BALANCED HERMITIAN METRICS FROM SU(2)-STRUCTURES

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Abstract. We study the intrinsic geometrical structure of hypersurfaces in 6-manifolds carrying a balanced Hermitian SU(3)-structure, which we call *balanced* SU(2)-*structure*. We provide sufficient conditions, in terms of suitable evolution equations, which imply that a 5-manifold with such structure can be isometrically embedded as a hypersurface in a balanced Hermitian SU(3)-manifold. Any 5-dimensional compact nilmanifold has an invariant balanced SU(2)-structure, and we show how some of them can be evolved to give new explicit examples of balanced Hermitian SU(3)-structures. Moreover, for $n = 3, 4$, we present examples of compact solvmanifolds endowed with a balanced SU(n)-structure such that the corresponding Bismut connection has holonomy equal to SU(n).

1. Introduction

Let (J, g) be a Hermitian structure on a $2n$-dimensional manifold M, with Kähler form F and Lee form θ. The 3-form JdF can be identified with the torsion T of the Bismut connection of (J, g), i.e. the unique Hermitian connection with totally skew-symmetric torsion [4], by $g(X, T(Y, Z)) = JdF(X, Y, Z)$. Hermitian structures for which the Lee form θ vanishes identically are called *balanced* [25]; equivalently, the form F^{n-1} given by the wedge product of the Kähler form $(n-1)$-times is closed. As it was observed by Fino and Grantcharov [10], the vanishing of the Ricci tensor of the Bismut connection of a Hermitian metric g on a compact complex manifold (M^{2n}, J) having a holomorphic non-vanishing $(n, 0)$-form implies that (M^{2n}, J, g) is conformally balanced, and therefore there is a balanced structure on M.

Balanced Hermitian structures on compact non-Kähler 6-dimensional manifolds have attracted much attention as models in the compactification of superstrings with fluxes. This was first considered by Strominger in [27], and since then many efforts have been done in order to construct compact solutions to the Strominger's system (see for example [2, 3, 9, 14, 15, 17] and the references therein). In fact, if Ψ is a holomorphic non-vanishing (3,0)-form on a Hermitian manifold (M^6, J, g), then the first equation in Strominger's system is equivalent to g be conformal to a balanced metric [14], namely $d(||\Psi||_F\, F \wedge F) = 0$. On the other hand, 6-dimensional supersymmetric non-compact solutions of the 10-dimensional heterotic supergravity may be interpreted in string theory as local models of a compact solution [13].

Several recent constructions have provided new balanced Hermitian manifolds. The existence of a Hermitian structure with restricted holonomy of the Bismut connection contained in SU(3) on a connected sum of at least two copies of $S^3 \times S^3$ was proved by Gutowski, Ivanov and Papadopoulos in [20], and recently on the connected sum $(k-1)(S^2 \times S^4)\#k(S^3 \times S^3)$ for $k \geq 1$ by D. Grantcharov, G. Grantcharov and Poon [18]. On the other hand, Goldstein and Prokushkin

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constructed balanced Hermitian metrics on torus bundles over $K3$ surfaces and over complex abelian surfaces [17], whereas Fu, Li and Yau construct in [12] balanced metrics on the family of non-Kähler Calabi-Yau threefolds obtained by smoothing after contracting $(-1,-1)$-rational curves on a Kähler Calabi-Yau threefold.

In this paper we consider *balanced* $\mathrm{SU}(n)$*-structures* (F,Ψ) on a $2n$-dimensional manifold M, i.e. $\mathrm{SU}(n)$-structures with Kähler form F satisfying $dF^{n-1}=0$ and with closed complex volume $(n,0)$-form Ψ. Notice that the latter condition implies that the underlying almost complex structure J is integrable, and therefore the pair $(J, g(\cdot,\cdot)=-F(\cdot,J\,\cdot))$ is a balanced Hermitian structure on M.

Our goal in this paper is to construct balanced SU(3)-structures by means of appropriate evolution equations starting from a suitable structure on a 5-manifold. Evolution equations to construct metrics of special holonomy G_2 in seven dimensions, resp. SU(3) in six dimensions, have been previously used by Hitchin [23], resp. Conti-Salamon [7].

