3 of [18]). The bidiagonal factorization of the collocation matrix of the Bernstein basis can be performed with HRA. In [20] accurate computations to solve algebraic problems associated to collocation matrices of Bernstein bases are shown. Urn models extend the binomial distribution. In [13] it is shown the connection between these models and CAGD and Approximation Theory, in particular with splines.

The *negative binomial distribution* is an appropriate model to treat those processes in which a certain trial is repeated until a certain number of favorable results are achieved for the first time. If the probability of failure is $t \in [0,1]$, r is the number of failures and k is the number of successes, then the probability of r failures up to obtain k successes (at least 1 success) is given by

$$P(r \text{ failures up to k successes}) = \binom{k+r-1}{r} t^r (1-t)^k.$$

Let us observe that, if $n = k + r - 1$, the negative binomial basis $(b_0, \dots, b_n)$ defined by $b_r(t) := \binom{n}{r} t^r (1-t)^{n-r+1}$, $r = 0, \dots, n$, can be considered as a weighted φ-transformed system from the Bernstein basis with $\varphi(t) = 1 - t$ and $d_i = 1$ for $i = 0, \dots, n$. Then, using Corollary 1, we deduce that the negative binomial basis is STP on $(0, 1)$ and, using Theorem 1 and the bidiagonal factorization (3),(4),(13) of the collocation matrix of the Bernstein basis at $0 < t_1 < \cdots < t_{n+1} < 1$, the corresponding bidiagonal factorization of the collocation matrix of the negative binomial basis is given by

$$\begin{aligned}
r_{i,j} &= \frac{(1-t_i)^{n-j+2}(1-t_{i-j})}{(1-t_{i-1})^{n-j+3}} \frac{\prod_{k=1}^{j-1} (t_i - t_{i-k})}{\prod_{k=2}^{j} (t_{i-1} - t_{i-k})}, \quad 1 \le j < i \le n+1, \\
\widehat{r}_{i,j} &= \frac{n-i+2}{i-1} \frac{t_j}{(1-t_j)}, \quad 1 \le j < i \le n+1, \\
q_{i,i} &= \binom{n}{i-1} \frac{(1-t_i)^{n-i+2}}{\prod_{k=1}^{i-1}(1-t_k)} \prod_{k=1}^{i-1} (t_i - t_k), \quad 1 \le i \le n+1.
\end{aligned} \tag{14}$$

Let us remark that the evaluation of $\varphi(t) = 1 - t$ does not include subtractions (except for the initial data). So, the bidiagonal factorization (3),(4),(14) of the collocation matrix of the negative binomial basis can be also performed with HRA. Section 6 will show accurate results obtained when computing their eigenvalues, singular values or the solutions of some linear systems associated to these collocation matrices, using the bidiagonal factorization (3),(4),(14) and the algorithms presented in [15] and [16].

The *geometric distribution* has applications in population and econometric models. If the probability of success is $t \in [0, 1]$ and k is the number of failures, then the probability of k failures up to obtain a success is given by

$$P(k \text{ failures until a success}) := (1-t)^k t.$$

For a given $n \in \mathbb{N}$, the geometric basis functions $b_k(t) := (1-t)^k t$, $k = 0, \dots, n$ can be considered as a weighted φ-transformed system from the basis $(1, 1-t, \dots, (1-t)^n)$ with $\varphi(t) = t$ and $d_i = 1$ for $i = 0, \dots, n$. The monomial basis $(1, t, \dots, t^n)$ is STP on $(0, \infty)$ and the bidiagonal factorization (3),(4) of its collocation matrix at $0 < t_0 < \ldots < t_{n+1} < 1$ is given by

$$\begin{aligned}
m_{i,j} &= \frac{\prod_{k=1}^{j-1}(t_i - t_{i-k})}{\prod_{k=2}^{j}(t_{i-1} - t_{i-k})}, \quad \widehat{m}_{i,j} = t_j, \quad 1 \le j < i \le n+1, \\
p_{i,i} &= \prod_{k=1}^{i-1} (t_i - t_k), \quad 1 \le i \le n+1
\end{aligned} \tag{15}$$

(see [17], [20] or Theorem 3 of [18]).

Using (15) and Theorem 1, it can be deduced that the bidiagonal decomposition of the collocation matrix of the geometric basis $(b_0, \ldots, b_n)$ is given by

$$r_{i,j} = \frac{t_i}{t_{i-1}} \frac{\prod_{k=1}^{j-1}(t_{i-k} - t_i)}{\prod_{k=2}^{j}(t_{i-k} - t_{i-1})}, \quad \widehat{r}_{i,j} = 1 - t_j, \quad 1 \leq j < i \leq n+1,$$

$$q_{i,i} = t_i \prod_{k=1}^{i-1}(t_k - t_i), \quad 1 \leq i \leq n+1. \tag{16}$$

Since the evaluation of $\varphi(t) = t$ does not include subtractions, the bidiagonal factorization (3),(4),(16) of the collocation matrix of the geometric basis for any sequence of parameters $0 < t_{n+1} < t_n < \ldots < t_1 < 1$ can be also performed with HRA. Section 6 will show accurate results obtained when computing their eigenvalues, singular values or the solutions of some linear systems associated to these collocation matrices using the bidiagonal factorization (3),(4),(16) and the algorithms presented in [15] and [16].

