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A singularly perturbed convection-diffusion parabolic problem with incompatible boundary/initial data

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ABSTRACT

A singularly perturbed parabolic problem of convection-diffusion type with incompatible inflow boundary and initial conditions is examined. In the case of constant coefficients, a set of singular functions are identified which match certain incompatibilities in the data and also satisfy the associated homogeneous differential equation. When the convective coefficient only depends on the time variable and the initial/boundary data is discontinuous, then a mixed analytical/numerical approach is taken. In the case of variable coefficients and the zero level of compatibility being satisfied (i.e. continuous boundary/initial data), a numerical method is constructed whose order of convergence is shown to depend on the next level of compatibility being satisfied by the data. Numerical results are presented to support the theoretical error bounds established for both of the approaches examined in the paper.

1. Introduction

In the case of sufficiently smooth and compatible data, the solution of the following singularly perturbed parabolic problem: Find u such that

$$Lu := -\varepsilon u_{xx} + a(x,t)u_x + u_t = f(x,t), \quad (x,t) \in G := (0,1) \times (0,T], \tag{1a}$$

$$u(0,t) = g_L(t),\ u(1,t) = g_R(t),\ t > 0, \quad u(x,0) = \phi(x), \quad 0 \le x \le 1; \tag{1b}$$

$$a(x,t) \ge \alpha > 0,\ (x,t) \in \bar{G}; \quad 0 < \varepsilon \le 1, \tag{1c}$$

will contain a boundary layer of width $O(\varepsilon)$ near the outflow boundary $x = 1$. Nevertheless, problems with incompatible data arise, for example, in geophysical fluid mechanics (see [5] and the references therein) and in the modeling of plasma sheaths [6]. In this paper, we examine the issues that arise when the problem data is not sufficiently compatible at the inflow point $(0,0)$.

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If the boundary and initial conditions are incompatible ($\phi(0) \neq g_L(0)$), a strong [8] interior layer will appear for small values of the singular perturbation parameter. If $\phi(0) = g_L(0)$, but the data are still not sufficiently compatible at $(0,0)$ then a weak [8] interior layer appears in the solution. The path of any interior layer is located along the characteristic curve $x = d(t)$, where $d(t)$ is implicitly defined by

$$d'(t) = a(d(t), t), \quad d(0) = 0.$$

To avoid the interior layer interacting with the outflow boundary, we assume that at the final time

$$d(T) < 1. \tag{1d}$$

See Remark 1 for necessary modifications when (1d) is not satisfied.

Let us recall the constraints on the data, a, f, ϕ, g_L and g_R, for the solution u to be sufficiently regular so that classical numerical analysis is applicable; i.e., for $u \in C^{4+\gamma}(\bar{G})$.[1] From [14] we have the following result: If $a, f \in C^{0+\gamma}(\bar{G})$, $\phi \in C^{2+\gamma}[0,1]$, $g_L, g_R \in C^{1+\gamma/2}[0,T]$ and

$$A_0 = 0 \text{ with } A_0 := g_L(0) - \phi(0), \tag{2a}$$

$$A_1 = 0 \text{ with } A_1 := -\varepsilon\phi''(0) + a(0,0)\phi'(0) + g_L'(0) - f(0,0), \tag{2b}$$

$$g_R(0) = \phi(1); \quad -\varepsilon\phi''(1) + a(1,0)\phi'(1) + g_R'(0) = f(1,0), \tag{2c}$$

then the solution of problem (1) satisfies $u \in C^{2+\gamma}(\bar{G})$. By differentiating with respect to the time variable the differential equation (1a) and applying the above conditions on the function $u_t(x,t)$, we arrive at the following result: If $a, f \in C^{2+\gamma}(\bar{G})$, $\phi \in C^{4+\gamma}[0,1]$, $g_l, g_R \in C^{2+\gamma/2}[0,T]$ and in addition to the constraints (2a), (2b) and (2c) we have

$$\begin{aligned} &A_2 = 0 \text{ with}\\ &A_2 := -\varepsilon^2\phi^{(iv)}(0) + 2\varepsilon a(0,0)\phi'''(0) - a^2(0,0)\phi''(0) + g_L''(0^-)\\ &\quad + \varepsilon(a_{xx}(0,0)\phi'(0) + 2a_x(0,0)\phi''(0)) + (a_t - aa_x)(0,0)\phi'(0)\\ &\quad - (f_t + \varepsilon f_{xx} - af_x)(0,0), \end{aligned} \tag{2d}$$

$$\begin{aligned} &-\varepsilon^2\phi^{(iv)}(1) + 2\varepsilon a(1,0)\phi'''(1) - a^2(1,0)\phi''(1) + g_R''(0^-)\\ &\quad + \varepsilon(a_{xx}(1,0)\phi'(1) + 2a_x(1,0)\phi''(1)) + (a_t - aa_x)(1,0)\phi'(1)\\ &\quad = (f_t + \varepsilon f_{xx} - af_x)(1,0), \end{aligned} \tag{2e}$$

then the solution of problem (1) satisfies $u \in C^{4+\gamma}(\bar{G})$.

The literature on numerical methods for singularly perturbed partial differential equations continues to rapidly expand since the publication of the books [7] and [21]. The bulk of this literature concentrates on linear problems with regular exponential layers (occurring near outflow boundaries) or characteristic layers (occurring near boundaries parallel to the characteristics of the reduced problem). In these publications, the solution is sufficiently smooth when the magnitude of the singular perturbation parameter is of order one. The absence of compatibility conditions for the parabolic problem (1) introduces new kinds of layer functions (see the discussion in [23, pp. 351–352] and the references therein). In the case of elliptic problems with incompatible data, we refer the reader to [2], [3] and [13], where the nature of the singularities (due to low compatibility) is identified.

In the case of problem (1) with constant coefficients and when (2a) is not satisfied, the discontinuous analytic solution in the quarter plane $x, t > 0$ is given, for example, in [20], whereas in [1] an asymptotic expansion is given in the domain $\bar{G}$. In the case of a variable coefficient $a(t)$ and when (2a) is satisfied, a uniformly valid asymptotic expansion to the continuous solution of the problem posed on the quarter plane is presented in [22].

[1] As in [9], we define the space $C^{0+\gamma}(D)$, where $D \subset \mathbf{R}^2$ is an open set, as the set of all functions that are Hölder continuous of degree $\gamma \in (0,1)$ with respect to the metric $\|\cdot\|$, where for all $\mathbf{p}_i = (x_i, t_i), \in \mathbf{R}^2, i = 1,2;$ $\|\mathbf{p}_1 - \mathbf{p}_2\|^2 = (x_1 - x_2)^2 + |t_1 - t_2|$. For f to be in $C^{0+\gamma}(D)$ the following semi-norm needs to be finite

$$\lceil f \rceil_{0+\gamma,D} = \sup_{\mathbf{p}_1 \neq \mathbf{p}_2,\ \mathbf{p}_1,\mathbf{p}_2 \in D} \frac{|f(\mathbf{p}_1) - f(\mathbf{p}_2)|}{\|\mathbf{p}_1 - \mathbf{p}_2\|^\gamma}.$$

The space $C^{n+\gamma}(D)$ is defined by

$$C^{n+\gamma}(D) = \left\{ z : \frac{\partial^{i+j} z}{\partial x^i \partial t^j} \in C^{0+\gamma}(D),\ 0 \le i + 2j \le n \right\},$$

and $\|\cdot\|_{n+\gamma}$, $\lceil\cdot\rceil_{n+\gamma}$ are the associated norms and semi-norms.

In this paper, we consider the convection-diffusion problem (1) where the compatibility conditions (2a), (2b) and (2d) at $(0,0)$ are not all imposed. To avoid additional regularity issues with the data we will assume that $a, f \in C^{5+\gamma}(\bar{G})$, $\phi \in C^7[0,1]$, $g_L \in C^5[0,T]$, $g_R \in C^3[0,T]$ and that the compatibility conditions (2c) and (2e) at $(1,0)$ are all satisfied.

We examine the problem where the initial and left boundary condition do not match, i.e., (2a) is not satisfied. In this case, we only study problem (1) when the convection coefficient $a(x,t)$ depends solely on the time variable. Moreover, we first separate off a singular function that matches the incompatibility at the point $(0,0)$ and then use a numerical method to approximate the difference between the solution u and this singular function.

We also examine the problem where the initial and boundary condition match, so that the zero level compatibility (2a) is satisfied; but the higher compatibility conditions (2b) and (2d) are not satisfied. As the solution is continuous, a numerical method can be applied directly to the problem. If (2a) is satisfied but the first level of compatibility (2b) is not satisfied, then the order of convergence of the standard numerical method constructed is shown to be 0.5. If the first level of compatibility (2a) and (2b) is satisfied, then that numerical method is essentially first order.

In §2 a set of functions $S_n(x,t), n \geq 0$ are constructed to model the nature of any singularity in the solution related to a lack of compatibility between the initial and boundary condition at the point $(0,0)$. Parameter-explicit pointwise bounds on the partial derivatives of these functions are also established in §2. In §3, the solution u of (1) is expanded in terms of these special functions $S_n(x,t)$ as follows:

$$u(x,t) = \sum_{i=0}^{1} A_i S_i(x,t) + \sum_{i=2}^{3} B_i S_i(x,t) + v(x,t) + w(x,t),$$

where the amplitudes A_i, $i=0,1$; B_i, $i=2,3$ are suitably chosen so that $v, w \in C^{4+\gamma}(\bar{G})$, where v is the regular component and w is the boundary layer component of the solution u. With the aid of this expansion, a numerical method is constructed in §4 to generate a numerical approximation to $u - A_0 S_0$ (including also the case $A_0 = 0$). The order of convergence of this method depends on whether A_1 is zero or not. In §5, numerical results are presented for sample test problems to illustrate the performance of the method and to validate the orders of convergence established in the two main Theorems 3 and 4 in §4. Technical details associated with establishing bounds on the derivatives of the functions $S_n(x,t)$ are given in the appendix.

Notation: Throughout the paper, C denotes a generic constant that is independent of the singular perturbation parameter ε and all the discretization parameters. The L_∞ norm on the domain D will be denoted by $\|\cdot\|_D$. We also define the following interior layer function

$$E_\gamma(x,t) := e^{-\frac{\gamma(x-d(t))^2}{4\varepsilon t}}, \quad 0 < \gamma \leq 1.$$

If $\gamma = 1$, we simply write $E_1(x,t) = E(x,t)$.