In [7] the notion of *hypo structure* on a 5-manifold N is introduced as the natural structure corresponding to the restriction to an oriented hypersurface N of an integrable SU(3)-structure on M. Hypo structures are a generalization in dimension 5 of Sasakian-Einstein metrics; in fact, Sasakian-Einstein metrics correspond to Killing spinors and hypo structures are induced by *generalized* Killing spinors. In terms of differential forms, a hypo structure on a 5-manifold N is an SU(2)-structure given by a quadruplet $(\eta,\omega_i, 1\leq i\leq 3)$ of differential forms, where $\omega_1\wedge\eta$, $\omega_2\wedge\eta$ and ω_3 are closed. Conti and Salamon prove that a hypo structure $(\eta,\omega_i,1\leq i\leq 3)$ on N can be lifted to an integrable SU(3)-structure on $N\times I$, for some open interval I, if it belongs to a one-parameter family of hypo structures $(\eta(t),\omega_i(t),1\leq i\leq 3)$ satisfying the evolution equations (9) given in Section 3; moreover, this always holds if the hypo structure is real analytic [7]. In particular, Sasakian-Einstein hypo structures can be lifted to $N\times\mathbb{R}^+$ with the cone metric, which is Kähler and Ricci flat [5].

In Section 2 we study the geometrical structure induced naturally on any oriented hypersurface N of a 6-manifold M equipped with a balanced SU(3)-structure. We shall refer to such a structure as a *balanced* SU(2)*-structure*. It turns out that any compact nilmanifold of dimension 5 has an invariant balanced SU(2)-structure, althought some 5-nilmanifolds do not admit invariant hypo structures [7]. We construct balanced SU(2)-structures on certain circle bundles $S^1\subset N\to X$ over a holomorphic symplectic manifold X. Balanced SU(2)-structures on total spaces of circle bundles over the Kodaira-Thurston manifold arising in this way are described in detail.

In Section 3 we provide sufficient conditions, in terms of appropriate evolution equations, under which a balanced SU(2)-structure on N can be lifted to a balanced SU(3)-structure on $N\times I$, $I\subset\mathbb{R}$ being some open interval. We find explicit solutions of the *balanced evolution equations* for compact 5-nilmanifolds not having a hypo structure, which allow us to exhibit new balanced Hermitian metrics in six dimensions.

Finally, Section 4 is devoted to a detailed description of several compact manifolds obtained as quotient of solvable Lie groups endowed with a balanced $\mathrm{SU}(n)$-structure for $n=3,4$, showing that the holonomy of their Bismut connection equals $\mathrm{SU}(n)$. Many of these examples are holomorphic parallelizable manifolds obtained in [26]. Recall that for any complex parallelizable manifold $M=\Gamma\backslash G$

of complex dimension n, any Hermitian left-invariant metric on the complex Lie group G is balanced [19], so it descends to M and by [10] the associated Bismut connection has (restricted) holonomy contained in $\mathrm{SU}(n)$.

2. Balanced SU(2)-structures

In this section we introduce a special type of SU(2)-structures on 5-manifolds, namely *balanced* SU(2)*-structures*, which allow us to construct new balanced Hermitian metrics by solving suitable evolution equations. First we recall some facts about SU(2)-structures on a 5-manifold. (For more details, we refer e.g. to [7].) Let N be a 5-manifold and let $L(N)$ be the principal bundle of linear frames on N. An SU(2)*-structure* on N is an SU(2)-reduction of $L(N)$. We have the following (see [7, Proposition 1])

Proposition 2.1. SU(2)*-structures on a* 5*-manifold* N *are in* $1:1$ *correspondence with quadruplets* $(\eta,\omega_1,\omega_2,\omega_3)$*, where* η *is a* 1*-form and* ω_i *are* 2*-forms on* N *satisfying*

$$\omega_i\wedge\omega_j=\delta_{ij}v,\quad v\wedge\eta\neq 0,$$

for some 4*-form* v*, and*

$$i_X\omega_3=i_Y\omega_1\Rightarrow\omega_2(X,Y)\geq 0,$$

where i_X *denotes the contraction by* X*.*
Equivalently, an SU(2)*-structure on* N *can be viewed as the datum of* (η,ω_3,Φ)*, where* η *is a* 1*-form,* ω_3 *is a* 2*-form and* $\Phi=\omega_1+i\,\omega_2$ *is a complex* 2*-form such that*

$$\eta\wedge\omega_3\wedge\omega_3\neq 0\,,\qquad \Phi^2=0\,,\qquad \omega_3\wedge\Phi=0\,,\qquad \Phi\wedge\overline{\Phi}=2\,\omega_3\wedge\omega_3\,,$$

and Φ *is of type* $(2,0)$ *with respect to* ω_3*.*

¿From now on, given a set $e^1,\dots,e^m$ of 1-forms on a differentiable manifold, we will write $e^{ij}=e^i\wedge e^j$, $e^{ijk}=e^i\wedge e^j\wedge e^k$, and so forth.