The *Poisson distribution* is popular for modelling the number of times an event occurs in an interval of time or space. An event can occur $k = 0, 1, 2, \ldots$ times in an interval. If the average number of events in an interval, also called the rate parameter, is designated by t, then the probability of observing k events in an interval is given by

$$P(k \text{ events in interval}) = \frac{t^k}{k!} e^{-t}.$$

The Poisson basis functions $b_k(t) := \frac{t^k}{k!} e^{-t}$, $k \in \mathbb{N}$, are the limit as n tends to infinity of the Bernstein basis of degree n over the interval $[0, n]$, that is,

$$b_k(t) = \lim_{n\to\infty} B_k^n\left(t/n\right), \quad B_k^n\left(t\right) = \binom{n}{k} t^k (1-t)^{n-k}, \quad t \in [0, 1],$$

and they also play a useful role in CAGD (see [22]). For a given $n \in \mathbb{N}$, the Poisson basis $(b_0, \ldots, b_n)$ can be considered as a weighted φ-transformed system from the monomial basis $(1, t, \ldots, t^n)$ with $\varphi(t) = e^{-t}$ and $d_i = 1/i!$, $i = 0, \ldots, n$. Then, using Corollary 1, we deduce that the Possion basis is STP on $(0, \infty)$ and, taking into account (15) and Theorem 1, we deduce that the bidiagonal factorization (3),(4) of the collocation matrix of the Possion basis at positive values $t_1 < \cdots < t_{n+1}$ is given by

$$r_{i,j} = e^{t_{i-1} - t_i} \frac{\prod_{k=1}^{j-1}(t_i - t_{i-k})}{\prod_{k=2}^{j}(t_{i-1} - t_{i-k})}, \quad \widehat{r}_{i,j} = \frac{1}{i-1} t_j, \quad 1 \leq j < i \leq n+1,$$

$$q_{i,i} = \frac{e^{-t_i}}{(i-1)!} \prod_{k=1}^{i-1}(t_i - t_k), \quad 1 \leq i \leq n+1. \tag{17}$$

Let us observe that the computation with HRA of the bidiagonal decomposition (3),(4),(17) should require the evaluation with HRA of the involved exponential function. Although this cannot be guaranteed, Section 6 will show that accurate algebraic computations with the collocation matrices associated to these non-polynomial bases can be performed.

5 Rational weighted φ-transformed systems

Given a system $(u_0,\dots,u_n)$ of functions defined on I and positive values $d_0,\dots,d_n$ such that $\sum_{k=0}^n d_k u_k(t)\neq 0$, for all $t\in I$, the system $(r_0,\dots,r_n)$ defined by

$$r_i(t):=\frac{d_i u_i(t)}{\sum_{k=0}^n d_k u_k(t)},\quad i=0,\dots,n,$$

satisfies $\sum_{i=0}^n r_i(t) = 1$, $\forall t \in I$, and generates a new space of rational functions. If $(u_0,\dots,u_n)$ is TP then $\sum_{k=0}^n d_k u_k(t)>0$, $\forall t\in I$, and $(r_0,\dots,r_n)$ can be considered as a particular weighted φ-transformed system with

$$\varphi(t):=\frac{1}{\sum_{k=0}^n d_k u_k(t)},\quad t\in I. \tag{18}$$

Given $t_1<\cdots<t_{n+1}$ in I such that the corresponding collocation matrix of $(u_0,\dots,u_n)$ is STP, by Theorem 1, we deduce that the corresponding collocation matrix of $(r_0,\dots,r_n)$ is also STP and, by considering (18) in Theorem 1, we can obtain its bidiagonal factorization (3),(4) from the corresponding bidiagonal factorization of the collocation matrix of $(u_0,\dots,u_n)$. It is important to notice that, by Remark 1, this bidiagonal factorization can be frequently used to perform algebraic calculations and interpolation with HRA. The particular cases of rational Bernstein bases and rational Said-Ball bases were considered in [5].

Now we shall consider nested spaces generated by a general class of rational weighted φ-transformed systems admitting degree elevation and de Casteljau-type evaluation algorithms.

Let us suppose that $I=[a,b]$ and $f,g:I\to\mathbb{R}$ are nonnegative continuous functions such that $f(t)\neq 0$, $g(t)\neq 0$, $\forall t\in(a,b)$. Let us define the system

$$(u_0^n,\dots,u_n^n),\quad u_i^n(t):=\binom{n}{i}f^i(t)g^{n-i}(t),\quad t\in[a,b],\quad i=0,\dots,n. \tag{19}$$

Following the approach of [25], for the particular case of rational Bernstein functions, let us consider linear factors $L_i(t) = a_i g(t) + b_i f(t)$ defined by positive values a_i and b_i, $i \in \mathrm{Z}_+$, and

$$\omega^n(t) := L_1(t) \cdot \ldots \cdot L_n(t). \tag{20}$$

It can be easily checked that $\omega^n(t) = \sum_{i=0}^n w_i^n u_i^n(t)$ where

$$w_i^n = \frac{1}{\binom{n}{i}} \left(\sum_{\substack{K \cup L = \{1,\ldots,n\} \\ |K|=(n-i), |L|=i}} \prod_{k \in K} a_k \prod_{l \in L} b_l \right). \tag{21}$$

