INVARIANT COMPLEX STRUCTURES ON 6-NILMANIFOLDS: CLASSIFICATION, FRÖLICHER SPECTRAL SEQUENCE AND SPECIAL HERMITIAN METRICS

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Abstract. We classify complex structures on 6-dimensional nilpotent Lie algebras up to equivalence. As an application, the behaviour of the associated Frölicher sequence is studied as well as its relation to the existence of strongly Gauduchon metrics. We also show that the strongly Gauduchon property and the balanced property are not closed under holomorphic deformation.

1. Introduction

Let $\mathfrak{g}$ be a Lie algebra endowed with an endomorphism $J\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ such that $J^2 = -\mathrm{Id}$. The endomorphism J is a *complex structure* if the integrability condition

$$[JX, JY] = J[JX, Y] + J[X, JY] + [X, Y]$$

is satisfied for any $X, Y \in \mathfrak{g}$; equivalently, the i-eigenspace $\mathfrak{g}_{1,0}$ of J in $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is a complex subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Nilpotent Lie algebras $\mathfrak{g}$ admitting a complex structure were classified by Salamon [27] up to dimension 6. More recently, Andrada, Barberis and Dotti classified in [2] the 6-dimensional Lie algebras $\mathfrak{g}$ having a complex structure J of abelian type, that is, the complex subalgebra $\mathfrak{g}_{1,0}$ is abelian, or equivalently $[JX, JY] = [X, Y]$ for any $X, Y \in \mathfrak{g}$.

A related question is to determine the complex structures on a given Lie algebra $\mathfrak{g}$ up to isomorphism in the following sense. Two complex structures J and J' on $\mathfrak{g}$ are *equivalent* if there exists an automorphism $F\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ of the Lie algebra such that $J = F^{-1} \circ J' \circ F$. The latter condition is equivalent to say that F, extended to $\mathfrak{g}_{\mathbb{C}}$, satisfies $F(\mathfrak{g}^J_{1,0}) \subset \mathfrak{g}^{J'}_{1,0}$. If $\mathcal{C}(\mathfrak{g})$ denotes the space of complex structures on $\mathfrak{g}$ then $\mathcal{C}(\mathfrak{g})/\mathrm{Aut}(\mathfrak{g})$ parametrizes the equivalence classes of complex structures on $\mathfrak{g}$.

A classification of abelian complex structures in dimension 6 is given in [2]. Some partial results on nilpotent Lie algebras can be found in several papers [6, 19, 29, 30], although to our knowledge there is no complete classification of complex structures on 6-dimensional nilpotent Lie algebras. This is our first goal here.

The classification of complex structures on nilpotent Lie algebras provides a classification of *invariant* complex structures on nilmanifolds. Let $M = \Gamma\backslash G$ be a nilmanifold, i.e. a compact quotient of a simply-connected nilpotent Lie group G by a lattice Γ of maximal rank. If J is a complex structure on the Lie algebra $\mathfrak{g}$ of G, then it gives rise to a left-invariant complex structure on G which descends to a complex structure on the quotient M in a natural way. Several interesting aspects of this complex geometry have been investigated, as for instance the Dolbeault cohomology [7, 13, 25], complex deformations [6, 8, 20, 26] or the existence of special Hermitian metrics [16, 29]. Recently, it is proved in [4] that the canonical bundle of any complex nilmanifold is holomorphically trivial and some applications to hypercomplex geometry are given.

As a first application of the classification of complex structures we study the behaviour of the Frölicher sequence [17]. Recall that the Frölicher sequence $E_r(M, J)$ of a complex manifold (M, J) is the spectral sequence associated to the double complex $(\Omega^{p,q}(M, J), \partial, \bar{\partial})$, where $\partial + \bar{\partial} = d$ is the decomposition, with respect to J, of the exterior differential d. The first term $E_1(M, J)$ is precisely the Dolbeault cohomology of (M, J) and after a finite number of steps the sequence converges to the de Rham cohomology of M. The first examples of compact complex manifolds for which $E_2 \not\cong E_\infty$ were independently found in [9] and [21]. The examples in [9] are complex nilmanifolds of complex dimension 3, which is the lowest possible dimension for which the Frölicher sequence can be non-degenerate at E_2. More recently, Rollenske has constructed in [24] complex nilmanifolds for which the sequence $\{E_r\}$ can be arbitrarily non-degenerate. The behaviour of the Frölicher sequence has been studied for some other complex manifolds [14, 28], but as far as we know its general behaviour for complex nilmanifolds has not been studied, although some partial results can be found in [10, 11, 12]. Here we study the Frölicher spectral sequence for general invariant complex structures on a 6-dimensional nilmanifold. A remarkable consequence of this study is the existence of a compact complex manifold on which the $\partial\bar{\partial}$-lemma fails but $E_1 \cong E_\infty$ and the Hodge diamond is symmetric.

As a second application of the classification of complex structures we consider strongly Gauduchon (sG for short) metrics in the sense of Popovici [22]. Any balanced Hermitian metric is sG and any sG metric is a Gauduchon metric [18]. In [23] the relation between the degeneration of the Frölicher sequence at E_1 and the existence of sG metrics is studied, showing that these two notions are unrelated. We study the existence of sG or balanced metrics on 6-nilmanifolds in relation to the general behaviour of the Frölicher sequence. Moreover, Popovici proved in [22] that the sG property of compact complex manifolds is open under holomorphic deformations, and conjectured in [23] that the sG property and the balanced property of compact complex manifolds are closed under holomorphic deformations. We construct a counterexample to both closedness conjectures.

The paper is structured as follows. In Section 2 we first review some general facts about complex structures on a 6-dimensional nilpotent Lie algebra $\mathfrak{g}$. By [27] such $\mathfrak{g}$ must be isomorphic to $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}$, $\mathfrak{h}_{19}^-$ or $\mathfrak{h}_{26}^+$ (see Theorem 2.1 for a description of the Lie algebras). Of special interest is $\mathfrak{h}_5$ because it corresponds to the real Lie algebra underlying the Iwasawa manifold, whose complex geometry is studied in [19]. For the first sixteen classes the complex structure is necessarily of *nilpotent* type in the sense of [13]. We classify the non-abelian nilpotent complex structures on 2-step and 3-step nilpotent Lie algebras in Sections 2.1 and 2.2, respectively. Then, using the classification of non-nilpotent complex structures obtained in [30] as well as the classification of abelian structures given in [2], we present in Tables 1 and 2 of Section 3 the complete classification of complex structures on 6-dimensional nilpotent Lie algebras up to equivalence.

Since J equivalent to J' implies that the terms in the associated Frölicher sequences are isomorphic, as an application we study the general behaviour of the Frölicher sequence $E_r(\Gamma\backslash G, J)$ in Section 4 (see Theorem 4.1 for details). We find that $E_2 \not\cong E_\infty$ if and only if the underlying Lie algebra $\mathfrak{g} \cong \mathfrak{h}_{13}, \mathfrak{h}_{14}$ or $\mathfrak{h}_{15}$. Moreover, $E_1 \cong E_2 \not\cong E_3 \cong E_\infty$ for any J when $\mathfrak{g} \cong \mathfrak{h}_{13}$ or $\mathfrak{h}_{14}$. In contrast, $\mathfrak{h}_{15}$

has a rich complex geometry with respect to Frölicher sequence because it admits complex structures for which $E_1 \ncong E_2 \cong E_\infty$, $E_1 \cong E_2 \ncong E_3 \cong E_\infty$ or even $E_1 \ncong E_2 \ncong E_3 \cong E_\infty$. In Example 4.6 we give a family J_t of non-equivalent complex structures on $\mathfrak{h}_{15}$ along which the Frölicher sequence has these three behaviours. We also show that a nilmanifold with underlying Lie algebra $\mathfrak{h}_6$ has a complex structure with degenerate Frölicher sequence and satisfying $h^{p,q}_{\bar\partial} = h^{q,p}_{\bar\partial}$ for every $p, q \in \mathbb{N}$, which provides an answer to a question recently posed in [3] (see Proposition 4.2).

In section 5 we study the existence of sG metrics on 6-dimensional nilmanifolds endowed with an invariant complex structure and show that the underlying Lie algebra must be isomorphic to $\mathfrak{h}_1, \ldots, \mathfrak{h}_6$ or $\mathfrak{h}_{19}^-$. It is also proved that the existence of sG metric implies the degeneration of the Frölicher sequence at E_2. Using [31] we give in Proposition 5.5 a classification of complex structures having sG metrics but not admitting any balanced metric, as well as a classification of nilpotent complex structures admitting balanced metric (see Table 3). Based on the complex geometry of the Lie algebra $\mathfrak{h}_4$, in Theorem 5.9 we show that neither the sG property nor the balanced property of compact complex manifolds are closed under holomorphic deformation.

2. NILPOTENT COMPLEX STRUCTURES ON 6-DIMENSIONAL NILPOTENT LIE ALGEBRAS

Given a Lie algebra $\mathfrak{g}$, let $\mathfrak{g}^*_{\mathbb{C}}$ be the dual of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. If $J\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ is an endomorphism such that $J^2 = -\mathrm{Id}$, then there is a natural bigraduation induced on $\bigwedge^* \mathfrak{g}^*_{\mathbb{C}} = \oplus_{p,q} \bigwedge^{p,q}(\mathfrak{g}^*)$, where the spaces $\bigwedge^{1,0}(\mathfrak{g}^*)$ and $\bigwedge^{0,1}(\mathfrak{g}^*)$, which we shall also denote by $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$, are the eigenspaces of the eigenvalues $\pm i$ of J as an endomorphism of $\mathfrak{g}^*_{\mathbb{C}}$, respectively. Now, if $d\colon \bigwedge^* \mathfrak{g}^*_{\mathbb{C}} \longrightarrow \bigwedge^{*+1} \mathfrak{g}^*_{\mathbb{C}}$ is the extension to the complexified exterior algebra of the usual Chevalley-Eilenberg differential, then it is well known that J is a complex structure if and only if $\pi_{0,2} \circ d|_{\mathfrak{g}^{1,0}} \equiv 0$, where $\pi_{0,2}\colon \bigwedge^2 \mathfrak{g}^*_{\mathbb{C}} \longrightarrow \bigwedge^{0,2}(\mathfrak{g}^*)$ denotes the canonical projection.

We shall focus on *nilpotent* Lie algebras (NLA for short). Salamon has proved in [27] the following equivalent condition for the integrability of J on a $2n$-dimensional NLA $\mathfrak{g}$: J is a complex structure on $\mathfrak{g}$ if and only if $\mathfrak{g}^{1,0}$ has a basis $\{\omega^j\}_{j=1}^n$ such that $d\omega^1 = 0$ and

$$d\omega^j \in \mathcal{I}(\omega^1, \ldots, \omega^{j-1}), \qquad \text{for } j = 2, \ldots, n,$$

where $\mathcal{I}(\omega^1, \ldots, \omega^{j-1})$ is the ideal in $\bigwedge^* \mathfrak{g}^*_{\mathbb{C}}$ generated by $\{\omega^1, \ldots, \omega^{j-1}\}$.

Recall that a complex structure J on a $2n$-dimensional NLA $\mathfrak{g}$ is *nilpotent* [13] if there exists a basis $\{\omega^j\}_{j=1}^n$ for $\mathfrak{g}^{1,0}$ satisfying $d\omega^1 = 0$ and

$$(1) \qquad d\omega^j \in \bigwedge^2 \langle \omega^1, \ldots, \omega^{j-1}, \omega^{\bar 1}, \ldots, \omega^{\overline{j-1}} \rangle, \qquad \text{for } j = 2, \ldots, n.$$

An important special class of nilpotent complex structures is the *abelian* class consisting of those structures J satisfying $[JX, JY] = [X, Y]$, for all $X, Y \in \mathfrak{g}$, or equivalently $d(\mathfrak{g}^{1,0}) \subset \bigwedge^{1,1}(\mathfrak{g}^*)$. They are also characterized by the fact that the subalgebra $\mathfrak{g}^{1,0}$ is abelian.

In six dimensions, the classification of NLAs in terms of the different types of complex structures that they admit is as follows.

Theorem 2.1. [27, 29] *Let* $\mathfrak{g}$ *be an NLA of dimension* 6*. Then,* $\mathfrak{g}$ *has a complex structure if and only if it is isomorphic to one of the following Lie algebras:*

$\mathfrak{h}_1 = (0,0,0,0,0,0)$,
$\mathfrak{h}_2 = (0,0,0,0,12,34)$,
$\mathfrak{h}_3 = (0,0,0,0,0,12+34)$,
$\mathfrak{h}_4 = (0,0,0,0,12,14+23)$,
$\mathfrak{h}_5 = (0,0,0,0,13+42,14+23)$,
$\mathfrak{h}_6 = (0,0,0,0,12,13)$,
$\mathfrak{h}_7 = (0,0,0,12,13,23)$,
$\mathfrak{h}_8 = (0,0,0,0,0,12)$,
$\mathfrak{h}_9 = (0,0,0,0,12,14+25)$,
$\mathfrak{h}_{10} = (0,0,0,12,13,14)$,
$\mathfrak{h}_{11} = (0,0,0,12,13,14+23)$,
$\mathfrak{h}_{12} = (0,0,0,12,13,24)$,
$\mathfrak{h}_{13} = (0,0,0,12,13+14,24)$,
$\mathfrak{h}_{14} = (0,0,0,12,14,13+42)$,
$\mathfrak{h}_{15} = (0,0,0,12,13+42,14+23)$,
$\mathfrak{h}_{16} = (0,0,0,12,14,24)$,
$\mathfrak{h}_{19}^{-} = (0,0,0,12,23,14-35)$,
$\mathfrak{h}_{26}^{+} = (0,0,12,13,23,14+25)$.

Moreover:

(*a*) *Any complex structure on* $\mathfrak{h}_{19}^{-}$ *and* $\mathfrak{h}_{26}^{+}$ *is non-nilpotent;*
(*b*) *For* $1 \leq k \leq 16$*, any complex structure on* $\mathfrak{h}_k$ *is nilpotent;*
(*c*) *Any complex structure on* $\mathfrak{h}_1$*,* $\mathfrak{h}_3$*,* $\mathfrak{h}_8$ *and* $\mathfrak{h}_9$ *is abelian;*
(*d*) *There exist both abelian and non-abelian nilpotent complex structures on* $\mathfrak{h}_2$*,* $\mathfrak{h}_4$*,* $\mathfrak{h}_5$ *and* $\mathfrak{h}_{15}$*;*
(*e*) *Any complex structure on* $\mathfrak{h}_6$*,* $\mathfrak{h}_7$*,* $\mathfrak{h}_{10}$*,* $\mathfrak{h}_{11}$*,* $\mathfrak{h}_{12}$*,* $\mathfrak{h}_{13}$*,* $\mathfrak{h}_{14}$ *and* $\mathfrak{h}_{16}$ *is not abelian.*

Remark 2.2. Here we use the usual notation, i.e. for instance $\mathfrak{h}_2 = (0,0,0,0,12,34)$ means that there is a basis $\{e^j\}_{j=1}^6$ satisfying $de^1 = de^2 = de^3 = de^4 = 0$, $de^5 = e^1 \wedge e^2$, $de^6 = e^3 \wedge e^4$; equivalently, the Lie bracket is given in terms of its dual basis $\{e_j\}_{j=1}^6$ by $[e_1, e_2] = -e_5$, $[e_3, e_4] = -e_6$.

Let $\mathfrak{g}$ be a Lie algebra endowed with two complex structures J and J'. We recall that J and J' are said to be *equivalent* if there is an automorphism $F \colon \mathfrak{g} \longrightarrow \mathfrak{g}$ of the Lie algebra such that $J' = F^{-1} \circ J \circ F$, that is, F is a linear automorphism such that $F^* \colon \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ commutes with the Chevalley-Eilenberg differential d and F commutes with the complex structures J and J'. The latter condition is equivalent to say that F^*, extended to the complexified exterior algebra, preserves the bigraduations induced by J and J'.