As a corollary of Proposition 2.1, we obtain the local characterization of SU(2)-structures (see [7]):

Corollary 2.2. *If* $(\eta,\omega_1,\omega_2,\omega_3)$ *is an* SU(2)*-structure on a* 5*-manifold* N*, then locally, there exists a basis of* 1*-forms* $\{e^1,\dots,e^5\}$ *such that*

$$\eta=e^1,\qquad \omega_1=e^{24}+e^{53},\qquad \omega_2=e^{25}+e^{34},\qquad \omega_3=e^{23}+e^{45}\,.$$

As a consequence, SU(2)-structures naturally arise on hypersurfaces of 6-manifolds with an SU(3)-structure. Indeed, let $f:N\longrightarrow M$ be an oriented hypersurface of a 6-manifold M endowed with an SU(3)-structure (F,Ψ_+,Ψ_-) and denote by $\mathbb{U}$ the unitary normal vector field. Then N inherits an SU(2)-structure $(\eta,\omega_1,\omega_2,\omega_3)$ given by

$$\eta=-i_{\mathbb{U}}F,\quad \omega_1=i_{\mathbb{U}}\Psi_-,\quad \omega_2=-i_{\mathbb{U}}\Psi_+,\quad \omega_3=f^*F. \tag{1}$$

Conversely, an SU(2)-structure $(\eta,\omega_1,\omega_2,\omega_3)$ on N induces an SU(3)-structure (F,Ψ_+,Ψ_-) on $N\times\mathbb{R}$ given by

$$F=\omega_3+\eta\wedge dt,\qquad \Psi=\Psi_++i\Psi_-=(\omega_1+i\omega_2)\wedge(\eta+idt), \tag{2}$$

where t is a coordinate on $\mathbb{R}$.

Definition 2.3. *An* SU(2)*-structure* $(\eta, \omega_1, \omega_2, \omega_3)$ *on a* 5*-manifold* N *is called* balanced *if it satisfies the following equations*

$$d(\omega_1 \wedge \eta) = 0, \qquad d(\omega_2 \wedge \eta) = 0, \qquad d(\omega_3 \wedge \omega_3) = 0. \tag{3}$$

In [7], an SU(2)-structure is said to be *hypo* if

$$d(\omega_1 \wedge \eta) = d(\omega_2 \wedge \eta) = d\omega_3 = 0. \tag{4}$$

Hence, it is obvious that any hypo structure is balanced. However, there are nilmanifolds admitting no invariant hypo structure, but having invariant balanced SU(2)-structures. In fact, if the Lie algebra underlying a 5-dimensional compact nilmanifold N is isomorphic to $(0,0,0,12,14)$, $(0,0,12,13,23)$ or $(0,0,12,13,14+23)$, then there is no invariant hypo structure on N [7]. (Here, for instance $(0,0,12,13,14+23)$ means that there is a basis $e^1,\dots,e^5$ such that the Chevalley-Eilenberg differential d is given by $de^1 = 0$, $de^2 = 0$, $de^3 = e^{12}$, $de^4 = e^{13}$, $de^5 = e^{14} + e^{23}$, and similarly for the other Lie algebras.) It is easy to check that the SU(2)-structure given by

$$\eta = e^1, \qquad \omega_1 = e^{24} + e^{53}, \qquad \omega_2 = e^{25} + e^{34}, \qquad \omega_3 = e^{23} + e^{45},$$

satisfies (3) on each one of these three Lie algebras. Therefore, we get the following

Proposition 2.4. *Any 5-dimensional compact nilmanifold has an invariant balanced* SU(2)*-structure.*

There exist also 5-dimensional solvable non-nilpotent Lie algebras with no invariant hypo structure, but having an invariant balanced SU(2)-structure. For example, let us consider the Lie algebra $\mathfrak{g}$ whose dual is spanned by $\{e^1,\dots,e^5\}$ such that

$$de^1 = 0\,,\ de^2 = 0\,,\ de^3 = e^{13}\,,\ de^4 = -e^{14}\,,\ de^5 = e^{34}\,.$$

