STRONG KÄHLER WITH TORSION STRUCTURES FROM ALMOST CONTACT MANIFOLDS

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Abstract. For an almost contact metric manifold N, we find conditions for which either the total space of an S^1-bundle over N or the Riemannian cone over N admits a strong Kähler with torsion (SKT) structure. In this way we construct new 6-dimensional SKT manifolds. Moreover, we study the geometric structure induced on a hypersurface of an SKT manifold, and use such structures to construct new SKT manifolds via appropriate evolution equations. Hyper-Kähler with torsion (HKT) structures on the total space of an S^1-bundle over manifolds with three almost contact structures are also studied.

1. Introduction

On any Hermitian manifold (M^{2n}, J, h) there exists a unique Hermitian connection ∇^B with totally skew-symmetric torsion, called in the literature as Bismut connection [4]. The torsion 3-form $h(X, T^B(Y, Z))$ of ∇^B can be identified with the 3-form

$$-JdF(\cdot,\cdot,\cdot) = -dF(J\cdot, J\cdot, J\cdot),$$

where $F(\cdot,\cdot) = h(\cdot, J\cdot)$ is the fundamental 2-form associated to the Hermitian structure (J, h).

Hermitian structures with closed JdF are called *strong Kähler with torsion* (shortly *SKT*) or also *pluriclosed* [9]. Since $\partial\bar{\partial}$ acts as $\frac{1}{2}dJd$ on forms of bidegree $(1,1)$, the latter condition is equivalent to $\partial\bar{\partial}F = 0$. SKT structures have been recently studied by many authors and they have also applications in type II string theory and in 2-dimensional supersymmetric σ-models [18, 26, 22].

The class of SKT metrics includes of course the Kähler metrics, but as in [12] we are interested on non-Kähler geometry, so for SKT metrics we will mean Hermitian metrics h such that its fundamental 2-form F is $\partial\bar{\partial}$-closed but not d-closed.

Gauduchon in [19] showed that on a compact complex surface an SKT metric can be found in the conformal class of any given Hermitian metric, but in higher dimensions the situation is more complicated.

SKT structures on 6-dimensional nilmanifolds, i.e. on compact quotients of nilpotent Lie groups by discrete subgroups, were classified in [12, 28]. Simply-connected examples of 6-dimensional SKT manifolds have been found in [17] by using torus bundles and recently Swann in [27] has reproduced them via the twist construction, by extending them to higher dimensions, and finding new other compact simply-connected SKT manifolds. Moreover, in [14] it has been showed that the SKT condition is preserved by the blow-up construction.

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The odd dimensional analog of Hermitian structures are given by normal almost contact metric structures. Indeed, on the product $N^{2n+1} \times \mathbb{R}$ of a $(2n+1)$-dimensional almost contact metric manifold N^{2n+1} by the real line $\mathbb{R}$ it is possible to define a natural almost complex structure, which is integrable if and only if the almost contact metric structure on N^{2n+1} is normal [25]. More in general, it is possible to construct Hermitian manifolds starting from an almost contact metric manifold N^{2n+1} by considering a principal fibre bundle P with base space N^{2n+1} and structural group S^1, i.e. an S^1-bundle over N^{2n+1} (see [24]). Indeed, in [24] by using the almost contact metric structure on N^{2n+1} and the connection 1-form θ, Ogawa constructed an almost Hermitian structure (J, h) on P and found conditions for which J is integrable and (J, h) is Kähler.

In Section 2 we determine conditions for which in general an S^1-bundle over an almost contact metric $(2n+1)$-dimensional manifold N^{2n+1} is SKT (Theorem 2.3). We study the particular case when N^{2n+1} is quasi-Sasakian, i.e. it has an almost contact metric structure for which the fundamental form is closed (Corollary 2.4). In this way we are able to construct some new 6-dimensional SKT examples, starting from 5-dimensional quasi-Sasakian Lie algebras and also from Sasakian ones.

A Sasakian structure can be also seen as the analog in odd dimensions of a Kähler structure. Indeed, by [7] a Riemannian manifold (N^{2n+1}, g) of odd dimension $2n+1$ admits a compatible Sasakian structure if and only if the Riemannian cone $N^{2n+1} \times \mathbb{R}^+$ is Kähler. In Section 3 we study which conditions has to satisfy the compatible almost contact metric structure on a Riemannian manifold (N^{2n+1}, g) in order to the Riemannian cone $N^{2n+1} \times \mathbb{R}^+$ to be SKT (Theorem 3.1). An example of an SKT manifold constructed as Riemannian cone is provided and the particular case that the Riemannian cone is 6-dimensional is considered in Section 4. This case is interesting since one can impose that the SKT structure is in addition an SKT SU(3)-structure and one can find relations with the SU(2)-structures studied by Conti and Salamon in [8].

In Section 5 we study the geometric structure induced naturally on any oriented hypersurface N^{2n+1} of a $(2n+2)$-dimensional manifold M^{2n+2} carrying an SKT structure and in Section 6 we use such structures to construct new SKT manifolds via appropriate evolution equations [20, 8], starting from a 5-dimensional manifold endowed with an SU(2)-structure (Theorem 6.4).

A good quaternionic analog of Kähler geometry is given by *hyper-Kähler with torsion* (shortly *HKT*) geometry. An HKT manifold is a hyper-Hermitian manifold $(M^{4n}, J_1, J_2, J_3, h)$ admitting a hyper-Hermitian connection with totally skew-symmetric torsion, i.e. for which the three Bismut connections associated to the three Hermitian structures (J_r, h), $r = 1, 2, 3$, coincide. This geometry was introduced by Howe and Papadopoulos [21] and later studied for instance in [16, 11, 2, 3, 27].

A particular interesting case is when the torsion 3-form of such hyper-Hermitian connection is closed. In this case the HKT manifold is called *strong*.

In the last section we find conditions for which an S^1-bundle over a $(4n+3)$-dimensional manifold endowed with three almost contact metric structures is HKT and in particular when it is strong HKT (Theorem 7.1).

2. SKT structures arising from S^1-bundles

Consider a $(2n+1)$-manifold N^{2n+1} with an almost contact metric structure (I, ξ, η, g), that is, I is a tensor field of type $(1,1)$, ξ is a vector field, η is a 1-form and g is a Riemannian metric on N^{2n+1} satisfying the following conditions:

$$I^2 = -Id + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(IU, IV) = g(U, V) - \eta(U)\eta(V),$$

for any vector fields U, V on N^{2n+1}. Denote by ω the fundamental 2-form of (I, ξ, η, g), i.e. ω is the 2-form on N^{2n+1} given by

$$\omega(.,.) = g(., I.).$$

Given the tensor field I consider its Nijenhuis torsion $[I, I]$ defined by

$$[I, I](X, Y) = I^2[X, Y] + [IX, IY] - I[IX, Y] - I[X, IY]. \tag{1}$$

On the product $N^{2n+1} \times \mathbb{R}$ it is possible to define a natural almost complex structure

$$J\left(X, f\frac{d}{dt}\right) = \left(IX + f\xi, -\eta(X)\frac{d}{dt}\right),$$

where f is a $\mathcal{C}^\infty$-function on $N^{2n+1} \times \mathbb{R}$ and t is the coordinate on $\mathbb{R}$.

We recall the following

Definition 2.1. [25] *An almost contact metric structure (I, ξ, η, g) on N^{2n+1} is called* normal *if the almost complex structure J on $N^{2n+1} \times \mathbb{R}$ is integrable, or equivalently if*

$$[I, I](X, Y) + 2d\eta(X, Y)\xi = 0,$$

for any vector fields X, Y on N^{2n+1}.

By [5, Lemma 2.1] for a normal almost contact metric structure (I, ξ, η, g), one has that $i_\xi d\eta = 0$.

Remark 2.2. The normality of the almost contact structure implies also that $Id\eta = d\eta$. Indeed, we have that $d(\eta - idt) = d\eta$ has no $(0,2)$-part and therefore it has also no $(2,0)$-part since $d\eta$ is real. Thus $Jd\eta = d\eta$, but we have also that $Jd\eta = Id\eta$ since $i_\xi d\eta = 0$.

We recall that a Hermitian manifold (M, J, h) is SKT if and only if the 3-form JdF is closed, where F is the fundamental 2-form of (J, h). In the paper we will use the convention that J acts on r-forms β as

$$(J\beta)(X_1, \ldots, X_r) = \beta(JX_1, \ldots, JX_r),$$

for any vector fields $X_1, \ldots, X_r$.

We now show conditions for which in general an S^1-bundle over an almost contact metric $(2n+1)$-dimensional manifold is SKT.

Let (N^{2n+1}, I, ξ, η) be a $(2n+1)$-dimensional almost contact manifold, and let Ω be a closed 2-form on N^{2n+1} which represents an integral cohomology class on N^{2n+1}. From the well-known result of Kobayashi [23], we can consider the circle bundle $S^1 \hookrightarrow P \to N^{2n+1}$, with connection 1-form θ on P whose curvature form is $d\theta = \pi^*(\Omega)$, where $\pi : P \to N^{2n+1}$ is the projection.

By using the almost contact structure (I, ξ, η) and the connection 1-form θ, one can define an almost complex structure J on P as follows (see [24]). For any right-invariant vector field X on P, JX is given by

$$\begin{aligned} \theta(JX) &= -\pi^*(\eta(\pi_* X)), \\ \pi_*(JX) &= I(\pi_* X) + \tilde{\theta}(X)\xi, \end{aligned} \tag{2}$$

where $\tilde{\theta}(X)$ is the unique function on N^{2n+1} such that

$$\pi^*\tilde{\theta}(X) = \theta(X). \tag{3}$$

The above definition can be extended to arbitrary vector fields X on P, since X can be written in the form

$$X = \sum_j f_j X_j,$$

with f_j smooth functions on P and X_j right-invariant vector fields. Then $JX = \sum_j f_j JX_j$.

In [24] it has been showed that if (N^{2n+1}, I, ξ, η) is normal, then the almost complex structure J on P defined by (2) is integrable if and only if $d\theta$ is J-invariant, that is,

$$J(d\theta) = d\theta,$$

or equivalently

$$d\theta(JX, Y) + d\theta(X, JY) = 0,$$

for any vector fields X, Y on P, i.e. $d\theta$ is a complex 2-form on P having bidegree $(1,1)$ with respect to J.