Notice that if $\mathfrak{g}_J^{1,0}$ and $\mathfrak{g}_{J'}^{1,0}$ denote the $(1,0)$-subspaces of $\mathfrak{g}_{\mathbb{C}}^*$ associated to J and J', then the complex structures J and J' are equivalent if and only if there is a $\mathbb{C}$-linear isomorphism $F^* \colon \mathfrak{g}_J^{1,0} \longrightarrow \mathfrak{g}_{J'}^{1,0}$ such that $d \circ F^* = F^* \circ d$.

In dimension 6, by Theorem 2.1, if the NLA $\mathfrak{g}$ admits complex structures then all of them are either nilpotent or non-nilpotent. The classification of abelian complex structures up to equivalence is obtained in [2], whereas the non-nilpotent complex structures are classified in [30] (see Section 3 for details). Therefore, it remains to study the equivalence classes of non-abelian nilpotent complex structures. In order to provide such classification, our starting point is the following reduction of the nilpotent condition (1).

Proposition 2.3. [29] *Let* J *be a nilpotent complex structure on an NLA* $\mathfrak{g}$ *of dimension* 6*. There is a basis* $\{\omega^j\}_{j=1}^3$ *for* $\mathfrak{g}^{1,0}$ *satisfying*

$$(2) \qquad \left\{ \begin{aligned} d\omega^1 &= 0, \\ d\omega^2 &= \epsilon\, \omega^{1\bar{1}}, \\ d\omega^3 &= \rho\, \omega^{12} + (1-\epsilon) A\, \omega^{1\bar{1}} + B\, \omega^{1\bar{2}} + C\, \omega^{2\bar{1}} + (1-\epsilon) D\, \omega^{2\bar{2}}, \end{aligned} \right.$$

where $A, B, C, D \in \mathbb{C}$ *and* $\epsilon, \rho \in \{0, 1\}$.

Here ω^{jk} (resp. $\omega^{j\bar{k}}$) means the wedge product $\omega^j \wedge \omega^k$ (resp. $\omega^j \wedge \omega^{\bar{k}}$), where $\omega^{\bar{k}}$ indicates the complex conjugation of ω^k. From now on, we shall use a similar abbreviated notation for "basic" forms of arbitrary bidegree.

Notice that in the equations (2) the complex structure is not abelian if and only if $\rho = 1$. Next we study the 2-step and 3-step cases in Sections 2.1 and 2.2, respectively.

2.1. **Non-abelian complex structures in the 2-step case.** Any 6-dimensional 2-step NLA $\mathfrak{g}$ has first Betti number at least 3, and if it is equal to 3 then necessarily the coefficient ϵ in (2) is non-zero. We consider firstly $\epsilon = 0$, i.e. the Lie algebra has first Betti number ≥ 4, and we will finish the section by considering the remaining case $\epsilon = 1$.

The following proposition provides a further reduction of the equations (2) when $\epsilon = 0$ and the structure is not complex-parallelizable. Recall that J is *complex-parallelizable* if $[JX, Y] = J[X, Y]$, for all $X, Y \in \mathfrak{g}$, or equivalently $d(\mathfrak{g}^{1,0}) \subset \bigwedge^{2,0}(\mathfrak{g}^*)$. These structures are the natural complex structures of *complex Lie algebras*, and in six dimensions they correspond to $\epsilon = A = B = C = D = 0$ and the possible Lie algebras are $\mathfrak{h}_1$ (for $\rho = 0$) and $\mathfrak{h}_5$ (for $\rho = 1$).

Proposition 2.4. *Let J be a complex structure on a 2-step NLA $\mathfrak{g}$ of dimension 6 with first Betti number ≥ 4. If J is not complex-parallelizable, then there is a basis $\{\omega^j\}_{j=1}^3$ for $\mathfrak{g}^{1,0}$ such that*

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \rho\,\omega^{12} + \omega^{1\bar{1}} + \lambda\,\omega^{1\bar{2}} + D\,\omega^{2\bar{2}}, \tag{3}$$

where $\rho \in \{0, 1\}$, $\lambda \in \mathbb{R}$ such that $\lambda \geq 0$, and $D \in \mathbb{C}$ with $\mathfrak{Im}\, D \geq 0$. Moreover, if we denote $x = \mathfrak{Re}\, D$ and $y = \mathfrak{Im}\, D$, then:

- (i) *If $\lambda = \rho$, then the Lie algebra $\mathfrak{g}$ is isomorphic to*
  - (i.1) $\mathfrak{h}_2$, *for $y > 0$;*
  - (i.2) $\mathfrak{h}_3$, *for $\rho = y = 0$ and $x \neq 0$;*
  - (i.3) $\mathfrak{h}_4$, *for $\rho = 1$, $y = 0$ and $x \neq 0$;*
  - (i.4) $\mathfrak{h}_6$, *for $\rho = 1$ and $x = y = 0$;*
  - (i.5) $\mathfrak{h}_8$, *for $\rho = x = y = 0$.*
- (ii) *If $\lambda \neq \rho$, then the Lie algebra $\mathfrak{g}$ is isomorphic to*
  - (ii.1) $\mathfrak{h}_2$, *for $4y^2 > (\rho - \lambda^2)(4x + \rho - \lambda^2)$;*
  - (ii.2) $\mathfrak{h}_4$, *for $4y^2 = (\rho - \lambda^2)(4x + \rho - \lambda^2)$;*
  - (ii.3) $\mathfrak{h}_5$, *for $4y^2 < (\rho - \lambda^2)(4x + \rho - \lambda^2)$.*

Proof. In [29, Lemma 11] it is proved that under these conditions there is a basis $\{\sigma^j\}_{j=1}^3$ for $\mathfrak{g}^{1,0}$ such that

$$d\sigma^1 = d\sigma^2 = 0, \; d\sigma^3 = \rho\,\sigma^{12} + \sigma^{1\bar{1}} + B\,\sigma^{1\bar{2}} + D\,\sigma^{2\bar{2}}, \tag{4}$$

where $B, D \in \mathbb{C}$ and $\rho \in \{0, 1\}$.

If $B \neq 0$ then we can take any non-zero solution z of $\bar{z}\frac{B}{|B|} = z$, and the equations (4) reduce to (3) with $\lambda = |B|$ with respect to the new basis $\{\omega^1 = z\,\sigma^1, \omega^2 = \bar{z}\,\sigma^2, \omega^3 = |z|^2\,\sigma^3\}$.

Consider now $B = \lambda$ with $\lambda \in \mathbb{R}^{\geq 0}$ in (4). If $D \neq 0$, then with respect to the new basis $\{\omega^1 = -\bar{D}\,\sigma^2, \omega^2 = \sigma^1 + \lambda\,\sigma^2, \omega^3 = \bar{D}\,\sigma^3\}$ we get (3) with $\bar{D}$ instead of D.

Finally, the second part of the proposition follows directly from [29, Proposition 13]. □

From now on we consider $\rho = 1$. By Proposition 2.4 any two complex structures on the Lie algebra $\mathfrak{h}_6$ are equivalent. Thus, it remains to classify up to equivalence the non-abelian structures J on $\mathfrak{h}_2$, $\mathfrak{h}_4$ and $\mathfrak{h}_5$. Any such J is identified with a triple $(1, \lambda, D)$ through equations (3) with $\rho = 1$, $\lambda \geq 0$ and $\mathfrak{Im}\, D \geq 0$.

We will say that two triples $(1, \lambda, D)$ and $(1, \lambda', D')$ are equivalent, denoted by $(1, \lambda, D) \sim (1, \lambda', D')$, if the corresponding structures J and J' are equivalent. So, the problem reduces to classify triples $(1, \lambda, D)$ up to equivalence.

Lemma 2.5. *Let us consider two triples* $(1, \lambda, D)$ *and* $(1, t, E)$ *as above.*

(i) *If* $D = 0$ *then,* $(1, t, E) \sim (1, \lambda, 0)$ *if and only if* $t = \lambda$ *and* $E = 0$.

(ii) *If* $D \neq 0$ *then,* $(1, t, E) \sim (1, \lambda, D)$ *if and only if there exist non-zero complex numbers* e, f *such that* $E = De/\bar{e}$ *and*

$$\left(\frac{|f|^2}{\bar{e}} - 1\right)(\bar{D}\bar{e} - De)^2 = (\lambda \bar{f} - t f)(\lambda \bar{D} \bar{e} f - t D e \bar{f}). \tag{5}$$

Proof. The structure equations corresponding to the triples $(1, \lambda, D)$ and $(1, t, E)$ are

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + \lambda\omega^{1\bar{2}} + D\omega^{2\bar{2}},$$
$$d\sigma^1 = d\sigma^2 = 0, \quad d\sigma^3 = \sigma^{12} + \sigma^{1\bar{1}} + t\sigma^{1\bar{2}} + E\sigma^{2\bar{2}},$$

where $\lambda, t \geq 0$ and $\mathfrak{Im}\, D, \mathfrak{Im}\, E \geq 0$. Then $(1, t, E) \sim (1, \lambda, D)$ if and only if there exists an automorphism of the Lie algebra preserving the complex equations, i.e. there is $(m_{ij}) \in \mathrm{GL}(3, \mathbb{C})$ such that $\sigma^i = \sum_{j=1}^{3} m_{ij}\, \omega^j$ and

$$d\sigma^i = \sum_{j=1}^{3} m_{ij}\, d\omega^j, \quad i = 1, 2, 3.$$

These conditions are equivalent to

$$\sigma^1 = a\,\omega^1 + b\,\omega^2, \quad \sigma^2 = c\,\omega^1 + f\,\omega^2, \quad \sigma^3 = m_{31}\,\omega^1 + m_{32}\,\omega^2 + e\,\omega^3,$$

and

$$\begin{cases} \text{(I)} & e = af - bc, \\ \text{(II)} & e = |a|^2 + t\,a\bar{c} + E|c|^2, \\ \text{(III)} & \lambda e = a\bar{b} + t\,a\bar{f} + Ec\bar{f}, \\ \text{(IV)} & 0 = \bar{a}b + t\,b\bar{c} + E\bar{c}f, \\ \text{(V)} & De = |b|^2 + t\,b\bar{f} + E|f|^2. \end{cases} \tag{6}$$

Notice that $m_{13} = m_{23} = 0$, $e \neq 0$ and the coefficients m_{31} and m_{32} are not relevant.

It is straightforward to see that coefficient f must be non-zero (otherwise $\lambda = t$ and $D = E$) and so we can express a as

$$a = \frac{e + bc}{f}.$$

First of all, let us suppose that $D = 0$. Replacing a in (IV) and using (V) we obtain that $b = 0$ and therefore $E = 0$ by equation (V). Combining (I) and (III) we get that $\lambda f = t\bar{f}$. Since λ and t are real non-negative numbers, we conclude that $\lambda = t$, i.e. $(1, \lambda, 0)$ defines an equivalence class for every $\lambda \geq 0$. This completes the proof of (i).

We suppose next that $D \neq 0$. In order to solve (6) we transform it into an equivalent system by doing the following substitutions. Replacing a in equation (IV) and using (V) we can express

$$\bar{c} = -\frac{b\bar{e}}{De}.$$

Next, in (II) we can substitute a and c and use again (V) to obtain that

$$De = E\bar{e},$$

which implies in particular $|D| = |E|$. Notice that since $D \neq 0$ we can assume $E \neq \bar{D}$ by Proposition 2.4. Now, $\bar{c} = -b/E$. Proceeding in a similar way in equation (III) we get

$$\bar{b} = \frac{\lambda f - t\bar{f}}{1 - D/\bar{E}}.$$

Finally, using the expressions of a, b, c above, equation (V) is equivalent to (5). Therefore, given $e, f \in \mathbb{C} - \{0\}$ satisfying $De = E\bar{e}$ and (5), it is always possible to find $a, b, c \in \mathbb{C}$ such that system (6) is satisfied. □

Remark 2.6. As a consequence of Lemma 2.5 (ii), when $D \neq 0$ a necessary condition for $(1, t, E)$ to be equivalent to $(1, \lambda, D)$ is that $|D| = |E|$. Moreover, to find an equivalent complex structure $(1, t, E)$ it suffices to find $t \geq 0$ and $e, f \in \mathbb{C} - \{0\}$ satisfying (5), because E is necessarily given by $E = De/\bar{e}$.

Corollary 2.7. *Let $E \neq \bar{D}$. If $(1, t, E) \sim (1, \lambda, D)$ then, $t = \lambda$ if and only if $E = D$.*

Proof. By hypothesis D cannot be zero, so we are in case (ii) of Lemma 2.5. Suppose first that $\lambda = t$ in (5), i.e.

$$(\bar{D}\bar{e} - De)^2 \left(\frac{|f|^2}{\bar{e}} - 1 \right) = \lambda^2 (\bar{f} - f)(\bar{D}\bar{e}f - De\bar{f}).$$

The right hand side of the previous equality is a real number. If it is zero then $e = |f|^2$ (otherwise $De = \bar{D}\bar{e}$ would imply $E = \bar{D}$); thus, e is a real number and since $E = De/\bar{e}$ we conclude that $D = E$. On the other hand, if it is a non-zero real number, then $\frac{|f|^2}{\bar{e}} - 1$ must be a real number and then $e \in \mathbb{R}$ and again $D = E$.

Conversely, let us suppose that $E = D \neq 0$. In this case $e \in \mathbb{R}$ and by (5) we can express it as

$$e = |f|^2 - \frac{(\lambda\bar{f} - tf)(\lambda\bar{D}f - tD\bar{f})}{(\bar{D} - D)^2}.$$

Notice that by hypothesis $D \neq \bar{E} = \bar{D}$. To ensure that $e \in \mathbb{R}$ it must happen that $(\lambda\bar{f} - tf)(\lambda\bar{D}f - tD\bar{f}) \in \mathbb{R}$ or equivalently,

$$|f|^2(\lambda^2 - t^2)(\bar{D} - D) = 0.$$

As $f(\bar{D} - D) \neq 0$ the only possibility to solve the previous equation is $\lambda = t$. □

From the previous results it follows that it remains to consider the case when $D \neq 0$ and $\lambda \neq t$. The next lemma provides a simplification of equation (5).

Lemma 2.8. *Let us suppose that $\lambda \neq t$, $D = x + iy \neq 0$ and $e \in \mathbb{C} - \{0\}$. Then, $(1, \lambda, D) \sim (1, t, De/\bar{e})$ if and only if*

$$4y^2 - (t^2 - \lambda^2)(4x + t^2 - \lambda^2) \geq 0. \tag{7}$$

Proof. By Lemma 2.5 (ii), we know that $(1,\lambda,D)\sim(1,t,De/\bar e)$ if and only if (5) is satisfied. This condition reads, with respect to $H=De$, as

$$(\bar H-H)^2\left(\bar D|f|^2-\bar H\right)=\bar H(\lambda\bar f-tf)(\lambda f\bar H-t\bar f H).$$

Taking real and imaginary parts in the expression above we obtain

$$\tag{8}\begin{cases} 4H_2^2(H_1-x|f|^2)= & |f|^2(t^2-\lambda^2)H_2^2+|f|^2(t^2+\lambda^2)H_1^2 \\ & -2\lambda t(f_1^2-f_2^2)H_1^2-4\lambda tH_1H_2f_1f_2, \\ 4H_2^2(y|f|^2-H_2)= & 2\lambda H_2\left[tH_1(f_1^2-f_2^2)+2tH_2f_1f_2-\lambda|f|^2H_1\right], \end{cases}$$

where $H=H_1+iH_2$ and $f=f_1+if_2$. Observe that $H_2\neq 0$, otherwise we get a contradiction using the first equation of (8).

Substituting the second equation of (8) in the first one and replacing H by De, we can express the system (8) as

$$\tag{9}\begin{cases} e_1^2(t^2-\lambda^2)+4ye_1e_2+e_2^2(t^2-\lambda^2+4x)=0, \\ 2H_2(y|f|^2-H_2)=\lambda\left[tH_1(f_1^2-f_2^2)+2tH_2f_1f_2-\lambda|f|^2H_1\right], \end{cases}$$

where $e=e_1+ie_2$.