2. A set of singular functions with incompatibilities

In [11] and [12], the parabolic problem (1) with compatible boundary/initial data is examined, but with a discontinuity in the initial condition $\phi(x)$ at some internal point $x = d$, $0 < d < 1$. In this case, the interior layer function

$$0.5\,\mathrm{erfc}\left(\frac{d(t)-x}{2\sqrt{\varepsilon t}}\right), \; 0 < d(0) < 1, \quad \mathrm{erfc}(z) := \frac{2}{\sqrt{\pi}} \int_{r=z}^{\infty} e^{-r^2} dr,$$

captures the nature of the singularity. We now define a set of related singular functions $S_n(x,t)$, which will form a basis for the regularity expansion of the solution $u(x,t)$ of problem (1). The regularity expansion is constructed in Theorem 1 (for $A_0 \neq 0$ and $a = a(t)$) and Theorem 2 (for $A_0 = 0$ and $a = a(x,t)$). For all $n \geq 0$:

$$S_n(x,t) := \frac{\psi_n^+(x,t) + (-1)^n \psi_n^-(x,t)}{a^n(0,0)}, \tag{3}$$

where the functions $\psi_n^\pm(x,t), n \geq 0$ are defined by

$$\psi_n^-(x,t) := (-1)^n 2^{n-1} n! (\varepsilon t)^{n/2} \mathrm{erfc}_n(\chi^-(x,t)), \tag{4a}$$

$$\psi_n^+(x,t) := (-1)^n 2^{n-1} n! (\varepsilon t)^{n/2} e^{\frac{xd(t)}{\varepsilon t}} \mathrm{erfc}_n(\chi^+(x,t)), \tag{4b}$$

$$\chi^\pm(x,t) := \frac{x \pm d(t)}{2\sqrt{\varepsilon t}}, \tag{4c}$$

and the iterated complementary error functions are

$$\mathrm{erfc}_{-1}(x) := \frac{2}{\sqrt{\pi}} e^{-x^2}, \quad \mathrm{erfc}_n(x) := \int_{s=x}^{\infty} \mathrm{erfc}_{n-1}(s)\, ds, \quad n \geq 0.$$

Observe that the first function S_0 is discontinuous and

$$S_1 \in C^{0+\gamma}(\bar{G}), \qquad S_{2n}, S_{2n+1} \in C^{2n+\gamma}(\bar{G}), \quad n \geq 1.$$

In the next lemma, we establish bounds on the derivatives of the first three functions S_n, $n = 0, 1, 2$. These bounds indicate both the strength of the singularity at $t = 0$ and how certain derivatives can depend on inverse powers of ε.

Lemma 1. *The function $S_0(x,t)$ satisfies the bounds*

$$|S_0| \leq C, \quad \left|\frac{\partial^i S_0}{\partial t^i}\right| \leq C\left[\frac{1}{t}\left(1+\sqrt{\frac{t}{\varepsilon}}\right)\right]^i E_\gamma(x,t), \quad i = 1, 2, \tag{5a}$$

$$\left|\frac{\partial^i S_0}{\partial x^i}\right| \leq \frac{C}{\varepsilon^i}\left(\frac{\varepsilon}{t} + \left(\frac{\varepsilon}{t}\right)^{i/2}\right) E_\gamma(x,t), \quad i = 1, 2, 3; \tag{5b}$$

the function $S_1(x,t)$ satisfies

$$|S_1| \leq C, \quad \left|\frac{\partial S_1}{\partial t}\right| \leq C, \ \left|\frac{\partial^2 S_1}{\partial t^2}\right| \leq C\frac{1}{t}\left(1+\sqrt{\frac{t}{\varepsilon}}\right) E_\gamma(x,t) + C, \tag{6a}$$

$$\left|\frac{\partial S_1}{\partial x}\right| \leq C, \ \left|\frac{\partial^2 S_1}{\partial x^2}\right| \leq \frac{C}{\varepsilon} E_\gamma(x,t) + C, \ \left|\frac{\partial^3 S_1}{\partial x^3}\right| \leq \frac{C}{\varepsilon\sqrt{\varepsilon t}} E_\gamma(x,t) + C; \tag{6b}$$

and the function $S_2(x,t)$ satisfies

$$|S_2| \leq C, \quad \left|\frac{\partial S_2}{\partial x}\right| \leq C, \tag{7a}$$

$$\left|\frac{\partial S_2}{\partial t}\right| \leq C, \ \left|\frac{\partial^2 S_2}{\partial t^2}\right| \leq C\left(1+\frac{\varepsilon}{t}\right)\left(1+\sqrt{\frac{t}{\varepsilon}}\right) E_\gamma(x,t) + C, \tag{7b}$$

$$\left|\frac{\partial^2 S_2}{\partial x^2}\right| \leq C\left(1+\sqrt{\frac{t}{\varepsilon}}\right) E_\gamma(x,t), \tag{7c}$$

$$\left|\frac{\partial^3 S_2}{\partial x^3}\right| \leq \frac{C}{\varepsilon}\left(1+\sqrt{\frac{t}{\varepsilon}}+\sqrt{\frac{\varepsilon}{t}}\right) E_\gamma(x,t). \tag{7d}$$

Proof. In the appendix, bounds on the partial derivatives of the functions $\psi_n^{\pm}(x,t)$ are established. These are used to prove the bounds on S_n. The bounds on S_0 follow directly from (27) and (31). Using (24a) and the recurrence relation (25) we deduce the following

$$\frac{\partial S_1}{\partial t} = \frac{a(d(t),t)}{a(0,0)} S_0 + x\frac{p(t)\psi_1^+}{\varepsilon t^2}, \quad p(t) := t a(d(t),t) - d(t);$$

$$a^2(0,0)\frac{\partial S_2}{\partial t} = 2\left(\varepsilon S_0 - a(0,0)a(d(t),t)S_1 + \frac{d(t)}{t}\psi_1^+\right) - \frac{p(t)}{\varepsilon t^2}(d(t)\psi_2^+ - \psi_3^+).$$

The bounds on the time derivatives of S_1 and S_2 follow. To deduce the bounds on the space derivatives of these components, we first note that from (25), we have

$$a(0,0)\frac{\partial S_1}{\partial x} = \psi_0^+ - \psi_0^- + \frac{d(t)}{\varepsilon t}\psi_1^+, \quad a(0,0)\frac{\partial^2 S_1}{\partial x^2} = \frac{d(t)}{t\varepsilon}\left(2\psi_0^+ + \frac{d(t)}{t\varepsilon}\psi_1^+\right),$$

$$a(0,0)\frac{\partial^3 S_1}{\partial x^3} = \frac{d(t)}{t\varepsilon}\left(2\frac{\partial \psi_0^+}{\partial x} + \frac{d(t)}{t\varepsilon}\left(\psi_0^+ + \frac{d(t)}{t\varepsilon}\psi_1^+\right)\right),$$

and from (32) the bounds on the space derivatives of S_1 follow. Next, we deduce bounds on the space derivatives of S_2. From the definitions in (4), the following bounds are obtained

$$\left|\frac{\partial^i \psi_j^-}{\partial x^i}\right| \le C(\sqrt{\varepsilon t})^{j-i} E_\gamma(x,t) + C, \quad j=1,2; \quad i=1,2,3,$$

and using the recurrence relation (25) we get the bounds

$$\left|\frac{\partial \psi_1^+}{\partial x}\right| \le C E_\gamma(x,t), \; \left|\frac{\partial^2 \psi_1^+}{\partial x^2}\right| \le \frac{C}{\varepsilon}\left(1+\sqrt{\frac{\varepsilon}{t}}\right) E_\gamma(x,t), \tag{8a}$$

$$\left|\frac{\partial^3 \psi_1^+}{\partial x^3}\right| \le \frac{C}{\varepsilon t}\left(1+\sqrt{\frac{t}{\varepsilon}}\right) E_\gamma(x,t), \tag{8b}$$

$$\left|\frac{\partial \psi_2^+}{\partial x}\right| \le C\sqrt{\varepsilon t} E_\gamma(x,t), \quad \left|\frac{\partial^2 \psi_2^+}{\partial x^2}\right| \le C\left(1+\sqrt{\frac{t}{\varepsilon}}\right) E_\gamma(x,t), \tag{8c}$$

$$\left|\frac{\partial^3 \psi_2^+}{\partial x^3}\right| \le \frac{C}{\varepsilon}\left(1+\sqrt{\frac{t}{\varepsilon}}+\sqrt{\frac{\varepsilon}{t}}\right) E_\gamma(x,t). \tag{8d}$$

The bounds on the space derivatives of S_2 follow immediately from the bounds above on the space derivatives of the singular functions ψ_i^- and ψ_i^+. $\square$

Observe that the strength of the singularity at $(0,0)$ in each of the functions S_n weakens as n increases. Using the identities in (25), we can deduce bounds on the remaining functions S_n, $n \ge 3$:

$$\left|\frac{\partial S_n}{\partial x}\right| \le C, \; n \ge 2, \quad \left|\frac{\partial^2 S_n}{\partial x^2}\right| \le C, \; n \ge 4, \quad \left|\frac{\partial^3 S_n}{\partial x^3}\right| \le C, \; n \ge 6, \tag{9a}$$

$$\left|\frac{\partial^2 S_3}{\partial x^2}\right| \le C\left(1+t\left(1+\sqrt{\frac{t}{\varepsilon}}\right) E_\gamma(x,t)\right), \tag{9b}$$

$$\left|\frac{\partial^3 S_{3+n}}{\partial x^3}\right| \le C\left(1+t^n\left(1+\sqrt{\frac{t}{\varepsilon}}\right)^{3-n} E_\gamma(x,t)\right), \quad n=0,1,2, \tag{9c}$$

$$\left|\frac{\partial S_n}{\partial t}\right| \le C, \; n \ge 2, \; \left|\frac{\partial^2 S_3}{\partial t^2}\right| \le C\left(1+\sqrt{\frac{\varepsilon}{t}} E_\gamma(x,t)\right), \left|\frac{\partial^2 S_n}{\partial t^2}\right| \le C, \; n \ge 4. \tag{9d}$$

3. The continuous problem

In the following result the asymptotic behaviour of the solution u to problem (1) is given when the convective coefficient a depends only on the time variable and u is discontinuous at $(0,0)$.