In terms of the 2-form Ω whose lifting to P is the curvature of the circle bundle $S^1 \hookrightarrow P \to N^{2n+1}$, the previous condition means that Ω is I-invariant, i.e. $I(\Omega) = \Omega$, and therefore $i_\xi \Omega = 0$.

If $\{e^1, \dots, e^{2n}, \eta\}$ is an adapted coframe on a neighborhood U on N^{2n+1}, i.e. such that

$$Ie^{2j-1} = -e^{2j}, \quad Ie^{2j} = e^{2j-1}, \quad 1 \leq j \leq n,$$

then we can take $\{\pi^*e^1, \dots, \pi^*e^{2n}, \pi^*\eta, \theta\}$ as a coframe in $\pi^{-1}(U)$. By using the coframe $\{\pi^*e^1, \dots, \pi^*e^{2n}\}$, we may write

$$d\theta = \pi^*\alpha + \pi^*\beta \wedge \pi^*\eta,$$

where α is a 2-form in $\bigwedge^2 < e^1, \dots, e^{2n} >$ and $\beta \in \bigwedge^1 < e^1, \dots, e^{2n} >$.

Next, suppose that N^{2n+1} has a normal almost contact metric structure (I, ξ, η, g). We consider a principal S^1-bundle P with base space N^{2n+1} and connection 1-form θ, and endow P with the almost complex structure J (associated to θ) defined by (2). Since N^{2n+1} has a Riemannian metric g, a Riemannian metric h on P compatible with J (see [24]) is given by

$$h(X, Y) = \pi^*g(\pi_* X, \pi_* Y) + \theta(X)\theta(Y), \tag{4}$$

for any right-invariant vector fields X, Y. The above definition can be extended to any vector field on P.

Theorem 2.3. *Let $(N^{2n+1}, I, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact metric manifold and let Ω be a closed 2-form on N^{2n+1} which represents an integral cohomology class. Consider the circle bundle $S^1 \hookrightarrow P \to N^{2n+1}$ with connection 1-form θ whose curvature form is $d\theta = \pi^*(\Omega)$, where $\pi : P \to N^{2n+1}$ is the projection.*

Then, the almost Hermitian structure (J, h) on P, defined by (2) and (4), is SKT *if and only if (I, ξ, η, g) is normal, $d\theta$ is J-invariant and such that*

$$d(\pi^*(I(i_\xi d\omega))) = 0, \qquad (5)$$
$$d(\pi^*(I(d\omega) - d\eta \wedge \eta)) = (-\pi^*(I(i_\xi d\omega)) + \pi^*\Omega) \wedge \pi^*\Omega,$$

where ω denotes the fundamental form of the almost contact metric structure (I, ξ, η, g).

Proof. As we mentioned previously, a result of Ogawa [24] asserts that the almost complex structure J is integrable if and only if (g, I, ξ, η) is normal and $J(d\theta) = d\theta$. Thus (J, h) is SKT if and only if the 3-form JdF is closed. By using the first equality of (2), we have that the fundamental 2-form F on P is

$$\begin{aligned} F(X, Y) &= h(X, JY) = \pi^*g(\pi_*X, \pi_*JY) + \theta(X)\theta(JY) \\ &= \pi^*g(\pi_*X, \pi_*JY) - \theta(X)\pi^*\eta(\pi_*Y). \end{aligned}$$

Therefore, taking into account that we are working with a circle bundle, and so its fibre is 1-dimensional, we have

$$F = \pi^*\omega + \pi^*\eta \wedge \theta.$$

Thus,

$$dF = \pi^*(d\omega) + \pi^*(d\eta) \wedge \theta - \pi^*\eta \wedge d\theta,$$

and

$$JdF = J(\pi^*(d\omega)) - J(\pi^*(d\eta)) \wedge \pi^*\eta - \theta \wedge d\theta, \qquad (6)$$

since $J(\pi^*\eta) = \theta$ and J is integrable, so $J(d\theta) = d\theta$.

Moreover, we have

$$J(\pi^*(d\omega)) = \pi^*(I(d\omega)) + \pi^*(I(i_\xi d\omega)) \wedge \theta. \qquad (7)$$

Indeed, locally and in terms of the adapted basis $\{e^1, \dots, e^{2n+1}\}$ such that

$$Ie^{2j-1} = -e^{2j}, \quad 1 \le j \le n, \quad Ie^{2n+1} = 0, \quad \eta = e^{2n+1},$$

we can write

$$d\omega = \alpha + \beta \wedge \eta,$$

where the local forms $\alpha \in \Lambda^3 < e^1, \dots, e^{2n} >$ and $\beta \in \Lambda^2 < e^1, \dots, e^{2n} >$ are generated only by $e^1, \dots, e^{2n}$. Furthermore, we have

$$I\alpha = I(d\omega), \quad \beta = i_\xi d\omega.$$

Thus,

$$J(\pi^*(d\omega)) = J(\pi^*(\alpha)) + J(\pi^*(i_\xi d\omega)) \wedge \theta.$$

Now, by using (2) and (3), we see that $J(\pi^*(\alpha)) = \pi^*(I\alpha)$ and $J(\pi^*(i_\xi d\omega)) = \pi^*(I(i_\xi d\omega))$, which proves (7). As a consequence of Remark 2.2 we have

$$J(\pi^*(d\eta)) = \pi^*(I(d\eta)) - \pi^*(I(i_\xi d\eta)) \wedge \theta = \pi^*(d\eta), \qquad (8)$$

since $i_\xi d\eta = 0$ and $Id\eta = d\eta$.

By using (7) and (8) we get

$$JdF = \pi^*(I(d\omega)) + \pi^*(I(i_\xi d\omega)) \wedge \theta - \pi^*(d\eta) \wedge \pi^*\eta - \theta \wedge d\theta. \qquad (9)$$

Therefore

$$\begin{aligned} d(JdF) &= d\left(\pi^*(I(d\omega))\right) + d(\pi^*\{I(i_\xi d\omega)\}) \wedge \theta + \pi^*(I(i_\xi d\omega)) \wedge d\theta \\ &\quad - d(\pi^*(d\eta)) \wedge \pi^*\eta - \pi^*(d\eta) \wedge d\pi^*\eta - d\theta \wedge d\theta. \end{aligned}$$

Consequently, $d(JdF) = 0$ if and only if

$$d(\pi^*(I(i_\xi d\omega))) = 0,$$

and

$$d(\pi^*(I(d\omega) - d\eta \wedge \eta)) = (\pi^*(-I(i_\xi d\omega)) + d\theta) \wedge d\theta,$$

which completes the proof. □

We recall that an almost contact metric manifold $(N^{2n+1}, I, \xi, \eta, g)$ is *quasi-Sasakian* if it is normal and its fundamental form ω is closed. If, in particular, $d\eta = \alpha\,\omega$, then the almost contact metric structure is called *α-Sasakian*. When $\alpha = -2$, the structure is said to be *Sasakian*.

By [15, Theorem 8.2] an almost contact metric manifold $(N^{2n+1}, I, \xi, \eta, g)$ admits a connection ∇^c preserving the almost contact metric structure and with totally skew-symmetric torsion tensor if and only if the Nijenhuis tensor of I, given by (1), is skew-symmetric and ξ is a Killing vector field. Moreover, this connection is unique.

Then, in particular on any quasi-Sasakian manifold $(N^{2n+1}, I, \xi, \eta, g)$ there exists a unique connection ∇^c with totally skew-symmetric torsion such that

$$\nabla^c I = 0, \quad \nabla^c g = 0, \quad \nabla^c \eta = 0.$$

Such connection ∇^c is uniquely determined by

$$g(\nabla^c_X Y, Z) = g(\nabla^g_X Y, Z) + \frac{1}{2}(d\eta \wedge \eta)(X, Y, Z), \tag{10}$$

where ∇^g denotes the Levi-Civita connection and $\frac{1}{2}(d\eta \wedge \eta)$ is the torsion 3-form of ∇^c.

Corollary 2.4. *Let $(N^{2n+1}, I, \xi, \eta, g)$ be a quasi-Sasakian $(2n+1)$-manifold and let Ω be a closed 2-form on N^{2n+1} which represents an integral cohomology class. Consider the circle bundle $S^1 \hookrightarrow P \to N^{2n+1}$ with connection 1-form θ whose curvature form is $d\theta = \pi^*(\Omega)$, where $\pi : P \to N^{2n+1}$ is the projection. Then, the almost Hermitian structure (J, h) on P, defined by (2) and (4), is* SKT *if and only if Ω is I-invariant, $i_\xi \Omega = 0$ and*

$$d\eta \wedge d\eta = -\Omega \wedge \Omega. \tag{11}$$

Moreover, the Bismut connection ∇^B of (J, h) on P and the connection ∇^c on N given by (10) are related by

$$h(\nabla^B_X Y, Z) = \pi^* g(\nabla^c_{\pi_* X} \pi_* Y, \pi_* Z), \tag{12}$$

for any vector fields $X, Y, Z \in Ker\theta$.

Proof. Since $d\omega = 0$, if we impose the SKT condition, by using the previous theorem, we get the equation (11).

The Bismut connection ∇^B associated to the Hermitian structure (J, h) on P is given by:

$$h(\nabla^B_X Y, Z) = h(\nabla^h_X Y, Z) - \frac{1}{2}\, dF(JX, JY, JZ), \tag{13}$$

for any vector fields X, Y, Z on P, where ∇^h is the Levi-Civita connection associated to h. Then, for any X, Y, Z in the kernel of θ we have

$$h(\nabla^B_X Y, Z) = \pi^* g(\nabla^h_X Y, Z) + \frac{1}{2}(\pi^*(d\eta) \wedge \pi^* \eta)(X, Y, Z).$$

By [24, Lemma 3] and the definition of ∇^c we get

$$h(\nabla^B_X Y, Z) = \pi^* g(\nabla^g_{\pi_* X}\pi_* Y, \pi_* Z) + \frac{1}{2}(\pi^*(d\eta)\wedge\pi^*\eta)(X, Y, Z) = \pi^* g(\nabla^c_{\pi_* X}\pi_* Y, \pi_* Z),$$

for any X, Y, Z in the kernel of θ. □

Remark 2.5. If the structure (I, ξ, η, g) is α-Sasakian, equation (11) reads as

$$\Omega \wedge \Omega = -\alpha^2\, \omega \wedge \omega.$$

In the case of a trivial S^1-bundle, i.e. by considering the natural almost Hermitian structure on the product $N^{2n+1} \times \mathbb{R}$, we get the following

Corollary 2.6. *Let $(N^{2n+1}, I, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact metric manifold. Consider on the product $N^{2n+1} \times \mathbb{R}$ the almost complex structure J given by*

$$JX = IX, \quad X \in Ker\eta, \quad J\xi = -\frac{d}{dt},$$

and the product metric $h = g + (dt)^2$. The Hermitian structure (J, h) is SKT *if and only if (I, ξ, η, g) is normal and such that*

$$d(I(d\omega)) = d(d\eta \wedge \eta), \quad d(I(i_\xi d\omega)) = 0,$$

where ω denotes the fundamental 2-form of the almost contact metric structure (g, I, ξ, η).