To solve the first equation in (9) as a second degree equation in e_1 we need the discriminant to be greater than or equal to 0, i.e. $4y^2-(t^2-\lambda^2)(4x+t^2-\lambda^2)\geq 0$, which is precisely condition (7).

Now, suppose that (7) holds. Then we obtain that

$$e_1=\frac{e_2\beta}{\lambda^2-t^2},\qquad e=e_2\left(\frac{\beta}{\lambda^2-t^2}+i\right),$$

where $\beta=2y+\sqrt{4y^2-(t^2-\lambda^2)(4x+t^2-\lambda^2)}$ and e_2 is determined by the second equation in (9). $\square$

Corollary 2.9. *Let us suppose that $\lambda\neq t$ and $D=x+iy\neq 0$. If (7) holds then*

$$(1,\lambda,D)\sim\left(1,t,D\left(\frac{\beta^2-(\lambda^2-t^2)^2}{\beta^2+(\lambda^2-t^2)^2}+\frac{2\beta(\lambda^2-t^2)}{\beta^2+(\lambda^2-t^2)^2}\,i\right)\right),$$

where $\beta=2y+\sqrt{4y^2-(t^2-\lambda^2)(4x+t^2-\lambda^2)}$.

Comparing the inequalities (ii.1) and (ii.2) in Proposition 2.4 with the condition (7), we observe that for $\mathfrak{h}_2$ and $\mathfrak{h}_4$ it is possible to take $t=1$ in the previous corollary in order to get equivalences with the complex structures (i.1) and (i.3), respectively. Therefore, using Corollary 2.7, we conclude:

Proposition 2.10. *Let us consider the family of complex structures*

$$\tag{10} d\omega^1=d\omega^2=0,\quad d\omega^3=\omega^{12}+\omega^{1\bar1}+\omega^{1\bar2}+D\,\omega^{2\bar2},\quad \mathfrak{Im}\,D\geq 0.$$

Then:

- (i) *Any non-abelian complex structure on $\mathfrak{h}_2$ is equivalent to one and only one structure in (10) with $\mathfrak{Im}\,D>0$;*
- (ii) *Any non-abelian complex structure on $\mathfrak{h}_4$ is equivalent to one and only one structure in (10) with $D\in\mathbb{R}-\{0\}$.*

The classification of complex structures on $\mathfrak{h}_5$ requires a more subtle study.

Lemma 2.11. *Any non-abelian complex structure on* $\mathfrak{h}_5$ *which is not complex-parallelizable belongs to one of the following families:*

(I) $d\omega^1 = d\omega^2 = 0,\quad d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + \lambda\,\omega^{1\bar{2}} + iy\,\omega^{2\bar{2}},$ *where* $0 \leq 2y < |1-\lambda^2|$;

(II) $d\omega^1 = d\omega^2 = 0,\quad d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + (x+iy)\,\omega^{2\bar{2}},$ *where* $4y^2 < 1+4x$.

Moreover,

(i) *the structures in family* (I) *are non-equivalent;*

(ii) *the structures in family* (II) *are non-equivalent;*

(iii) *a structure* $(1,\lambda,iy)$ *in family* (I) *is equivalent to a structure in family* (II) *if and only if* $2\lambda^2 \in [0,1)$ *and* $2y \in [\lambda^2, 1-\lambda^2)$.

Proof. Let us consider a complex structure given by $(1,\lambda,D=x+i\,y)$ on $\mathfrak{h}_5$, i.e.

$$4y^2 < (1-\lambda^2)(4x+1-\lambda^2),$$

according to Proposition 2.4 (ii.3). If $\lambda^2 \geq 2x$, then $(1,\lambda,D) \sim (1,\sqrt{\lambda^2-2x}, i|D|)$ because (7) expresses simply as $4|D|^2 \geq 0$ and it trivially holds. On the other hand, if $\lambda^2 < 2x$, then $(1,\lambda,D) \sim (1,0,E)$, where E is given in Corollary 2.9, because in this case $4y^2 + \lambda^2(4x-\lambda^2) \geq 0$, that is, condition (7) is satisfied.

To study further equivalences, it is clear that structures in family (I) are non-equivalent and the same holds for structures in family (II). Now let us consider the triples $(1,\lambda,iy)$ and $(1,0,E)$. Then, (7) expresses simply as

$$4y^2 \geq \lambda^4. \tag{11}$$

Condition for family (I) implies that $4y^2 < (1-\lambda^2)^2$, which is equivalent to $4y^2 - \lambda^4 < 1 - 2\lambda^2$, so if $2\lambda^2 \geq 1$ then (11) does not hold. Now, if $0 \leq \lambda^2 < \frac{1}{2}$ then the condition for family (I) is equivalent to $y < \frac{1}{2} - \frac{\lambda^2}{2}$, and therefore when $2y \in [\lambda^2, 1-\lambda^2)$ the triple $(1,\lambda,iy)$ in family (I) is equivalent to the triple $(1,0,E = -\frac{1}{2}(\lambda^2 - \sqrt{4y^2-\lambda^4}\,i))$ in family (II). □

Proposition 2.12. *Any non-abelian complex structure on* $\mathfrak{h}_5$ *which is not complex-parallelizable is equivalent to one and only one structure in the following families:*

(I) $d\omega^1 = d\omega^2 = 0,\quad d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + \lambda\,\omega^{1\bar{2}} + D\,\omega^{2\bar{2}},$

$$\textit{where } \mathfrak{Re}\,D = 0 \textit{ and } \begin{cases} 0 \leq 2\,\mathfrak{Im}\,D < \lambda^2, & 0 < \lambda^2 < \frac{1}{2};\ \textit{or} \\ 0 \leq 2\,\mathfrak{Im}\,D < |1-\lambda^2|, & \frac{1}{2} \leq \lambda^2. \end{cases}$$

(II) $d\omega^1 = d\omega^2 = 0,\quad d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + D\,\omega^{2\bar{2}},$ *where* $4(\mathfrak{Im}\,D)^2 < 1 + 4\,\mathfrak{Re}\,D$.

To finish this section, it remains to study the case of 2-step NLAs $\mathfrak{g}$ with first Betti number equal to 3, which corresponds to $\epsilon = 1$ in (2).

Proposition 2.13. *Let* J *be a nilpotent complex structure on an NLA* $\mathfrak{g}$ *given by* (2) *with* $\epsilon = 1$*, i.e.*

$$d\omega^1 = 0, \quad d\omega^2 = \omega^{1\bar{1}}, \quad d\omega^3 = \rho\,\omega^{12} + B\,\omega^{1\bar{2}} + C\,\omega^{2\bar{1}},$$

with $\rho \in \{0,1\}$ *and* $B, C \in \mathbb{C}$ *such that* $(\rho, B, C) \neq (0,0,0)$*. Then* $\mathfrak{g}$ *is* 2*-step nilpotent if and only if* $B = \rho = 1$ *and* $C = 0$*. In such case* $\mathfrak{g}$ *is isomorphic to* $\mathfrak{h}_7$ *and all the complex structures are equivalent.*

Proof. Let Z_1, Z_2, Z_3 be the dual basis of $\omega^1, \omega^2, \omega^3$. It is clear that $[\mathfrak{g}, \mathfrak{g}]$ has dimension at least 2 and is contained in $\langle i(Z_2 - \bar{Z}_2), \mathfrak{Re}\, Z_3, \mathfrak{Im}\, Z_3\rangle$. Since $\mathfrak{Re}\, Z_3, \mathfrak{Im}\, Z_3$ are central elements and

$$[i(Z_2 - \bar{Z}_2), Z_1] = (\rho - B)i\, Z_3 + \bar{C}i\, \bar{Z}_3,$$

we conclude that $\mathfrak{g}$ is 2-step nilpotent if and only if $B = \rho$ and C vanishes.

Let $(\rho, B, C) = (1, 1, 0)$ and let us consider a basis $\{e^1, \ldots, e^6\}$ for $\mathfrak{g}^*$ given by $\omega^1 = \frac{1}{\sqrt{2}}(e^2 + ie^1)$, $\omega^2 = \frac{1}{\sqrt{2}}e^3 + ie^4$ and $\omega^3 = e^6 + ie^5$. Now, the Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_7$. □

2.2. **Nilpotent complex structures in the 3-step case.** In this section we classify, up to equivalence, nilpotent complex structures on 3-step NLAs $\mathfrak{g}$ of dimension 6. In this case the coefficient $\epsilon = 1$ in the equations (2) given in Proposition 2.3. The equivalence of complex structures in terms of the triple (ρ, B, C) is given in the following lemma.

Lemma 2.14. *Let $\mathfrak{g}$ be an NLA endowed with a nilpotent complex structure* (2) *with $\epsilon = 1$ and $(\rho, B, C) \neq (0, 0, 0)$. Then:*

(i) *If the structure is abelian, then there is a basis $\{\omega^j\}_{j=1}^3$ for $\mathfrak{g}^{1,0}$ satisfying either*

$$d\omega^1 = 0, \quad d\omega^2 = \omega^{1\bar{1}}, \quad d\omega^3 = \omega^{2\bar{1}}, \tag{12}$$

or

$$d\omega^1 = 0, \quad d\omega^2 = \omega^{1\bar{1}}, \quad d\omega^3 = \omega^{1\bar{2}} + c\,\omega^{2\bar{1}}, \tag{13}$$

where $c \in \mathbb{R}$, $c \geq 0$.

(ii) *In the non-abelian case there is a basis $\{\omega^j\}_{j=1}^3$ for $\mathfrak{g}^{1,0}$ satisfying*

$$d\omega^1 = 0, \quad d\omega^2 = \omega^{1\bar{1}}, \quad d\omega^3 = \omega^{12} + B\,\omega^{1\bar{2}} + c\,\omega^{2\bar{1}}, \tag{14}$$

where $B \in \mathbb{C}$ and $c \in \mathbb{R}$ such that $c \geq 0$.

Moreover, for any possible choice of parameters B and c, each structure in (12), (13) *and* (14) *defines an equivalence class of complex structures.*

Proof. If the complex structure is abelian then the pair $(B, C) \neq (0, 0)$ since $\rho = 0$. If $B = 0$ then it is clear that one arrives at equation (12). If $B \neq 0$ then with respect to the basis $\{z\,\omega^1, |z|^2\,\omega^2, \frac{z|z|^2}{B}\,\omega^3\}$, where z is any non-zero solution of $\frac{|C|}{|B|}\,\bar{z} = \frac{C}{B}\,z$, the equations (2) reduce to the form (13).

For the proof of (ii), we observe that with respect to $\{z\,\omega^1, |z|^2\,\omega^2, z|z|^2\,\omega^3\}$, where $z \neq 0$ satisfies $\bar{z}\,|C| = z\,C$, the equations (2) reduce to (14).

Finally, the non-equivalence of the different complex structures defined in (12), (13) and (14) follows by a similar argument to the first part of the proof of Lemma 2.5. □

The following result provides a classification of abelian structures in the 3-step case in a slightly more straightforward way than the one given in [2].

Corollary 2.15. *Let J be an abelian structure on an NLA $\mathfrak{g}$ given by* (12) *or* (13). *Then, $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_{15}$, except for $c = 1$ in which case $\mathfrak{g} \cong \mathfrak{h}_9$.*

Proof. For the equations (13), let us consider a basis $\{e^1, \ldots, e^6\}$ for $\mathfrak{g}^*$ given by $\omega^1 = -e^1 + i\, e^2$, $\omega^2 = 2e^3 + 2i\, e^4$ and $\omega^3 = 2e^5 + 2(c+1)i\, e^6$. Then, e^1, e^2, e^3 are closed, $de^4 = e^{12}$, $de^5 = (c-1)(e^{13} + e^{42})$ and $de^6 = e^{14} + e^{23}$. Thus, if $c \neq 1$ then the Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_{15}$; otherwise, $\mathfrak{g} \cong \mathfrak{h}_9$. Finally, it is easy to check that the Lie algebra $\mathfrak{g}$ underlying (12) is also isomorphic to $\mathfrak{h}_{15}$. ☐

Notice that the family (14) includes the case $\mathfrak{h}_7$ precisely for $\rho = B = 1$ and $c = 0$ as it is shown in Proposition 2.13. Next we determine the Lie algebras underlying the complex equations (14) in the remaining cases. They all have first Betti number equal to 3 and are nilpotent in step 3. Also notice that the dimension of their center is at least 2.

Proposition 2.16. *Let J be a nilpotent complex structure on a 3-step NLA $\mathfrak{g}$ given by* (14)*. Then $\mathfrak{g}$ has 3-dimensional center if and only if $|B| = 1$, $B \neq 1$ and $c = 0$. In such case $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_{16}$.*

Proof. Let Z_1, Z_2, Z_3 be the dual basis of $\omega_1, \omega_2, \omega_3$. Then, $\mathfrak{Re}\,(Z_3)$ and $\mathfrak{Im}\,(Z_3)$ are central elements. Let $T = \lambda_1 Z_1 + \bar\lambda_1 \bar Z_1 + \lambda_2 Z_2 + \bar\lambda_2 \bar Z_2$ be another non-zero element in the center of $\mathfrak{g}$, where $(\lambda_1, \lambda_2) \in \mathbb{C}^2 - \{(0,0)\}$. It follows from (14) that

$$0 = [T, Z_1] = \bar\lambda_1 Z_2 - \bar\lambda_1 \bar Z_2 - (\lambda_2 - B\bar\lambda_2) Z_3 - c\bar\lambda_2 \bar Z_3,$$

which implies $\lambda_1 = 0$, $c\lambda_2 = 0$ and $\lambda_2 = B\bar\lambda_2$. Therefore, $c = 0$ and $|B| = 1$ in order the center to be 3-dimensional, because otherwise the equation $\lambda_2 = B\bar\lambda_2$ would have trivial solution. Moreover, $B \neq 1$ because $\mathfrak{g}$ is nilpotent in step 3.

Finally, since $|B| = 1$ and $B \neq 1$, let us consider the basis $\{e^1, \ldots, e^6\}$ for $\mathfrak{g}^*$ given by: $e^1 + i\, e^2 = i(B-1)\omega^1$, $e^3 = \omega^2 + \omega^{\bar 2}$, $e^4 = \frac{1-\mathfrak{Re}\, B}{1-B} i(\omega^2 + B\, \omega^{\bar 2})$, $e^5 + i\, e^6 = (1 - \mathfrak{Re}\, B)\omega^3$. Then, we can write the differential of ω^3 in the form

$$d\omega^3 = \omega^1 \wedge (\omega^2 + B\, \omega^{\bar 2}) = \left(\frac{i(B-1)}{1-\mathfrak{Re}\, B} \omega^1 \right) \wedge \left(\frac{1-\mathfrak{Re}\, B}{1-B} i(\omega^2 + B\, \omega^{\bar 2}) \right),$$

which implies that e^1, e^2, e^3 are closed, $de^4 = e^{12}$, $de^5 = e^{14}$ and $de^6 = e^{24}$, i.e. $\mathfrak{g} \cong \mathfrak{h}_{16}$. ☐

Next we establish the conditions for the coefficients B and c in terms of the dimension of $\mathfrak{g}^2 = [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]$.

Lemma 2.17. *Let J be a complex structure on a 3-step NLA $\mathfrak{g}$ given by* (14)*. Then:*

(i) *If $c = |B-1| \neq 0$, then* $\dim \mathfrak{g}^2 = 1$.
(ii) *If $c \neq |B-1|$, then* $\dim \mathfrak{g}^2 = 2$.