Theorem 1. *Assume that $a(x,t)=a(t)$, $\forall (x,t) \in \bar{G}$ and $a_t(0)=0$. The solution u of (1) can be expanded as follows*

$$u(x,t) = \sum_{i=0}^{1} A_i S_i(x,t) + \sum_{i=2}^{3} B_i S_i(x,t) + v(x,t) + w(x,t), \tag{10a}$$

where the constants A_i are defined in (2a) and (2b). The constants B_2, B_3 are defined such that for $0 \le i+2j \le 4$

$$\left|\frac{\partial^i v}{\partial x^i}\right| \le C(1+\varepsilon^{2-i}), \quad \left|\frac{\partial^j v}{\partial t^j}\right| \le C; \tag{10b}$$

$$\left|\frac{\partial^{i+j} w}{\partial x^i \partial t^j}\right| \le C\varepsilon^{-i}(1+\varepsilon^{1-j}) e^{-\alpha(1-x)/\varepsilon}. \tag{10c}$$

Proof. With the assumptions $a=a(t)$, $a_t(0)=0$ and noting (24d), we have

$$LS_i = p(t)\frac{\psi_{i+1}^+}{\varepsilon t^2}, \; p(t)=t^3P(t), \; |P(t)| \le C \quad \text{and} \quad LS_0 \in C^{2+\gamma}(\bar{G}). \tag{11}$$

We identify the remainder R by

$$R := u(x,t) - \sum_{i=0}^{1} A_i S_i(x,t), \quad R(0,t) = g_L(t) - C_L(t), \; R(x,0) = \phi(x),$$

where

$$C_L(t) := \sum_{i=0}^{1} A_i S_i(0,t) = \sum_{i=0}^{1} A_i \left(\frac{d(t)}{a(0)}\right)^i.$$

Then $C_L(0) = A_0, \; C_L'(0) = A_1$ and the remainder function R satisfies

$$LR = f - \frac{p(t)}{\varepsilon t^2}\left(A_0\psi_1^+ + \frac{A_1}{a(0)}\psi_2^+\right) \in C^{2+\gamma}(\bar{G}). \tag{12}$$

Hence, $R \in C^{2+\gamma}(\bar{G})$ as the amplitudes $A_i, i = 0, 1$ have been chosen so that the compatibility conditions (2a) and (2b) are satisfied by the problem data defining R.

The remainder R is further decomposed as follows

$$R(x,t) = \sum_{n=2}^{3} B_n S_n(x,t) + v(x,t) + w(x,t), \quad v := \sum_{n=4}^{5} B_n S_n(x,t) + z + v_S, \tag{13}$$

with $v, z, v_S, w \in C^{4+\gamma}(\bar{G})$. The functions z and v_S of the regular component v of R are required in our decomposition due to the weak singular right-hand side of the differential equation (12). The boundary layer function w of R will satisfy the problem $Lw = 0$, $w(0,t) = w(x,0) = 0$, $w(1,t) \neq 0$. All these functions and the constants B_n are specified below.

Consider the following function

$$z(x,t) := \phi(x) + z_0(x,t) + \varepsilon z_1(x,t) + \varepsilon^2 R_z(x,t);$$

where

$$L_0 z_0 = f + \varepsilon\phi''(x) - a(t)\phi'(x), \; 0 < x \leq 1, \; t > 0,$$

$$z_0(0,t) = g_L(t) - C_L(t) - \phi(0) - \sum_{n=2}^{5} B_n S_n(0,t), \; t > 0, \; z_0(x,0) = 0, \; 0 \leq x \leq 1,$$

$$L_0 z_1 = \frac{\partial^2 z_0}{\partial x^2}, \; 0 < x \leq 1, \; t > 0, \; z_1(0,t) = 0, \; t > 0, \; z_1(x,0) = 0, \; 0 \leq x \leq 1,$$

$$L R_z = \frac{\partial^2 z_1}{\partial x^2}, \; (x,t) \in G, \; R_z(0,t) = R_z(1,t) = 0, \; t > 0, \; R_z(x,0) = 0, \; 0 \leq x \leq 1,$$

with the reduced differential operator

$$L_0 := \frac{\partial}{\partial t} + a(t)\frac{\partial}{\partial x}.$$

From this construction, $z(1,t) = z_0(1,t) + \varepsilon z_1(1,t)$ and

$$Lz = f, \quad z(0,t) = R(0,t) - \sum_{n=2}^{5} B_n S_n(0,t), \; z(x,0) = R(x,0).$$

Note that the Hölder space $C^{n+\gamma}(D)$ is the standard function space used for parabolic problems. In the case of the first order problem $L_0 z_0 = f$, a different function space $\mathcal{C}^{n,\gamma}(D)$ is required. This function space is defined by

$$\mathcal{C}^{n,\gamma}(D) := \left\{ z : \frac{\partial^{i+j} z}{\partial x^i \partial t^j} \in \mathcal{C}^{0+\gamma}(D), \; 0 \leq i + j \leq n \right\}.$$

Observe that the function z_0 satisfies $z_0(0^+, 0) = z_0(0, 0^+)$ and the first level compatibility condition

$$f(0,0) + \varepsilon\phi''(0) - a(0)\phi'(0) = g_L'(0) - C_L'(0),$$

is satisfied automatically. Hence, from [4] and [16, Theorem 4.1], the function z_0 belongs to the space $\mathcal{C}^{1,\gamma}(\bar{G})$. Now the parameters $B_n, n = 2, 3, 4, 5$ are chosen so that the necessary compatibility conditions on the reduced solution z_0 are imposed in order that $z_0 \in \mathcal{C}^{5,\gamma}(\bar{G})$. Then, $z_1 \in \mathcal{C}^{4,\gamma}(\bar{G})$ and $R_z \in C^{4+\gamma}(\bar{G})$. Hence, $z \in C^{4+\gamma}(\bar{G})$ and

$$\left|\frac{\partial^{i+j}z}{\partial x^i\partial t^j}\right| \le C\left(1+\varepsilon^{2-i}\right), \quad 0\le i+2j\le 4.$$

We next define the component v_S as the solution of the initial-boundary value problem

$$Lv_S = LR - Lz - \sum_{n=2}^{5} B_n LS_n = -\sum_{n=0}^{1} A_n LS_n - \sum_{n=2}^{5} B_n LS_n, \quad (x,t)\in G,$$
$$v_S(0,t)=v_S(1,t)=0,\ t>0, \quad v_S(x,0)=0,\ 0\le x\le 1.$$

Observe that $v_S \in C^{4+\gamma}(\bar{G})$ as

$$(Lv_S)(0,0)=(Lv_S)_x(0,0)=(Lv_S)_{xx}(0,0)=(Lv_S)_t(0,0)=0,$$

and $Lv_S\in C^{2+\gamma}(\bar{G})$. To deduce bounds on the derivatives of the component v_S consider the stretched variables

$$\tau=\frac{t}{\varepsilon},\ \zeta=\frac{x}{\varepsilon} \tag{14}$$

and we denote $\tilde{g}(\zeta,\tau):=g(x,t)$ for any function g. Then, we have

$$\frac{x+d(t)}{2\sqrt{\varepsilon t}} = \frac{\zeta+\tilde{d}_1(\tau)}{2\sqrt{\tau}}, \text{ with } \tilde{d}_1(\tau):=\frac{1}{\varepsilon}\int_{s=0}^{\varepsilon\tau} a(s)\,ds = \int_{s=0}^{\tau}\tilde{a}(s)\,ds.$$

Using (11), we have that

$$-\frac{\partial^2\tilde{v}_S}{\partial\zeta^2}+\tilde{a}(\tau)\frac{\partial\tilde{v}_S}{\partial\zeta}+\frac{\partial\tilde{v}_S}{\partial\tau} = -\varepsilon^2\tilde{\Phi}^+(\zeta,\tau),$$

where

$$\tilde{\Phi}^+(\zeta,\tau):=\tau\tilde{P}(\tau)\left(\sum_{n=0}^{1}\frac{A_n\tilde{\psi}^+_{n+1}}{\varepsilon}+\sum_{n=2}^{5}\frac{B_n\tilde{\psi}^+_{n+1}}{\varepsilon}\right).$$

Then, from the definition (4) of the basic functions ψ_n^- and ψ_n^+, we have

$$\left|\frac{\partial^{i+j}\tilde{\Phi}^+}{\partial\zeta^i\partial\tau^j}\right|\le C, \quad 0\le i+2j\le 2.$$

From [9] and [14], we have the following estimates for the partial derivatives of $\tilde{v}_S$

$$\left|\frac{\partial^{i+j}\tilde{v}_S}{\partial\zeta^i\partial\tau^j}\right|\le C\varepsilon^2, \quad 0\le i+2j\le 4.$$

Returning to the original variables, we get that

$$\left|\frac{\partial^{i+j}v_S}{\partial x^i\partial t^j}\right|\le C\left(1+\varepsilon^{2-(i+j)}\right), \quad 0\le i+2j\le 4.$$

The regular component v, which is defined in (13), satisfies $v\in C^{4+\gamma}(\bar{G})$ and the bounds (10b) follow by its construction.

Finally, consider the boundary layer component w; it is the solution of

$$Lw=0, \quad (x,t)\in G, \quad w(x,0)=0,\ 0\le x\le 1,$$
$$w(0,t)=0,\ w(1,t)=\left(R-v-\sum_{n=2}^{3}B_nS_n\right)(1,t),\ t>0.$$

The bounds on w are established as in [10, Theorem 1]. We note that we require the assumption (1d) to establish the bounds on the derivatives of the boundary layer function w. This assumption guarantees that $d(T)<1$ and then the interior and boundary layers do not interact with each other. □

In the next theorem, we consider the case where (2a) is satisfied and the solution is continuous. In this case we can relax the constraints on the coefficient $a(x,t)$ and allow this coefficient to vary in both space and time.