As a consequence of previous results we get

Corollary 2.7. *Let $(N^{2n+1}, I, \xi, \eta, g)$ be a $(2n+1)$-dimensional quasi-Sasakian manifold such that $d\eta \wedge d\eta = 0$. Then, the Hermitian structure (J, h) on $N^{2n+1} \times \mathbb{R}$ is* SKT. *Moreover, its Bismut connection ∇^B coincides with the unique connection ∇^c on N^{2n+1} given by* (10).

Proof. In this case, since $d\omega = 0$ we get

$$d(JdF) = -d(d\eta \wedge \eta).$$

Moreover, by using (12)

$$h(\nabla^B_X Y, Z) = g(\nabla^c_X Y, Z),$$

for any vector fields X, Y, Z on N^{2n+1}. □

2.1. **Examples.** We will start presenting three examples of quasi-Sasakian Lie algebras satisfying the condition $d\eta \wedge d\eta = 0$. By applying Corollary 2.7 one gets an SKT structure on the product of the corresponding simply-connected Lie group by $\mathbb{R}$.

Example 2.8. Let $\mathfrak{s}$ be the 5-dimensional Lie algebra with structure equations

$$\begin{cases} de^1 = e^{13} + e^{23} + e^{25} - e^{34} + e^{35}, \\ de^2 = 2e^{12} - 2e^{13} + e^{14} - e^{15} - e^{24} + e^{34} + e^{45}, \\ de^3 = -e^{12} + e^{13} + e^{14} - e^{15} + 2e^{24} - 2e^{34} + e^{45}, \\ de^4 = -e^{12} - e^{23} + e^{24} - e^{25} - e^{35}, \\ de^5 = e^{12} - e^{13} - e^{24} + e^{34}, \end{cases}$$

where by e^{ij} we denote $e^i \wedge e^j$.

Consider on $\mathfrak{s}$ the quasi-Sasakian structure (I, ξ, η, g) given by

$$\eta = e^5, \quad Ie^1 = -e^2, \quad Ie^3 = -e^4, \quad \omega = -e^{12} - e^{34}, \quad g = \sum_{j=1}^{5} (e^j)^2. \tag{14}$$

We have that the above quasi-Sasakian structure satisfies the condition $d(d\eta \wedge \eta) = 0$.

The Lie algebra $\mathfrak{s}$ is 2-step solvable since the commutator

$$\mathfrak{s}^1 = [\mathfrak{s}, \mathfrak{s}] = \mathbb{R} < e_1 - e_4,\, e_2 + e_3,\, e_1 - e_2 + 2e_3 - e_5 >$$

is abelian, where $\{e_1, \dots, e_5\}$ denotes the dual basis of $\{e^1, \dots, e^5\}$. Moreover $\mathfrak{s}$ has trivial center, it is irreducible and non unimodular, since we have that the trace of ad_{e_1} is equal to -3.

Example 2.9. Consider the family of 2-step solvable Lie algebras $\mathfrak{s}_a$, $a \in \mathbb{R} - \{0\}$, given by

$$\begin{cases} de^1 = a\, e^{23} + 3\, e^{25}, \\ de^2 = -a\, e^{13} - 3\, e^{15}, \\ de^3 = a\, e^{34}, \\ de^4 = 0, \\ de^5 = -\frac{a^2}{3}\, e^{34}. \end{cases}$$

The almost contact metric structure (I, ξ, η, g) given by (14) is quasi-Sasakian and satisfies the condition $d\eta \wedge d\eta = 0$. Moreover, the second cohomology group of $\mathfrak{s}_a$ is generated by e^{12} and e^{45}.

Example 2.10. Another example of family of quasi-Sasakian Lie algebras satisfying the condition $d\eta \wedge d\eta = 0$ is $\mathfrak{g}_b$, $b \in \mathbb{R} - \{0\}$, with structure equations

$$\begin{cases} de^1 = b\,(e^{13} + e^{14} - e^{23} + e^{24}) + e^{25}, \\ de^2 = b\,(-e^{13} + e^{14} - e^{23} - e^{24}) - e^{15}, \\ de^3 = 2\, e^{45}, \\ de^4 = -2\, e^{35}, \\ de^5 = -4b^2\, e^{34}, \end{cases}$$

and endowed with the quasi-Sasakian structure given by (14). The second cohomology group of $\mathfrak{g}_b$ is generated by e^{12}. The Lie algebras $\mathfrak{g}_b$ are not solvable since for the commutator we have $[\mathfrak{g}_b, \mathfrak{g}_b] = \mathfrak{g}_b$.

The Lie groups underlying examples 2.9 and 2.10 satisfy also the conditions of Corollary 2.4 with $\Omega \wedge \Omega = 0$ just by considering as connection 1-form the 1-form e^6 such that $de^6 = \lambda e^{12}$ and then $\Omega = \lambda e^{12}$. With this expression of de^6 we have that: $d^2 e^6 = 0$, $J(de^6) = de^6$ and $de^6 \wedge de^6 = 0$, and therefore equation (11) is satisfied. Observe that $\lambda = 0$ provides examples of trivial S^1-bundles.

We can recover also one of the 6-dimensional nilmanifolds found in [12].

Example 2.11. Consider the 5-dimensional nilpotent Lie algebra with structure equations

$$\begin{cases} de^j = 0, \quad j = 1, \dots, 4, \\ de^5 = e^{12} + e^{34}, \end{cases}$$

and endowed with the quasi-Sasakian structure given by (14). If we consider the closed 2-form $\Omega = e^{13} + e^{24}$ and we apply Corollary 2.4 we have that there exists a non trivial S^1-bundle over the corresponding 5-dimensional nilmanifold. Moreover, since $de^5 \wedge de^5 = -\Omega \wedge \Omega \neq 0$, the total space of this S^1-bundle is an SKT nilmanifold. More precisely, according to the classification given in [12] (see also [28]), the nilmanifold is the one with underlying Lie algebra isomorphic to $\mathfrak{h}_3 \oplus \mathfrak{h}_3$, where by $\mathfrak{h}_3$ we denote the real 3-dimensional Heisenberg Lie algebra.

Since the starting Lie algebra in Example 2.11 is Sasakian, it is natural to start with other 5-dimensional Sasakian Lie algebras to construct new SKT structures in dimension 6. A classification of 5-dimensional Sasakian Lie algebras was obtained in [1].

Example 2.12. Consider the 5-dimensional Lie algebra $\mathfrak{k}_3$ with structure equations

$$\begin{cases} de^j = 0, \quad j = 1, 4, \\ de^2 = -e^{13}, \\ de^3 = e^{12}, \\ de^5 = \lambda\, e^{14} + \mu\, e^{23}, \end{cases}$$

where $\lambda, \mu < 0$. By [1] $\mathfrak{k}_3$ admits the Sasakian structure given by

$$\begin{aligned} &Ie^1 = e^4, \quad Ie^2 = e^3, \quad \eta = e^5, \\ &g = -\tfrac{\lambda}{2}\, e_1 \otimes e_1 - \tfrac{\lambda}{2}\, e_2 \otimes e_2 - \tfrac{\mu}{2}\, e_3 \otimes e_3 - \tfrac{\mu}{2}\, e_4 \otimes e_4 + \, e_5 \otimes e_5, \end{aligned}$$

and it is isomorphic to $\mathbb{R} \ltimes (\mathfrak{h}_3 \times \mathbb{R})$. Moreover, by [1] the corresponding solvable simply-connected Lie group admits a compact quotient by a discrete subgroup.

Consider on $\mathfrak{k}_3$ the closed 2-form $\Omega = \lambda\, e^{14} - \mu\, e^{23}$. Ω is I-invariant and satisfies $\Omega \wedge \Omega = -2\lambda\mu\, e^{1234}$. Since e^5 is the contact form and $de^5 \wedge de^5 = 2\lambda\mu\, e^{1234}$, again we get by Corollary 2.4 an SKT structure on a non trivial S^1-bundle over the 5-dimensional solvmanifold. We will denote by e^6 the connection 1-form.

The orthonormal basis $\{\alpha^1 = e^1, \alpha^2 = e^4, \alpha^3 = e^2, \alpha^4 = e^3, \alpha^5 = e^5, \alpha^6 = \theta\}$ for the SKT metric satisfies the equations

$$d\alpha^1 = d\alpha^2 = 0, \quad d\alpha^3 = -\alpha^{14}, \quad d\alpha^4 = \alpha^{13},$$

$$d\alpha^5 = \lambda\, \alpha^{12} + \mu\, \alpha^{34}, \quad d\alpha^6 = \lambda\, \alpha^{12} - \mu\, \alpha^{34},$$

and the complex structure is given by $J(X_1) = X_2, J(X_3) = X_4, J(X_5) = X_6$, where $\{X_i\}_{i=1}^6$ denotes the basis dual to $\{\alpha^i\}_{i=1}^6$. Since the fundamental 2-form is $F = \alpha^{12} + \alpha^{34} + \alpha^{56}$, one has that the 3-form torsion T of the SKT structure is

$$T = \lambda\, \alpha^{12}(\alpha^5 + \alpha^6) + \mu\, \alpha^{34}(\alpha^5 - \alpha^6).$$

Moreover, $*T = \lambda\, \alpha^{12}(\alpha^5 + \alpha^6) - \mu\, \alpha^{34}(\alpha^5 - \alpha^6)$, where $*$ denotes the Hodge operator of the metric, which implies that the torsion form is also coclosed.

The only nonzero curvature forms $(\Omega^B)^i_j$ of the Bismut connection ∇^B are

$$(\Omega^B)^1_2 = -2\, \lambda^2 \alpha^{12}, \qquad (\Omega^B)^3_4 = -2\, \mu^2 \alpha^{34}.$$

A direct calculation shows that the 1-forms α^5, α^6 and the 2-forms α^{12}, α^{34} are parallel with respect to the Bismut connection, which implies that $\nabla^B T = 0$.