Proof. From (14) we have that

$$\mathfrak{g}^2 = [Z_2 - \bar Z_2, \mathfrak{g}] = \langle (1-B)Z_3 + c\,\bar Z_3,\ c\,Z_3 + (1-\bar B)\bar Z_3 \rangle.$$

It is clear that $\dim \mathfrak{g}^2 = 2$ if and only if $(1-B)(1-\bar B) - c^2 \neq 0$. ☐

Notice that if $c = |B-1| \neq 0$ then $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_{10}$, $\mathfrak{h}_{11}$ or $\mathfrak{h}_{12}$. Since the case $c = 0 \neq |B-1|$, $|B| = 1$ corresponds to $\mathfrak{g} \cong \mathfrak{h}_{16}$ by Proposition 2.16, we conclude that for $c \neq |B-1|$ and $(c, |B|) \neq (0,1)$ the Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_{13}$, $\mathfrak{h}_{14}$ or $\mathfrak{h}_{15}$.

In order to distinguish the underlying Lie algebras, we use the following argument for $\mathfrak{g} = \mathfrak{h}_k$, $10 \leq k \leq 15$. Let $\alpha(\mathfrak{g})$ be the number of linearly independent elements

τ in $\bigwedge^2(\mathfrak{g}^*)$ such that $\tau \in d(\mathfrak{g}^*)$ and $\tau \wedge \tau = 0$. This number can be identified with the number of linearly independent exact 2-forms which are decomposable, that is, $\alpha(\mathfrak{h}_k) = 3$ for $k = 10, 12, 13$, $\alpha(\mathfrak{h}_k) = 2$ for $k = 11, 14$ and $\alpha(\mathfrak{h}_k) = 1$ for $k = 15$.

If τ is any exact element in $\bigwedge^2(\mathfrak{g}^*)$ then $\tau = \mu\, d\omega^2 + \bar\mu\, d\omega^{\bar 2} + \nu\, d\omega^3 + \bar\nu\, d\omega^{\bar 3}$, for some $\mu, \nu \in \mathbb{C}$, and by (14) we have

$$\tau = (\mu - \bar\mu)\,\omega^{1\bar 1} + \nu\,\omega^{12} + (\nu B - \bar\nu c)\,\omega^{1\bar 2} + (\nu c - \bar\nu \bar B)\,\omega^{2\bar 1} + \bar\nu\,\omega^{\bar 1\bar 2}.$$

A direct calculation shows that

$$\tau \wedge \tau = 2\left(|\nu|^2(1 - |B|^2 - c^2) + c\,\left(\nu^2 B + \bar\nu^2 \bar B\right)\right)\,\omega^{12\bar 1\bar 2}.$$

Thus, if we denote $p = \mathfrak{Re}\ \nu$ and $q = \mathfrak{Im}\ \nu$, then $\tau \wedge \tau = 0$ if and only if

$$(15)\quad \left(1 - |B|^2 - c^2 + 2c\,\mathfrak{Re}\ B\right) p^2 - (4c\,\mathfrak{Im}\ B)\, pq + \left(1 - |B|^2 - c^2 - 2c\,\mathfrak{Re}\ B\right) q^2 = 0.$$

Observe that the trivial solution $p = q = 0$ corresponds to $\tau = 2i\,\mathfrak{Im}\ \mu\,\omega^{1\bar 1}$, according to the fact that $\alpha(\mathfrak{g}) \geq 1$.

Proposition 2.18. *Let J be a complex structure on a 3-step NLA $\mathfrak{g}$ given by* (14) *with $c = |B - 1| \neq 0$. Then:*

(i) $\mathfrak{g} \cong \mathfrak{h}_{10}$ *if and only if $B = 0$;*
(ii) $\mathfrak{g} \cong \mathfrak{h}_{11}$ *if and only if $B \in \mathbb{R} - \{0, 1\}$;*
(iii) $\mathfrak{g} \cong \mathfrak{h}_{12}$ *if and only if $\mathfrak{Im}\, B \neq 0$.*

In particular, all the complex structures on $\mathfrak{h}_{10}$ are equivalent.

Proof. Since $c = |B - 1| \neq 0$, it follows from Lemma 2.17 that $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_{10}$, $\mathfrak{h}_{11}$ or $\mathfrak{h}_{12}$.

Firstly, $\mathfrak{g} \cong \mathfrak{h}_{10}$ if and only if the coefficients in equation (15) vanish. In fact, for $\mathfrak{h}_{10}$ we have by Theorem 2.1 that $\nu\, d\omega^3 + \bar\nu\, d\omega^{\bar 3} \in \langle e^{12}, e^{13}, e^{14} \rangle$ for any $\nu \in \mathbb{C}$ so any pair $(p, q) \in \mathbb{R}^2$ solves the equation (15), which implies the vanishing of its coefficients. Conversely, if the coefficients $1 - |B|^2 - c^2 + 2c\,\mathfrak{Re}\ B$, $c\,\mathfrak{Im}\ B$ and $1 - |B|^2 - c^2 - 2c\,\mathfrak{Re}\ B$ are all zero then neccesarily $B = 0$ and $c = 1$, that is, $d\omega^1 = 0$, $d\omega^2 = \omega^{1\bar 1}$ and $d\omega^3 = (\omega^1 - \omega^{\bar 1}) \wedge \omega^2$, and therefore the Lie algebra is isomorphic to $\mathfrak{h}_{10}$.

On the other hand, notice that if $c = |B - 1| \neq 0$ and $(B, c) \neq (0, 1)$ then (15) is a second degree equation in p or q. Since its discriminant is a positive multiple of $(\mathfrak{Im}\ B)^2$, if $\mathfrak{Im}\ B \neq 0$ then we get two independent solutions and $\alpha(\mathfrak{g}) = 3$, that is, $\mathfrak{g} \cong \mathfrak{h}_{12}$. Finally, for $\mathfrak{Im}\ B = 0$ the equation (15) provides one solution and $\alpha(\mathfrak{g}) = 2$, so $\mathfrak{g} \cong \mathfrak{h}_{11}$. □

Proposition 2.19. *Let J be a complex structure on a 3-step NLA $\mathfrak{g}$ given by* (14) *with $c \neq |B - 1|$ such that $(c, |B|) \neq (0, 1)$. Then:*

(i) $\mathfrak{g} \cong \mathfrak{h}_{13}$ *if and only if $c^4 - 2(|B|^2 + 1)c^2 + (|B|^2 - 1)^2 < 0$;*
(ii) $\mathfrak{g} \cong \mathfrak{h}_{14}$ *if and only if $c^4 - 2(|B|^2 + 1)c^2 + (|B|^2 - 1)^2 = 0$;*
(iii) $\mathfrak{g} \cong \mathfrak{h}_{15}$ *if and only if $c^4 - 2(|B|^2 + 1)c^2 + (|B|^2 - 1)^2 > 0$.*

Proof. Since $c \neq |B - 1|$ and $(c, |B|) \neq (0, 1)$, it follows from Lemma 2.17 and Proposition 2.16 that $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_{13}$, $\mathfrak{h}_{14}$ or $\mathfrak{h}_{15}$.

Notice that the condition $(c, |B|) \neq (0, 1)$ implies that the coefficients of p^2 and q^2 in equation (15) cannot be both zero, so (15) is always a second degree equation. Let

$$\Delta = c^4 - 2(|B|^2 + 1)c^2 + (|B|^2 - 1)^2.$$

Since the discriminant as a second degree equation in p is equal to $-4q^2\Delta$ and the discriminant as a second degree equation in q equals $-4p^2\Delta$, the number of independent solutions of equation (15) depends on the sign of Δ. Thus, for $\Delta < 0$ there exist two such solutions and thus $\mathfrak{g} \cong \mathfrak{h}_{13}$, for $\Delta = 0$ there exists only one such solution and $\mathfrak{g} \cong \mathfrak{h}_{14}$, and finally for $\Delta > 0$ there is no solution and $\alpha(\mathfrak{g}) = 1$, which implies that $\mathfrak{g} \cong \mathfrak{h}_{15}$. ☐

3. Classification of complex structures

As a consequence of our previous study, in this section we present in Table 1 the classification of nilpotent complex structures up to equivalence on 6-dimensional NLAs. In the table the closed (1,0)-form ω^1 does not appear, and the coefficients $c, \lambda \in \mathbb{R}^{\geq 0}$ and $B, D \in \mathbb{C}$ with $\mathfrak{Im}\, D \geq 0$.

In Table 1 we have also included the classification of abelian structures J on 6-dimensional NLAs obtained in [2]. In the 3-step case we use directly the equations given in Lemma 2.14 and Corollary 2.15, but in the 2-step case we have written the complex structure equations of any abelian J in a form that fits in our Proposition 2.4. More precisely, in the 2-step case we consider first the following reduction of the equations (3) of any abelian complex structure.

Corollary 3.1. *If J is abelian and $\mathfrak{g}$ is 2-step then there is a basis $\{\omega^j\}_{j=1}^3$ for $\mathfrak{g}^{1,0}$ satisfying one of the following equations:*

(i) $d\omega^1 = d\omega^2 = d\omega^3 = 0$;

(ii) $d\omega^1 = d\omega^2 = 0,\ d\omega^3 = \omega^{1\bar{1}} + D\,\omega^{2\bar{2}}$, *with* $D \in \mathbb{C},\ |D| = 1,\ \mathfrak{Im}\, D \geq 0$;

(iii) $d\omega^1 = d\omega^2 = 0,\ d\omega^3 = \omega^{1\bar{1}} + \omega^{1\bar{2}} + D\,\omega^{2\bar{2}}$, *with* $D \in \mathbb{C},\ \mathfrak{Im}\, D \geq 0$.

Proof. Suppose $\rho = 0$ in (3). If in addition $\lambda = 0$, then in terms of the basis $\{\sqrt{|D|}\ \omega^1,\ |D|\,\omega^2,\ |D|\,\omega^3\}$ we obtain (i) or (ii), whereas if $\lambda \neq 0$ then we get equations (iii) with respect to $\{\omega^1,\ \lambda\,\omega^2,\ \omega^3\}$. ☐

Next we illustrate how to rewrite the complex structure equations of any abelian J on the Lie algebra $\mathfrak{h}_5$ in a form that fits in our Corollary 3.1. By [2, Theorem 3.5] there is, up to isomorphism, one family J_t, $t \in (0,1]$, of abelian structures given by

$$J_t e^1 = e^3, \quad J_t e^2 = e^4, \quad J_t e^5 = \frac{1}{t}\, e^6.$$

With respect to the (1,0)-basis $\{\sigma^1 = e^1 - i\, e^3, \sigma^2 = e^2 - i\, e^4, \sigma^3 = -2i\, e^5 - \frac{2}{t}\, e^6\}$, the complex structure equations for J_t are

$$d\sigma^1 = d\sigma^2 = 0, \quad d\sigma^3 = \sigma^{1\bar{1}} - \frac{i}{t}\sigma^{1\bar{2}} - \frac{i}{t}\sigma^{2\bar{1}} - \sigma^{2\bar{2}}.$$

Now, by [29, Lemma 11] there exists a (1,0)-basis $\{\omega^j\}_{j=1}^3$ satisfying

$$d\omega^1 = d\omega^2 = 0, \qquad d\omega^3 = \omega^{1\bar{1}} + \omega^{1\bar{2}} + D\,\omega^{2\bar{2}},$$

with $D = \frac{1-t^2}{4}$. Notice that $D \in [0, \frac{1}{4})$ because $t \in (0,1]$. Therefore, any abelian complex structure on $\mathfrak{h}_5$ is given, up to isomorphism, as in Table 1.

For completeness we include Table 2 with the classification of non-nilpotent complex structures on 6-dimensional NLAs obtained in [30].

Table 1: Classification of nilpotent complex structures

$\mathfrak{g}$	Abelian structures ($\rho = 0$)	Non-abelian Nilpotent structures ($\rho = 1$)
$\mathfrak{h}_1$	$d\omega^2 = 0,\ d\omega^3 = 0$	—
$\mathfrak{h}_2$	$d\omega^2 = 0,\ d\omega^3 = \omega^{1\bar{1}} + D\,\omega^{2\bar{2}},$ $\mathfrak{Im}\, D = 1$	$d\omega^2 = 0,\ d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + \omega^{1\bar{2}} + D\,\omega^{2\bar{2}},$ $\mathfrak{Im}\, D > 0$
$\mathfrak{h}_3$	$d\omega^2 = 0,\ d\omega^3 = \omega^{1\bar{1}} \pm \omega^{2\bar{2}}$	—
$\mathfrak{h}_4$	$d\omega^2 = 0,$ $d\omega^3 = \omega^{1\bar{1}} + \omega^{1\bar{2}} + \frac{1}{4}\omega^{2\bar{2}}$	$d\omega^2 = 0,\ d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + \omega^{1\bar{2}} + D\,\omega^{2\bar{2}},$ $D \in \mathbb{R} - \{0\}$
$\mathfrak{h}_5$	$d\omega^2 = 0,$ $d\omega^3 = \omega^{1\bar{1}} + \omega^{1\bar{2}} + D\,\omega^{2\bar{2}},$ $D \in [0, \frac{1}{4})$	$d\omega^2 = 0,\ d\omega^3 = \omega^{12}$
		$d\omega^2 = 0,\ d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + \lambda\,\omega^{1\bar{2}} + D\,\omega^{2\bar{2}},$ with (λ, D) satisfying one of: • $\lambda = 0 \leq \mathfrak{Im}\, D,\ 4(\mathfrak{Im}\, D)^2 < 1 + 4\,\mathfrak{Re}\, D;$ • $0 < \lambda^2 < \frac{1}{2},\ 0 \leq \mathfrak{Im}\, D < \frac{\lambda^2}{2},\ \mathfrak{Re}\, D = 0;$ • $\frac{1}{2} \leq \lambda^2 < 1,\ 0 \leq \mathfrak{Im}\, D < \frac{1-\lambda^2}{2},\ \mathfrak{Re}\, D = 0;$ • $\lambda^2 > 1,\ 0 \leq \mathfrak{Im}\, D < \frac{\lambda^2-1}{2},\ \mathfrak{Re}\, D = 0.$
$\mathfrak{h}_6$	—	$d\omega^2 = 0,\ d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + \omega^{1\bar{2}}$
$\mathfrak{h}_7$	—	$d\omega^2 = \omega^{1\bar{1}},\ d\omega^3 = \omega^{12} + \omega^{1\bar{2}}$
$\mathfrak{h}_8$	$d\omega^2 = 0,\ d\omega^3 = \omega^{1\bar{1}}$	—
$\mathfrak{h}_9$	$d\omega^2 = \omega^{1\bar{1}},\ d\omega^3 = \omega^{1\bar{2}} + \omega^{2\bar{1}}$	—
$\mathfrak{h}_{10}$	—	$d\omega^2 = \omega^{1\bar{1}},\ d\omega^3 = \omega^{12} + \omega^{2\bar{1}}$
$\mathfrak{h}_{11}$	—	$d\omega^2 = \omega^{1\bar{1}},\ d\omega^3 = \omega^{12} + B\,\omega^{1\bar{2}} + \lvert B-1\rvert\,\omega^{2\bar{1}},$ $B \in \mathbb{R} - \{0, 1\}$
$\mathfrak{h}_{12}$	—	$d\omega^2 = \omega^{1\bar{1}},\ d\omega^3 = \omega^{12} + B\,\omega^{1\bar{2}} + \lvert B-1\rvert\,\omega^{2\bar{1}},$ $\mathfrak{Im}\, B \neq 0$
$\mathfrak{h}_{13}$	—	$d\omega^2 = \omega^{1\bar{1}},\ d\omega^3 = \omega^{12} + B\,\omega^{1\bar{2}} + c\,\omega^{2\bar{1}},$ $c \neq \lvert B - 1\rvert, \quad (c, \lvert B\rvert) \neq (0, 1),$ $c^4 - 2(\lvert B\rvert^2 + 1)c^2 + (\lvert B\rvert^2 - 1)^2 < 0$
$\mathfrak{h}_{14}$	—	$d\omega^2 = \omega^{1\bar{1}},\ d\omega^3 = \omega^{12} + B\,\omega^{1\bar{2}} + c\,\omega^{2\bar{1}},$ $c \neq \lvert B - 1\rvert, \quad (c, \lvert B\rvert) \neq (0, 1),$ $c^4 - 2(\lvert B\rvert^2 + 1)c^2 + (\lvert B\rvert^2 - 1)^2 = 0$
$\mathfrak{h}_{15}$	$d\omega^2 = \omega^{1\bar{1}},\ d\omega^3 = \omega^{2\bar{1}}$	$d\omega^2 = \omega^{1\bar{1}},\ d\omega^3 = \omega^{12} + B\,\omega^{1\bar{2}} + c\,\omega^{2\bar{1}},$ $c \neq \lvert B - 1\rvert, \quad (c, \lvert B\rvert) \neq (0, 1),$ $c^4 - 2(\lvert B\rvert^2 + 1)c^2 + (\lvert B\rvert^2 - 1)^2 > 0$
	$d\omega^2 = \omega^{1\bar{1}},\ d\omega^3 = \omega^{1\bar{2}} + c\,\omega^{2\bar{1}},$ $c \neq 1$	
$\mathfrak{h}_{16}$	—	$d\omega^2 = \omega^{1\bar{1}},\ d\omega^3 = \omega^{12} + B\,\omega^{1\bar{2}},$ $\lvert B\rvert = 1,\ B \neq 1$