Theorem 2. *Assume that* $a_x(0,0)=0$ *and* $g_L(0)=\phi(0)$. *The solution* u *of (1) can be expanded as follows:*

$$u(x,t)=A_1S_1(x,t)+\sum_{i=2}^{3}B_iS_i(x,t)+v(x,t)+w(x,t), \tag{15a}$$

where A_1 *is defined in* (2b) *and for* $0\le i+2j\le 4$

$$\left|\frac{\partial^i v}{\partial x^i}\right|\le C(1+\varepsilon^{2-i}),\qquad \left|\frac{\partial^j v}{\partial t^j}\right|\le C, \tag{15b}$$

$$\left|\frac{\partial^{i+j} w}{\partial x^i\partial t^j}\right|\le C\varepsilon^{-i}(1+\varepsilon^{1-j})e^{-\alpha(1-x)/\varepsilon}. \tag{15c}$$

Proof. Follow the argument in Theorem 2, but now we define the remainder to be $R:=u(x,t)-A_1S_1(x,t)$ which satisfies

$$LR=f-\frac{p(t)}{\varepsilon t^2}\frac{A_1}{a(0,0)}\psi_2^+-(a(x,t)-a(d(t),t))A_1\frac{\partial S_1}{\partial x}. \tag{16}$$

We examine the regularity of the function R. Compare (16) with (12). As in Theorem 1, the term $f-\frac{p(t)}{\varepsilon t^2}\frac{A_1}{a(0,0)}\psi_2^+$ in the right-hand side of (16) belongs to $C^{2+\gamma}(\bar G)$ since $\psi_2^+(x,t)\in C^{2+\gamma}(\bar G)$. We now consider the other term $(a(x,t)-a(d(t),t))A_1\frac{\partial S_1}{\partial x}$ of (16). Observe that, if $a_x(0,0)=0$ then

$$a(x,t)-a(d(t),t)=\int_{s=d(t)}^{x}a_x(s,t)\,ds=\int_{s=d(t)}^{x}\int_{r=0}^{t}a_{xt}(s,r)\,dr\,ds+\int_{s=d(t)}^{x}\int_{r=0}^{s}a_{xx}(r,0)\,dr\,ds.$$

Hence,

$$\begin{aligned}|a(x,t)-a(d(t),t)|&\le Ct|x-d(t)|+C|(x-d(t))|\,|(x+d(t))|\\&\le Ct|x-d(t)|+C(x-d(t))^2.\end{aligned}$$

In addition, we have that

$$\begin{aligned}a(0,0)\frac{\partial}{\partial t}\left(\frac{\partial S_1}{\partial x}\right)&=\frac{\partial}{\partial t}\left(\psi_0^+-\psi_0\right)+\frac{\partial}{\partial t}\left(\frac{d(t)}{\varepsilon t}\psi_1^+\right)\quad\text{and}\\ \frac{\partial}{\partial t}\left(\psi_0^+-\ \psi_0\right)&=\frac{d(t)-2ta(d(t),t)}{2t\sqrt{\varepsilon\pi t}}E(x,t)+\frac{p(t)x}{\varepsilon t^2}\psi_0^+,\\ \frac{\partial}{\partial t}\left(\frac{d(t)}{\varepsilon t}\psi_1^+\right)&=\psi_1^+\frac{\partial}{\partial t}\left(\frac{d(t)}{\varepsilon t}\right)+\frac{d(t)}{\varepsilon t}\frac{\partial\psi_1^+}{\partial t}.\end{aligned}$$

Recall (24c) and observe also that

$$(x-d(t))(x+d(t))\frac{1}{t}\psi_0^+\in C^{0+\gamma}(\bar G).$$

Together, these imply that

$$(x-d(t))t\frac{\partial\psi_1^+}{\partial t},\ (x-d(t))^2\frac{\partial\psi_1^+}{\partial t}\in C^{0+\gamma}(\bar G).$$

Therefore,

$$(x-d(t))t\frac{\partial^2 S_1}{\partial x\partial t},\ (x-d(t))^2\frac{\partial^2 S_1}{\partial x\partial t}\in C^{0+\gamma}(\bar G).$$

Thus, it follows that $LR\in C^{2+\gamma}(\bar G)$ if we assume that $a_x(0,0)=0$. Hence, $R\in C^{2+\gamma}(\bar G)$. Furthermore, using the stretched variables τ and ζ defined in (14), we note that if

$$\Phi_1(x,t):=\frac{1}{a(0,0)}(a(x,t)-a(d(t),t))\left((\psi_0^+-\psi_0^-)+\frac{d(t)}{\varepsilon}\psi_1^+\right),$$

then, as $|\tilde a(\zeta,\tau)-\tilde a(d(\tau),\tau)|\le C\varepsilon^2(\zeta+\tau)^2$, it follows that

$$|\tilde{\Phi}_1(\zeta,\tau)| \le C\varepsilon^2(\zeta+\tau)^2\left(|\tilde{\psi}_0^+|+|\tilde{\psi}_0^-|+\tau|\tilde{\psi}_1^+|\right).$$

Use this expression and (32) to deduce bounds on the derivatives of the corresponding component v_S of the solution u. The argument is then completed as in the proof of the previous theorem. □

In the next section, we describe a numerical method that will generate a numerical approximation to $y = u - A_0 S_0$. If $A_0 = 0$ (i.e., the zero level compatibility is satisfied), note that $y = u$. The function y satisfies the singularly perturbed problem

$$Ly = f - A_0 LS_0, \quad (x,t) \in G, \tag{17a}$$

$$y(0,t) = g_L(t) - A_0,\ y(1,t) = g_R(t) - A_0 S_0(1,0), \quad t > 0, \tag{17b}$$

$$y(x,0) = \phi(x), \quad 0 \le x \le 1. \tag{17c}$$

4. Numerical method

Let N and $M = O(N)$ be two positive integers. We approximate problem (17) with a finite difference scheme on a mesh $\bar{G}^{N,M} = \{x_i\}_{i=0}^N \times \{t_j\}_{j=0}^M$. We denote by $\partial G^{N,M} := \bar{G}^{N,M} \backslash G$. The mesh $\bar{G}^{N,M}$ incorporates a uniform mesh ($t_j := kj$ with $k = T/M$) for the time variable and a piecewise-uniform mesh for the space variable with $h_i := x_i - x_{i-1}$. The piecewise uniform mesh $\{x_i\}_{i=0}^N$ is a Shishkin mesh [7] which splits the interval $[0,1]$ into the two subintervals

$$[0, 1-\sigma] \cup [1-\sigma, 1], \quad \text{where} \quad \sigma := \min\left\{0.5, \frac{\varepsilon}{\alpha}\ln N\right\}.$$

The $N+1$ space mesh points are distributed in the ratio $N/2 : N/2$ across the two subintervals. The discrete problem[2] is: Find Y such that

$$L^{N,M}Y := -\varepsilon\delta_x^2 Y + aD_x^- Y + D_t^- Y = f - A_0 LS_0, \quad t_j > 0, \tag{18a}$$

$$Y(x_i, 0) = y(x_i, 0),\ 0 < x_i < 1, \tag{18b}$$

$$Y(0,t_j) = y(0,t_j), \quad Y(1,t_j) = y(1,t_j),\ t_j \ge 0. \tag{18c}$$

We form a global approximation $\bar{Y}$ using simple bilinear interpolation:

$$\bar{Y}(x,t) := \sum_{i=0,j=1}^{N,M} Y(x_i,t_j)\varphi_i(x)\eta_j(t),$$

where $\varphi_i(x)$ is the standard hat function centered at $x = x_i$ and $\eta_j(t) := (t - t_{j-1})/k$, $t \in (t_{j-1}, t_j]$, $\eta_j(t) := 0$ otherwise.

In the next theorem, a convergence result is given in the particular case of $a = a(t)$ and $g_L(0) \ne \phi(0)$. In this case, the solution of problem (1) is decomposed as in Theorem 1 and after separating off the singular function S_0, the numerical method (18) is applied to approximate $y = u - A_0 S_0$.

Theorem 3. *Assume that $a(x,t) = a(t)$, $\forall (x,t) \in \bar{G}$, $a_t(0) = 0$, and $M = O(N)$. If Y is the solution of (18) and y is the solution of (17), then*

$$\|\bar{Y} - y\|_{\bar{G}} \le C|A_1|N^{-1/2} + CN^{-1}\ln N.$$

Proof. As in the case of the continuous problem, the discrete solution can be decomposed into the sum $Y = A_1 S_1^N + A_2 S_2^N + V + W$, where

$$L^{N,M}V = Lv,\ (x_i,t_j) \in G^{N,M} \text{ and } V = v,\ (x_i,t_j) \in \partial G^{N,M};$$

$$L^{N,M}W = 0,\ (x_i,t_j) \in G^{N,M} \text{ and } W = w,\ (x_i,t_j) \in \partial G^{N,M};$$

$$L^{N,M}S_k^N = LS_k,\ (x_i,t_j) \in G^{N,M} \text{ and } S_k^N = S_k,\ (x_i,t_j) \in \partial G^{N,M},\ k = 1,2.$$

[2] We use the following notation for the finite difference approximations of the derivatives:

$$D_t^- Y(x_i,t_j) := \frac{Y(x_i,t_j) - Y(x_i,t_{j-1})}{k}, \quad D_x^- Y(x_i,t_j) := \frac{Y(x_i,t_j) - Y(x_{i-1},t_j)}{h_i},$$

$$D_x^+ Y(x_i,t_j) := \frac{Y(x_{i+1},t_j) - Y(x_i,t_j)}{h_{i+1}}, \quad \delta_x^2 Y(x_i,t_j) := \frac{2}{h_i + h_{i+1}}(D_x^+ Y(x_i,t_j) - D_x^- Y(x_i,t_j)).$$

Using the bounds on the derivatives (10c) of the component w to obtain appropriate truncation error estimates, the discrete maximum principle with a suitable discrete barrier function and following the arguments in [17], we can establish the following bounds

$$|(w-W)(x_i,t_j)| \le CN^{-1}\ln N, \quad (x_i,t_j) \in \bar{G}^{N,M}. \tag{19}$$

The error due to the regular component v can be bounded in a classical way [18] to deduce that

$$|(v-V)(x_i,t_j)| \le CN^{-1}, \quad (x_i,t_j) \in \bar{G}^{N,M}. \tag{20}$$

Let us now consider the two weakly singular functions S_1, S_2 and their numerical approximations S_1^N, S_2^N. For both functions, the truncation error is denoted by

$$\mathcal{T}_{S_k;i,j} := L^{N,M}(S_k - S_k^N)(x_i,t_j),$$

then

$$|\mathcal{T}_{S_k;i,j}| \le C\varepsilon(h_i+h_{i+1}) \left\| \frac{\partial^3 S_k(x,t_j)}{\partial x^3} \right\|_{(x_{i-1},x_{i+1})} + C\min\left\{ h_i \left\| \frac{\partial^2 S_k(x,t_j)}{\partial x^2} \right\|_{(x_{i-1},x_i)}, \left\| \frac{\partial S_k(x,t_j)}{\partial x} \right\|_{(x_{i-1},x_i)} \right\}$$
$$+ C\min\left\{ \frac{1}{k} \int_{w=t_{j-1}}^{t_j} \int_{r=w}^{t_j} \left| \frac{\partial^2 S_k(x_i,r)}{\partial t^2} \right| dr\, dw, \left\| \frac{\partial S_k(x_i,t)}{\partial t} \right\|_{(t_{j-1},t_j)} \right\},$$

as

$$|D_t^- S_k(x_i,t_j)| \le \frac{1}{k} \int_{r=t_{j-1}}^{t_j} \left| \frac{\partial S_k(x_i,r)}{\partial r}\, dr \right| \le C \left\| \frac{\partial S_k(x_i,t)}{\partial t} \right\|_{(t_{j-1},t_j)}.$$

Note also that at each time level,

$$\left(-\varepsilon\delta_x^2 + aD_x^- + \frac{1}{k}I\right)(S_k - S_k^N)(x_i,t_j) = \mathcal{T}_{S_k;i,j} + \frac{1}{k}(S_k - S_k^N)(x_i,t_{j-1}),\ t_j > 0.$$