The positivity of a_i and b_i guarantees that $\omega_i^n > 0$ and $\omega^n(t) > 0$, $\forall t \in (a,b)$. Let us now denote by $(\rho_0^n, \ldots, \rho_n^n)$ the weighted $1/\omega^n$-transformed system corresponding to the weights $w_0^n, \ldots, w_n^n$ given in (21)

$$\rho_i^n(t) := w_i^n \frac{1}{\omega^n(t)} u_i^n(t), \quad i = 0, \ldots, n. \tag{22}$$

This system spans the space of rational functions with common denominator $\omega^n(t)$,

$$\mathcal{R}^n := \operatorname{span}\{\rho_i^n(t) \mid i = 0, \ldots, n\} = \{u(t)/\omega^n(t) \mid u(t) \in \mathcal{U}^n\},$$

where $\mathcal{U}^n$ is the space of functions generated by the basis (19).

Let us observe that Proposition 2 of [25] establishes the following recurrence relation satisfied by the weights (21)

$$w_i^n = a_n \frac{(n-i)}{n} w_i^{n-1} + b_n \frac{i}{n} w_{i-1}^{n-1}, \quad 0 \le i \le n. \tag{23}$$

On the other hand, by replacing in Propositions 3 and 4 of [25] the functions t and $1-t$ by $f(t)$ and $g(t)$, respectively, one can easily deduce the following relations satisfied by the functions of weighted $1/\omega^n$-transformed systems

$$\begin{aligned}
\rho_i^n(t) &= a_n \frac{g(t)}{L_n(t)} \rho_i^{n-1}(t) + b_n \frac{f(t)}{L_n(t)} \rho_{i-1}^{n-1}(t), \; i = 0, \ldots, n, \\
\rho_i^n(t) &= a_{n+1} \frac{n+1-i}{n+1} \frac{w_i^n}{w_i^{n+1}} \rho_i^{n+1}(t) + b_{n+1} \frac{i+1}{n+1} \frac{w_i^n}{w_{i+1}^{n+1}} \rho_{i+1}^{n+1}(t), \; i = 0, \ldots, n.
\end{aligned}$$

These relations guarantee the nested nature of the generated spaces, i.e. $\mathcal{R}^n \subset \mathcal{R}^{n+1}$, and allow the definition of degree elevation and de Casteljau-type algorithms for the evaluation of parametric curves

$$\gamma(t) = \sum_{i=0}^{n} P_i \rho_i^n(t), \quad t \in [a, b].$$

Theorem 2 of [18] proves that, given nonnegative functions $f, g : I \to \mathbb{R}$ such that $f(t) \neq 0$, $g(t) \neq 0$, $\forall t \in (a, b)$ and f/g is a strictly increasing function, then

$$A := \left(\binom{n}{j-1} f^{j-1}(t_i) g^{n-j+1}(t_i) \right)_{1 \le i,j \le n+1}, \quad a < t_1 < \cdots < t_{n+1} < b, \tag{24}$$

is STP. Moreover, in Theorem 3 of [18], the following bidiagonal decomposition (3) of the collocation matrices (24) was deduced

$$A = F_n F_{n-1} \cdots F_1 D G_1 \cdots G_{n-1} G_n, \tag{25}$$

where F_i and G_i, $1 \le i \le n$, are the lower and upper triangular bidiagonal matrices of the form (4) and $D = \operatorname{diag}(p_{1,1}, \ldots, p_{n+1,n+1})$. The entries $m_{i,j}, \widehat{m}_{i,j}$ and $p_{i,i}$ are given by

$$\begin{aligned}
m_{i,j} &= \frac{g^{n-j+1}(t_i) g(t_{i-j})}{g^{n-j+2}(t_{i-1})} \frac{\prod_{k=1}^{j-1} \big(f(t_i) g(t_{i-k}) - f(t_{i-k}) g(t_i) \big)}{\prod_{k=2}^{j} \big(f(t_{i-1}) g(t_{i-k}) - f(t_{i-k}) g(t_{i-1}) \big)}, \quad 1 \le j < i \le n+1, \\
\widehat{m}_{i,j} &= \frac{n-i+2}{i-1} \frac{f(t_j)}{g(t_j)}, \quad 1 \le j < i \le n+1, \\
p_{i,i} &= \binom{n}{i-1} \frac{g^{n-i+1}(t_i)}{\prod_{k=1}^{i-1} g(t_k)} \prod_{k=1}^{i-1} \big(f(t_i) g(t_k) - f(t_k) g(t_i) \big), \quad 1 \le i \le n+1.
\end{aligned} \tag{26}$$

Using Corollary 1 we deduce that the corresponding weighted $1/\omega^n$-transformed systems (22) are STP on (a, b) and then are of interest in CAGD and have shape preserving properties. According to Theorem 1, the collocation matrix $\widetilde{A}$ of the weighted $1/\omega^n$-transformed systems (22) corresponding to $a < t_1 < \cdots < t_{n+1} < b$ is STP and can be factorized as

$$\widetilde{A} = \widetilde{F}_n \widetilde{F}_{n-1} \cdots \widetilde{F}_1 \widetilde{D} \widetilde{G}_1 \cdots \widetilde{G}_{n-1} \widetilde{G}_n, \tag{27}$$

where $\widetilde{F}_i$ and $\widetilde{G}_i$, $1 \le i \le n$, are the lower and upper triangular bidiagonal matrices of the form (10) and $\widetilde{D} = \operatorname{diag}(q_{1,1}, \ldots, q_{n+1,n+1})$. The off-diagonal entries $r_{i,j}$, $\widehat{r}_{i,j}$ of $\widetilde{F}_i$ and $\widetilde{G}_i$, respectively, and the diagonal entries $q_{i,i}$ of