Then $\mathfrak{g}$ is a 5-dimensional solvable non-nilpotent Lie algebra. The simply-connected Lie group G associated with $\mathfrak{g}$ has a uniform discrete subgroup Γ, so that $N = \Gamma\backslash G$ is a compact solvmanifold. In fact, the manifold N is the topological product of the unit circle by the compact solvable 4-dimensional manifold studied in [1], which is a circle bundle over the compact solvmanifold $Sol(3)$. A straightforward calculation shows that the following forms

$$\eta = e^1, \qquad \omega_1 = e^{24} + e^{53}, \qquad \omega_2 = e^{25} + e^{34}, \qquad \omega_3 = e^{23} + e^{45}\,.$$

satisfy

$$d(\omega_1 \wedge \eta) = d(\omega_3 \wedge \eta) = d(\omega_2 \wedge \omega_2) = 0,$$

and thus they define a balanced SU(2)-structure on N. However, N has not invariant hypo SU(2)-structures. First, using Hattori's theorem [22], we have that the real cohomology groups of N of degree ≤ 2 are

$$H^0(N) = \langle 1\rangle, \quad H^1(N) = \langle [e^1], [e^2]\rangle, \quad H^2(N) = \langle [e^{12}]\rangle.$$

Now, let us suppose that N has an invariant hypo SU(2)-structure $(\eta, \omega_1, \omega_2, \omega_3)$. Then

$$\omega_3 = ae^{12} + be^{13} + ce^{14} + fe^{34},$$

for some real numbers a, b, c and f. Therefore, $\omega_3 \wedge \omega_3 = 2afe^{1234}$, and so $\eta = e^5 + \sum_{i=1}^4 \lambda_i e^i$ since $\omega_3 \wedge \omega_3 \wedge \eta$ is a volume form. On the other hand,

$$\omega_1 = \sum_{i,j=1}^{4} a_{ij} e^{ij}, \qquad \omega_2 = \sum_{i,j=1}^{4} b_{ij} e^{ij},$$

for some real numbers a_{ij} and b_{ij}. Now, the conditions $d(\omega_1 \wedge \eta) = d(\omega_2 \wedge \eta) = 0$ imply that

$$\omega_1 = a_{13}e^{13} + a_{14}e^{14} + a_{34}e^{34},$$

and

$$\omega_2 = b_{13}e^{13} + b_{14}e^{14} + b_{34}e^{34},$$

which implies that $\omega_1 \wedge \omega_1 = \omega_2 \wedge \omega_2 = 0$. This is not possible for an SU(2)-structure on N.

Balanced SU(2)-structures on 5-manifolds are related to balanced Hermitian structures in six dimensions as the next proposition shows. First, recall that a *balanced* SU(3)*-structure* on a 6-manifold M is an SU(3)-structure $(F, \Psi = \Psi_+ + i\Psi_-)$ (where F is the Kähler form of an almost Hermitian structure and $\Psi = \Psi_+ + i\Psi_-$ is a complex volume form) such that F^2 and Ψ are closed. The latter condition implies that the underlying almost complex structure is integrable. Notice that any balanced SU(3)-structure is in particular half-flat [6, 23, 8], that is, it satisfies $dF^2 = d\Psi_+ = 0$.

Proposition 2.5. *Let $f\colon N \longrightarrow M$ be an immersion of an oriented 5-manifold into a 6-manifold with an* SU(3)*-structure. If the* SU(3)*-structure is balanced then the* SU(2)*-structure on N given by (1) is balanced.*

Proof. From (1) it follows that $\omega_1 \wedge \eta = f^*\Psi_+$, $\omega_2 \wedge \eta = f^*\Psi_-$ and $\omega_3 \wedge \omega_3 = f^*F^2$. Now, if F^2 and Ψ are closed then the induced structure is balanced. □

Let (X, J) be a complex surface. By definition, a *holomorphic symplectic structure* on X is the datum of a d-closed and non-degenerate $(2,0)$-form $\omega = \omega_1 + i\,\omega_2$ on X.