In the case of the weaker singular function S_2, we use the bounds (7) so that the truncation error at the first time level $t = t_1$ is

$$|\mathcal{T}_{S_2;i,1}| \le 2\left(\varepsilon \left\| \frac{\partial^2 S_2}{\partial x^2} \right\|_{(x_{i-1},x_{i+1})} + a \left\| \frac{\partial S_2}{\partial x} \right\|_{(x_{i-1},x_i)} + \left\| \frac{\partial S_2}{\partial t} \right\|_{(t_0,t_1)} \right) \le C(\varepsilon + \sqrt{\varepsilon t_1}) e^{-\gamma \frac{(x_i - at_1)^2}{4\varepsilon T}} \le C.$$

At the next time levels t_n, $n \ge 2$, we again use the bounds (7) to deduce the truncation error bounds

$$|\mathcal{T}_{S_2;i,j}| \le C\varepsilon(h_i+h_{i+1}) \left\| \frac{\partial^3 S_2}{\partial x^3} \right\|_{(x_{i-1},x_{i+1})} + ah_i \left\| \frac{\partial^2 S_2}{\partial x^2} \right\|_{(x_{i-1},x_i)} + Ck \left\| \frac{\partial^2 S_2}{\partial t^2} \right\|_{(t_{j-1},t_j)}$$
$$\le \left(CN^{-1}\left(1 + \sqrt{\frac{\varepsilon}{t_j}} + \sqrt{\frac{t_j}{\varepsilon}}\right) + CM^{-1}\left(1 + \sqrt{\frac{\varepsilon}{t_{j-1}}} + \frac{\varepsilon}{t_{j-1}} + \sqrt{\frac{t_j}{\varepsilon}}\right)\right) E_\gamma(x,t)$$
$$\le C\left(\frac{M^{-1}}{\sqrt{\varepsilon}} + \frac{M^{-1/2}}{\sqrt{j-1}} + \frac{\varepsilon}{j-1} \right) E_\gamma(x,t), \quad j \ge 2,$$

as $M = CN$. Then, we deduce the error bound

$$\left|(S_2 - S_2^N)(x_i,t_j)\right| \le CM^{-1} \sum_{n=1}^{j} |\mathcal{T}_{S_2;i,n}| \le CM^{-1} + CM^{-1} \sum_{n=2}^{j} |\mathcal{T}_{S_2;i,n}|$$
$$\le CM^{-1} + CM^{-1} \sum_{n=2}^{j} \frac{M^{-1}}{\sqrt{\varepsilon}} E_\gamma(x,t) + CM^{-3/2} \sum_{n=2}^{j} \frac{1}{\sqrt{n-1}} + CM^{-1}\varepsilon \sum_{n=2}^{j} \frac{1}{n-1}$$
$$\le CM^{-1} + CM^{-3/2} \int_{s=1}^{j} \frac{ds}{\sqrt{s}} + CM^{-1}\varepsilon \int_{s=1}^{j} \frac{ds}{s}$$
$$\le CM^{-1} + CM^{-1}\varepsilon \ln M \le CM^{-1}.$$

Finally, we consider the error due to the singular component S_1. The argument splits into the two cases of $\varepsilon \le CM^{-1}$ and $\varepsilon \ge CM^{-1}$. If $M\varepsilon \ge C$, from (6) we obtain the following truncation errors bounds at the first time level $t = t_1$

$$|\mathcal{T}_{S_1;i,1}| \le 2\left(\varepsilon \left\| \frac{\partial^2 S_1}{\partial x^2}\right\|_{(x_{i-1},x_{i+1})} + a\left\|\frac{\partial S_1}{\partial x}\right\|_{(x_{i-1},x_i)} + \left\|\frac{\partial S_1}{\partial t}\right\|_{(t_0,t_1)}\right) \le C.$$

At the next time levels t_n, $n \ge 2$, use again (6) to deduce the truncation error bounds

$$\begin{aligned}
|\mathcal{T}_{S_1;i,j}| &\le C\varepsilon(h_i + h_{i+1})\left\|\frac{\partial^3 S_1}{\partial x^3}\right\|_{(x_{i-1},x_{i+1})} + ah_i\left\|\frac{\partial^2 S_1}{\partial x^2}\right\|_{(x_{i-1},x_i)} + Ck\left\|\frac{\partial^2 S_1}{\partial t^2}\right\|_{(t_{j-1},t_j)}\\
&\le C\left(\frac{N^{-1}}{\varepsilon} + \frac{N^{-1}+M^{-1}}{\sqrt{\varepsilon t_{j-1}}} + \frac{M^{-1}}{t_{j-1}}\right)E_\gamma(x,t) + CN^{-1}\\
&\le C\left(\frac{M^{-1}}{\varepsilon} + \frac{M^{-1/2}}{\sqrt{\varepsilon(j-1)}} + \frac{1}{j-1}\right)E_\gamma(x,t) + CN^{-1}, \quad j \ge 2,
\end{aligned}$$

as $M = CN$. Then,

$$\begin{aligned}
\left|(S_1 - S_1^N)(x_i,t_j)\right| &\le CM^{-1}\sum_{n=1}^{j}|\mathcal{T}_{S_1;i,n}| \le CM^{-1} + CM^{-1}\sum_{n=2}^{j}|\mathcal{T}_{S_1;i,n}|\\
&\le CM^{-1} + C\frac{M^{-1}}{\sqrt{\varepsilon}}\sum_{n=2}^{j}\frac{M^{-1}}{\sqrt{\varepsilon}}E_\gamma(x,t) + C\frac{M^{-3/2}}{\sqrt{\varepsilon}}\sum_{n=2}^{j}\frac{1}{\sqrt{n-1}} + CM^{-1}\sum_{n=2}^{j}\frac{1}{n-1}\\
&\le C\frac{M^{-1}}{\sqrt{\varepsilon}} + C\frac{M^{-3/2}}{\sqrt{\varepsilon}}\int_{s=1}^{j}\frac{ds}{\sqrt{s}} + CM^{-1}\int_{s=1}^{j}\frac{ds}{s},\\
&\le CM^{-1/2} + CM^{-1}\ln M \le CM^{-1/2}.
\end{aligned} \tag{21}$$

In the other case of $M\varepsilon \le C$, from (8a) we first note the following bounds

$$\left|\frac{\partial \psi_1^+}{\partial x}\right| + \left|\frac{\partial^2\psi_1^+}{\partial x^2}\right| \le \frac{C}{\varepsilon}E_\gamma(x,t),$$

and from (26b) and (32)

$$\left|\frac{\partial \psi_1^+}{\partial t}\right| \le |L\psi_1^+| + \varepsilon\left|\frac{\partial^2\psi_1^+}{\partial x^2}\right| + a(t)\left|\frac{\partial\psi_1^+}{\partial x}\right| \le CE_\gamma(x,t),$$

which should be compared to (6). Also, $a(0)LS_1 = L\psi_1^+$. Hence,

$$|\mathcal{T}_{S_1;i,j}| \le C\varepsilon\left\|\frac{\partial^2\psi_1^+}{\partial x^2}\right\|_{(x_{i-1},x_{i+1})} + C\left\|\frac{\partial\psi_1^+}{\partial x}\right\|_{(x_{i-1},x_i)} + C\left\|\frac{\partial\psi_1^+}{\partial t}\right\|_{(t_{j-1},t_j)} \le CE_\gamma(x,t).$$

We now have

$$\begin{aligned}
\left|(S_1 - S_1^N)(x_i,t_j)\right| &\le CM^{-1}\sum_{n=1}^{j}|\mathcal{T}_{S_1;i,n}| \le C\sqrt{\varepsilon}\sum_{n=1}^{j}\frac{M^{-1}}{\sqrt{\varepsilon}}E_\gamma(x,t)\\
&\le C\sqrt{\varepsilon}\sum_{n=1}^{j}\frac{M^{-1}}{\sqrt{\varepsilon}}e^{-\gamma\frac{(x_i-at_j)^2}{4\varepsilon T}} \le C\sqrt{\varepsilon} \le CM^{-1/2},
\end{aligned} \tag{22}$$

where we have used that $\int_{r=-\infty}^{\infty}\frac{1}{p}e^{-\frac{r^2}{p}}dr = \sqrt{\pi}$. From (21) and (22), we deduce

$$\left|(S_1 - S_1^N)(x_i,t_j)\right| \le CM^{-1/2}.$$

Combining all of the bounds above, we deduce the nodal error bound

$$\|Y - y\|_{\bar{G}^{N,M}} \le C|A_1|N^{-1/2} + CN^{-1}\ln N.$$

Use the arguments in [11] to extend this nodal error bound to the global error bound. □

Table 1
Maximum two-mesh global differences and orders of convergence for Example 1.

	N=M=16	N=M=32	N=M=64	N=M=128	N=M=256	N=M=512	N=M=1024
$\varepsilon = 2^{0}$	2.593E-03	1.306E-03	6.567E-04	3.285E-04	1.643E-04	8.212E-05	4.106E-05
	0.989	0.992	1.000	1.000	1.000	1.000	
$\varepsilon = 2^{-6}$	3.004E-02	1.768E-02	1.014E-02	5.522E-03	2.888E-03	1.467E-03	7.321E-04
	0.764	0.802	0.877	0.935	0.977	1.003	
$\varepsilon = 2^{-12}$	3.547E-02	2.356E-02	1.598E-02	1.103E-02	7.581E-03	5.175E-03	3.442E-03
	0.590	0.560	0.535	0.541	0.551	0.588	
$\varepsilon = 2^{-18}$	3.551E-02	2.363E-02	1.609E-02	1.118E-02	7.818E-03	5.502E-03	3.877E-03
	0.588	0.554	0.526	0.516	0.507	0.505	
$\varepsilon = 2^{-24}$	3.551E-02	2.363E-02	1.609E-02	1.118E-02	7.820E-03	5.505E-03	3.882E-03
	0.588	0.554	0.525	0.516	0.506	0.504	
$\varepsilon = 2^{-30}$	3.551E-02	2.363E-02	1.609E-02	1.118E-02	7.820E-03	5.506E-03	3.882E-03
	0.588	0.554	0.525	0.516	0.506	0.504	
$D^{N,M}$	3.551E-02	2.363E-02	1.609E-02	1.118E-02	7.820E-03	5.506E-03	3.882E-03
$P^{N,M}$	0.588	0.554	0.525	0.516	0.506	0.504	

Remark 1. If the convective coefficient $a(t)$ only depends on the time variable and the constraint (1d) is not imposed on the final time T, then the interior layer will interact with the boundary layer (see [1,11]) in an $O(\sqrt{\varepsilon})$ neighbourhood of the point $(1, T_*)$, where $d(T_*) = 1$. To retain the parameter uniform error bound (as stated in Theorem 3), an additional piecewise uniform Shishkin mesh in time should be used either side of $t = T_*$. See [11] for details of the mesh and the associated proof of uniform convergence. Minor modifications to the proof of the error bound are required to deal with the presence of additional terms involving $\psi_i^+(1,t), i = 0, 1, 2, 3, 4$. Example 3 in the numerical section deals with this case of the interior layer and boundary layer interacting.