$\widetilde{D}$ are

$$r_{i,j} = \frac{\omega^n(t_{i-1})}{\omega^n(t_i)} m_{i,j}, \quad \widehat{r}_{i,j} = \frac{w_{i-1}^n}{w_{i-2}^n} \widehat{m}_{i,j}, \quad 1 \le j < i \le n+1,$$

$$q_{i,i} = \frac{w_{i-1}^n}{\omega^n(t_i)} p_{i,i}, \quad 1 \le i \le n+1, \tag{28}$$

where ω^n and w_i^n are defined in (20) and (21), respectively, and $m_{i,j}$, $\widehat{m}_{i,j}$, $p_{i,i}$ are the entries given in (26).

Let us observe that by Remark 1, if the evaluation of f and g does not require subtractions (except for the initial data) and the computation of (26) can be performed with HRA, then weighted $1/\omega^n$-transformed systems guarantee excellent computational properties since many algebraic computations associated to $\widetilde{A}$ can be performed with HRA.

Let us now consider some interesting examples that can be obtained by considering $f(t) = t$, $g(t) = 1-t$, $t \in [0,1]$. In this case the functions u_i^n defined in (19) coincide with the Bernstein polynomials

$$u_i^n(t) = B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}, \quad i = 0, \dots, n. \tag{29}$$

For the choice $a_i = a$, $b_i = b$, $1 \le i \le n$, we have $w_i^n = a^{n-i} b^i$, $0 \le i \le n$. In this case, $\omega^n(t) = (a(1-t) + bt)^n$ and the corresponding weighted $1/\omega^n$-transformed systems are Bernstein polynomials composed with a rational reparametrization of degree 1 that maps the boundaries of the interval $[0,1]$ onto itself (see [25]). In fact

$$\rho_i^n(t) = B_i^n\left(\frac{bt}{a(1-t)+bt}\right), \quad i = 0, \dots, n.$$

Using the bidiagonal factorization given in (26) and (28), we can obtain the coefficients of the bidiagonal factorization (27) of the collocation matrices of this basis.

$$\begin{aligned}
r_{i,j} &= \frac{(a(1-t_{i-1}) + bt_{i-1})^n}{(a(1-t_i) + bt_i)^n} \frac{(1-t_i)^{n-j+1}(1-t_{i-j})}{(1-t_{i-1})^{n-j+2}} \frac{\prod_{k=1}^{j-1}(t_i - t_{i-k})}{\prod_{k=2}^{j}(t_{i-1} - t_{i-k})}, \\
\widehat{r}_{i,j} &= \frac{n-i+2}{i-1} \frac{b}{a} \frac{t_j}{1-t_j}, \quad 1 \le j < i \le n+1, \\
q_{i,i} &= a^{n-i+1} b^{i-1} \binom{n}{i-1} \frac{(1-t_i)^{n-i+1}}{(a(1-t_i) + bt_i)^n} \prod_{k=1}^{i-1} \frac{(t_i - t_k)}{(1-t_k)}, \quad 1 \le i \le n+1.
\end{aligned}$$

Let us recall that given a real number $q > 0$ and any non-negative integer k, the q-integer $[k]$ is defined as

$$[k] := \begin{cases} (1-q^k)/(1-q), & q \neq 1, \\ k, & q = 1, \end{cases}$$

and the q-factorial $[k]!$ as

$$[k]! := \begin{cases} [k][k-1]\cdots[1], & k \geq 1, \\ 1, & k = 0. \end{cases}$$

For integers $0 \leq k \leq n$, the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$ is defined by

$$\begin{bmatrix} n \\ 0 \end{bmatrix} := 1, \quad \begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n][n-1]\cdots[n-k+1]}{[k]!} = \frac{[n]!}{[k]![n-k]!} \quad k > 0.$$

The Lupaş q-analogues of the Bernstein functions of degree n (cf. [14]) are the rational Bernstein functions

$$\rho_i^n(t) := \frac{a_{n,i}(t)}{w_n(t)}, \quad i = 0, \ldots, n$$

with

$$a_{n,i}(t) := \begin{bmatrix} n \\ i \end{bmatrix} q^{i(i-1)/2} t^i (1-t)^{n-i}, \quad w_n(t) := \sum_{i=0}^{n} a_{n,i}(t) = \prod_{j=1}^{n} (1 - t + q^{j-1} t).$$

Clearly, this basis is a weighted $1/\omega^n$-transformed system (22) where the weigths $w_i^n = \begin{bmatrix} n \\ i \end{bmatrix} q^{i(i-1)/2}$ can be obtained from (21) for the particular choice $a_i = 1$ and $b_i = q^{i-1}$, $i = 1, \ldots, n$ (see [25]). The bidiagonal factorization (28) of its collocation matrices coincides with the obtained in [9].