$d\omega^1 = 0;\ \ \lambda, c \geq 0;\ \ B, D \in \mathbb{C}.$

Table 2: Classification of non-nilpotent complex structures

$\mathfrak{g}$	Complex structures
$\mathfrak{h}_{19}^{-}$	$d\omega^1 = 0, \quad d\omega^2 = \omega^{13} + \omega^{1\bar{3}}, \quad d\omega^3 = \pm i(\omega^{1\bar{2}} - \omega^{2\bar{1}})$
$\mathfrak{h}_{26}^{+}$	$d\omega^1 = 0, \quad d\omega^2 = \omega^{13} + \omega^{1\bar{3}}, \quad d\omega^3 = i\,\omega^{1\bar{1}} \pm i(\omega^{1\bar{2}} - \omega^{2\bar{1}})$

4. Frölicher spectral sequence

In this section we study the behaviour of the Frölicher sequence for 6-nilmanifolds endowed with an invariant complex structure. Recall that given a complex manifold M, the Frölicher spectral sequence $E_r^{p,q}(M)$ is the spectral sequence associated to the double complex $(\Omega^{p,q}(M), \partial, \bar{\partial})$, where ∂ and $\bar{\partial}$ come from the well-known decomposition $d = \partial + \bar{\partial}$ of the exterior differential d on M [17].

The first term $E_1(M)$ in the sequence is precisely the Dolbeault cohomology of M, that is, $E_1^{p,q}(M) \cong H_{\bar{\partial}}^{p,q}(M)$, and after a finite number of steps this sequence converges to the de Rham cohomology of M. More concretely, for each $r \geq 1$ there is a sequence of homomorphisms d_r

$$\cdots \longrightarrow E_r^{p-r,q+r-1}(M) \xrightarrow{d_r} E_r^{p,q}(M) \xrightarrow{d_r} E_r^{p+r,q-r+1}(M) \longrightarrow \cdots$$

such that $d_r \circ d_r = 0$ and $E_{r+1}^{p,q}(M) = \operatorname{Ker} d_r / \operatorname{Im} d_r$. The homomorphisms d_r are induced from ∂. When $r = 1$ the homomorphism $d_1 \colon H_{\bar{\partial}}^{p,q}(M) \longrightarrow H_{\bar{\partial}}^{p+1,q}(M)$ is given by $d_1([\alpha_{p,q}]) = [\partial \alpha_{p,q}]$, for $[\alpha_{p,q}] \in H_{\bar{\partial}}^{p,q}(M)$. We will also use that

$$E_2^{p,q}(M) = \frac{\{\alpha_{p,q} \in \Omega^{p,q}(M) \,|\, \bar{\partial}\alpha_{p,q} = 0,\ \partial\alpha_{p,q} = -\bar{\partial}\alpha_{p+1,q-1}\}}{\{\bar{\partial}\beta_{p,q-1} + \partial\gamma_{p-1,q} \,|\, \bar{\partial}\gamma_{p-1,q} = 0\}},$$

and the homomorphism $d_2 \colon E_2^{p,q}(M) \longrightarrow E_2^{p+2,q-1}(M)$ is given by $d_2([\alpha_{p,q}]) = [\partial\alpha_{p+1,q-1}]$, for $[\alpha_{p,q}] \in E_2^{p,q}(M)$ (see for example [11] for general descriptions of d_r and $E_r^{p,q}$).

Let $M = \Gamma\backslash G$ be a nilmanifold endowed with an invariant complex structure J, and let $\mathfrak{g}$ be the Lie algebra of G. In dimension 6, Rollenske proved in [26, Section 4.2] that if $\mathfrak{g} \ncong \mathfrak{h}_7$ then the natural inclusion

$$\left(\bigwedge\nolimits^{p,\bullet}(\mathfrak{g}^*), \bar{\partial}\right) \hookrightarrow (\Omega^{p,\bullet}(M), \bar{\partial})$$

induces an isomorphism

$$\iota \colon H_{\bar{\partial}}^{p,q}(\mathfrak{g}) \longrightarrow H_{\bar{\partial}}^{p,q}(M)$$

between the Lie-algebra Dolbeault cohomology of $(\mathfrak{g}, J)$ and the Dolbeault cohomology of M. Thus, an inductive argument [12, Theorem 4.2] implies that the natural map $\iota \colon E_r^{p,q}(\mathfrak{g}) \longrightarrow E_r^{p,q}(M)$ is also an isomorphism, and therefore $E_r^{p,q}(M) \cong E_r^{p,q}(\mathfrak{g})$ for any p, q and any $r \geq 1$, whenever $\mathfrak{g} \ncong \mathfrak{h}_7$. Using this, in the next result we show the general behaviour of the Frölicher sequence in dimension 6.

Theorem 4.1. *Let $M = \Gamma\backslash G$ be a 6-dimensional nilmanifold endowed with an invariant complex structure J such that the underlying Lie algebra $\mathfrak{g} \ncong \mathfrak{h}_7$. Then the Frölicher spectral sequence $E_r^{p,q}(M, J)$ behaves as follows:*

- (a) *If $\mathfrak{g} \cong \mathfrak{h}_1$, $\mathfrak{h}_3$, $\mathfrak{h}_6$, $\mathfrak{h}_8$, $\mathfrak{h}_9$, $\mathfrak{h}_{10}$, $\mathfrak{h}_{11}$, $\mathfrak{h}_{12}$ or $\mathfrak{h}_{19}^{-}$, then $E_1 \cong E_\infty$ for any J.*
- (b) *If $\mathfrak{g} \cong \mathfrak{h}_2$ or $\mathfrak{h}_4$, then $E_1 \cong E_\infty$ if and only if J is non-abelian; moreover, any abelian complex structure on $\mathfrak{h}_2$ or $\mathfrak{h}_4$ satisfies $E_1 \ncong E_2 \cong E_\infty$.*
- (c) *If $\mathfrak{g} \cong \mathfrak{h}_5$ and J is a complex structure on $\mathfrak{h}_5$ given in Table 1, then:*
  - (c.1) *$E_1 \ncong E_2 \cong E_\infty$ when J is complex-parallelizable;*

(c.2) $E_1 \cong E_\infty$ *if and only if* J *is not complex-parallelizable and* $\rho D \neq 0$*; moreover,* $E_1 \ncong E_2 \cong E_\infty$ *when* $\rho D = 0$.

(d) *If* $\mathfrak{g} \cong \mathfrak{h}_{16}$ *or* $\mathfrak{h}_{26}^{+}$*, then* $E_1 \ncong E_2 \cong E_\infty$ *for any* J.

(e) *If* $\mathfrak{g} \cong \mathfrak{h}_{13}$ *or* $\mathfrak{h}_{14}$*, then* $E_1 \cong E_2 \ncong E_3 \cong E_\infty$ *for any* J.

(f) *If* $\mathfrak{g} \cong \mathfrak{h}_{15}$ *and* J *is a complex structure on* $\mathfrak{h}_{15}$ *given in Table 1, then:*

(f.1) $E_1 \ncong E_2 \cong E_\infty$*, when* $c = 0$ *and* $|B - \rho| \neq 0$*;*

(f.2) $E_1 \cong E_2 \ncong E_3 \cong E_\infty$*, when* $\rho = 1$ *and* $|B - 1| \neq c \neq 0$*;*

(f.3) $E_1 \ncong E_2 \ncong E_3 \cong E_\infty$*, when* $\rho = 0$ *and* $|B| \neq c \neq 0$.

Proof. The proof is straightforward and we only give it explicitly for the case (f), that is, $\mathfrak{g} \cong \mathfrak{h}_{15}$, because it is the most intriguing case where different non-trivial behaviours can be produced.

We will use the notation $E_r^{|k|} = \oplus_{p+q=k} E_r^{p,q}$. Since $E_\infty^{|k|} \cong H^k_{\mathrm{dR}}$, it is clear that $\dim E_r^{|k|} \geq b_k = \dim H^k_{dR}$ for all k, and the equalities hold if and only if $E_r \cong E_\infty$.

For the calculation of the first term E_1, that is, the Dolbeault cohomology, by the Serre duality it suffices to study the spaces $E_1^{p,q} = H^{p,q}_{\bar\partial}$ for $(p,q) = (1,0), (0,1), (2,0), (1,1), (0,2), (3,0)$ and $(2,1)$.

Let J be a complex structure on $\mathfrak{h}_{15}$ given in Table 1. If J is abelian then $(B, c) = (0, 1)$ or $(1, c)$ with $c \neq 1$, therefore

$$
\begin{aligned}
&H^{1,0}_{\bar\partial} = \langle [\omega^1] \rangle, \qquad H^{2,0}_{\bar\partial} = \langle [\omega^{12}], \delta_0^c[\omega^{13}] \rangle, \qquad H^{3,0}_{\bar\partial} = \langle [\omega^{123}] \rangle,\\
&H^{0,1}_{\bar\partial} = \langle [\omega^{\bar1}], [\omega^{\bar2}], [\omega^{\bar3}] \rangle, \qquad H^{0,2}_{\bar\partial} = \langle [\omega^{\bar1\bar2}], [\omega^{\bar1\bar3}], [\omega^{\bar2\bar3}] \rangle,\\
&H^{1,1}_{\bar\partial} = \langle (1-\delta_0^c)[\omega^{1\bar2}], [\omega^{1\bar3}], \delta_0^c[\omega^{2\bar1}], [B\omega^{2\bar2} + \omega^{3\bar1}], \delta_0^c[\omega^{3\bar2}] \rangle,\\
&H^{2,1}_{\bar\partial} = \langle \delta_0^c[\omega^{12\bar1}], [\omega^{12\bar2}], [\omega^{12\bar3}], [B\omega^{13\bar2} - c\omega^{23\bar1}], \delta_0^c[\omega^{13\bar3}] \rangle,
\end{aligned}
\tag{16}
$$

where δ_0^c is equal to 0 if $c \neq 0$, and equals 1 if $c = 0$. Since $\dim E_1^{|1|} = 4 > 3 = b_1(\mathfrak{h}_{15})$ we get that $E_1 \ncong E_\infty$ for any abelian J.

When J is not abelian, i.e. $\rho = 1$, the Dolbeault cohomology groups are

$$
\begin{aligned}
&H^{1,0}_{\bar\partial} = \langle [\omega^1], \delta_0^B\delta_0^c[\omega^3] \rangle, \qquad H^{2,0}_{\bar\partial} = \langle [\omega^{12}], \delta_0^c[\omega^{13}] \rangle, \qquad H^{3,0}_{\bar\partial} = \langle [\omega^{123}] \rangle,\\
&H^{0,1}_{\bar\partial} = \langle [\omega^{\bar1}], [\omega^{\bar2}] \rangle, \qquad H^{0,2}_{\bar\partial} = \langle [\omega^{\bar1\bar3}], [\omega^{\bar2\bar3}] \rangle,\\
&H^{1,1}_{\bar\partial} = \langle (Bc + \delta_0^B)[\omega^{1\bar2}], [\omega^{1\bar3} + \omega^{2\bar2}], [B\omega^{1\bar3} - \omega^{3\bar1}], \delta_0^c[\omega^{2\bar1}], \delta_0^c[\omega^{3\bar2}] \rangle,\\
&H^{2,1}_{\bar\partial} = \langle \delta_0^c[\omega^{12\bar1}], [\omega^{12\bar2}], [c\,\omega^{12\bar3} + \omega^{13\bar2}], [B\omega^{12\bar3} + \omega^{23\bar1}], \delta_0^c[\omega^{13\bar3} + \omega^{23\bar2}] \rangle,
\end{aligned}
\tag{17}
$$

where δ_0^B has a similar definition as for δ_0^c above. Notice that the coefficient $Bc + \delta_0^B$ is non-zero except for $B \neq 0$ and $c = 0$. Thus, $\dim E_1^{|2|} \geq 6 > 5 = b_2(\mathfrak{h}_{15})$ and so $E_1 \ncong E_\infty$ also for any non-abelian J.

In order to prove (f.1) we need to study independently the abelian and the non-abelian complex structures with $c = 0$ and $B \neq \rho$ on $\mathfrak{h}_{15}$. We start with the abelian ones. In this case, by Table 1 we can suppose $B = 1$ and from (16) it follows that the dimensions of $E_1^{|2|}$ and $E_1^{|3|}$ are

$$\dim E_1^{|2|} = 9 > 5 = b_2(\mathfrak{h}_{15}), \qquad \dim E_1^{|3|} = 12 > 6 = b_3(\mathfrak{h}_{15}).$$

For the following d_1-homomorphisms

$$E_1^{0,1} \xrightarrow{d_1} E_1^{1,1} \xrightarrow{d_1} E_1^{2,1} \xrightarrow{d_1} E_1^{3,1}, \qquad E_1^{0,2} \xrightarrow{d_1} E_1^{1,2} \xrightarrow{d_1} E_1^{2,2} \xrightarrow{d_1} E_1^{3,2},$$

the classes $[\omega^{\bar3}]$, $[\omega^{1\bar3}]$, $[\omega^{3\bar2}]$, $[\omega^{13\bar3}]$, $[\omega^{\bar2\bar3}]$, $[\omega^{3\bar2\bar3}]$, $[\omega^{2\bar2\bar3}+\omega^{3\bar1\bar3}]$ and $[\omega^{13\bar2\bar3}]$ have linearly independent images. Counting dimensions for $E_2^{|k|}$ we get that

$\dim E_2^{|1|} \leq \dim E_1^{|1|} - 1 = 3 = b_1(\mathfrak{h}_{15})$, $\quad \dim E_2^{|2|} \leq \dim E_1^{|2|} - 4 = 5 = b_2(\mathfrak{h}_{15})$,
$\dim E_2^{|3|} \leq \dim E_1^{|3|} - 6 = 6 = b_3(\mathfrak{h}_{15})$, $\quad \dim E_2^{|4|} \leq \dim E_1^{|4|} - 4 = 5 = b_4(\mathfrak{h}_{15})$,
$\dim E_2^{|5|} \leq \dim E_1^{|5|} - 1 = 3 = b_5(\mathfrak{h}_{15})$.

This implies that $E_2 \cong E_\infty$ because necessarily $\dim E_2^{|k|} = b_k(\mathfrak{h}_{15})$ for all k.

If $\rho = 1$ and $c = 0$, then $B \neq 1$ and by (17) we have $\dim E_1^{|1|} = b_1(\mathfrak{h}_{15}) + \delta_0^B$. So $E_1^{|1|} \cong E_\infty^{|1|}$ when $B \neq 0$. For $B = 0$, since $d_1([\omega^3]) \neq 0$ and $d_1([\omega^{3\bar1\bar2\bar3}]) \neq 0$, we conclude that $\dim E_2^{|1|} \leq \dim E_1^{|1|} - 1 = 3 = b_1(\mathfrak{h}_{15})$ and $\dim E_2^{|5|} \leq \dim E_1^{|5|} - 1 = 3 = b_1(\mathfrak{h}_{15})$, and therefore, $E_2^{|k|} \cong E_\infty^{|k|}$ if $k = 1$ or $k = 5$.