Lemma 2.6. *Let (X, J) an almost complex 4-dimensional manifold, and let $\omega = \omega_1 + i\,\omega_2$ be a form of type $(2,0)$ with respect to J such that $\omega \wedge \overline{\omega}$ does not vanish at any point of X. Then, there is a 2-form ω_3 on X such that*

$$\omega_i \wedge \omega_j = \delta_{ij}\, \omega_3 \wedge \omega_3 \qquad (i, j = 1, 2, 3).$$

Proof. Since $\omega \wedge \omega$ vanishes identically, we have $\omega_1 \wedge \omega_1 = \omega_2 \wedge \omega_2$ and $\omega_1 \wedge \omega_2 = 0$. Let g be a Riemannian metric on X compatible with J and let $F(\cdot,\cdot) = g(\cdot, J\,\cdot)$ be the fundamental 2-form of the almost Hermitian structure (J, g). Then, on a neighbourhood U of each point there is a basis of (1,0)-forms a_1, a_2 such that $F_{|U} = \frac{i}{2}(a_1 \wedge \bar{a}_1 + a_2 \wedge \bar{a}_2)$. Since ω has type (2,0) with respect to J, there is a non-vanishing complex function f on U such that $\omega_{|U} = f\, a_1 \wedge a_2$. Therefore,

$$(F \wedge F)_{|U} = \frac{1}{2} a_1 \wedge a_2 \wedge \bar{a}_1 \wedge \bar{a}_2,$$

and

$$(\omega_1 \wedge \omega_1)_{|U} = \frac{1}{2}(\omega \wedge \overline{\omega})_{|U} = \frac{|f|^2}{2} a_1 \wedge a_2 \wedge \bar{a}_1 \wedge \bar{a}_2,$$

But $F\wedge F$ and $\omega_1\wedge\omega_1$ are global (real) 4-forms on X which do not vanish at any point, so there is a non-vanishing (real) function h on X such that $F\wedge F = h\,\omega_1\wedge\omega_1$. The previous identities show that $h>0$. Now, the conformal metric $g' = h^{-1/2}g$ is compatible with J, and the fundamental 2-form $\omega_3 = h^{-1/2}F$ of (J,g') satisfies $\omega_1\wedge\omega_1 = \omega_3\wedge\omega_3$ and $\omega_1\wedge\omega_3 = 0 = \omega_2\wedge\omega_3$. □

Notice that the previous lemma holds in particular on a complex surface equipped with a holomorphic symplectic form. Then we have the following

Proposition 2.7. *Let (X,J) be a complex surface equipped with a holomorphic symplectic structure $\omega = \omega_1 + i\,\omega_2$. Then, for any integral closed 2-form Ω on X annihilating $\cos\theta\,\omega_1 + \sin\theta\,\omega_2$ and $\sin\theta\,\omega_1 - \cos\theta\,\omega_2$ for some θ, there is a principal circle bundle $\pi\colon N \longrightarrow X$ with connection form ρ such that Ω is the curvature of ρ and the quadruplet $(\eta, \omega_1^\theta, \omega_2^\theta, \omega_3^\theta)$ on N given by*

$$\begin{aligned}
&\eta = \rho,\\
&\omega_1^\theta = \pi^*(\cos\theta\,\omega_1 + \sin\theta\,\omega_2),\\
&\omega_2^\theta = \pi^*(-\sin\theta\,\omega_1 + \cos\theta\,\omega_2),\\
&\omega_3^\theta = \pi^*(\omega_3)
\end{aligned}$$

is a balanced SU(2)*-structure, where ω_3 is any 2-form as in Lemma 2.6.*

Proof. Since $d\omega_1 = d\omega_2 = 0$ and $d\eta = \pi^*(\Omega)$, a simple calculation shows that

$$d(\omega_1^\theta\wedge\eta) = \omega_1^\theta\wedge d\eta = \pi^*((\cos\theta\,\omega_1 + \sin\theta\,\omega_2)\wedge\Omega) = 0,$$

and

$$d(\omega_2^\theta\wedge\eta) = \omega_2^\theta\wedge d\eta = \pi^*((-\sin\theta\,\omega_1 + \cos\theta\,\omega_2)\wedge\Omega) = 0.$$

The existence of a principal circle bundle in the conditions above follows from a well known result by Kobayashi [24]. □

Remark 2.8. Notice that $\Omega = 0$ satisfies the hypothesis in the previous proposition for each θ and one gets the trivial circle bundle $N = X\times\mathbb{R}$ with the balanced SU(2)-structure which is the natural extension to N of the holomorphic symplectic structure on X.

Remark 2.9. Following [16, Definition 1.1], a *symplectic couple* on an oriented 4-manifold X is a pair of symplectic forms (ω_1,ω_2) such that $\omega_1\wedge\omega_2 = 0$ and $\omega_1\wedge\omega_1, \omega_2\wedge\omega_2$ are volume forms defining the positive orientation. A symplectic couple is called *conformal* if $\omega_1\wedge\omega_1 = \omega_2\wedge\omega_2$. By [16, Theorem 1.3] it follows that X admits a conformal symplectic couple if and only if X is diffeomorphic to (a) a complex torus, (b) a $K3$ surface or (c) a primary Kodaira surface. In the hypothesis of Proposition 2.7, (ω_1,ω_2) defines a conformal symplectic couple on the 4-manifold X and ω_3 is a non-degenerate 2-form on X such that $\omega_i\wedge\omega_3 = 0$, $i=1,2$ and $\omega_1\wedge\omega_1 = \omega_2\wedge\omega_2 = \omega_3\wedge\omega_3$.