In the final theorem, we consider the case of $g_L(0) = \phi(0)$, where the solution of (1) is continuous. In this case the solution of problem (1) with $a = a(x,t)$ can be decomposed as in Theorem 2 and the numerical method (18) is applied directly to the problem without separating off the singular function S_0. The proof of Theorem 3 is also valid for the following result.

Theorem 4. *Assume that $a_x(0,0) = 0$, $g_L(0) = \phi(0)$ and $M = O(N)$. If Y is the solution of (18) and u is the solution of (1), then*

$$\|\bar{Y} - y\|_{\bar{G}} \le C|A_1|N^{-1/2} + CN^{-1}\ln N.$$

Remark 2. From the hypothesis of Theorem 4, it is satisfied that $A_0 = g_L(0) - \phi(0) = 0$ and then $y = u$ and $Y = U$, where U is the numerical approximation of u at the mesh points of $\bar{G}^{N,M}$.

5. Numerical experiments

The solution of all the test examples presented below is unknown and the global orders of convergence are estimated using the two-mesh method [7, Chapter 8]. In this particular section, the computed solutions with (18) on the Shishkin meshes $\bar{G}^{N,M}$ and $\bar{G}^{2N,2M}$ are denoted, respectively, by $Y^{N,M}$ and $Y^{2N,2M}$. Let $\bar{Y}^{N,M}$ be the bilinear interpolation of the discrete solution $Y^{N,M}$ on the mesh $\bar{G}^{N,M}$. Then, compute the maximum two-mesh global differences

$$D_\varepsilon^{N,M} := \|\bar{Y}^{N,M} - \bar{Y}^{2N,2M}\|_{\bar{G}^{N,M} \cup \bar{G}^{2N,2M}}$$

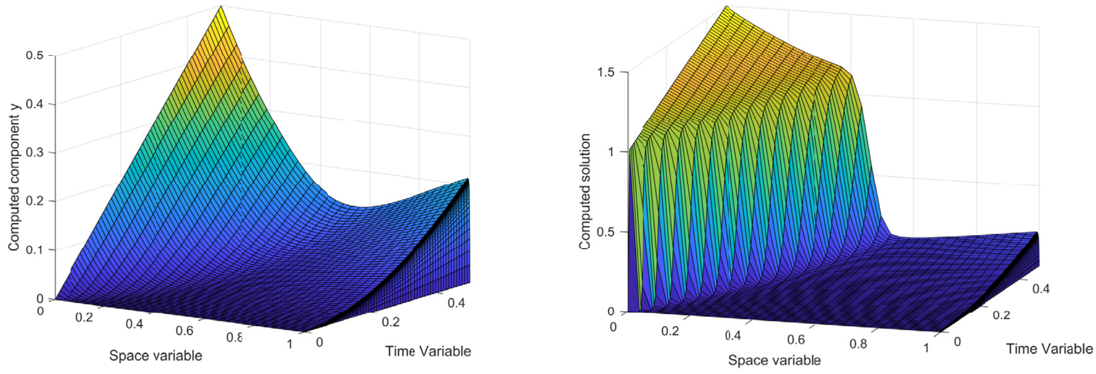
and use these values to estimate the orders of global convergence $P_\varepsilon^{N,M}$

$$P_\varepsilon^{N,M} := \log_2\left(\frac{D_\varepsilon^{N,M}}{D_\varepsilon^{2N,2M}}\right).$$

The uniform two-mesh global differences $D^{N,M}$ and the uniform orders of global convergence $P^{N,M}$ are calculated by

$$D^{N,M} := \max_{\varepsilon \in S} D_\varepsilon^{N,M}, \quad P^{N,M} := \log_2\left(\frac{D^{N,M}}{D^{2N,2M}}\right),$$

where $S = \{2^0, 2^{-1}, \ldots, 2^{-30}\}$. In all of the tables below we display the maximum and uniform two-mesh global differences and the corresponding orders of convergence for $N = 16, 32, \ldots, 1024$ and $N = M$. For the sake of brevity, we display the results in the tables for a smaller representative set of values of ε. Note that in the first three examples, the convective coefficient $a(t)$ does not depend on the spatial variable.



(a) Computed component Y (b) Numerical approximation U

Fig. 1. Example 1 with $\varepsilon = 2^{-10}$: Computed component Y with the scheme (18) for $N = M = 64$ and the numerical approximation U.

Table 2
Maximum two-mesh global differences and orders of convergence for Example 2.

	N=M=16	N=M=32	N=M=64	N=M=128	N=M=256	N=M=512	N=M=1024
$\varepsilon = 2^{0}$	6.959E-03	3.209E-03	1.541E-03	7.536E-04	3.725E-04	1.852E-04	9.233E-05
	1.117	1.058	1.032	1.016	1.008	1.004	
$\varepsilon = 2^{-6}$	5.202E-02	2.956E-02	1.666E-02	9.232E-03	5.067E-03	2.751E-03	1.484E-03
	0.816	0.827	0.852	0.865	0.881	0.891	
$\varepsilon = 2^{-12}$	6.938E-02	3.622E-02	2.037E-02	1.125E-02	6.138E-03	3.329E-03	1.792E-03
	0.938	0.830	0.856	0.875	0.883	0.894	
$\varepsilon = 2^{-18}$	6.968E-02	3.634E-02	2.044E-02	1.130E-02	6.161E-03	3.342E-03	1.799E-03
	0.939	0.830	0.855	0.875	0.882	0.894	
$\varepsilon = 2^{-24}$	6.969E-02	3.634E-02	2.044E-02	1.130E-02	6.162E-03	3.343E-03	1.799E-03
	0.939	0.830	0.855	0.875	0.882	0.894	
$\varepsilon = 2^{-30}$	6.969E-02	3.634E-02	2.044E-02	1.130E-02	6.163E-03	3.340E-03	1.802E-03
	0.939	0.830	0.855	0.874	0.884	0.890	
$D^{N,M}$	6.969E-02	3.634E-02	2.044E-02	1.130E-02	6.163E-03	3.343E-03	1.802E-03
$p^{N,M}$	0.939	0.830	0.855	0.874	0.883	0.891	

Example 1. We consider the following initial-boundary value problem

$$u_t - \varepsilon u_{xx} + (1 - t^2)u_x = 2tx, \quad (x,t) \in (0,1) \times (0,0.5],$$
$$u(x,0) = 0, \quad x \in (0,1),$$
$$u(0,t) = 1 + t,\ u(1,t) = 0, \quad t \in [0,0.5].$$

Note that $a'(0) = 0$ and $A_1 = 1 \neq 0$ in this example. In Fig. 1 the computed component Y with the scheme (18) for $\varepsilon = 2^{-10}$ and $N = M = 64$ is shown. The approximation U to the solution of Example 1 also appears in that figure; the interior layer emanating from the point $(0,0)$ and the boundary layer in the outflow boundary are observed. The numerical results are in Table 1 and they indicate that the numerical method (18) converges uniformly and globally with order $O(N^{-1/2})$ in agreement with Theorem 3.

Example 2. Consider the example

$$u_t - \varepsilon u_{xx} + (1 - t^2)u_x = 2tx, \quad (x,t) \in (0,1) \times (0,0.5],$$
$$u(x,0) = x^3, \quad x \in (0,1),$$
$$u(0,t) = 1 + t^2,\ u(1,t) = 1, \quad t \in [0,0.5].$$

In this example the data problem satisfy that $a'(0) = 0$ but $A_1 = 0$. The numerical results obtained for Example 2 with the numerical method (18) are given in Table 2 and they indicate that the method converges with order $O(N^{-1}\ln N)$ (see [7, p. 169]) as stated in Theorem 3.

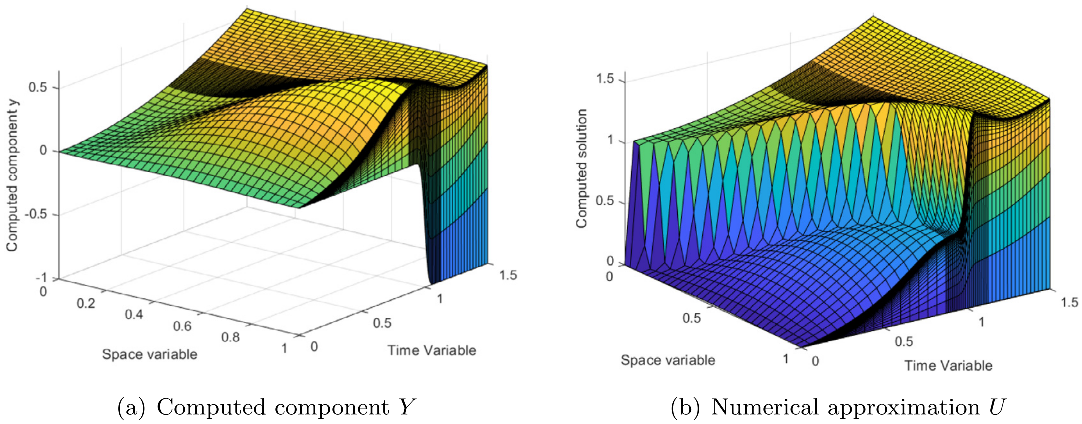


Fig. 2. Example 3 with $\varepsilon = 2^{-10}$: Computed component Y with the scheme (18) for $N = M = 64$ and the numerical approximation U.

Table 3
Maximum two-mesh global differences and orders of convergence for Example 3.