Now, for $B \neq 1$ we have that $\dim E_1^{|2|} = 8 + \delta_0^B > 5 = b_2(\mathfrak{h}_{15})$, $\dim E_1^{|3|} = 12 > 6 = b_3(\mathfrak{h}_{15})$, $\dim E_1^{|4|} = 8 + \delta_0^B > 5 = b_4(\mathfrak{h}_{15})$. In order to conclude that $E_2 \cong E_\infty$ it suffices to observe that for the following homomorphisms

$$E_1^{1,1} \xrightarrow{d_1} E_1^{2,1} \xrightarrow{d_1} E_1^{3,1}, \qquad E_1^{0,2} \xrightarrow{d_1} E_1^{1,2} \xrightarrow{d_1} E_1^{2,2}$$

the classes $[\omega^{1\bar3}+\omega^{2\bar2}]$, $[\omega^{3\bar2}]$, $[\omega^{13\bar3}+\omega^{23\bar2}]$, $[\omega^{\bar2\bar3}]$, $[\omega^{3\bar2\bar3}]$ and $[B\omega^{2\bar2\bar3}+\omega^{3\bar1\bar3}]$ have linearly independent images.

For case (f.2), we consider $\rho = 1$ and $|B-1| \neq c \neq 0$. As $\dim E_1^{|1|} = 3 = b_1(\mathfrak{h}_{15})$, we get that $E_1^{|1|} \cong E_\infty^{|1|}$. Now, for the map $E_2^{0,2} \xrightarrow{d_2} E_2^{2,1}$ we have $d_2([\omega^{\bar2\bar3}]) = \left[\partial\left(\omega^{2\bar3} + \frac{1-\bar B}{c}\,\omega^{3\bar2}\right)\right] = \frac{|B-1|^2-c^2}{c}\,[\omega^{12\bar2}] \neq 0$, because $\omega^{12\bar2} \neq \bar\partial\beta_{2,0} + \partial\gamma_{1,1}$ for any $\beta_{2,0}$ and any $\bar\partial$-closed $\gamma_{1,1}$. Hence,

$$b_2(\mathfrak{h}_{15}) \leq \dim E_3^{|2|} \leq \dim E_2^{|2|} - 1 \leq \dim E_1^{|2|} - 1 = 6 - 1 = 5 = b_2(\mathfrak{h}_{15})$$

and we conclude that $E_\infty^{|2|} \cong E_3^{|2|} \ncong E_2^{|2|} \cong E_1^{|2|}$.

Similarly, $d_2\colon E_2^{1,2} \longrightarrow E_2^{3,1}$ is non-zero (for instance, $d_2([\omega^{3\bar1\bar3} + B\omega^{2\bar2\bar3}]) \neq 0$). Thus,

$$b_3(\mathfrak{h}_{15}) \leq \dim E_3^{|3|} \leq \dim E_2^{|3|} - 2 \leq \dim E_1^{|3|} - 2 = 8 - 2 = 6 = b_3(\mathfrak{h}_{15})$$

and we conclude that $E_\infty^{|3|} \cong E_3^{|3|} \ncong E_2^{|3|} \cong E_1^{|3|}$. By the same argument

$$b_4(\mathfrak{h}_{15}) \leq \dim E_3^{|4|} \leq \dim E_2^{|4|} - 1 \leq \dim E_1^{|4|} - 1 = 6 - 1 = 5 = b_4(\mathfrak{h}_{15})$$

and therefore $E_\infty^{|4|} \cong E_3^{|4|} \ncong E_2^{|4|} \cong E_1^{|4|}$. Summing up all the information, we conclude that $E_1 \cong E_2 \ncong E_3 \cong E_\infty$ in case (f.2).

For the last case (f.3), we first observe that $d_1([\omega^{\bar3}]) = -c[\omega^{1\bar2}] - \bar B[\omega^{2\bar1}]$. Since this class is zero if and only if $c\,\omega^{1\bar2} + \bar B\omega^{2\bar1} \in \bar\partial(\Lambda^{1,0}) = \langle \omega^{1\bar1}, B\omega^{1\bar2} + c\,\omega^{2\bar1}\rangle$, i.e. $|B| = c$, the map $d_1\colon E_1^{0,1} \longrightarrow E_1^{1,1}$ is non-zero. Therefore, $\dim E_2^{|1|} \leq \dim E_1^{|1|} - 1 = 3$, i.e. $E_1^{|1|} \ncong E_2^{|1|} \cong E_\infty^{|1|}$. Moreover, since $d_2([\omega^{\bar2\bar3}]) \neq 0$, we deduce that

$$b_2(\mathfrak{h}_{15}) \leq \dim E_3^{|2|} \leq \dim E_2^{|2|} - 1 \leq \dim E_1^{|2|} - 2 = 7 - 2 = 5 = b_2(\mathfrak{h}_{15}),$$

so $E_\infty^{|2|} \cong E_3^{|2|} \ncong E_2^{|2|} \ncong E_1^{|2|}$. Analogously, $d_2([\omega^{3\bar1\bar3} + B\omega^{2\bar2\bar3}]) \neq 0$, which implies

$$b_3(\mathfrak{h}_{15}) \leq \dim E_3^{|3|} \leq \dim E_2^{|3|} - 2 \leq \dim E_1^{|3|} - 2 = 8 - 2 = 6 = b_3(\mathfrak{h}_{15}),$$

and we conclude that $E_\infty^{|3|} \cong E_3^{|3|} \ncong E_2^{|3|} \cong E_1^{|3|}$. We also have

$$b_4(\mathfrak{h}_{15}) \leq \dim E_3^{|4|} \leq \dim E_2^{|4|} - 1 \leq \dim E_1^{|4|} - 2 = 7 - 2 = 5 = b_4(\mathfrak{h}_{15}),$$

and therefore $E_\infty^{|4|} \cong E_3^{|4|} \ncong E_2^{|4|} \ncong E_1^{|4|}$. Consequently, $E_1 \ncong E_2 \ncong E_3 \cong E_\infty$ in case (f.3). □

In [3] the authors posed the following problem: to construct a compact complex manifold such that $E_1 \cong E_\infty$ and $h_{\bar\partial}^{p,q} = h_{\bar\partial}^{q,p}$ for every $p, q \in \mathbb{N}$ but for which the $\partial\bar\partial$-lemma does not hold. Since nilmanifolds do not satisfy the $\partial\bar\partial$-lemma, unless they are complex tori, the following result provides a solution.

Proposition 4.2. *Let J be any invariant complex structure on a nilmanifold M with underlying Lie algebra isomorphic to $\mathfrak{h}_6$. Then $E_1(M) \cong E_\infty(M)$ and the Hodge numbers satisfy*

$$\begin{gathered} h_{\bar\partial}^{0,0}(M) = 1, \\ h_{\bar\partial}^{1,0}(M) = 2, \quad h_{\bar\partial}^{0,1}(M) = 2, \\ h_{\bar\partial}^{2,0}(M) = 2, \quad h_{\bar\partial}^{1,1}(M) = 5, \quad h_{\bar\partial}^{0,2}(M) = 2, \\ h_{\bar\partial}^{3,0}(M) = 1, \quad h_{\bar\partial}^{2,1}(M) = 5, \quad h_{\bar\partial}^{1,2}(M) = 5, \quad h_{\bar\partial}^{0,3}(M) = 1, \\ h_{\bar\partial}^{3,1}(M) = 2, \quad h_{\bar\partial}^{2,2}(M) = 5, \quad h_{\bar\partial}^{1,3}(M) = 2, \\ h_{\bar\partial}^{3,2}(M) = 2, \quad h_{\bar\partial}^{2,3}(M) = 2, \\ h_{\bar\partial}^{3,3}(M) = 1. \end{gathered}$$

Proof. Any complex structure J on $\mathfrak{h}_6$ is equivalent to the complex structure given in Table 1, that is, $\rho = \lambda = 1$ and $D = 0$. Its Dolbeault cohomology groups $H_{\bar\partial}^{p,q}$ for $(p,q) = (1,0), (0,1), (2,0), (1,1), (0,2), (3,0)$ and $(2,1)$ are

$$\begin{aligned} H_{\bar\partial}^{1,0} &= \langle [\omega^1], [\omega^2] \rangle, \qquad H_{\bar\partial}^{2,0} = \langle [\omega^{12}], [\omega^{13}] \rangle, \qquad H_{\bar\partial}^{3,0} = \langle [\omega^{123}] \rangle, \\ H_{\bar\partial}^{0,1} &= \langle [\omega^{\bar1}], [\omega^{\bar2}] \rangle, \qquad H_{\bar\partial}^{0,2} = \langle [\omega^{\bar1\bar3}], [\omega^{\bar2\bar3}] \rangle, \\ H_{\bar\partial}^{1,1} &= \langle [\omega^{1\bar2}], [\omega^{2\bar1}], [\omega^{2\bar2}], [\omega^{1\bar3} + \omega^{3\bar2}], [\omega^{3\bar1} + \omega^{3\bar2}] \rangle, \\ H_{\bar\partial}^{2,1} &= \langle [\omega^{12\bar2}], [\omega^{13\bar1}], [\omega^{12\bar3} + \omega^{23\bar1}], [\omega^{12\bar3} - \omega^{23\bar2}], [\omega^{13\bar2}] \rangle. \end{aligned}$$

By Serre duality we get the above Hodge diamond which is symmetric. Moreover,

$$\dim E_1^{|1|} = 4 = b_1(\mathfrak{h}_6), \quad \dim E_1^{|2|} = 9 = b_2(\mathfrak{h}_6), \quad \dim E_1^{|3|} = 12 = b_3(\mathfrak{h}_6),$$

so the Frölicher spectral sequence degenerates at the first step. □

The following result shows that there are many complex nilmanifolds for which the Frölicher spectral sequence is stable under small deformations of the complex structure.

Proposition 4.3. *Let $M = \Gamma\backslash G$ be a 6-dimensional nilmanifold endowed with an invariant complex structure J, and let $\mathfrak{g}$ be the Lie algebra of G. If $\mathfrak{g} \cong \mathfrak{h}_1, \mathfrak{h}_3, \mathfrak{h}_6, \mathfrak{h}_8, \mathfrak{h}_9, \mathfrak{h}_{10}, \mathfrak{h}_{11}, \mathfrak{h}_{12}, \mathfrak{h}_{13}, \mathfrak{h}_{14}, \mathfrak{h}_{16}, \mathfrak{h}_{19}^-$ or $\mathfrak{h}_{26}^+$, then $\dim E_r^{p,q}(M, J)$ is stable under small deformations of J for any p, q and any $r \geq 1$.*

Proof. By [25, Theorem 2.6], all small deformations of the complex structure J are again invariant complex structures. Proceeding as in the proof of Theorem 4.1, it can be proved that if $\mathfrak{g} \ncong \mathfrak{h}_2, \mathfrak{h}_4, \mathfrak{h}_5$ or $\mathfrak{h}_{15}$, then $\dim E_r^{p,q}(M)$ does not depend on the invariant complex structure on M for any p, q and any $r \geq 1$, so it is stable under small deformation of J. □

Next we provide some examples of explicit families of complex structures on nilmanifolds corresponding to $\mathfrak{h}_5$ and $\mathfrak{h}_{15}$ along which the Frölicher sequence varies. In Corollaries 5.11 and 5.12 bellow, further properties of the Frölicher spectral sequence on nilmanifolds are shown.

Example 4.4. Let J be a non complex-parallelizable and non-abelian complex structure on $\mathfrak{h}_5$ given in Table 1 with non-degenerate Frölicher sequence. According to Theorem 4.1, J has complex structure equations of the form

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + \lambda\,\omega^{1\bar{2}},$$

for some non-negative $\lambda \neq 1$. With respect to the real basis $\{e^1, \ldots, e^6\}$ given by

$$e^1 + i\,e^2 = \omega^1, \quad \frac{1}{1+\lambda}(e^3 - e^1) + \frac{i}{1-\lambda}(e^2 + e^4) = \omega^2, \quad e^5 + ie^6 = \omega^3,$$

the complex structure J expresses as

$$\begin{array}{lll} Je^1 = -e^2, & Je^3 = -\frac{2}{1-\lambda}\,e^2 - \frac{1+\lambda}{1-\lambda}\,e^4, & Je^5 = -e^6, \\ Je^2 = e^1, & Je^4 = -\frac{2}{1+\lambda}\,e^1 + \frac{1-\lambda}{1+\lambda}\,e^3, & Je^6 = e^5. \end{array}$$

For any $t \in [0, \frac{1}{2})$, consider the complex structure J_t given by

$$\begin{aligned} J_t e^1 &= \tfrac{4d(1-\lambda)}{\alpha^2}\,e^1 - \tfrac{1-\lambda^2}{\alpha}\,e^2 - \tfrac{2d(1-\lambda)^2}{\alpha^2}\,e^3 + \tfrac{8d^2(1-\lambda)}{\alpha^3}\,e^4, \\ J_t e^2 &= \tfrac{1-\lambda^2}{\alpha}\,e^1 + \tfrac{2d(1-\lambda^2)}{\alpha^2}\,e^4, \\ J_t e^3 &= -\tfrac{2d}{(1-\lambda)^2}\,e^1 - \tfrac{2\alpha}{(1-\lambda^2)(1-\lambda)}\,e^2 - \tfrac{(1+\lambda)^2}{\alpha}\,e^4, \\ J_t e^4 &= -\tfrac{2(1-\lambda)}{\alpha}\,e^1 + \tfrac{2d}{1-\lambda^2}\,e^2 + \tfrac{(1-\lambda)^2}{\alpha}\,e^3 - \tfrac{4d(1-\lambda)}{\alpha^2}\,e^4, \\ J_t e^5 &= \tfrac{2d}{1-\lambda^2}\,e^5 - \tfrac{4d^2+(1-\lambda^2)^2}{\alpha(1-\lambda^2)}\,e^6, \\ J_t e^6 &= \tfrac{\alpha}{1-\lambda^2}\,e^5 - \tfrac{2d}{1-\lambda^2}\,e^6, \end{aligned}$$

where $\alpha = \sqrt{(1-\lambda^2)^2 - 4d^2}$, and

$$d(t,\lambda) = \begin{cases} t, & \text{if } \lambda = 0, \\ t\lambda^2/4, & \text{if } \lambda^2 \in (0, 1/2), \\ t(1-\lambda^2)/4, & \text{if } \lambda^2 \in [1/2, 1), \\ -t(1-\lambda^2)/4, & \text{if } \lambda^2 > 1. \end{cases}$$

Notice that $J_0 = J$. Now, the $(1,0)$-basis

$$\begin{aligned} \omega^1 &= \tfrac{1-\lambda^2}{\alpha}\,e^1 + \tfrac{2d(1-\lambda^2)}{\alpha^2}\,e^4 + i\,e^2, \\ \omega^2 &= \tfrac{1-\lambda}{\alpha}(e^3 - e^1) - \tfrac{2d(1-\lambda)}{\alpha^2}e^4 + \tfrac{i}{1-\lambda}\left(\tfrac{2d}{\alpha}e^1 + e^2 + \tfrac{(1-\lambda^2)^2}{\alpha^2}e^4\right), \\ \omega^3 &= e^5 - \tfrac{2d}{\alpha}e^6 + i\tfrac{1-\lambda^2}{\alpha}e^6, \end{aligned}$$

satisfies

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + \lambda\,\omega^{1\bar{2}} + D\,\omega^{2\bar{2}},$$

with $D = i\,d(t,\lambda)$. According to Theorem 4.1, the Frölicher spectral sequence degenerates if and only if $D \neq 0$, i.e. if and only if $t > 0$. In conclusion, J can be deformed into a non-abelian complex structure with $E_1 \cong E_\infty$.