Finally, since $\nabla^B \alpha^i \neq 0$ for $i = 1, 2, 3, 4$, we conclude that $Hol(\nabla^B) = U(1) \times U(1) \subset U(3)$.

3. SKT structures arising from Riemannian cones

Let N^{2n+1} be a $(2n+1)$-dimensional manifold endowed with an almost contact metric structure (I, ξ, η, g) and denote by ω its fundamental 2-form.

The Riemannian cone of N^{2n+1} is defined as the manifold $N^{2n+1} \times \mathbb{R}^+$ equipped with the cone metric:

$$h = t^2 g + (dt)^2. \tag{15}$$

The cone $N^{2n+1} \times \mathbb{R}^+$ has a natural almost Hermitian structure defined by

$$F = t^2 \omega + t\eta \wedge dt. \tag{16}$$

The almost complex structure J on $N^{2n+1} \times \mathbb{R}^+$ defined by (F, h) is given by

$$JX = IX, \; X \in \operatorname{Ker} \eta, \quad J\xi = -t\frac{d}{dt}.$$

In terms of a local orthonormal adapted coframe $\{e^1, \dots, e^{2n}\}$ for g such that

$$\omega = -\sum_{j=1}^{n} e^{2j-1} \wedge e^{2j}, \tag{17}$$

we have

$$\begin{array}{l} Je^{2j-1} = -e^{2j}, \quad Je^{2j} = e^{2j-1}, \; j = 1, \dots, n, \\ J(te^{2n+1}) = dt, \quad J(dt) = -te^{2n+1}. \end{array} \tag{18}$$

The almost Hermitian structure (J, h) on $N^{2n+1} \times \mathbb{R}^+$ is Kähler if and only if the almost contact metric structure (I, ξ, η, g) on N^{2n+1} is Sasakian, i.e. a normal contact metric structure.

If we impose that the almost Hermitian structure (J, h) on $N^{2n+1} \times \mathbb{R}^+$ is SKT, we can prove the following

Theorem 3.1. *Let $(N^{2n+1}, I, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact metric manifold. The almost Hermitian structure (J, h) on the Riemannian cone $(N^{2n+1} \times \mathbb{R}^+, h)$, given by* (15) *and* (16)*, is SKT if and only if (I, ξ, η, g) is normal and*

$$-4\eta \wedge \omega + 2I(d\omega) - 2d\eta \wedge \eta = d(I(i_\xi d\omega)), \tag{19}$$

where ω denotes the fundamental 2-form of the almost contact metric structure (I, ξ, η, g).

Proof. J is integrable if and only if the almost contact metric structure is normal. Now we compute JdF. We have that

$$dF = 2tdt \wedge \omega + t^2 d\omega + td\eta \wedge dt,$$

and

$$JdF = -2t^2 \eta \wedge \omega + t^2 J(d\omega) - t^2 d\eta \wedge \eta,$$

since

$$J\omega = \omega, \quad J(dt) = -t\eta, \quad Jd\eta = d\eta.$$

Moreover, with respect to an adapted basis $\{e^1, \dots, e^{2n+1}\}$ we may prove, in a similar way as in the proof of Theorem 2.3, that

$$Jd\omega = I(d\omega) + I(i_\xi d\omega) \wedge J\eta. \tag{20}$$

As a consequence we get

$$JdF = -2t^2 \eta \wedge \omega + t^2 I(d\omega) + tdt \wedge I(i_\xi d\omega) - t^2 d\eta \wedge \eta.$$

Therefore, by imposing $d(JdF) = 0$ we obtain the two equations

$$\begin{cases} -4\eta \wedge \omega + 2I(d\omega) - 2d\eta \wedge \eta - d(I(i_\xi d\omega)) = 0, \\ -2d(\eta \wedge \omega) + d(I(d\omega)) - d(d\eta \wedge \eta) = 0. \end{cases}$$

Since the second equation is consequence of the first one, we have that the Hermitian structure (F, h) on the Riemannian cone $N^{2n+1} \times \mathbb{R}^+$ is SKT if and only if the almost contact metric structure $(I, \eta, \xi, g, \omega)$ on N^{2n+1} satisfies the equation (19). □

Remark 3.2. As a consequence of previous theorem we have that, if $n = 1$, equation (19) is satisfied if and only if the 3-dimensional manifold N is Sasakian. On the other hand, if $n > 1$ and the almost contact metric structure on N^{2n+1} is quasi-Sasakian (i.e. $d\omega = 0$), then the structure has to be Sasakian, i.e. $d\eta = -2\omega$.

Example 3.3. Consider the 5-dimensional Lie algebras $\mathfrak{g}_{a,b,c}$ with structure equations

$$\begin{cases} de^1 = a\, e^{23} + 2\, e^{25} + \left(-\frac{1}{2}ab + \frac{b^3}{2a} + 2\frac{b}{a}\right) e^{34} + b\, e^{45}, \\ de^2 = -a\, e^{13} - 2\, e^{15} - \frac{1}{2}bc\, e^{34} - b\, e^{35}, \\ de^3 = \left(-\frac{4}{a} - \frac{b^2}{a}\right) e^{34}, \\ de^4 = c\, e^{34}, \\ de^5 = 2\, e^{12} + b\, e^{14} - b\, e^{23} + (2 + b^2)\, e^{34}, \end{cases}$$

where $a, b, c \in \mathbb{R}$ and $a \neq 0$, endowed with the normal almost contact metric structure $(I, \xi, \eta, g, \omega)$ with

$$Ie^1 = -e^2, \quad Ie^3 = -e^4, \quad \eta = e^5, \quad \omega = -e^{12} - e^{34}.$$

This structure satisfies (19) and therefore, the Riemannian cones over the corresponding simply-connected Lie groups are SKT.

4. SKT SU(3)-structures

Let (M^6, J, h) be a 6-dimensional almost Hermitian manifold. An $SU(3)$–structure on M^6 is determined by the choice of a $(3,0)$-form $\Psi = \Psi_+ + i\Psi_-$ of unit norm. If Ψ is closed, then the underlying almost complex structure J is integrable and the manifold is Hermitian. We will denote the $SU(3)$-structure (J, h, Ψ) simply by (F, Ψ), where F is the fundamental 2-form, since from F and Ψ we can reconstruct the almost Hermitian structure.

We can give the following

Definition 4.1. *We say that an* SU(3)*-structure* (F, Ψ) *on* M^6 *is* SKT *if*

$$d\Psi = 0, \qquad d(JdF) = 0, \tag{21}$$

where J *is the associated complex structure.*

We will see the relation between SKT $SU(3)$-structures in dimension 6 and $SU(2)$-structures in dimension 5.

First we recall some facts about SU(2)-structures on a 5-dimensional manifold. An SU(2)-*structure* on a 5-dimensional manifold N^5 is an SU(2)-reduction of the principal bundle of linear frames on N^5. By [8, Proposition 1], these structures are in 1 : 1 correspondence with quadruplets $(\eta, \omega_1, \omega_2, \omega_3)$, where η is a 1-form and ω_i are 2-forms on N^5 satisfying

$$\omega_i \wedge \omega_j = \delta_{ij} v, \quad v \wedge \eta \neq 0,$$

for some 4-form v, and

$$i_X \omega_3 = i_Y \omega_1 \;\Rightarrow\; \omega_2(X, Y) \geq 0,$$

where i_X denotes the contraction by X. Equivalently, an SU(2)-structure on N^5 can be viewed as the datum of (η, ω_1, Φ), where η is a 1-form, ω_1 is a 2-form and $\Phi = \omega_2 + i\,\omega_3$ is a complex 2-form such that

$$\eta \wedge \omega_1 \wedge \omega_1 \neq 0, \qquad \Phi \wedge \Phi = 0, \qquad \omega_1 \wedge \Phi = 0, \qquad \Phi \wedge \overline{\Phi} = 2\,\omega_1 \wedge \omega_1,$$

and Φ is of type $(2, 0)$ with respect to ω_1.

SU(2)-structures are locally characterized as follows (see [8]): If $(\eta, \omega_1, \omega_2, \omega_3)$ is an SU(2)-structure on a 5-manifold N^5, then locally, there exists an orthonormal basis of 1-forms $\{e^1, \ldots, e^5\}$ such that

$$\omega_1 = e^{12} + e^{34}, \qquad \omega_2 = e^{13} - e^{24}, \qquad \omega_3 = e^{14} + e^{23}, \qquad \eta = e^5\,.$$

We can also consider the local tensor field I given by

$$Ie^1 = -e^2, \quad Ie^2 = e^1, \quad Ie^3 = -e^4, \quad Ie^4 = e^3, \quad Ie^5 = 0.$$

This tensor gives rise to a global tensor field of type $(1, 1)$ on the manifold N^5 defined by $\omega_1(X, Y) = g(X, IY)$, for any vector fields X, Y on N^5, where g is the Riemannian metric on N^5 underlying the $SU(2)$-structure. The tensor field I satisfies

$$I^2 = -Id + \eta \otimes \xi,$$

where ξ is the vector field on N^5 dual to the 1-form η.

Therefore, given an $SU(2)$-structure $(\eta, \omega_1, \omega_2, \omega_3)$ we also have an almost contact metric structure (I, ξ, η, g) on the manifold, where ω_1 is the fundamental form.

Remark 4.2. Notice that we have two more almost contact metric structures when one considers ω_2 and ω_3 as fundamental forms.

If N^5 has an $SU(2)$-structure $(\eta, \omega_1, \omega_2, \omega_3)$, the product $N^5 \times \mathbb{R}$ has a natural $SU(3)$-structure given by

$$\begin{aligned} F &= \omega_1 + \eta \wedge dt, \\ \Psi &= (\omega_2 + i\omega_3) \wedge (\eta - i dt). \end{aligned} \tag{22}$$

Moreover, by Corollary 2.6 the previous $SU(3)$-structure is SKT if and only if

$$\begin{aligned} &d(I(d\omega_1)) = d(d\eta \wedge \eta), \quad d(I(i_\xi d\omega_1)) = 0, \\ &d\omega_2 = -3\,\omega_3 \wedge \eta, \quad d\omega_3 = 3\,\omega_2 \wedge \eta. \end{aligned} \tag{23}$$

Then we have proved the following

Theorem 4.3. *Let N^5 be a 5-dimensional manifold endowed with an $SU(2)$-structure $(\eta, \omega_1, \omega_2, \omega_3)$. The $SU(3)$-structure (F, Ψ), given by* (22)*, on the product $N^5 \times \mathbb{R}$ is SKT if and only if the equations* (23) *are satisfied.*

Example 4.4. Consider on the 5-dimensional Lie algebras, introduced in Examples 2.8, 2.9 and 2.10, the $SU(2)$-structure given by

$$\omega = \omega_1 = e^{12} + e^{34}, \quad \omega_2 = e^{13} - e^{24}, \quad \omega_3 = e^{14} + e^{23}.$$

For the example 2.8 we have:

$$d\omega_2 = -2\,\omega_3 \wedge \eta - 4(e^{124} - e^{134}),$$

$$d\omega_3 = 2\,\omega_2 \wedge \eta + 4(e^{123} + e^{234}).$$

For the examples 2.9 and 2.10 we get $d\omega_2 = -3\,\omega_3 \wedge \eta$ and $d\omega_3 = 3\,\omega_2 \wedge \eta$, therefore on the product of the corresponding simply-connected Lie groups by $\mathbb{R}$ one gets an SKT $SU(3)$-structure.