Now, by considering positive weights $d_0, \ldots, d_n$, we can define the weighted Lupaş q-analogue of Bernstein functions of degree n as

$$r_{n,i}(t;q) := \frac{d_i a_{n,i}(t)}{\sum_{k=0}^{n} d_k a_{n,k}(t)}, \quad i = 0, \ldots, n.$$

Using the bidiagonal factorization given in (26) and (28), we can obtain the coefficients of the bidiagonal factorization (27) of the collocation matrices of weighted Lupaş q-analogue of Bernstein functions as follows:

$$
\begin{aligned}
r_{i,j} &= \frac{\sum_{k=0}^{n} d_k a_{n,k}(t_{i-1})}{\sum_{k=0}^{n} d_k a_{n,k}(t_i)} \frac{(1-t_i)^{n-j+1}(1-t_{i-j})}{(1-t_{i-1})^{n-j+2}} \frac{\prod_{k=1}^{j-1}(t_i - t_{i-k})}{\prod_{k=2}^{j}(t_{i-1} - t_{i-k})}, \\
\widehat{r}_{i,j} &= \frac{d_{i-1}}{d_{i-2}} \frac{n-i+2}{i-1} \frac{1-q^{n-i+2}}{1-q^{i-1}} q^{i-2} \frac{t_j}{1-t_j}, \quad 1 \le j < i \le n+1, \\
q_{i,i} &= d_{i-1} \begin{bmatrix} n \\ i-1 \end{bmatrix} q^{(i-1)(i-2)/2} \frac{(1-t_i)^{n-i+1}}{\sum_{k=0}^{n} d_k a_{n,k}(t_i)} \prod_{k=1}^{i-1} \frac{(t_i - t_k)}{(1-t_k)}, \quad 1 \le i \le n+1.
\end{aligned}
$$

Finally, let us observe that there are other interesting choices of functions $f(t)$ and $g(t)$ satisfying conditions of Theorem 2 of [18]. We can consider $f(t) := t^2$ and $g(t) := 1 - t^2$, $t \in [0,1]$. In this case, the basis (19) is the basis with optimal shape preserving properties of the space $\langle 1, t^2, \ldots, t^{2n} \rangle$ of even polynomials of degree less than or equal to $2n$ on $[0,1]$.

Another particular case can be given by considering the functions

$$
f(t) = sin^2(t/2) = \frac{1 - cos(t)}{2}, \quad g(t) = cos^2(t/2) = \frac{1 + cos(t)}{2}, \quad t \in I = [0, \pi].
$$

In [23] it was proved that the system (19) is the basis with optimal shape preserving properties of the space of even trigonometric polynomials given by $\langle 1, cos(t), cos(2t), \ldots, cos(nt) \rangle$.

Now, let us consider $0 < \Delta < \pi/2$ and

$$
f(t) := sin\left(\frac{\Delta + t}{2}\right), \quad g(t) := sin\left(\frac{\Delta - t}{2}\right), \quad t \in I = [-\Delta, \Delta].
$$

For $n = 2m$, the system (19) is a basis that coincides, up to a positive scaling, with the basis with optimal shape preserving properties of the space $\langle 1, cos(t), sin(t), \ldots, cos(mt), sin(mt) \rangle$ of trigonometric polynomials of degree less than or equal to m on I (see Section 3 of [24]).

Finally, for any $\Delta > 0$, we can also consider

$$
f(t) = sinh\left(\frac{\Delta + t}{2}\right), \quad g(t) := sinh\left(\frac{\Delta - t}{2}\right), \quad t \in I = [-\Delta, \Delta].
$$

For $n = 2m$, the system (19) is a basis with shape preserving properties of the space $\langle 1, e^t, e^{-t}, \ldots, e^{mt}, e^{-mt} \rangle$ of hyperbolic polynomials of degree less than or equal to m on I.

Taking into account that curves generated by the corresponding weighted $1/\omega^n$-transformed systems also inherit algorithms of the traditional rational Bézier curves, they can be considered as modeling tools in CAD/CAM systems. Trigonometric and hyperbolic curves are getting considerable importance since they provide the opportunity to construct conics, cylinders and surfaces of revolution, catenary, etc. Shape preserving rational trigonometric interpolation is very important in scientific data visualization and has been applied in other fields such as Engineering, Biology, Chemistry, Medical and social sciences (see [2] and the references therein).

In the next section we are going to illustrate accurate computations with collocation matrices of the considered weighted φ-transformed systems.

6 Numerical experiments

In [15], assuming that the multipliers and diagonal pivots of the Neville elimination of a nonsingular $n \times n$ TP matrix A and its transpose are known with HRA, Koev presents algorithms for computing with HRA its eigenvalues, singular values and the solution of linear systems of equations $Ax = c$ where the entries of the vector c have alternating signs. In [16] Koev implemented these algorithms with the Matlab or Octave functions `TNSolve`, `TNEigenvalues` and `TNSingularvalues`. The computational cost of the function `TNSolve` is $\mathcal{O}(n^2)$ elementary operations and it requires as input arguments the bidiagonal factorization (3) of the matrix A and the vector c of the linear system $Ax = c$. The computational cost of `TNEigenvalues` and `TNSingularvalues` is $\mathcal{O}(n^3)$. These functions also requiere as input argument the bidiagonal factorization (3) of the matrix A (see [17]).