	N=M=16	N=M=32	N=M=64	N=M=128	N=M=256	N=M=512	N=M=1024
$\varepsilon = 2^0$	4.394E-02	1.659E-02	8.327E-03	4.169E-03	2.087E-03	1.044E-03	5.220E-04
	1.405	0.995	0.998	0.999	0.999	1.000	
$\varepsilon = 2^{-6}$	1.814E-01	1.194E-01	6.717E-02	4.134E-02	2.517E-02	1.435E-02	8.148E-03
	0.604	0.830	0.700	0.716	0.811	0.816	
$\varepsilon = 2^{-12}$	1.805E-01	1.190E-01	6.693E-02	4.124E-02	2.514E-02	1.433E-02	8.143E-03
	0.601	0.830	0.699	0.714	0.811	0.816	
$\varepsilon = 2^{-18}$	1.802E-01	1.187E-01	6.676E-02	4.117E-02	2.511E-02	1.432E-02	8.135E-03
	0.603	0.830	0.698	0.713	0.811	0.815	
$\varepsilon = 2^{-24}$	1.802E-01	1.187E-01	6.674E-02	4.116E-02	2.511E-02	1.431E-02	8.134E-03
	0.603	0.830	0.697	0.713	0.811	0.815	
$\varepsilon = 2^{-30}$	1.802E-01	1.187E-01	6.673E-02	4.116E-02	2.508E-02	1.432E-02	8.149E-03
	0.603	0.830	0.697	0.715	0.808	0.813	
$D^{N,M}$	1.814E-01	1.231E-01	8.014E-02	4.140E-02	2.520E-02	1.437E-02	8.165E-03
$p^{N,M}$	0.560	0.619	0.953	0.716	0.811	0.815	

Example 3. Consider the example

$$u_t - \varepsilon u_{xx} + (1 + 3t^2 - 2t)u_x = 4x(1 - x), \quad (x, t) \in (0, 1) \times (0, 1.5],$$
$$u(x, 0) = x^3(1 - x)^3, \quad x \in (0, 1),$$
$$u(0, t) = 1 + 0.25t^2, \; u(1, t) = 0, \quad t \in [0, 1.5].$$

Note that $a'(0) \neq 0$ and $A_1 = 0$. As regards the outflow point $(1, 0)$, the compatibility condition (2c) is satisfied, but not (2e). Furthermore, condition (1d) is not fulfilled; then the interior and boundary layers interact with each other. This effect is observed in Fig. 2 where the approximations to the component y and the solution u are shown. The numerical results obtained with the scheme (18) combined with a modification to the mesh in time [11, (19)] (see also Remark 1) are given in Table 3. These results suggest that the method converges globally and uniformly with order $O(N^{-1} \ln N)$.

Example 4. Consider the example

$$u_t - \varepsilon u_{xx} + (1 + x^2)u_x = 4x(1 - x), \quad (x, t) \in (0, 1) \times (0, 0.5],$$
$$u(x, 0) = 0, \quad x \in (0, 1),$$
$$u(0, t) = u(1, t) = t^2, \quad t \in [0, 0.5].$$

Note that $a_x(0, 0) = 0$ and $A_0 = A_1 = 0$. The numerical results obtained with the scheme (18) are given in Table 4 and they indicate that the method converges with order $O(N^{-1} \ln N)$ as stated in Theorem 4.

Example 5. Consider the example

$$u_t - \varepsilon u_{xx} + (1 + x)u_x = 4x(1 - x), \quad (x, t) \in (0, 1) \times (0, 0.5],$$

Table 4
Maximum two-mesh global differences and orders of convergence for Example 4.

	N=M=16	N=M=32	N=M=64	N=M=128	N=M=256	N=M=512	N=M=1024
$\varepsilon=2^{0}$	2.960E-03 0.967	1.515E-03 0.992	7.615E-04 0.996	3.819E-04 0.998	1.912E-04 0.999	9.567E-05 1.000	4.785E-05
$\varepsilon=2^{-6}$	2.590E-02 0.656	1.643E-02 0.742	9.829E-03 0.837	5.503E-03 0.863	3.026E-03 0.889	1.634E-03 0.900	8.757E-04
$\varepsilon=2^{-12}$	3.115E-02 0.636	2.004E-02 0.739	1.201E-02 0.843	6.698E-03 0.865	3.677E-03 0.884	1.992E-03 0.897	1.070E-03
$\varepsilon=2^{-18}$	3.126E-02 0.637	2.011E-02 0.739	1.205E-02 0.843	6.718E-03 0.865	3.689E-03 0.884	1.999E-03 0.897	1.073E-03
$\varepsilon=2^{-24}$	3.126E-02 0.637	2.011E-02 0.739	1.205E-02 0.843	6.719E-03 0.865	3.689E-03 0.884	1.999E-03 0.897	1.073E-03
$\varepsilon=2^{-30}$	3.126E-02 0.637	2.011E-02 0.739	1.205E-02 0.843	6.718E-03 0.865	3.690E-03 0.885	1.998E-03 0.894	1.075E-03
$D^{N,M}$ $p^{N,M}$	3.126E-02 0.637	2.011E-02 0.739	1.205E-02 0.843	6.719E-03 0.865	3.690E-03 0.884	1.999E-03 0.895	1.075E-03

Table 5
Maximum two-mesh global differences and orders of convergence for Example 5.

	N=M=16	N=M=32	N=M=64	N=M=128	N=M=256	N=M=512	N=M=1024
$\varepsilon=2^{0}$	1.426E-03 0.782	8.292E-04 0.863	4.557E-04 0.932	2.388E-04 0.967	1.222E-04 0.985	6.173E-05 0.994	3.099E-05
$\varepsilon=2^{-6}$	2.563E-02 0.714	1.562E-02 0.772	9.145E-03 0.839	5.113E-03 0.820	2.896E-03 0.856	1.600E-03 0.879	8.695E-04
$\varepsilon=2^{-12}$	3.099E-02 0.550	2.117E-02 0.559	1.437E-02 0.535	9.917E-03 0.520	6.914E-03 0.528	4.794E-03 0.557	3.258E-03
$\varepsilon=2^{-18}$	3.108E-02 0.547	2.128E-02 0.554	1.449E-02 0.525	1.007E-02 0.501	7.115E-03 0.492	5.061E-03 0.490	3.603E-03
$\varepsilon=2^{-24}$	3.108E-02 0.547	2.128E-02 0.554	1.449E-02 0.525	1.007E-02 0.501	7.119E-03 0.491	5.065E-03 0.489	3.609E-03
$\varepsilon=2^{-30}$	3.108E-02 0.547	2.128E-02 0.554	1.449E-02 0.525	1.007E-02 0.501	7.119E-03 0.491	5.065E-03 0.489	3.609E-03
$D^{N,M}$ $p^{N,M}$	3.108E-02 0.547	2.128E-02 0.554	1.449E-02 0.525	1.007E-02 0.501	7.119E-03 0.491	5.065E-03 0.489	3.609E-03

$$u(x,0)=0, \quad x\in(0,1),$$
$$u(0,t)=t;\ u(1,t)=t^2, \quad t\in[0,0.5].$$

Note that $a_x(0,0)\neq 0$ and $A_0=0$, $A_1\neq 0$. The numerical results obtained with the scheme (18) are given in Table 5. They suggest that the method converges globally and uniformly with order $O(N^{-1/2})$, but the theoretical justification of these results remains open, as the proof in Theorem 4 requires $a_x(0,0)=0$.

Remark 3. In the numerical experiments performed, we have considered (28) when evaluating ψ_0^+ and $\theta := d(t_j)x_i/(t_j\varepsilon)$ is a large number with $(x_i,t_j)\in G^{N,M}$ to prevent overflow problems. If $\theta\geq 300$, the value of the Mill's ratio H in Examples 1, 2 and 3 has been computed by using that

$$H(r)\sim \frac{1}{r\sqrt{\pi}}\left(1+\sum_{m=1}^{\infty}(-1)^m\frac{1.3\ldots(2m-1)}{(2r^2)^m}\right), \text{ as } r\to\infty.$$

This series has been approximated by the n-th partial sum and the maximum two-mesh global differences in Tables 1, 2 and 3 have been obtained with $n=5$. Similar results have been obtained if larger values of n are considered.

Acknowledgements

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Appendix A. Bounds on the derivatives of the functions $\psi_i^{\pm}(x,t)$

The functions $\psi_i^{\pm}(x,t)$ are defined by means of the iterated integrals of the complementary error function. Define

$$\mathrm{erfc}_{-1}(x) := \frac{2}{\sqrt{\pi}} e^{-x^2}, \quad \mathrm{erfc}_n(x) := \int_{s=x}^{\infty} \mathrm{erfc}_{n-1}(s)\,ds, \quad n \geq 0.$$

Note that $\mathrm{erfc}_0(x) = \mathrm{erfc}(x)$[3] and

$$\mathrm{erfc}_n(x) = \frac{2}{\sqrt{\pi}} \int_{s=x}^{\infty} \frac{(s-x)^n}{n!} e^{-s^2}\,ds.$$

In addition, we have the following identities [19]

$$n\,\mathrm{erfc}_n(x) + x\,\mathrm{erfc}_{n-1}(x) = \frac{1}{2}\mathrm{erfc}_{n-2}(x); \quad n \geq 1; \tag{23a}$$

$$(-1)^n\,\mathrm{erfc}_n(x) + \mathrm{erfc}_n(-x) = \frac{i^{-n}}{2^{n-1}n!} H_n(ix); \quad n \geq 0, \tag{23b}$$

where $i^2 = -1$ and H_n is the Hermite polynomial of degree n.

Recall the definitions in (4) and note that

$$\psi_1^{\pm}(x,t) = (x \pm d(t))\psi_0^{\pm}(x,t) + 2\varepsilon t\psi_{-1}^{\pm}(x,t), \tag{24a}$$

$$2t\frac{\partial \psi_1^-}{\partial t} = \psi_1^- - \big(2ta(d(t),t) + (x-d(t))\big)\psi_0^-, \tag{24b}$$

$$2t\frac{\partial \psi_1^+}{\partial t} = \left(1 + \frac{2xp(t)}{\varepsilon t}\right)\psi_1^+ + \big(2ta(d(t),t) - (x+d(t))\big)\psi_0^+, \tag{24c}$$

where

$$\psi_{-1}^{\pm}(x,t) := -\frac{E(x,t)}{2\sqrt{\varepsilon\pi t}},$$

and

$$p(t) := ta(d(t),t) - d(t) = \int_{s=0}^{t} a(d(t),t) - a(d(s),s)\,ds. \tag{24d}$$

Observe that $p(t) \equiv 0$, when $a(x,t) = a$ is a constant.