Next we illustrate the construction given in Proposition 2.7 by showing principal circle bundles over holomorphic symplectic manifolds in the cases (a) and (c) of Remark 2.9. Let us consider the closed 4-manifold $X = \Gamma\backslash G$, where the Lie algebra $\mathfrak{g}$ of G has the following structure equations

$$de^1 = 0\,,\quad de^2 = 0\,,\quad de^3 = 0\,,\quad de^4 = -\epsilon\, e^{23}\qquad (\epsilon = 0,1). \tag{5}$$

Clearly X is the Kodaira-Thurston manifold for $\epsilon = 1$ and a 4-torus for $\epsilon = 0$. Consider the complex structure J on X defined by the complex $(1,0)$-forms

$$\varphi^1 = e^1 + ie^4\,, \quad \varphi^2 = e^2 + ie^3,$$

so that

$$\omega = \varphi^1 \wedge \varphi^2 = \left(e^{12} + e^{34}\right) + i\left(e^{13} - e^{24}\right) = \omega_1 + i\,\omega_2$$

is a holomorphic symplectic structure on X. The metric g on X given by $g = \sum_{i=1}^4 e^i \otimes e^i$ is J-Hermitian and the fundamental form of (J, g) is precisely $\omega_3 = e^{14} + e^{23}$, so we are in the conditions of Proposition 2.7. Observe that ω_3 is closed only when X is a 4-torus, i.e. $\epsilon = 0$.

Now let Ω be a closed 2-form on X such that

$$\Omega \wedge (\cos\theta\,\omega_1 + \sin\theta\,\omega_2) = 0 = \Omega \wedge (\sin\theta\,\omega_1 - \cos\theta\,\omega_2), \tag{6}$$

for some θ. A direct calculations shows that

$$\Omega = a(e^{12} - e^{34}) + b(e^{13} + e^{24}) + (\epsilon - 1)c_1\,e^{14} + c_2\,e^{23}$$

satisfies (6) for all θ, where a, b, c_1, c_2 are constant. Applying Proposition 2.7, we have a balanced SU(2)-structure on the total space N, which is non-hypo if $\epsilon = 1$. In this case it is easy to see that N is a compact 5-nilmanifold with underlying Lie algebra isomorphic to either $(0, 0, 0, 0, 12)$ or $(0, 0, 0, 12, 13 + 24)$.

3. Evolution equations and balanced Hermitian metrics

Next we establish evolution equations that allow the construction of new balanced Hermitian metrics in dimension six from balanced SU(2)-structures in dimension five.

Theorem 3.1. *Let $(\eta(t), \omega_1(t), \omega_2(t), \omega_3(t))$ be a family of* SU(2)*-structures on a 5-manifold N, for $t \in I = (a, b)$. Then, the* SU(3)*-structure on $M = N \times I$ given by*

$$F = \omega_3(t) + \eta(t) \wedge dt, \qquad \Psi = (\omega_1(t) + i\omega_2(t)) \wedge (\eta(t) + idt), \tag{7}$$

is balanced Hermitian if and only if $(\eta(t), \omega_1(t), \omega_2(t), \omega_3(t))$ is a balanced SU(2)*-structure for any t in the open interval I, and the following evolution equations*

$$\begin{cases} \partial_t(\omega_1 \wedge \eta) = -d\omega_2, \\ \partial_t(\omega_2 \wedge \eta) = d\omega_1, \\ \partial_t(\omega_3 \wedge \omega_3) = -2\,d(\omega_3 \wedge \eta), \end{cases} \tag{8}$$

are satisfied.