Corollary 4.5. *Let $M = \Gamma\backslash G$ be the nilmanifold underlying the Iwasawa manifold, i.e. $\mathfrak{g} \cong \mathfrak{h}_5$. Let J be a non complex-parallelizable and non-abelian complex structure on M given in Table 1 with $E_1 \ncong E_\infty$. Then, J can be deformed into an invariant complex structure with degenerate Frölicher spectral sequence.*

The Lie algebra $\mathfrak{h}_{15}$ has a rich complex geometry with respect to the Frölicher sequence and in the next example we construct a family J_t along which the three cases in (f) of Theorem 4.1 are realized.

Example 4.6. On $\mathfrak{h}_{15}$, let us consider the following family of complex structures

$$\begin{aligned} J_t e^1 &= -\sqrt{\frac{3(3-\sin t)(7+3\sin t)}{(5+\sin t)(11-\sin t)}}\, e^2, \\ J_t e^3 &= \sqrt{\frac{3(3-\sin t)(11-\sin t)}{(5+\sin t)(7+3\sin t)}}\, e^4, \\ J_t e^5 &= -\sqrt{\frac{(11-\sin t)(7+3\sin t)}{3(3-\sin t)(5+\sin t)}}\, e^6, \end{aligned}$$

where $t \in \mathbb{R}$. Let

$$\begin{aligned} 4\,\omega^1 &= \sqrt{(11-\sin t)(5+\sin t)}\, e^1 + i\,\sqrt{3(3-\sin t)(7+3\sin t)}\, e^2, \\ 8\,\omega^2 &= (5+\sin t)(7+3\sin t)\, e^3 - i\,\sqrt{3(5+\sin t)(3-\sin t)(11-\sin t)(7+3\sin t)}\, e^4, \end{aligned}$$

and

$$\begin{aligned} 128\,\omega^3 = (5+\sin t)(7+3\sin t)\,\Big[&3(3-\sin t)\sqrt{(11-\sin t)(5+\sin t)}\, e^5 \\ &+ i\,(11-\sin t)\sqrt{3(3-\sin t)(7+3\sin t)}\, e^6\Big]. \end{aligned}$$

Then, $\{\omega^1, \omega^2, \omega^3\}$ is a $(1,0)$-basis for J_t satisfying

$$d\omega^1 = 0, \quad d\omega^2 = \omega^{1\bar{1}}, \quad d\omega^3 = \frac{1-\sin t}{2}\,\omega^{12} + 2\,\omega^{1\bar{2}} + \frac{1+\sin t}{4}\,\omega^{2\bar{1}}.$$

It is clear that the complex structure J_t is abelian if and only if $t = \frac{4k+1}{2}\pi$, $k \in \mathbb{Z}$. For any $t \neq \frac{4k+1}{2}\pi$ the complex structure equations can be written as

$$d\omega^1 = 0, \quad d\omega^2 = \omega^{1\bar{1}}, \quad d\omega^3 = \omega^{12} + \frac{4}{1-\sin t}\,\omega^{1\bar{2}} + \frac{1+\sin t}{2(1-\sin t)}\,\omega^{2\bar{1}}.$$

Thus, concerning the Frölicher spectral sequence for the family $\{J_t\}_{t\in\mathbb{R}}$, by Theorem 4.1 (f) we get:

- $E_1 \ncong E_2 \ncong E_3 \cong E_\infty$, for $t = \frac{4k+1}{2}\pi$, $k \in \mathbb{Z}$;
- $E_1 \ncong E_2 \cong E_\infty$, for $t = \frac{4k-1}{2}\pi$, $k \in \mathbb{Z}$;
- $E_1 \cong E_2 \ncong E_3 \cong E_\infty$, for any other value of t.

As a consequence of this example, in the following result we show that for $r \geq 2$ the dimension of the term $E_r^{p,q}(J_t)$ in general is neither upper nor lower semi-continuous function of t. This is in deep contrast with the case $r = 1$, as it is well known the upper semicontinuity of the Hodge numbers $\dim H^{p,q}_{\bar{\partial}}(J_t)$ with respect to t along a deformation.

Corollary 4.7. *Let M be a nilmanifold with underlying Lie algebra $\mathfrak{h}_{15}$ endowed with the invariant complex structures J_t given in Example 4.6. Then,*

$$\dim E_2^{0,2}(J_{\frac{\pi}{2}}) = 3 > 2 = \dim E_2^{0,2}(J_t), \quad \dim E_2^{1,1}(J_{\frac{\pi}{2}}) = 2 < 3 = \dim E_2^{1,1}(J_t),$$

and

$$\dim E_3^{0,2}(J_{\frac{\pi}{2}}) = 2 > 1 = \dim E_3^{0,2}(J_t), \quad \dim E_3^{1,1}(J_{\frac{\pi}{2}}) = 2 < 3 = \dim E_3^{1,1}(J_t),$$

for any $t \in (\frac{\pi}{2}, \frac{3\pi}{2})$. Therefore, the dimensions of the terms $E_2^{1,1}(J_t)$ and $E_3^{1,1}(J_t)$ are not upper semi-continuous functions of t, and the dimensions of the terms $E_2^{0,2}(J_t)$ and $E_3^{0,2}(J_t)$ are not lower semi-continuous functions of t.

Proof. It follows directly from the proof of Theorem 4.1 taking into account that for $t = \frac{\pi}{2}$ the complex structure lies in case (f.3) and for any $t \in (\frac{\pi}{2}, \frac{3\pi}{2})$ the structures J_t lie in case (f.2). □

5. STRONGLY GAUDUCHON AND BALANCED HERMITIAN METRICS

Let (M, J) be a complex manifold of complex dimension n. A Hermitian structure Ω is *strongly Gauduchon* (*sG* for short) if $\partial\Omega^{n-1}$ is $\bar\partial$-exact [22]. In particular, any balanced Hermitian structure (i.e. $d\Omega^{n-1} = 0$) is sG, and any sG metric is a *Gauduchon* metric [18], that is, Ω^{n-1} is $\partial\bar\partial$-closed or equivalently the Lee form is co-closed.

Next we suppose that $(M = \Gamma\backslash G, J)$ is a nilmanifold endowed with an invariant complex structure. By using the symmetrization process given in [5] (see also [15], [29] and [31, Proposition 3.2]) one easily arrives at:

Proposition 5.1. *$(M = \Gamma\backslash G, J)$ has an sG metric if and only if it has an invariant one.*

Therefore, the existence of sG metrics on $(M = \Gamma\backslash G, J)$ is reduced to the existence at the Lie algebra level $\mathfrak{g}$ of G.

Corollary 5.2. *Let Ω be an invariant Hermitian structure on $(M = \Gamma\backslash G, J)$. If J is abelian then, Ω is sG if and only if it is balanced.*

Proof. It follows directly from the fact that $\bar\partial(\bigwedge^{n,n-2}(\mathfrak{g}^*)) = 0$ for any abelian complex structure. □

From now on we consider $n = 3$.

Proposition 5.3. *Let $M = \Gamma\backslash G$ be a 6-dimensional nilmanifold endowed with an invariant complex structure J. There exists an sG metric on $(M = \Gamma\backslash G, J)$ if and only if the Lie algebra $\mathfrak{g}$ of G is isomorphic to $\mathfrak{h}_1, \ldots, \mathfrak{h}_6$ or $\mathfrak{h}_{19}^-$.*

Proof. By Proposition 5.1 it suffices to study the invariant case. Let us start with the non-nilpotent case. The fundamental 2-form of any J-Hermitian metric is given by

$$2\,\Omega = i\,(r^2\omega^{1\bar1} + s^2\omega^{2\bar2} + t^2\omega^{3\bar3}) + u\omega^{1\bar2} - \bar u\omega^{2\bar1} + v\omega^{2\bar3} - \bar v\omega^{3\bar2} + z\omega^{1\bar3} - \bar z\omega^{3\bar1}, \tag{18}$$

where coefficients r^2, s^2, t^2 are non-zero real numbers and u, v, $z \in \mathbb{C}$ satisfy $r^2s^2 > |u|^2$, $s^2t^2 > |v|^2$, $r^2t^2 > |z|^2$ and $r^2s^2t^2 + 2\mathfrak{Re}\,(i\bar u\bar v z) > t^2|u|^2 + r^2|v|^2 + s^2|z|^2$. From Table 2 and using the calculations in the proof of [29, Proposition 25] we have

$$4\partial\Omega\wedge\Omega = \big(i\epsilon(s^2t^2 - |v|^2) \pm (t^2u + t^2\bar u + iv\bar z - i\bar v z)\big)\,\omega^{123\bar1\bar2} + \big(uv - is^2z\big)\,\omega^{123\bar1\bar3}.$$

Since $\bar\partial(\bigwedge^{3,1}(\mathfrak{g}^*)) = \langle \omega^{123\bar1\bar3}\rangle$, if the Hermitian structure (J,Ω) is sG then

$$\mp i\epsilon(s^2t^2-|v|^2) = t^2(u+\bar u) + iv\bar z - i\bar v z.$$

Since the left-hand side is purely imaginary and the right-hand side is real, we get that $\epsilon = 0$ and therefore $\mathfrak{g}\cong\mathfrak{h}_{19}^-$.

For the nilpotent case, let us consider the general complex equations (2). Now, the fundamental 2-form of any J-Hermitian metric is given also by (18). Using again [29, Proposition 25], we get

$$\begin{aligned}4\partial\Omega\wedge\Omega = &\big((1-\epsilon)\bar A(s^2t^2-|v|^2) + \bar B(it^2u+\bar v z) - \bar C(it^2\bar u - v\bar z)\\ &+ (1-\epsilon)\bar D(r^2t^2-|z|^2)\big)\,\omega^{123\bar1\bar2} - \epsilon(s^2t^2-|v|^2)\,\omega^{123\bar1\bar3}.\end{aligned}$$

Since $\bar\partial(\bigwedge^{3,1}(\mathfrak{g}^*)) = \langle \rho\,\omega^{123\bar1\bar2}\rangle$, if the Hermitian structure (J,Ω) is sG then $\epsilon = 0$, i.e. $\mathfrak{g}\cong\mathfrak{h}_i$ for $i=1,\dots,6$. Moreover, if in addition $\rho = 1$, then any J-Hermitian structure is sG.

In conclusion, if there exists an sG metric then $\mathfrak{g}\cong\mathfrak{h}_1,\dots,\mathfrak{h}_6$ or $\mathfrak{h}_{19}^-$. The converse follows directly from [29, Theorem 26] because these Lie algebras admit balanced Hermitian metrics. □

Remark 5.4. From the proof of the previous proposition it follows that on $\mathfrak{h}_2,\mathfrak{h}_4,\mathfrak{h}_5$ and $\mathfrak{h}_6$, if J is a non-abelian nilpotent complex structure then any invariant J-Hermitian metric is sG. This is in contrast with $\mathfrak{h}_{19}^-$, where for any complex structure the space of balanced metrics is strictly contained in the space of sG metrics, and moreover there are Hermitian metrics which are not sG. For instance, consider a Hermitian metric on $\mathfrak{h}_{19}^-$ given by

$$\Omega = \frac{i}{2}\,\omega^{1\bar1} + (u^2+z^2+1)i\,\omega^{2\bar2} + (u^2+z^2+1)i\,\omega^{3\bar3} + \frac{u}{2}(\omega^{1\bar2}-\omega^{2\bar1}) + \frac{z}{2}(\omega^{1\bar3}-\omega^{3\bar1}),$$

that is, in (18) we take $r=1$, $v=0$, u and z real and $s^2=t^2=2(u^2+z^2+1)$:

- if $u=z=0$ then the metric is balanced;
- if $u=0$ and $z\neq 0$ then the metric is sG but not balanced;
- if $u\neq 0$ then the metric is not sG.

Notice that this indicates a contrast between the sG and SKT geometries, since by [16] the existence of an SKT structure on a 6-dimensional nilpotent Lie algebra depends only on the complex structure.

There exist compact complex manifolds having sG metrics but not admitting any balanced metric [23, Theorem 1.8]. Next we show the general situation for nilmanifolds in dimension 6.

Proposition 5.5. *Let $M=\Gamma\backslash G$ be a 6-dimensional nilmanifold with an invariant complex structure J such that $(M=\Gamma\backslash G, J)$ does not admit balanced metrics. If $(M=\Gamma\backslash G, J)$ has sG metric, then J is non-abelian nilpotent and $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_2$, $\mathfrak{h}_4$ or $\mathfrak{h}_5$. Moreover, according to the classification in Table 1, such a J is given by: $\mathfrak{Re}\,D + (\mathfrak{Im}\,D)^2\geq\frac14$ on $\mathfrak{h}_2$; $\mathfrak{Re}\,D\geq\frac14$ on $\mathfrak{h}_4$; and $\lambda=0, \mathfrak{Im}\,D\neq 0$ or $\lambda=\mathfrak{Im}\,D=0, \mathfrak{Re}\,D\geq 0$ on $\mathfrak{h}_5$.*

Proof. Any complex structure on $\mathfrak{h}_6$ or $\mathfrak{h}_{19}^-$ admits balanced metrics. From [31] we have that only $\mathfrak{h}_3$ and $\mathfrak{h}_5$ have abelian complex structures J admitting balanced metric. In fact, any such J on $\mathfrak{h}_5$ admits balanced Hermitian metrics, whereas for $\mathfrak{h}_3$ the complex structure must be equivalent to the choice of $(-)$-sign in Table 1. From Corollary 5.2, it remains to study the non-abelian nilpotent complex structures J

on $\mathfrak{h}_2$, $\mathfrak{h}_4$ and $\mathfrak{h}_5$. Since any such J admits sG metrics by Remark 5.4, next we show which of them do not admit balanced metrics.

In the three cases the complex equations are of the form

$$d\omega^1 = d\omega^2 = 0, \qquad d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + \lambda\,\omega^{1\bar{2}} + D\,\omega^{2\bar{2}}. \tag{19}$$

A similar argument as in the proof of [31, Proposition 2.3] shows that, up to equivalence, the fundamental 2-form of any J-Hermitian metric is given by

$$2\,\Omega = i\,(\omega^{1\bar{1}} + s^2\,\omega^{2\bar{2}} + t^2\,\omega^{3\bar{3}}) + u\,\omega^{1\bar{2}} - \bar{u}\,\omega^{2\bar{1}},$$

where $s^2 > |u|^2$ and $t^2 > 0$.

If $D = x + iy$ and $u = u_1 + iu_2$, the balanced condition is

$$s^2 + x + iy = u_2\lambda + iu_1\lambda. \tag{20}$$

We distinguish several cases depending on the values of λ.

If $\lambda \neq 0$ then Ω is balanced if and only if $u_1 = y/\lambda$ and $u_2 = (s^2 + x)/\lambda$. The condition $s^2 > |u|^2$ is equivalent to $s^4 + (2x - \lambda^2)s^2 + x^2 + y^2 < 0$ and it is easy to see that a non-zero s satisfying this condition exists if and only if

$$\lambda^4 - 4x\lambda^2 - 4y^2 > 0. \tag{21}$$

From Table 1, for $\mathfrak{h}_2$ we get any J such that $x + y^2 \geq \frac{1}{4}$ has no balanced metrics. Similarly, for $\mathfrak{h}_4$ any J such that $x \geq \frac{1}{4}$ does not admit balanced metric.

For $\mathfrak{h}_5$ and $\lambda \neq 0$ we have that $x = 0$ by Table 1. Thus, there is no balanced metrics if and only if $\lambda^4 \leq 4y^2$. Since $y \geq 0$, this is equivalent to $\lambda^2 \leq 2y$. But from Table 1 we get that this cannot happen, therefore for $\lambda \neq 0$ the complex structures admit balanced metric.