Some recurrence relations are given below which are useful when bounding the derivatives of the functions $\psi_n^{\pm}(x,t)$. For all $n \geq 1$

$$\frac{\partial \psi_n^-}{\partial x} = n\psi_{n-1}^-, \quad \frac{\partial \psi_n^+}{\partial x} = n\psi_{n-1}^+ + \frac{d(t)}{\varepsilon t}\psi_n^+, \tag{25a}$$

and for all $n \geq 2$ (using (23a)) we have

[3] The first three iterated integrals of the complementary error function are

$$\mathrm{erfc}_1(x) = \frac{e^{-x^2}}{\sqrt{\pi}} - x\,\mathrm{erfc}(x) = e^{-x^2}\left(\frac{1}{\sqrt{\pi}} - xe^{x^2}\,\mathrm{erfc}(x)\right),$$

$$\mathrm{erfc}_2(x) = \frac{1}{4}\left((1+2x^2)\,\mathrm{erfc}(x) - \frac{2xe^{-x^2}}{\sqrt{\pi}}\right),$$

$$\mathrm{erfc}_3(x) = \frac{1}{6}\left(\frac{(1+x^2)e^{-x^2}}{\sqrt{\pi}} - \frac{(3x+2x^3)}{2}\,\mathrm{erfc}(x)\right).$$

$$\psi_n^{\pm}(x,t) = (x \pm d(t))\psi_{n-1}^{\pm}(x,t) + 2(n-1)\varepsilon t\psi_{n-2}^{\pm}(x,t), \tag{25b}$$
$$\frac{\partial \psi_n^-}{\partial t} = \varepsilon n(n-1)\psi_{n-2}^- - a(d(t),t)n\psi_{n-1}^-,$$
$$\frac{\partial \psi_n^+}{\partial t} = \varepsilon n(n-1)\psi_{n-2}^+ + \left(2\frac{d(t)}{t} - a(d(t),t)\right)n\psi_{n-1}^+ - \frac{p(t)}{\varepsilon t^2}(d(t)\psi_n^+ - \psi_{n+1}^+).$$

In the case of constant coefficients one has $L\psi_n^- = L\psi_n^+ = 0$, but for variable $a(x,t)$, by using (23a) we have that for all $n \geq 0$

$$L\psi_n^- = (a(d(t),t) - a(x,t))\frac{\partial \psi_n^-}{\partial x}, \tag{26a}$$

$$L\psi_n^+ = (a(d(t),t) - a(x,t))\frac{\partial \psi_n^+}{\partial x} + p(t)\frac{\psi_{n+1}^+}{\varepsilon t^2}. \tag{26b}$$

Using the inequality $\mathrm{erfc}(z) \leq Ce^{-z^2} \leq Ce^{\gamma^2/4}e^{-\gamma z}, \forall z \geq 0$ it follows (see [11] and [12]) that

$$\left|\frac{\partial^j}{\partial t^j}\psi_0^-(x,t)\right|, \left|\frac{\partial^j}{\partial t^j}E(x,t)\right| \leq C\left(\frac{1}{t} + \frac{1}{\sqrt{\varepsilon t}}\right)^j E_\gamma(x,t); \quad j = 1,2, \tag{27a}$$

$$\left|\psi_0^-(x,t)\right| \leq C \quad \text{and} \quad |\psi_0^-(x,t)| \leq CE(x,t), \quad \text{if} \quad x \geq d(t), \tag{27b}$$

$$\left|\frac{\partial^i}{\partial x^i}\psi_0^-(x,t)\right|, \left|\frac{\partial^i}{\partial x^i}E(x,t)\right| \leq C\left(\frac{1}{\sqrt{\varepsilon t}}\right)^i E_\gamma(x,t), \quad 1 \leq i \leq 4. \tag{27c}$$

The following remark is used to prove bounds on the derivatives of the singular function ψ_0^+ and to compute the numerical results presented in Section §5 (see Remark 3).

Remark 4. The function ψ_0^+ can be written as

$$\psi_0^+(x,t) = \frac{1}{2}e^{\frac{d(t)x}{t\varepsilon}}\,\mathrm{erfc}\left(\frac{x+d(t)}{2\sqrt{\varepsilon t}}\right) = \frac{1}{2}E(x,t)\,H\left(\frac{x+d(t)}{2\sqrt{\varepsilon t}}\right), \tag{28}$$

where H is the Mill's ratio and it is defined by $H(x) := e^{x^2}\mathrm{erfc}(x)$. From [15], we have the inequality

$$\frac{1}{\frac{\pi-1}{\sqrt{\pi}}x + \sqrt{1+\frac{x^2}{\pi}}} \leq H(x) \leq \frac{1}{\frac{2}{\sqrt{\pi}}x + \sqrt{1+\frac{(\pi-2)^2x^2}{\pi}}}. \tag{29}$$

Hence, for all $x > 0$

$$\left(1 - \sqrt{\pi}xH(x)\right) \leq \min\left\{\frac{1}{2x^2}, \frac{1}{\sqrt{\pi}x}\right\},$$

and

$$e^{\frac{d(t)x}{t\varepsilon}}\,\mathrm{erfc}_1\left(\frac{x+d(t)}{2\sqrt{\varepsilon t}}\right) = \frac{E(x,t)}{\sqrt{\pi}}\left(1 - \sqrt{\pi}\left(\frac{x+d(t)}{2\sqrt{\varepsilon t}}\right)H\left(\frac{x+d(t)}{2\sqrt{\varepsilon t}}\right)\right).$$

Hence,

$$\frac{1}{\sqrt{\varepsilon t}}\left(e^{\frac{d(t)x}{t\varepsilon}}\,\mathrm{erfc}_1\left(\frac{x+d(t)}{2\sqrt{\varepsilon t}}\right)\right) \leq C\frac{E(x,t)}{x+d(t)}\min\left\{1, \frac{\sqrt{\varepsilon t}}{x+d(t)}\right\}. \tag{30}$$

In the next lemma bounds on the derivatives of the function ψ_0^+ are deduced.

Lemma 2. *For the singular function ψ_0^+, we have the following bounds*

$$|\psi_0^+(x,t)| \leq C\min\left\{1, \frac{\sqrt{\varepsilon t}}{x+d(t)}\right\}E(x,t), \tag{31a}$$

$$\left|\frac{\partial}{\partial t}\psi_0^+(x,t)\right| \leq \frac{C}{t}E_\gamma(x,t), \tag{31b}$$

$$\left|\frac{\partial^2}{\partial t^2}\psi_0^+(x,t)\right| \le \frac{C}{t^2}\left(1+\sqrt{\frac{t}{\varepsilon}}\right)E_\gamma(x,t), \tag{31c}$$

$$\left|\frac{\partial}{\partial x}\psi_0^+(x,t)\right| \le \frac{C}{x+d(t)}E_\gamma(x,t), \tag{31d}$$

$$\left|\frac{\partial^2}{\partial x^2}\psi_0^+(x,t)\right| \le \frac{C}{\varepsilon t}E_\gamma(x,t), \tag{31e}$$

$$\left|\frac{\partial^3}{\partial x^3}\psi_0^+(x,t)\right| \le \frac{C}{\varepsilon^2 t}\left(1+\sqrt{\frac{\varepsilon}{t}}\right)E_\gamma(x,t). \tag{31f}$$

Proof. Note first that

$$p(t) = ta(d(t),t) - d(t).$$

Hence, $|p(t)| \le Ct^2$.[4] To prove (31a), we use that $|H(r)| \le C$ and $rH(r) \le C$ for all $r \ge 0$, where H is defined in Remark 4. Then,

$$(x+d(t))\psi_0^+(x,t) \le C\sqrt{\varepsilon t}E(x,t) \quad \text{and} \quad \psi_0^+(x,t) \le CE(x,t).$$

Using (30), (31a) and the identity (25b) we easily establish the following bounds

$$|\psi_i^+(x,t)| \le C(\sqrt{\varepsilon t})^i \min\left\{1, \sqrt{\frac{\varepsilon}{t}}\right\}E(x,t), \quad i=1,2. \tag{32}$$

To prove (31b), observe that

$$\begin{aligned}(x+d(t))\frac{\partial\psi_0^+(x,t)}{\partial t} &= \frac{xp(t)}{t}\left(\frac{(x+d(t))}{t\varepsilon}\psi_0^+(x,t) - \frac{E(x,t)}{\sqrt{\varepsilon\pi t}}\right)\\ &+ (4xp(t) + (x+d(t))((x+d(t)-2ta(t)))\frac{E(x,t)}{4t\sqrt{\varepsilon\pi t}}\\ &= \frac{xp(t)}{\varepsilon t^2}\psi_1^+(x,t) + (x-d(t))^2\frac{E(x,t)}{4t\sqrt{\varepsilon\pi t}} + a(d(t),t)(x-d(t))\frac{E(x,t)}{2\sqrt{\varepsilon\pi t}},\end{aligned}$$

and use (30) and $|p(t)| \le Ct^2$. Next, we prove (31c). We have that

$$(x+d(t))\frac{\partial^2\psi_0^+(x,t)}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{xp(t)}{\varepsilon t^2}\psi_1^+(x,t) + (x+d(t)+2p(t))\frac{(x-d(t))E(x,t)}{4t\sqrt{\varepsilon\pi t}}\right) - a(d(t),t)\frac{\partial\psi_0^+(x,t)}{\partial t},$$

and

$$\begin{aligned}&\left|\frac{\partial}{\partial t}\left(\frac{p(t)}{t^2}\right)\right| \le C + C\frac{\|\nabla a(0,0)\|}{t} \le \frac{C}{t},\\ &\left|\frac{(x-d(t))}{2\sqrt{\varepsilon t}}\frac{\partial E(x,t)}{\partial t}\right| \le \frac{C}{t}\left(1+\sqrt{\frac{t}{\varepsilon}}\right)E_\gamma(x,t),\\ &\left|\frac{\psi_1^+(x,t)}{\varepsilon t}\right| \le \frac{C}{t}\min\left\{1,\sqrt{\frac{t}{\varepsilon}}\right\}E_\gamma(x,t),\\ &\frac{\partial\psi_1^+(x,t)}{\partial t} = \left(\frac{1}{2t} + \frac{xp(t)}{\varepsilon t^2}\right)\psi_1^+(x,t) + \frac{2p(t)-(x-d(t))}{2t}\psi_0^+(x,t).\end{aligned}$$

Collecting all of these bounds yields (31c). From

$$(x+d(t))\frac{\partial}{\partial x}\psi_0^+(x,t) = d(t)\left(\frac{x+d(t)}{t\varepsilon}\psi_0^+(x,t) - \frac{E(x,t)}{\sqrt{\varepsilon\pi t}}\right) - \frac{(x-d(t))E(x,t)}{2\sqrt{\varepsilon\pi t}},$$

and (30), we have (31d). Note also that for $i = 2, 3$

[4] In the particular case of $\nabla a(0,0) = (0,0)$, one has $|p(t)| \le Ct^3$.

$$\frac{\partial^i}{\partial x^i}\psi_0^+(x,t) = \frac{d(t)}{t\varepsilon}\frac{\partial^{i-1}}{\partial x^{i-1}}\psi_0^+(x,t) - \frac{1}{2\sqrt{\varepsilon\pi t}}\frac{\partial^{i-1}}{\partial x^{i-1}}E(x,t),$$

from which (31e) and (31f) follows. □

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