Proof. A direct calculation shows that the SU(3)-structure given by (7) satisfies

$$dF^2 = d(\omega_3 \wedge \omega_3) + (\partial_t(\omega_3 \wedge \omega_3) + 2\,d(\omega_3 \wedge \eta)) \wedge dt,$$

and

$$d\Psi = d(\omega_1 \wedge \eta) - (\partial_t(\omega_1 \wedge \eta) + d\omega_2) \wedge dt + i\,d(\omega_2 \wedge \eta) - i\,(\partial_t(\omega_2 \wedge \eta) - d\omega_1) \wedge dt.$$

The forms F^2 and Ψ are both closed if and only if $(\eta(t), \omega_1(t), \omega_2(t), \omega_3(t))$ is a balanced SU(2)-structure for any $t \in I$, and satisfies the equations (8). $\square$

Remark 3.2. Notice that in the special case when the balanced SU(2)-structures in the family are hypo, the last equation in (8) reduces to

$$\omega_3 \wedge (\partial_t \omega_3 + d\eta) = 0.$$

Hence, any solution of the hypo evolution equations introduced in [7], namely

$$\partial_t(\omega_1 \wedge \eta) = -d\omega_2, \quad \partial_t(\omega_2 \wedge \eta) = d\omega_1, \quad \partial_t \omega_3 = -d\eta, \tag{9}$$

is trivially a solution of (8), and the resulting SU(3)-structure is integrable because F in (7) is closed.

Next we give some explicit solutions to the balanced evolution equations (8). Our first example is the 5-manifold $N = X \times \mathbb{R}$, where X is the Kodaira-Thurston manifold. Let us consider the non-hypo balanced SU(2)-structure given by (5) for $\epsilon = 1$ and with zero curvature, namely

$$\eta = e^5, \qquad \omega_1 = e^{12} + e^{34}, \qquad \omega_2 = e^{13} - e^{24}, \qquad \omega_3 = e^{14} + e^{23}.$$

The family of non-hypo balanced SU(2)-structures on N given by $\eta(t) = e^5$ and

$$\omega_1(t) = e^{12} + e^3(e^4 - t\,e^5),\ \omega_2(t) = e^{13} + (e^4 - t\,e^5)e^2,\ \omega_3(t) = e^1(e^4 - t\,e^5) + e^{23},$$

coincides with the previous one for $t = 0$ and satisfies the evolution equations (8) for any $t \in \mathbb{R}$.

Denote by G the simply-connected nilpotent Lie group with Lie algebra $(0,0,0,0,12)$, so that $N = \Gamma\backslash G$ for some lattice Γ of maximal rank in G. It follows from Proposition 3.1 that the SU(3)-structure on $G \times \mathbb{R}$ given by

$$\begin{aligned} F &= e^{14} + e^{23} - t\,e^{15} + e^5 \wedge dt, \\ \Psi_+ &= e^{125} + e^{345} - \big(e^{13} - e^{24} + t\,e^{25}\big) \wedge dt, \\ \Psi_- &= e^{135} - e^{245} + \big(e^{12} + e^{34} - t\,e^{35}\big) \wedge dt, \end{aligned} \tag{10}$$

is balanced Hermitian.

Next we find explicit solutions of (8) for the non-hypo nilpotent Lie algebras $(0,0,0,12,14)$ and $(0,0,12,13,23)$.

Lie algebra (0,0,0,12,14): It is straightforward to check that the family of balanced SU(2)-structures given by

$$\begin{aligned} &\eta(t) = \sqrt[3]{\tfrac{2-3t}{2}}\,e^1, \\ &\omega_1(t) = \tfrac{1}{2}\left(\sqrt[3]{\tfrac{2}{2-3t}} - \tfrac{2-3t}{2}\right)e^{23} + \sqrt[3]{\tfrac{2-3t}{2}}\,e^{24} - \sqrt[3]{\tfrac{2}{2-3t}}\,e^{35}, \\ &\omega_2(t) = \sqrt[3]{\tfrac{2}{2-3t}}\,e^{25} + \sqrt[3]{\tfrac{2-3t}{2}}\,e^{34}, \\ &\omega_3(t) = e^{23} - \tfrac{1}{2}\left(1 - \tfrac{2-3t}{2}\sqrt[3]{\tfrac{2-3t}{2}}\right)e^{24} + e^{45}, \end{aligned}$$

satisfies the evolution equations (8) for $t \in \mathbb{R} - \{2/3\}$. Observe that the volume form of the associated Riemannian metric along the family is given by

$$\omega_1(t) \wedge \omega_1(t) \wedge \eta(t) = 2\sqrt[3]{\frac{2-3t}{2}}\,e^{12345},$$

so the orientation for $t \in (-\infty, 2/3)$ is opposite to the orientation for $t \in (2/3, \infty)$.