We will study the existence of SKT $SU(3)$-structures on a Riemannian cone over a 5-dimensional manifold N^5 endowed with an $SU(2)$-structure $(\eta, \omega_1, \omega_2, \omega_3)$. Then N^5 has an induced almost contact metric structure (I, ξ, η, g) and ω_1 is its fundamental form.

The Riemannian cone $(N^5 \times \mathbb{R}^+, h)$ of (N^5, g) has a natural $SU(3)$-structure defined by

$$\begin{aligned} F &= t^2\omega_1 + t\eta \wedge dt, \\ \Psi &= t^2(\omega_2 + i\omega_3) \wedge (t\eta - i dt). \end{aligned}$$

In terms of a local orthonormal coframe $\{e^1, \dots, e^5\}$ for g such that

$$\omega_1 = -e^{12} - e^{34}, \quad \omega_2 = -e^{13} + e^{24}, \quad \omega_3 = -e^{14} - e^{23}, \quad \eta = e^5,$$

we have that

$$Je^1 = -e^2, \quad Je^2 = e^1, \quad Je^3 = -e^4, \quad Je^4 = e^3, \quad J(te^5) = dt, \quad J(dt) = -te^5.$$

We recall that the $SU(3)$-structure (F, Ψ) on $N^5 \times \mathbb{R}^+$ is integrable if and only if the $SU(2)$-structure $(\eta, \omega_1, \omega_2, \omega_3)$ on N^5 is Sasaki-Einstein, or equivalently if and only if

$$d\eta = -2\,\omega_1, \quad d\omega_2 = -3\,\omega_3 \wedge \eta, \quad d\omega_3 = 3\,\omega_2 \wedge \eta.$$

For the Riemannian cones we can prove the following

Corollary 4.5. *Let N^5 be a 5-dimensional manifold endowed with an* SU(2)*-structure $(\eta, \omega_1, \omega_2, \omega_3)$. The* SU(3)*-structure (F, Ψ) on the Riemannian cone $(N^5 \times \mathbb{R}^+, h)$ is* SKT *if and only if*

$$\begin{cases} -4\eta \wedge \omega_1 + 2I(d\omega_1) - 2d\eta \wedge \eta = d(I(i_\xi d\omega_1)), \\ d\omega_2 = 3\,\omega_3 \wedge \eta, \\ d\omega_3 = -3\,\omega_2 \wedge \eta. \end{cases} \tag{24}$$

Proof. By imposing that $d\Psi = 0$ we get the conditions

$$d\omega_2 = -3\,\omega_3 \wedge \eta, \quad d\omega_3 = 3\,\omega_2 \wedge \eta.$$

By imposing $d(JdF) = 0$, we obtain, as in the proof of Theorem 3.1, the equation (19) for $\omega = \omega_1$.

□

5. Almost contact metric structure induced on a hypersurface

Here we study the almost contact metric structure induced naturally on any oriented hypersurface N^{2n+1} of a $(2n+2)$-manifold M^{2n+2} equipped with an SKT structure.

Let $f\colon N^{2n+1} \longrightarrow M^{2n+2}$ be an oriented hypersurface of a $(2n+2)$-dimensional manifold M^{2n+2} endowed with an SKT structure (J,h,F) and denote by $\mathbb{U}$ the unitary normal vector field. It is well known that N^{2n+1} inherits an almost contact metric structure (I,ξ,η,g) such that η and the fundamental 2-form ω are given by

$$\eta = -f^*(i_{\mathbb{U}}F), \quad \omega = f^*F, \tag{25}$$

where F is the fundamental 2-form of the almost Hermitian structure (see for instance [6]).

Proposition 5.1. *Let $f\colon N^{2n+1} \longrightarrow M^{2n+2}$ be an immersion of an oriented $(2n+1)$-dimensional manifold into a $(2n+2)$-dimensional Hermitian manifold (M^{2n+2},J,h). If the Hermitian structure (J,h) is* SKT*, then the induced almost contact metric structure (I,ξ,η,g) on N^{2n+1}, with η and ω given by* (25)*, satisfies*

$$d\big(Id\omega - I(f^*(i_{\mathbb{U}}dF))\wedge\eta\big) = 0. \tag{26}$$

Proof. We can choose locally an adapted coframe $\{e^1,\dots,e^{2n+2}\}$ for the Hermitian structure such that the unitary normal vector field $\mathbb{U}$ is dual to e^{2n+2}. Since the almost complex structure J is given in this adapted basis by

$$Je^{2j-1} = -e^{2j}, \quad Je^{2j} = e^{2j-1},\ j=1,\dots,n,$$
$$Je^{2n+1} = e^{2n+2}, \quad Je^{2n+2} = -e^{2n+1},$$

the tensor field I on N^{2n+1} satisfies that $If^*e^i = f^*Je^i$, $i=1,\dots,2n+1$, that is,

$$If^*e^{2j-1} = -f^*e^{2j}, \quad If^*e^{2j} = f^*e^{2j-1},\ j=1,\dots,n, \quad If^*e^{2n+1} = 0.$$

However, $If^*e^{2n+2} = 0 \neq f^*e^{2n+1} = -f^*Je^{2n+2}$.

Now we compute f^*JdF. First we decompose (locally and in terms of the adapted basis) the differential of F as follows:

$$dF = \alpha + \beta\wedge e^{2n+1} + \gamma\wedge e^{2n+2} + \mu\wedge e^{2n+1}\wedge e^{2n+2},$$

where the local forms $\alpha \in \bigwedge^3 < e^1,\dots,e^{2n} >$, $\beta,\gamma \in \bigwedge^2 < e^1,\dots,e^{2n} >$ and $\mu \in \bigwedge^1 < e^1,\dots,e^{2n} >$ are generated only by $e^1,\dots,e^{2n}$. Then,

$$JdF = J\alpha + J\beta\wedge e^{2n+2} - J\gamma\wedge e^{2n+1} + J\mu\wedge e^{2n+1}\wedge e^{2n+2}.$$

Since $f^*e^{2n+2} = 0$ and using that $f^*e^{2n+1} = \eta$, we get

$$f^*JdF = f^*J\alpha - (f^*J\gamma)\wedge\eta.$$

But $f^*(i_{\mathbb{U}}dF) = f^*\gamma + f^*\mu\wedge\eta$, which implies that

$$I(f^*(i_{\mathbb{U}}dF)) = If^*\gamma = f^*J\gamma.$$

On the other hand,

$$Id\omega = Idf^*F = If^*dF = If^*\alpha = f^*J\alpha.$$

We conclude that

$$f^*JdF = f^*J\alpha - (f^*J\gamma)\wedge\eta = Id\omega - I(f^*(i_{\mathbb{U}}dF))\wedge\eta.$$

Now, if the Hermitian structure is SKT, then JdF is closed and the induced structure satisfies (26). □

Remark 5.2. Notice that using that $i_{\mathbb{U}}dF = \mathcal{L}_{\mathbb{U}}F - di_{\mathbb{U}}F$ we can write (26) as

$$d\big(Id\omega - I(f^*(\mathcal{L}_{\mathbb{U}}F) + d\eta) \wedge \eta\big) = 0.$$

Therefore, if $f^*(\mathcal{L}_{\mathbb{U}}F) = 0$, the induced almost contact metric structure has to satisfy the equation

$$d\big(Id\omega - I(d\eta) \wedge \eta\big) = 0.$$

In the case of the product $N^{2n+1} \times \mathbb{R}$ the condition $f^*(\mathcal{L}_{\mathbb{U}}F) = 0$ is satisfied.

In the case of the Riemannian cone we have that

$$\mathcal{L}_{\frac{d}{dt}}F = 2t\omega + dt \wedge \eta,$$

and therefore we get $f^*(\mathcal{L}_{\frac{d}{dt}}F) = 2\omega$.

In this way we recover some of the equations obtained in Corollary 2.6 and in Theorem 3.1.

Now we study the structure induced naturally on any oriented hypersurface N^5 of a 6-manifold M^6 equipped with an SKT SU(3)-structure.