Finally, in the case $\lambda = 0$ on $\mathfrak{h}_5$ we get that the balanced condition (20) reduces to $y = 0$ and $s^2 = -x > 0$. From Table 1 we have that $0 < 1 + 4x$, i.e. $x \in (-\frac{1}{4}, \infty)$. Therefore, if $y \neq 0$ or $y = 0, x \geq 0$ then there are no balanced metrics. □

As pointed out by Popovici [23], the degeneration of the Frölicher sequence at E_1 and the existence of sG metrics are unrelated. From the study of the sG geometry above and from Theorem 4.1 we get:

Theorem 5.6. *Let $M = \Gamma\backslash G$ be a 6-dimensional nilmanifold endowed with an invariant complex structure J. If there exists an sG metric then the Frölicher spectral sequence degenerates at the second level, i.e. $E_2(M) \cong E_\infty(M)$. Moreover, if there exists an sG metric and $\mathfrak{g} \not\cong \mathfrak{h}_5$, then $E_1(M) \cong E_\infty(M)$.*

Proof. By Proposition 5.3, the Lie algebra $\mathfrak{g}$ underlying $M = \Gamma\backslash G$ must be isomorphic to $\mathfrak{h}_1, \ldots, \mathfrak{h}_6$ or $\mathfrak{h}_{19}^-$, so Theorem 4.1 implies that the Frölicher sequence degenerates at the second level. The last assertion follows directly by taking into account Corollary 5.2 and Table 3 below. □

It is interesting whether this result holds in general, that is:

Question 5.7. *Does the Frölicher spectral sequence degenerate at the second step for any compact complex manifold M of complex dimension 3 admitting an sG metric?*

In the following table we show the complex structures J, up to equivalence, on $\mathfrak{h}_1, \ldots, \mathfrak{h}_6$ that admit balanced Hermitian metrics. The classification follows from the proof of Proposition 5.5.

Table 3: Classification of nilpotent complex structures admitting balanced metrics

$\mathfrak{g}$	Abelian structures	Non-Abelian Nilpotent structures
$\mathfrak{h}_1$	$d\omega^2=0,\ d\omega^3=0$	—
$\mathfrak{h}_2$	—	$d\omega^2=0,\ d\omega^3=\omega^{12}+\omega^{1\bar{1}}+\omega^{1\bar{2}}+(x+iy)\,\omega^{2\bar{2}}$, $y>0,\quad x+y^2<\frac{1}{4}$
$\mathfrak{h}_3$	$d\omega^2=0,\ d\omega^3=\omega^{1\bar{1}}-\omega^{2\bar{2}}$	—
$\mathfrak{h}_4$	—	$d\omega^2=0,\ d\omega^3=\omega^{12}+\omega^{1\bar{1}}+\omega^{1\bar{2}}+x\,\omega^{2\bar{2}}$, $x<\frac{1}{4},\ x\neq 0$
$\mathfrak{h}_5$	$d\omega^2=0$, $d\omega^3=\omega^{1\bar{1}}+\omega^{1\bar{2}}+x\,\omega^{2\bar{2}}$, $0\le x<\frac{1}{4}$	$d\omega^2=0,\ d\omega^3=\omega^{12}$
		$d\omega^2=0,\ d\omega^3=\omega^{12}+\omega^{1\bar{1}}+\lambda\,\omega^{1\bar{2}}+(x+iy)\,\omega^{2\bar{2}}$, with (λ,x,y) satisfying one of: • $\lambda=y=0$, $x\in\left(-\frac{1}{4},0\right)$; • $0<\lambda^2<\frac{1}{2},\ 0\le y<\frac{\lambda^2}{2},\ x=0$; • $\frac{1}{2}\le\lambda^2<1,\ 0\le y<\frac{1-\lambda^2}{2},\ x=0$; • $\lambda^2>1,\ 0\le y<\frac{\lambda^2-1}{2},\ x=0$.
$\mathfrak{h}_6$	—	$d\omega^2=0,\ d\omega^3=\omega^{12}+\omega^{1\bar{1}}+\omega^{1\bar{2}}$

Motivated by [23, Theorem 1.9] next we study the relation between the degeneration of the Frölicher spectral sequence and the existence of sG or balanced metrics. The possibilities are well illustrated in the following deformations of the complex structure corresponding to $\lambda=x=y=0$ on a nilmanifold with underlying Lie algebra $\mathfrak{h}_5$.

Example 5.8. Let $J_{\lambda,D=x+iy}$ be a non-abelian nilpotent complex structure on $\mathfrak{h}_5$ as in Table 1.

If $\lambda=0$ and $D=0$ then there are sG metrics, there do not exist balanced metrics and $E_1\ncong E_2\cong E_\infty$. In terms of the standard real basis $\{e^1,\dots,e^6\}$ on $\mathfrak{h}_5$ the complex structure $J_{0,0}$ is given by

$$\begin{aligned} &J_{0,0}\,e^1=-e^2, && J_{0,0}\,e^3=-2e^2-e^4, && J_{0,0}\,e^5=-e^6,\\ &J_{0,0}\,e^2=e^1, && J_{0,0}\,e^4=-2e^1+e^3, && J_{0,0}e^6=e^5. \end{aligned}$$

We consider the following deformations of $J_{0,0}$ in the x-direction:

$$\begin{aligned} &J_{0,x}\,e^1=\tfrac{1}{\sqrt{1+4x}}\left[(4x-1)e^2+2xe^4\right], && J_{0,x}\,e^2=\sqrt{1+4x}\,e^1+\tfrac{2x}{\sqrt{1+4x}}\,e^3,\\ &J_{0,x}\,e^3=-\sqrt{1+4x}\,(2e^2+e^4), && J_{0,x}\,e^4=-2\sqrt{1+4x}\,e^1+\tfrac{1-4x}{\sqrt{1+4x}}\,e^3,\\ &J_{0,x}\,e^5=\tfrac{-1}{\sqrt{1+4x}}\,e^6, && J_{0,x}e^6=\sqrt{1+4x}\,e^5. \end{aligned}$$

If $x\in(-\frac{1}{4},0)$ then there are balanced metrics and $E_1\cong E_\infty$.

If $x\in(0,\infty)$ then there are sG metrics, there do not exist balanced metrics and $E_1\cong E_\infty$.

Finally, let us consider the following deformation of $J_{0,0}$ in the λ-direction:

$$\begin{aligned} J_{\lambda,0}\, e^1 &= -e^2, & J_{\lambda,0}\, e^2 &= e^1, \\ J_{\lambda,0}\, e^3 &= \tfrac{-1}{1-\lambda}(2e^2 + (1+\lambda)\, e^4), & J_{\lambda,0}\, e^4 &= \tfrac{1}{1+\lambda}(-2e^1 + (1-\lambda)\, e^3), \\ J_{\lambda,0}\, e^5 &= -e^6, & J_{\lambda,0} e^6 &= e^5. \end{aligned}$$

If $\lambda^2 \in (0, \frac{1}{2})$ then there are balanced metrics and $E_1 \ncong E_2 \cong E_\infty$.

Next we address some problems on deformation openness or closedness of several properties. Let Δ be an open disc around the origin in $\mathbb{C}$. Following [23, Definition 1.12], a given property $\mathcal{P}$ of a compact complex manifold is said to be *open* under holomorphic deformations if for every holomorphic family of compact complex manifolds $(M, J_a)_{a\in\Delta}$ and for every $a_0 \in \Delta$ the following implication holds:

(M, J_{a_0}) has property $\mathcal{P} \Longrightarrow (M, J_a)$ has property $\mathcal{P}$ for all $a \in \Delta$ sufficiently close to a_0.

A given property $\mathcal{P}$ of a compact complex manifold is said to be *closed* under holomorphic deformations if for every holomorphic family of compact complex manifolds $(M, J_a)_{a\in\Delta}$ and for every $a_0 \in \Delta$ the following implication holds:

(M, J_a) has property $\mathcal{P}$ for all $a \in \Delta\backslash\{a_0\} \Longrightarrow (M, J_{a_0})$ has property $\mathcal{P}$.

Alessandrini and Bassanelli proved in [1] (see also [15]) that the balanced property of compact complex manifolds is not deformation open. In contrast, Popovici has shown in [22] that the sG property is open under holomorphic deformations, and conjectured in [23, Conjectures 1.21 and 1.23] that both the sG and the balanced properties of compact complex manifolds are closed under holomorphic deformation.

The following result provides a counterexample to both conjectures. We consider a nilmanifold M with underlying Lie algebra isomorphic to $\mathfrak{h}_4$. The abelian complex structure J_0 on M does not admit sG metrics, so it is sufficient to deform holomorphically J_0 in an open disc Δ around the origin such that J_a admits balanced metric for any $a \neq 0$. Using the Kuranishi's method, Maclaughlin, Pedersen, Poon and Salamon proved in [20] that J_0 has a locally complete family of deformations consisting entirely of invariant complex structures and obtained the deformation parameter space in terms of invariant forms. We will combine this result with our existence result of balanced metrics in Table 3.

Theorem 5.9. *Let (M, J_0) be a nilmanifold with underlying Lie algebra $\mathfrak{h}_4$ endowed with abelian complex structure J_0. Then, there is a holomorphic family of compact complex manifolds $(M, J_a)_{a\in\Delta}$, where $\Delta = \{a \in \mathbb{C} \mid |a| < 1\}$, such that (M, J_a) has a balanced metric for each $a \in \Delta\backslash\{0\}$.*

Proof. Let us consider the structure equations of the abelian complex structure J_0 as

$$d\eta^1 = d\eta^2 = 0, \quad d\eta^3 = \frac{i}{2}\eta^{1\bar{1}} + \frac{1}{2}\eta^{1\bar{2}} + \frac{1}{2}\eta^{2\bar{1}}.$$

For each $a \in \mathbb{C}$ such that $|a| < 1$, we consider the basis of (1,0)-forms $\{\mu^1, \mu^2, \mu^3\}$ given by

$$\mu^1 = \eta^1 + a\eta^{\bar{1}} - ia\eta^{\bar{2}}, \quad \mu^2 = \eta^2, \quad \mu^3 = \eta^3.$$

Notice that this corresponds to $\Phi^1_1 = a$, $\Phi^1_2 = -ia$ and $\Phi^2_1 = \Phi^2_2 = \Phi^3_1 = \Phi^3_2 = \Phi^3_3 = 0$ in the parameter space for J_0 obtained in [20, Example 8].

A direct calculation shows that

$$d\mu^1 = d\mu^2 = 0, \quad 2(1-|a|^2)d\mu^3 = 2\bar{a}\mu^{12} + i\mu^{1\bar{1}} + \mu^{1\bar{2}} + \mu^{2\bar{1}} - i|a|^2\mu^{2\bar{2}}, \tag{22}$$

so the equations define a complex structure J_a for each $a \in \Delta$. If $a = 0$ then the complex nilmanifold (M, J_0) does not admit sG metrics because J_0 is abelian.

For any $a \in \mathbb{C}$ such that $0 < |a| < 1$ the complex structure is nilpotent but not abelian. In this case we can normalize the coefficient of μ^{12} by taking $\frac{1-|a|^2}{\bar{a}}\mu^3$ instead of μ^3, so we can suppose that the complex structure equations are

$$d\mu^1 = d\mu^2 = 0, \quad d\mu^3 = \mu^{12} + \frac{i}{2\bar{a}}\mu^{1\bar{1}} + \frac{1}{2\bar{a}}(\mu^{1\bar{2}} + \mu^{2\bar{1}}) - \frac{ia}{2}\mu^{2\bar{2}}.$$

With respect to the (1,0)-basis $\{\omega^1 = \mu^1 - i\mu^2,\ \omega^2 = -2\bar{a}i\,\mu^2,\ \omega^3 = -2\bar{a}i\,\mu^3\}$, the structure equations for J_a become

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{12} + \omega^{1\bar{1}} - \frac{1}{a}\,\omega^{1\bar{2}} + \frac{1-|a|^2}{4|a|^2}\,\omega^{2\bar{2}}.$$

Now, as in the proof of Proposition 2.4 we can suppose that the coefficient of $\omega^{1\bar{2}}$ is equal to $1/|a|$.

In conclusion, for any $a \in \mathbb{C}$ such that $0 < |a| < 1$ there exists a (1,0)-basis for which the complex equations are of the form (19) with $\lambda = \frac{1}{|a|}$ and $D = \frac{1-|a|^2}{4|a|^2}$. Taking $x = \mathfrak{Re}\, D = \frac{1-|a|^2}{4|a|^2}$ and $y = \mathfrak{Im}\, D = 0$, one has $4x + \rho - \lambda^2 = 0$ according to Proposition 2.4 (ii.2). Now, following the proof of Proposition 5.5, since $\lambda \neq 0$ the complex structure J_a admits a balanced metric if and only if (21) is satisfied. But the latter condition reads

$$\lambda^2(\lambda^2 - 4x) = \frac{1}{|a|^2} > 0,$$

so there exists a balanced Hermitian metric for each $a \in \mathbb{C}$ such that $0 < |a| < 1$. $\square$

Remark 5.10. It is worth giving a closer look at the failure of the sG property at $a = 0$. Let J_a be the family of complex structures given by (22) for any $a \in \Delta = \{a \in \mathbb{C} \mid |a| < 1\}$, and let us consider the real 2-form Ω compatible with J_a given by

$$2\Omega = ir^2\,\mu^{1\bar{1}} + is^2\,\mu^{2\bar{2}} + it^2\,\mu^{3\bar{3}},$$

where $r, s, t \in \mathbb{R}$. Since

$$4\Omega \wedge d\Omega = \frac{it^2}{2(1-|a|^2)}(s^2 - |a|^2r^2)(\mu^{12\bar{1}\bar{2}\bar{3}} - \mu^{123\bar{1}\bar{2}}),$$

the 4-form Ω^2 is closed if and only if $s^2 = |a|^2r^2$, i.e. if and only if Ω is given by

$$2\Omega = ir^2\,\mu^{1\bar{1}} + i|a|^2r^2\,\mu^{2\bar{2}} + it^2\,\mu^{3\bar{3}}.$$

This defines a balanced J_a-Hermitian metric for any $r, t > 0$ and for any $0 < |a| < 1$; however, in the "central limit" $a = 0$ the form becomes degenerate, that is, the underlying metric is not positive definite.

It is well known that the property of *"the Frölicher spectral sequence degenerating at E_1"* is open under holomorphic deformations. In [14, Theorem 5.4] it is proved that this property is not closed under holomorphic deformations. As a consequence of Theorems 4.1 and 5.9 we obtain another example based on the complex geometry of $\mathfrak{h}_4$.

Corollary 5.11. *Let (M, J_0) be a nilmanifold with underlying Lie algebra $\mathfrak{h}_4$ endowed with abelian complex structure J_0. There is a holomorphic family of compact complex manifolds $(M, J_a)_{a\in\Delta}$, where $\Delta = \{a \in \mathbb{C} \mid |a| < 1\}$, such that $E_1(M, J_a) \cong E_\infty(M, J_a)$ for each $a \in \Delta\backslash\{0\}$, but $E_1(M, J_0) \ncong E_\infty(M, J_0)$.*

The upper semicontinuity of the Hodge numbers is crucial in the proof of the openness of the property "$E_1 \cong E_\infty$". Since we proved in Corollary 4.7 that the upper semicontinuity fails for $E_2^{p,q}$, the following result is not so unexpected.

Corollary 5.12. *The property of* "the Frölicher spectral sequence degenerating at E_2" *is not open.*

Proof. The family J_t given in Example 4.6 satisfies $E_2(J_{-\frac{\pi}{2}}) \cong E_\infty(J_{-\frac{\pi}{2}})$, because $J_{-\frac{\pi}{2}}$ is in case (f.1) of Theorem 4.1, but $E_2(J_t) \ncong E_\infty(J_t)$ for $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$. ☐

This result is relevant in relation to Question 5.7 since the existence of sG metric is an open property. Notice that there is no contradiction because the Lie algebra in Example 4.6 is $\mathfrak{h}_{15}$ which does not admit any sG metric.

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