We have implemented the Matlab or Octave function `TNBDA`, which takes as input arguments the bidiagonal factorization (3) of the collocation matrix at $t_1, \ldots, t_{n+1}$ of a system, positive values $d_0, \ldots, d_n$ and $\varphi(t_1), \ldots, \varphi(t_{n+1})$ for a given positive function φ. Using (11), `TNBDA` computes the bidiagonal factorization (3) of the collocation matrix at $t_1, \ldots, t_{n+1}$ of the corresponding weighted φ-transformed system. We have used this bidiagonal decomposition with `TNSolve`, `TNEigenValues`, `TNSingularValues` in order to obtain solutions for the above mentioned algebraic problems.

Now we include some numerical experiments considering collocation matrices of weighted φ-transformed systems. Due to the ill conditioning of these matrices, traditional methods do not achieve accurate solutions when solving the mentioned algebraic problems. The numerical results show this fact and the high accuracy of the algorithms that we have presented, even when the bidiagonal factorization of A is not computed with HRA.

6.1 Linear systems

Linear systems arise when solving interpolation problems. So, in this section, we shall illustrate the accuracy of the computed solutions of $Ax = c$ when using the function `TNSolve` with the bidiagonal factorization of A given by

`TNBDA`. We have obtained the solution of the systems using Mathematica with a precision of 100 digits and considered this solution exact. Then we have computed with Matlab two approximations, the first one using `TNBDA` and `TNSolve` and the second one using the Matlab command \.

First, we have considered collocation matrices of $(n+1)$-dimensional negative binomial bases, geometric bases and Poisson bases at equidistant parameters in $(0,1)$. Table 1 illustrates the condition number of these matrices.

n+1	$\kappa(\mathbf{A_n})$	$\kappa(\mathbf{A_n})$	$\kappa(\mathbf{A_n})$
10	1.0×10^{4}	8.5×10^{7}	6.0×10^{6}
20	1.8×10^{8}	6.7×10^{16}	3.6×10^{15}
25	2.5×10^{10}	2.0×10^{21}	8.0×10^{20}
50	1.4×10^{21}	5.4×10^{43}	1.4×10^{53}

Table 1: Condition number of collocation matrices of negative binomial bases (left), geometric bases (middle) and Poisson bases (right).

Relative errors solving $\mathbf{A}_n x = \mathbf{c}_n$ with $\mathbf{c}_n = ((-1)^{i+1}c_i)_{1\le i\le n+1}$ where c_i is a random integer value are shown in Table 2. The computed results confirm the accuracy of the proposed method that, clearly, keeps the accuracy when the dimension of the problem increases. In contrast, when n increases the condition number of the considered matrices considerably increases and that explains the bad results obtained with the Matlab command \.

n+1	$\mathbf{A_n} \setminus \mathbf{c_n}$	`TNBDA`	$\mathbf{A_n} \setminus \mathbf{c_n}$	`TNBDA`	$\mathbf{A_n} \setminus \mathbf{c_n}$	`TNBDA`
10	3.44798×10^{-13}	4.87933×10^{-16}	1.39105×10^{-11}	1.75715×10^{-16}	1.47267×10^{-10}	1.13940×10^{-15}
20	3.65455×10^{-10}	7.82315×10^{-16}	0.00173326	5.94029×10^{-16}	0.000338972	5.15797×10^{-16}
25	4.08151×10^{-8}	8.70322×10^{-16}	0.951529	8.85806×10^{-16}	0.999085	3.73189×10^{-16}
50	1.00833	5.31708×10^{-16}	1.000000	7.22416×10^{-16}	1.000000	2.51721×10^{-15}

Table 2: Relative errors when solving $\mathbf{A}_n x = \mathbf{c}_n$ with collocation matrices of negative binomial bases (left), geometric bases (middle) and Poisson bases (right).

For different values of n we have also considered collocation matrices at equidistant parameters in the interior of the interval domain of rational weighted $1/\omega^n$-transformed systems (22), obtained by considering factors $L_i(t) = a_i g(t) + b_i f(t)$ with $a_i = 2$ and $b_i = 5$, $i \in \mathbb{N}$. Tables 3 and 4 illustrate the condition number of all considered matrices.

n+1	$\kappa(\mathbf{A_n})$	$\kappa(\mathbf{A_n})$
10	1.7×10^{5}	1.4×10^{3}
20	1.2×10^{11}	4.7×10^{6}
25	1.1×10^{14}	2.8×10^{8}
50	6.2×10^{28}	1.9×10^{17}

Table 3: Condition number of collocation matrices of weighted φ-transformed systems with $f(t) = t$, $g(t) = 1 - t$ (left), with $f(t) = t^2$, $g(t) = 1 - t^2$ (right).

n+1	$\kappa(\mathbf{A_n})$	$\kappa(\mathbf{A_n})$	$\kappa(\mathbf{A_n})$
10	1.1×10^4	3.1×10^5	9.9×10^4
20	1.3×10^9	4.2×10^{11}	4.5×10^{10}
25	5.0×10^{11}	4.9×10^{14}	3.1×10^{13}
50	8.6×10^{24}	1.2×10^{30}	5.9×10^{27}

Table 4: Condition number of collocation matrices of weighted φ-transformed systems with $f(t) = sin^2\,(t/2)$, $g(t) = cos^2\,(t/2)$ (left), with $f(t) = sin\,((1+t)/2)$, $g(t) = sin\,((1-t)/2)$ (middle) with $f(t) = sinh\,((1+t)/2)$, $g(t) = sinh\,((1-t)/2)$ (right).