Let $f\colon N^5 \longrightarrow M^6$ be an oriented hypersurface of a 6-manifold M^6 endowed with an SU(3)-structure $(F, \Psi = \Psi_+ + i\,\Psi_-)$ and denote by $\mathbb{U}$ the unitary normal vector field. Then N^5 inherits an SU(2)-structure $(\eta, \omega_1, \omega_2, \omega_3)$ given by

$$\eta = -f^*(i_{\mathbb{U}}F), \quad \omega_1 = f^*F, \quad \omega_2 = -f^*(i_{\mathbb{U}}\Psi_-), \quad \omega_3 = f^*(i_{\mathbb{U}}\Psi_+). \tag{27}$$

As a consequence of Proposition 5.1 we have the following

Corollary 5.3. *Let $f\colon N^5 \longrightarrow M^6$ be an immersion of an oriented 5-dimensional manifold into a 6-dimensional manifold with an* SU(3)*-structure. If the* SU(3)*-structure is* SKT*, then the induced* SU(2)*-structure on N^5 given by* (27) *satisfies*

$$d\big(Id\omega_1 - If^*(i_{\mathbb{U}}dF) \wedge \eta\big) = 0, \tag{28}$$

and

$$d(\omega_2 \wedge \eta) = 0, \qquad d(\omega_3 \wedge \eta) = 0. \tag{29}$$

Proof. The equation (28) follows by Proposition 5.1 taking $\omega = \omega_1$. We can choose locally an adapted coframe $\{e^1, \dots, e^5, e^6\}$ for the $SU(3)$-structure such that the unitary normal vector field $\mathbb{U}$ is dual to e^6. From (27) it follows that $\omega_2 \wedge \eta = f^*\Psi_+$ and $\omega_3 \wedge \eta = f^*\Psi_-$. Now, if $\Psi = \Psi_+ + i\,\Psi_-$ is closed then the induced structure satisfies (29). □

5.1. **A simple example.** Consider the 6-dimensional nilmanifold M^6 whose underlying nilpotent Lie algebra has structure equations

$$\begin{cases} de^j = 0, j = 1, 2, 3, 6, \\ de^4 = e^{12}, \\ de^5 = e^{14}, \end{cases}$$

and it is endowed with the $SU(3)$-structure given by

$$F = -e^{14} - e^{26} - e^{53}, \quad \Psi = (e^1 - ie^4) \wedge (e^2 - ie^6) \wedge (e^5 - ie^3).$$

The oriented hypersurface with normal vector field dual to e^2 is a 5-dimensional nilmanifold N^5, which has by [8] no invariant hypo structures, but the SU(2)-structure on N^5

$$\eta = e^2, \qquad \omega_1 = -e^{14} - e^{53}, \qquad \omega_2 = -e^{15} - e^{34}, \qquad \omega_3 = -e^{13} - e^{45}, \tag{30}$$

satisfies (28) and (29). In section 6 we will show that by using this $SU(2)$-structure and appropriate evolution equations we can construct an SKT $SU(3)$-structure on the product of N^5 with an open interval.

6. SKT evolution equations

The goal here is to construct SKT SU(3)-structures by means of appropriate evolution equations starting from a suitable $SU(2)$-structure on a 5-dimensional manifold, following ideas of [20] and [8].

Lemma 6.1. *Let $(\eta(t), \omega_1(t), \omega_2(t), \omega_3(t))$ be a family of* SU(2)*-structures on a* 5*-dimensional manifold N^5, for $t \in (a, b)$. Then, the* SU(3)*-structure on $M^6 = N^5 \times (a, b)$ given by*

$$F = \omega_1(t) + \eta(t) \wedge dt, \qquad \Psi = (\omega_2(t) + i\omega_3(t)) \wedge (\eta(t) - idt),$$

satisfies the condition $d\Psi = 0$ if and only if $(\eta(t), \omega_1(t), \omega_2(t), \omega_3(t))$ is an SU(2)*-structure such that*

$$\begin{aligned} &\hat{d}(\omega_2(t) \wedge \eta(t)) = 0, \quad \hat{d}(\omega_3(t) \wedge \eta(t)) = 0, \\ &\partial_t(\omega_2(t) \wedge \eta(t)) = -\hat{d}\omega_3(t), \quad \partial_t(\omega_3(t) \wedge \eta(t)) = \hat{d}\omega_2(t), \end{aligned} \tag{31}$$

hold, for any t in the open interval (a, b).

Here $\hat{d}$ denotes the exterior differential on N^5 and d the exterior differential on M^6. Now we show which are the additional evolution equations to add to the last two equations of (31) to ensure that $dJdF = 0$.

Proposition 6.2. *Let $(\eta(t), \omega_1(t), \omega_2(t), \omega_3(t))$ be a family of* SU(2)*-structures on N^5, for $t \in (a, b)$. Then, the* SU(3)*-structure on $M^6 = N^5 \times (a, b)$ given by*

$$F = \omega_1(t) + \eta(t) \wedge dt, \qquad \Psi = (\omega_2(t) + i\omega_3(t)) \wedge (\eta(t) - idt), \tag{32}$$

satisfies that JdF is closed if and only if $(\eta(t), \omega_1(t), \omega_2(t), \omega_3(t))$ satisfies the following evolution equations

$$\begin{cases} \hat{d}\Big(I_t\hat{d}\omega_1(t) - I_t(\partial_t\omega_1(t) + \hat{d}\eta(t)) \wedge \eta(t)\Big) = 0, \\ \partial_t\Big(I_t\hat{d}\omega_1(t) - I_t(\partial_t\omega_1(t) + \hat{d}\eta(t)) \wedge \eta(t)\Big) = \\ -\hat{d}\Big(I_t(i_\xi\hat{d}\omega_1(t)) - I_t(i_\xi(\partial_t\omega_1(t) + \hat{d}\eta(t))) \wedge \eta(t)\Big), \end{cases} \tag{33}$$

where, for each $t \in (a, b)$, $\xi(t)$ denotes the vector field on N^5 dual to $\eta(t)$.

Proof. Since $F = \omega_1(t) + \eta(t) \wedge dt$, we have that

$$dF = \hat{d}\omega_1 + (\partial_t\omega_1 + \hat{d}\eta) \wedge dt.$$

Let $\{e^1(t), \dots, e^4(t), \eta(t)\}$ be a local adapted basis for the SU(2)-structure $(\eta(t), \omega_1(t), \omega_2(t), \omega_3(t))$. Then $\{e^1(t), \dots, e^4(t), \eta(t), dt\}$ is an adapted basis for the SU(3)-structure (32) and J is given by

$$Je^1(t) = -e^2(t),\ Je^2(t) = e^1(t),\ Je^3(t) = -e^4(t),\ Je^4(t) = e^3(t),$$
$$J\eta(t) = dt,\ Jdt = -\eta(t).$$

Then, the structures I_t induced on N^5 for each t are given by

$$I_te^1(t) = -e^2(t),\ I_te^2(t) = e^1(t),\ I_te^3(t) = -e^4(t),\ Ie^4(t) = e^3(t),\ I_t\eta(t) = 0.$$

Now, given $\tau(t) \in \Omega^k(N^5)$, $t \in (a,b)$, we can decompose it locally as

$$\tau(t) = \alpha(t) + \beta(t) \wedge \eta(t),$$

where $\alpha(t) \in \bigwedge^k < e^1(t), \dots, e^4(t) >$ and $\beta(t) \in \bigwedge^{k-1} < e^1(t), \dots, e^4(t) >$. Therefore

$$J\tau(t) = J\alpha(t) + J\beta(t) \wedge J\eta(t) = I_t\alpha(t) + I_t\beta(t) \wedge dt = I_t\tau(t) - (-1)^k I_t(i_{\xi(t)}\tau(t)) \wedge dt.$$

Applying this to JdF we get

$$\begin{aligned} JdF &= J\hat{d}\omega_1 - J(\partial_t\omega_1 + \hat{d}\eta) \wedge \eta(t) \\ &= I_t\hat{d}\omega_1 - I_t(\partial_t\omega_1 + \hat{d}\eta) \wedge \eta(t) + I_t(i_{\xi(t)}\hat{d}\omega_1) \wedge dt - I_t\Big(i_\xi(\partial_t\omega_1 + \hat{d}\eta)\Big) \wedge \eta(t) \wedge dt. \end{aligned}$$

Finally, taking the differential of JdF we get

$$\begin{aligned} dJdF &= \hat{d}\Big(I_t\hat{d}\omega_1 - I_t(\partial_t\omega_1 + \hat{d}\eta) \wedge \eta(t)\Big) + \partial_t\Big(I_t\hat{d}\omega_1 - I_t(\partial_t\omega_1 + \hat{d}\eta) \wedge \eta(t)\Big) \wedge dt \\ &\quad + \hat{d}\Big[I_t(i_{\xi(t)}\hat{d}\omega_1) - I_t\Big(i_\xi(\partial_t\omega_1 + \hat{d}\eta)\Big) \wedge \eta(t)\Big] \wedge dt. \end{aligned}$$

□

Remark 6.3. Observe that the first equation in (33) is exactly condition (28) for $F = \omega_1(t) + \eta(t) \wedge dt$ (see Remark 5.2).

As a consequence of Lemma 6.1 and Proposition 6.2, we get

Theorem 6.4. *Let $(\eta(t), \omega_1(t), \omega_2(t), \omega_3(t))$, $t \in (a,b)$, be a family of* SU(2)*-structures on a 5-dimensional manifold N^5, such that*

$$\hat{d}(\omega_2(t) \wedge \eta(t)) = 0, \quad \hat{d}(\omega_3(t) \wedge \eta(t)) = 0, \tag{34}$$

for any t. If the following evolution equations

$$\begin{cases} \hat{d}\Big(I_t\hat{d}\omega_1(t) - I_t(\partial_t\omega_1(t) + \hat{d}\eta(t)) \wedge \eta(t)\Big) = 0, \\ \partial_t\Big(I_t\hat{d}\omega_1(t) - I_t(\partial_t\omega_1(t) + \hat{d}\eta(t)) \wedge \eta(t)\Big) = \\ -\hat{d}\Big(I_t(i_\xi\hat{d}\omega_1(t)) - I_t(i_\xi(\partial_t\omega_1(t) + \hat{d}\eta(t))) \wedge \eta(t)\Big), \\ \partial_t(\omega_2(t) \wedge \eta(t)) = -\hat{d}\omega_3(t), \\ \partial_t(\omega_3(t) \wedge \eta(t)) = \hat{d}\omega_2(t), \end{cases} \tag{35}$$

are satisfied, then the SU(3)*-structure on $M = N \times (a,b)$ given by*

$$F = \omega_1(t) + \eta(t) \wedge dt, \qquad \Psi = (\omega_2(t) + i\omega_3(t)) \wedge (\eta(t) - idt), \tag{36}$$

is SKT.

Example 6.5. Let us consider the Lie algebra with structure equations

$$\begin{cases} de^j = 0, j = 1,2,3, \\ de^4 = e^{12}, \\ de^5 = e^{14}, \end{cases}$$

underlying the 5-dimensional nilmanifold N^5 considered in Example 5.1 and endowed with the $SU(2)$-structure given by (30). It is straight forward to verify that

$$d(\omega_2 \wedge \eta) = d(\omega_3 \wedge \eta) = d(\omega_1 \wedge \omega_1) = 0.$$

Let us evolve the previous SU(2)-structure in the following way:

$$\begin{aligned} \omega_1(t) &= -e^{14} - e^{53}, \\ \omega_2(t) &= -\left(1 + \tfrac{3}{2}t\right)^{1/3} e^{15} - \left(1 + \tfrac{3}{2}t\right)^{-1/3} e^{34}, \\ \omega_3(t) &= -\left(1 + \tfrac{3}{2}t\right)^{1/3} e^{13} - \left(1 + \tfrac{3}{2}t\right)^{-1/3} e^{45}, \\ \eta(t) &= \left(1 + \tfrac{3}{2}t\right)^{1/3} e^{2}, \end{aligned}$$

where $t \in (-2/3, \infty)$.