We have considered $\mathbf{c}_n = ((-1)^{i+1}c_i)_{1\le i\le n+1}$ where c_i is a nonnegative random real number. Table 5 shows the relative errors when $f(t) = t$, $g(t) = 1-t$ and the relative errors when $f(t) = t^2$, $g(t) = 1-t^2$, $t \in [0,1]$. Let us observe that, if $f(t) = t$ and $g(t) = 1-t$, then

$$f(t_i)g(t_j) - f(t_j)g(t_i) = t_i - t_j.$$

On the other hand, if $f(t) = t^2$ and $g(t) = 1-t^2$, then

$$f(t_i)g(t_j) - f(t_j)g(t_i) = (t_i - t_j)(t_i + t_j).$$

In both cases the parameters (28) of the bidiagonal factorization (27) can be obtained with HRA and then $\mathbf{A}_n x = \mathbf{c}_n$ can also be solved with HRA. The numerical experiments confirm this fact.

n+1	$\mathbf{A_n} \setminus \mathbf{c_n}$	TNBDA	$\mathbf{A_n} \setminus \mathbf{c_n}$	TNBDA
10	3.3579×10^{-13}	1.1191×10^{-15}	2.4173×10^{-14}	9.1615×10^{-16}
20	1.2413×10^{-9}	6.2974×10^{-16}	5.6324×10^{-11}	2.4457×10^{-15}
25	4.1424×10^{-7}	2.0843×10^{-15}	3.0923×10^{-9}	2.0016×10^{-15}
50	1.0000	7.5480×10^{-15}	0.9998	6.4231×10^{-15}

Table 5: Relative errors solving $\mathbf{A}_n x = \mathbf{c}_n$ with $f(t) = t$, $g(t) = 1-t$ (left), with $f(t) = t^2$, $g(t) = 1-t^2$ (right).

Finally, Table 6 shows the relative errors in the solution of $\mathbf{A}_n x = \mathbf{c}_n$ with other functions f and g. In these cases, the computation with HRA of the parameters (28) of the bidiagonal factorization (27) should require the evaluation with HRA of the involved trigonometric or hyperbolic functions. Although this cannot be guaranteed, the numerical experiments show again that accurate algebraic computations with the collocation matrices associated to these non-polynomial basis functions can be performed.

n+1	$\mathbf{A_n} \setminus \mathbf{c_n}$	TNBDA	$\mathbf{A_n} \setminus \mathbf{c_n}$	TNBDA	$\mathbf{A_n} \setminus \mathbf{c_n}$	TNBDA
10	1.54259×10^{-14}	3.21312×10^{-16}	2.6135×10^{-13}	1.8511×10^{-15}	5.1019×10^{-14}	1.9355×10^{-15}
20	2.33365×10^{-12}	5.28031×10^{-16}	3.3808×10^{-9}	1.4878×10^{-15}	2.8313×10^{-9}	2.1849×10^{-15}
25	3.55721×10^{-11}	3.10246×10^{-15}	1.8684×10^{-6}	2.3294×10^{-15}	2.0441×10^{-7}	2.9733×10^{-15}
50	0.00306142	3.25552×10^{-15}	1.0000	2.1001×10^{-14}	1.0000	8.1439×10^{-15}

Table 6: Relative errors solving $\mathbf{A}_n x = \mathbf{c}_n$ with $f(t) = sin^2(t/2)$, $g(t) = cos^2(t/2)$ (left), with $f(t) = sin((1+t)/2)$, $g(t) = sin((1-t)/2)$ (middle) with $f(t) = sinh((1+t)/2)$, $g(t) = sinh((1-t)/2)$ (right).

6.2 Eigenvalues and singular values

We have also used the bidiagonal decomposition provided by `TNBDA` for computing, with the Matlab functions `TNEigenValues` and `TNSingularValues`, the eigenvalues and the singular values, respectively, of the collocation matrices of weighted φ-transformed systems considered in the previous subsection. We have also computed their approximations with the Matlab functions `eig` and `svd`, respectively. In order to determine the accuracy of the approximations, we have calculated the eigenvalues and singular values of the matrices by using Mathematica with a precision of 100 digits and computed the relative errors corresponding to the approximations, considering the eigenvalues and singular values provided by Mathematica as exact.

The approximations of the eigenvalues and singular values obtained by means of `TNBDA` are very accurate for all considered n, whereas the approximations of the eigenvalues and singular values obtained with the Matlab commands `eig` and `svd` are not very accurate when n increases. Since these collocation matrices are all STP, let us recall that, by Theorem 6.2 of [1], all their eigenvalues are positive and distinct. Tables 7, 8, 9, 10, 11, 12, 13 and 14 show the relative errors of the approximations to the lowest eigenvalue and the lowest singular value obtained with both methods.

n+1	eig	TNBDA	svd	TNBDA
10	3.09244×10^{-13}	0.	1.13631×10^{-13}	7.82315×10^{-16}
20	1.98025×10^{-9}	7.45447×10^{-16}	6.52333×10^{-10}	9.63835×10^{-16}
25	4.32252×10^{-7}	1.36414×10^{-15}	2.53703×10^{-8}	2.01433×10^{-16}.
50	6784.57	1.17511×10^{-15}	416.354	6.06316×10^{-16}