It is immediate to observe that the family $(\omega_1(t), \omega_2(t), \omega_3(t), \eta(t))$ verifies equations (34) and the two last equations in (35) for any $t \in (-2/3, \infty)$. Moreover, it verifies the following conditions:

$$\partial_t \omega_1(t) = 0, \quad \hat{d}(\eta(t)) = 0, \quad i_\xi\left(\hat{d}(\omega_1(t))\right) = 0, \quad \partial_t\left(I_t(\hat{d}\omega_1(t))\right) = 0,$$

which implies that the evolution equations (33) are also satisfied.

On the product $N^5 \times \mathbb{R}$ let us consider the local basis of 1-forms given by

$$\begin{aligned} &\beta^1 = \left(1 + \tfrac{3}{2}t\right)^{1/3} e^1, \quad \beta^2 = \left(1 + \tfrac{3}{2}t\right)^{-1/3} e^4, \quad \beta^3 = e^5, \quad \beta^4 = e^3, \\ &\beta^5 = \left(1 + \tfrac{3}{2}t\right)^{1/3} e^2, \quad \beta^6 = dt. \end{aligned}$$

The structure equations are:

$$\begin{cases} d\beta^1 = -\frac{1}{2}\left(1 + \frac{3}{2}t\right)^{-1} \beta^{16}, \\ d\beta^2 = \left(1 + \frac{3}{2}t\right)^{-1} \left(\beta^{15} + \frac{1}{2}\beta^{26}\right), \\ d\beta^3 = \beta^{12}, \\ d\beta^4 = 0, \\ d\beta^5 = -\frac{1}{2}\left(1 + \frac{3}{2}t\right)^{-1} \beta^{56}, \\ d\beta^6 = 0. \end{cases}$$

J is given locally by $J\beta^1 = -\beta^2$, $J\beta^3 = -\beta^4$, $J\beta^5 = \beta^6$. The fundamental form $F = -\beta^{12} - \beta^{34} + \beta^{56}$ verifies that $d(JdF) = 0$ and the $(3,0)$-form $\Psi = (\beta^1 + i\,\beta^2) \wedge (\beta^3 + i\,\beta^4) \wedge (\beta^5 - i\,\beta^6)$ is closed. Therefore, (F, Ψ) is a local SKT SU(3)-structure on $N^5 \times \mathbb{R}$.

Remark 6.6. A Hermitian structure (J, h) on a 6-dimensional manifold M^6 is called *balanced* if $F \wedge F$ is closed, F being the associated fundamental 2-form. In [10] it was introduced the notion of balanced SU(2)-structures on 5-dimensional manifolds, together with appropriate evolution equations whose solution gives rise to a balanced SU(3)-structure in six dimensions.

If M^6 is compact, then a balanced structure cannot be SKT (see for instance [12]).

The SU(2)-structure (30) on the previous example is also balanced and it gives rise to a balanced metric on the product of N^5 with a open interval (see (11) in [10]). However one can check directly that this solution is not SKT.

Notice that if G is the nilpotent Lie group underlying N^5, the product $G \times \mathbb{R}$ has no left-invariant SKT structures and it does not admit any left-invariant complex structures; however we find a local SKT SU(3)-structure on it.

7. HKT structures

In this section we will find conditions for which an S^1-bundle over a $(4n+3)$-dimensional manifold endowed with three almost contact metric structures is hyper-Kähler with torsion (HKT for short). We recall that a $4n$-dimensional hyper-Hermitian manifold $(M^{4n}, J_1, J_2, J_3, h)$ is a hypercomplex manifold (M^{4n}, J_1, J_2, J_3) endowed with a Riemannian metric h which is compatible with the complex structures J_r, $r = 1, 2, 3$, i.e. such that

$$h(J_rX, J_rY) = h(X, Y),$$

for any $r = 1, 2, 3$ and any vector fields X, Y on M^{4n}.

A hyper-Hermitian manifold $(M^{4n}, J_1, J_2, J_3, h)$ is called HKT if and only if

$$J_1 dF_1 = J_2 dF_2 = J_3 dF_3, \tag{37}$$

where F_r denotes the fundamental 2-form associated to the Hermitian structure (J_r, h) (see [16]).

Let us consider a $(4n+3)$-dimensional manifold N^{4n+3} endowed with three almost contact metric structures (I_r, ξ_r, η_r, g), $r = 1, 2, 3$, such that

$$\begin{aligned} I_k &= I_iI_j - \eta_j \otimes \xi_i = -I_jI_i + \eta_i \otimes \xi_j, \\ \xi_k &= I_i\xi_j = -I_j\xi_i, \quad \eta_k = \eta_iI_j = -\eta_jI_i. \end{aligned} \tag{38}$$

By applying Theorem 2.3 we can construct hyper-Hermitian structures on S^1-bundles over N^{4n+3} and study when they are strong HKT.

Theorem 7.1. *Let N^{4n+3} be a $(4n+3)$-dimensional manifold with three normal almost contact metric structures (I_r, ξ_r, η_r, g), $r = 1, 2, 3$, satisfying (38), and let Ω be a closed 2-form on N^{4n+3} which represents an integral cohomology class and which is I_r-invariant for every $r = 1, 2, 3$. Consider the circle bundle $S^1 \hookrightarrow P \to N^{4n+3}$ with connection 1-form θ whose curvature form is $d\theta = \pi^*(\Omega)$, where $\pi : P \to N$ is the projection. Then, the hyper-Hermitian structure (J_1, J_2, J_3, h) on P, defined by (2) and (4), is* HKT *if and only if*

$$\begin{aligned} &\pi^*(I_1(d\omega_1)) - \pi^*(d\eta_1) \wedge \pi^*\eta_1 = \pi^*(I_2(d\omega_2)) - \pi^*(d\eta_2) \wedge \pi^*\eta_2 \\ &= \pi^*(I_3(d\omega_3)) - \pi^*(d\eta_3) \wedge \pi^*\eta_3, \\ &\pi^*(I_1(i_{\xi_1} d\omega_1)) = \pi^*(I_2(i_{\xi_2} d\omega_2)) = \pi^*(I_3(i_{\xi_3} d\omega_3)), \end{aligned} \tag{39}$$

where ω_r denotes the fundamental form of the almost contact structure (I_r, ξ_r, η_r, g). Moreover, the HKT *structure is strong if and only if*

$$\begin{aligned} &d(\pi^*(I_r(i_{\xi_r} d\omega_r))) = 0, \\ &d(\pi^*(I_r(d\omega_r) - d\eta_r \wedge \eta_r)) = (\pi^*(-I_r(i_{\xi_r} d\omega_r)) + \pi^*\Omega) \wedge \pi^*\Omega, \end{aligned} \tag{40}$$

for every $r = 1, 2, 3$.

Proof. The almost hyper-Hermitian structure (J_1, J_2, J_3, h) on P, defined by (2) and (4), is hyper-Hermitian if and only (I_r, ξ_r, η_r, g) is normal and $d\theta$ is J_r-invariant for every $r = 1, 2, 3$. The HKT condition is equivalent to (37). By (9) we have

$$J_r dF_r = \pi^*(I_r(d\omega_r)) + \pi^*(I_r(i_{\xi_r} d\omega_r)) \wedge \theta - \pi^*(d\eta_r) \wedge \pi^*\eta_r - \theta \wedge d\theta,$$

where F_r is the fundamental 2-form of (J_r, h). Therefore, the condition (37) is satisfied if and only if (39) holds. Finally, $J_r dF_r$ are closed forms if and only if (40) holds. □

Consider on $N^{4n+3} \times \mathbb{R}$ the almost Hermitian structures (J_r, F_r, h) defined by

$$h = g + (dt)^2, \quad F_r = \omega_r + \eta_r \wedge dt, \tag{41}$$

and

$$J_r(\eta_r) = dt, \quad J_r(X) = I_r(X),\ X \in \operatorname{Ker} \eta_r. \tag{42}$$

Moreover, by (38) we have:

$$\begin{aligned} &J_1 J_2 = J_3 = -J_2 J_1, \\ &J_1 \eta_2 = I_1 \eta_2 = -\eta_3, \quad J_2 \eta_3 = I_2 \eta_3 = -\eta_1, \quad J_3 \eta_1 = I_3 \eta_1 = -\eta_2. \end{aligned}$$

Therefore (J_r, F_r, h), $r = 1, 2, 3$, is a hyper-Hermitian structure on $N^{4n+3} \times \mathbb{R}$ if and only if the structures $(I_r,\ \xi_r,\ \eta_r)$ for $r = 1, 2, 3$ are normal.

Corollary 7.2. *Let N^{4n+3} be a $(4n+3)$-dimensional manifold endowed with three normal almost contact metric structures $(I_r,\ \xi_r,\ \eta_r, g)$, $r = 1, 2, 3$. Consider on the product $N^{4n+3} \times \mathbb{R}$ the hyper-Hermitian structure (J_1, J_2, J_3, h) defined by* (41) *and* (42)*. Then, (J_1, J_2, J_3, h) is* HKT *if and only if*

$$I_1(d\omega_1) - d\eta_1 \wedge \eta_1 = I_2(d\omega_2) - d\eta_2 \wedge \eta_2 = I_3(d\omega_3) - d\eta_3 \wedge \eta_3,$$

$$I_1(i_{\xi_1} d\omega_1) = I_2(i_{\xi_2} d\omega_2) = I_3(i_{\xi_3} d\omega_3).$$

The HKT structure is strong if and only if

$$d(I_r(i_{\xi_r} d\omega_r)) = 0, \quad d(I_r(d\omega_r) - d\eta_r \wedge \eta_r) = 0$$

for every $r = 1, 2, 3$.

Moreover, if (J_1, J_2, J_3, h) is such that

$$d\eta_1 \wedge \eta_1 = d\eta_2 \wedge \eta_2 = d\eta_3 \wedge \eta_3,$$

and one of the following conditions:

(a) *$d\omega_r = 0$ for any $r = 1, 2, 3$, i.e. $(I_r,\ \xi_r,\ \eta_r)$ is quasi-Sasakian for any $r = 1, 2, 3$ or*
(b) *$d\omega_i \wedge \eta_j \wedge \eta_k \neq 0$, where (i, j, k) is a permutation of $(1, 2, 3)$, and*

$$I_1(d\omega_1) = I_2(d\omega_2) = I_3(d\omega_3), \quad I_1(i_{\xi_1} d\omega_1) = I_2(i_{\xi_2} d\omega_2) = I_3(i_{\xi_3} d\omega_3),$$

is satisfied, then (J_1, J_2, J_3, h) is HKT*. In the case (a) the* HKT *structure is strong. In the case (b) the* HKT *structure is strong if and only if*

$$d\,(I_1(d\omega_1)) = d\,(I_1(i_{\xi_1} d\omega_1)) = 0.$$

Proof. By Theorem 7.1 the hyper-Hermitian structure (J_r, F_r, h), $r = 1, 2, 3$, is HKT if and only if

$$\begin{aligned} &I_1(d\omega_1) - d\eta_1 \wedge \eta_1 = I_2(d\omega_2) - d\eta_2 \wedge \eta_2 = I_3(d\omega_3) - d\eta_3 \wedge \eta_3, \\ &I_1(i_{\xi_1} d\omega_1) = I_2(i_{\xi_2} d\omega_2) = I_3(i_{\xi_3} d\omega_3). \end{aligned} \tag{43}$$

Let us express locally

$$d\omega_r = \alpha_r + \sum_{i=1}^{3} \beta_i^r \wedge \eta_i + \sum_{i<j=1}^{3} \gamma_{ij}^r \wedge \eta_i \wedge \eta_j + \rho_r\, \eta_1 \wedge \eta_2 \wedge \eta_3, \tag{44}$$

where α_r, β_i^r and γ_{ij}^r are 3-forms, 2-forms and 1-forms respectively in $\bigcap_{i=1}^{3} \operatorname{Ker} \eta_i$ and ρ_r are smooth functions.

By using the normality of the three almost contact metric structures, and then that $i_{\xi_r} d\eta_r = 0$ and $I_r(d\eta_r) = d\eta_r$, we can write locally:

$$\begin{array}{rcl} d\eta_1 &=& A_1 + B_1 \wedge \eta_2 - I_1 B_1 \wedge \eta_3 + C_1\, \eta_2 \wedge \eta_3, \\ d\eta_2 &=& A_2 + B_2 \wedge \eta_1 + I_2 B_2 \wedge \eta_3 + C_2\, \eta_1 \wedge \eta_3, \\ d\eta_3 &=& A_3 + B_3 \wedge \eta_1 - I_3 B_3 \wedge \eta_2 + C_3\, \eta_1 \wedge \eta_2, \end{array} \tag{45}$$

where $I_r A_r = A_r$. A_r and B_r are 2-forms and 1-forms respectively in $\bigcap_{i=1}^3 \operatorname{Ker} \eta_i$ and C_r are smooth functions.

We have

$$J_r(dF_r) = J_r(d\omega_r) + J_r(d\eta_r \wedge dt) = J_r(d\omega_r) - d\eta_r \wedge \eta_r.$$

Therefore, by using (44) and (45), we obtain

$$\begin{array}{rcl} J_1(dF_1) &=& I_1\alpha_1 + I_1\beta_1^1 \wedge dt - A_1 \wedge \eta_1 - I_1\beta_3^1 \wedge \eta_2 - I_1\beta_2^1 \wedge \eta_3 \\ && -I_1\gamma_{13}^1 \wedge \eta_2 \wedge dt + I_1\gamma_{12}^1 \wedge \eta_3 \wedge dt + B_1 \wedge \eta_1 \wedge \eta_2 - I_1 B_1 \wedge \eta_1 \wedge \eta_3 \\ && +I_1\gamma_{23}^1 \wedge \eta_2 \wedge \eta_3 + \rho_1\, \eta_2 \wedge \eta_3 \wedge dt - C_1\, \eta_1 \wedge \eta_2 \wedge \eta_3, \\ J_2(dF_2) &=& I_2\alpha_2 + I_2\beta_2^2 \wedge dt - I_2\beta_3^2 \wedge \eta_1 - A_2 \wedge \eta_2 + I_2\beta_1^2 \wedge \eta_3 \\ && +I_2\gamma_{23}^2 \wedge \eta_1 \wedge dt + I_2\gamma_{12}^2 \wedge \eta_3 \wedge dt - B_2 \wedge \eta_1 \wedge \eta_2 + I_2\gamma_{13}^2 \wedge \eta_1 \wedge \eta_3 \\ && +I_2 B_2 \wedge \eta_2 \wedge \eta_3 - \rho_2\, \eta_1 \wedge \eta_3 \wedge dt + C_2\, \eta_1 \wedge \eta_2 \wedge \eta_3, \\ J_3(dF_3) &=& I_3\alpha_3 + I_3\beta_3^3 \wedge dt + I_3\beta_2^3 \wedge \eta_1 - I_3\beta_1^3 \wedge \eta_2 - A_3 \wedge \eta_3 \\ && +I_3\gamma_{23}^3 \wedge \eta_1 \wedge dt - I_3\gamma_{13}^3 \wedge \eta_2 \wedge dt + I_3\gamma_{12}^3 \wedge \eta_1 \wedge \eta_2 - B_3 \wedge \eta_1 \wedge \eta_3 \\ && +I_3 B_3 \wedge \eta_2 \wedge \eta_3 + \rho_3\, \eta_1 \wedge \eta_2 \wedge dt - C_3\, \eta_1 \wedge \eta_2 \wedge \eta_3. \end{array}$$

The conditions (43) are satisfied if and only if

$$\begin{array}{l} \gamma_{12}^1 = \gamma_{13}^1 = \gamma_{12}^2 = \gamma_{23}^2 = \gamma_{13}^3 = \gamma_{23}^3 = 0, \quad \rho_r = 0, \quad C_1 = -C_2 = C_3, \\ I_1\alpha_1 = I_2\alpha_2 = I_3\alpha_3, \quad I_1\beta_1^1 = I_2\beta_2^2 = I_3\beta_3^3, \\ A_1 = I_2\beta_3^2 = -I_3\beta_2^3, \quad A_2 = -I_1\beta_3^1 = I_3\beta_1^3, \quad A_3 = I_1\beta_2^1 = -I_2\beta_1^2, \\ B_1 = -B_2 = I_3\gamma_{12}^3, \quad -I_1 B_1 = -B_3 = I_2\gamma_{13}^2, \quad I_2 B_2 = I_3 B_3 = I_1\gamma_{23}^1. \end{array} \tag{46}$$

Since $I_r A_r = A_r$ we obtain that the coefficients β_i^r for $r \neq i = 1, 2, 3$ must satisfy the following conditions:

$$I_i\left(\beta_j^i - I_k\beta_j^i\right) = 0, \quad \forall i, j, k = 1, 2, 3, \quad i \neq j,\ j \neq k,\ k \neq i.$$

The last three equations in (46) are satisfied if and only if $\gamma_{23}^1 = \gamma_{13}^2 = \gamma_{12}^3 = 0$.

Thus, finally, we obtain:

$$\begin{array}{l} d\omega_r = \alpha_r + \sum_{i=1}^3 \beta_i^r \wedge \eta_i, \quad d\eta_i = A_i + \lambda\, \eta_j \wedge \eta_k, \\ I_i\left(\beta_j^i - I_k\beta_j^i\right) = 0, \quad \forall i, j, k = 1, 2, 3, \quad i \neq j,\ j \neq k,\ k \neq i, \\ I_1\alpha_1 = I_2\alpha_2 = I_3\alpha_3, \\ A_1 = I_2\beta_3^2 = -I_3\beta_2^3, \quad A_2 = -I_1\beta_3^1 = I_3\beta_1^3, \quad A_3 = I_1\beta_2^1 = -I_2\beta_1^2. \end{array} \tag{47}$$

for any even permutation of $(1, 2, 3)$.

Now, the expression for $d(J_1 dF_1)$ is the following:

$$\begin{aligned} d(J_1 dF_1) &= d\big(I_1(d\omega_1) + I_1(i_{\xi_1} d\omega_1) \wedge dt\big) - d\left((d\eta_1) \wedge \eta_1\right) \\ &= d\left(I_1(d\omega_1)\right) + d\big(I_1(i_{\xi_1} d\omega_1)\big) \wedge dt - d\eta_1 \wedge d\eta_1 \\ &= d\left(I_1(d\omega_1) - d\eta_1 \wedge \eta_1\right) + d\big(I_1(i_{\xi_1} d\omega_1)\big) \wedge dt, \end{aligned}$$

and thus the HKT structure is strong if and only if

$$d(I_1(d\omega_1) - d\eta_1 \wedge \eta_1) = 0, \quad \text{and} \quad d(I_1(i_{\xi_1} d\omega_1)) = 0.$$

To prove the last part of the corollary it is sufficient to consider coefficients $\beta_r^i = 0$ if $r \neq i$ in expression (44).

□

Example 7.3. Consider the 7-dimensional Lie group $G = \mathrm{SU}(2) \ltimes \mathbb{R}^4$ with structure equations

$$\begin{cases} de^1 = -\frac{1}{2} e^{25} - \frac{1}{2} e^{36} - \frac{1}{2} e^{47}, \\ de^2 = \frac{1}{2} e^{15} + \frac{1}{2} e^{37} - \frac{1}{2} e^{46}, \\ de^3 = \frac{1}{2} e^{16} - \frac{1}{2} e^{27} + \frac{1}{2} e^{45}, \\ de^4 = \frac{1}{2} e^{17} + \frac{1}{2} e^{26} - \frac{1}{2} e^{35}, \\ de^5 = e^{67}, \\ de^6 = -e^{57}, \\ de^7 = e^{56}. \end{cases}$$

By [13] G admits a compact quotient $M^7 = \Gamma \backslash G$ by a uniform discrete subgroup Γ and it is endowed with a weakly generalized G_2-structure. Moreover, by [3] $M^7 \times S^1$ admits a strong HKT structure. We can show that M^7 has three normal almost contact metric structures (I_r, ξ_r, η_r, g) for $r = 1, 2, 3$ given by

$$\begin{array}{llll} I_1 e^1 = e^2, & I_1 e^3 = e^4, & I_1 e^5 = e^6, & \eta_1 = e^7, \\ I_2 e^1 = e^3, & I_2 e^2 = -e^4, & I_2 e^5 = -e^7, & \eta_2 = e^6, \\ I_3 e^1 = e^4, & I_3 e^2 = e^3, & I_3 e^6 = e^7, & \eta_3 = e^5, \end{array}$$

satisfying the conditions (a) of Corollary 7.2.

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