Table 7: Relative errors when computing the lowest eigenvalue (left) and the lowest singular value (right) of collocation matrices of binomial negatives basis functions.

n+1	eig	TNBDA	svd	TNBDA
10	3.31746×10^{-11}	3.50531×10^{-16}	9.946626×10^{-11}	2.99503×10^{-16}
20	0.00583381	6.47223×10^{-16}	0.00181714	1.70411×10^{-16}
25	86.706	4.24794×10^{-16}	89.422	7.48982×10^{-16}.
50	3.97376×10^{23}	1.37939×10^{-15}	1.37387×10^{23}	6.33473×10^{-16}

Table 8: Relative errors when computing the lowest eigenvalue (left) and the lowest singular value (right) of collocation matrices of geometric bases.

n+1	eig	TNBDA	svd	TNBDA
10	1.67788×10^{-10}	2.01781×10^{-15}	1.24564×10^{-10}	5.0222×10^{-16}
20	0.000067064	4.78068×10^{-16}	0.00187398	4.00738×10^{-16}
25	34.131	6.17955×10^{-16}	14.667	8.09263×10^{-16}.
50	2.1166×10^{18}	2.80909×10^{-15}	4.022298×10^{16}	8.40959×10^{-15}

Table 9: Relative errors when computing the lowest eigenvalue (left) and the lowest singular value (right) of collocation matrices of Poisson bases.

n+1	eig	TNBDA	svd	TNBDA
10	1.80908×10^{-13}	2.18488×10^{-16}	3.59637×10^{-12}	8.09628×10^{-16}
20	3.9324×10^{-8}	1.51895×10^{-16}	7.02917×10^{-7}	1.37421×10^{-15}
25	0.0000625168	1.06001×10^{-15}	0.000970628	1.93093×10^{-15}
50	2.042×10^{6}	6.71094×10^{-15}	6.97145×10^{10}	7.04631×10^{-15}

Table 10: Relative errors when computing the lowest eigenvalue (left) and the lowest singular value (right) of collocation matrices of weigthed φ-transformed bases (22) with $f(t) = t$, $g(t) = 1 - t$.

n+1	eig	TNBDA	svd	TNBDA
10	1.10761×10^{-14}	1.69399×10^{-15}	2.53164×10^{-15}	0.
20	3.21721×10^{-11}	1.5083×10^{-15}	2.47006×10^{-11}	2.52138×10^{-15}
25	6.63705×10^{-10}	1.11226×10^{-15}	5.01779×10^{-10}	0.
50	0.156406	4.19403×10^{-15}	0.328617	8.40959×10^{-15}

Table 11: Relative errors when computing the lowest eigenvalue (left) and the lowest singular value (right) of collocation matrices of weigthed φ-transformed bases (22) with $f(t) = t^2$, $g(t) = 1 - t^2$.

n+1	eig	TNBDA	svd	TNBDA
10	9.86257×10^{-14}	4.26029×10^{-16}	2.50339×10^{-13}	1.65788×10^{-15}
20	3.46514×10^{-9}	1.92921×10^{-15}	9.69498×10^{-9}	1.72095×10^{-15}
25	2.34348×10^{-6}	1.25379×10^{-15}	1.25441×10^{-6}	3.10795×10^{-15}
50	30.0203	6.69068×10^{-15}	7.3221×10^{6}	3.34347×10^{-15}

Table 12: Relative errors when computing the lowest eigenvalue (left) and the lowest singular value (right) of collocation matrices of weigthed φ-transformed bases (22) with $f(t) = sin^2\,(t/2)$, $g(t) = cos^2\,(t/2)$.

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n+1	eig	TNBDA	svd	TNBDA
10	1.44963×10^{-12}	5.72976×10^{-16}	3.64477×10^{-12}	$4,32152 \times 10^{-16}$
20	3.60789×10^{-7}	4.1266×10^{-15}	7.45664×10^{-7}	1.85645×10^{-15}
25	0.000101621	1.57645×10^{-15}	0.00653125	1.9671×10^{-15}
50	1.29153×10^{7}	3.9368×10^{-15}	6.15482×10^{11}	3.25117×10^{-16}

Table 13: Relative errors when computing the lowest eigenvalue (left) and the lowest singular value (right) of weigthed φ-transformed bases (22) with $f(t) = sin\,((1+t)/2)$, $g(t) = sin\,((1-t)/2)$.

n+1	eig	TNBDA	svd	TNBDA
10	4.60233×10^{-13}	2.70486×10^{-16}	8.45483×10^{-13}	6.82722×10^{-16}
20	9.05243×10^{-8}	2.80275×10^{-15}	1.04254×10^{-7}	4.425×10^{-16}
25	0.0000577108	1.05533×10^{-15}	0.000107575	2.95282×10^{-16}
50	71585.3	7.90656×10^{-15}	9.1323×10^{9}	7.45104×10^{-16}

Table 14: Relative errors when computing the lowest eigenvalue (left) and the lowest singular value (right) of collocation matrices of weigthed φ-transformed bases (22) with $f(t) = sinh\,((1+t)/2)$, $g(t) = sinh\,((1-t)/2)$.

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