Let $I=(-\infty,2/3)$ and denote by G the simply-connected nilpotent Lie group with Lie algebra $(0,0,0,12,14)$. The basis of 1-forms on the product manifold $G\times I$ given by

$$\alpha^1=e^2,\qquad \alpha^2=e^3,\quad \alpha^3=\sqrt[3]{\tfrac{2-3t}{2}}e^4,\quad \alpha^4=\tfrac12\sqrt[3]{\tfrac{2}{2-3t}}(e^2+2e^5)-\tfrac{2-3t}{4}e^2,$$
$$\alpha^5=\sqrt[3]{\tfrac{2-3t}{2}}e^1,\quad \alpha^6=dt,$$

is orthonormal with respect to the Riemannian metric associated to the balanced SU(3)-structure on $G\times I$. The balanced Hermitian structure on $G\times I$ is given by

$$\begin{aligned}
F &= e^{23}-\tfrac12 e^{24}+e^{45}+\tfrac{2-3t}{4}\sqrt[3]{\tfrac{2-3t}{2}}e^{24}+\sqrt[3]{\tfrac{2-3t}{2}}e^1\wedge dt,\\
\Psi_+ &= \tfrac12 e^{123}-e^{135}-\tfrac{2-3t}{4}\sqrt[3]{\tfrac{2-3t}{2}}e^{123}+\sqrt[3]{\tfrac{(2-3t)^2}{4}}e^{124}\\
&\quad -\Big(\sqrt[3]{\tfrac{2}{2-3t}}e^{25}+\sqrt[3]{\tfrac{2-3t}{2}}e^{34}\Big)\wedge dt,\\
\Psi_- &= e^{125}+\sqrt[3]{\tfrac{(2-3t)^2}{4}}e^{134}\\
&\quad +\Big(\tfrac12\sqrt[3]{\tfrac{2}{2-3t}}e^{23}-\tfrac{2-3t}{4}e^{23}+\sqrt[3]{\tfrac{2-3t}{2}}e^{24}-\sqrt[3]{\tfrac{2}{2-3t}}e^{35}\Big)\wedge dt.
\end{aligned}\tag{11}$$

Lie algebra (0,0,12,13,23): A direct calculation shows that the family of balanced SU(2)-structures given by

$$\begin{aligned}
\eta(t) &= \tfrac{2}{2-t}e^3,\\
\omega_1(t) &= \tfrac{2-t}{2}(e^{15}+e^{42}),\\
\omega_2(t) &= \tfrac{t(2-t)(t-4)}{4}e^{12}+\tfrac{2-t}{2}(e^{14}+e^{25}),\\
\omega_3(t) &= e^{12}-\tfrac{t(2-t)^2(t-4)}{8}e^{25}-\tfrac{(2-t)^2}{4}e^{45},
\end{aligned}$$

satisfies the evolution equations (8) for $t\in\mathbb{R}-\{2\}$. Observe that the volume form of the associated Riemannian metric along the family is given by

$$\omega_1(t)\wedge\omega_1(t)\wedge\eta(t)=-2e^{12345},$$

so it remains constant.

Let $I=(-\infty,2)$ and denote by G the simply-connected nilpotent Lie group with Lie algebra $(0,0,12,13,23)$. The basis of 1-forms on the product manifold $G\times I$ given by

$$\alpha^1=e^1,\qquad \alpha^2=e^2,\qquad \alpha^3=\tfrac{2-t}{2}e^5,\qquad \alpha^4=\tfrac{t(2-t)(t-4)}{4}e^2+\tfrac{2-t}{2}e^4,$$
$$\alpha^5=\tfrac{2}{2-t}e^3,\qquad \alpha^6=dt,$$

is orthonormal with respect to the Riemannian metric associated to the balanced SU(3)-structure on $G\times I$. The balanced Hermitian structure on $G\times I$ is given by

$$\begin{aligned}
F &= e^{12}-\tfrac{t(2-t)^2(t-4)}{8}e^{25}-\tfrac{(2-t)^2}{4}e^{45}+\tfrac{2}{2-t}e^3\wedge dt,\\
\Psi_+ &= -e^{135}+e^{234}-\tfrac{2-t}{2}\big(\tfrac{t(t-4)}{2}e^{12}+e^{14}+e^{25}\big)\wedge dt,\\
\Psi_- &= -e^{134}-e^{235}+\tfrac{t(t-4)}{2}e^{123}+\tfrac{2-t}{2}(e^{15}-e^{24})\wedge dt.
\end{aligned}\tag{12}$$

As a consequence of our previous examples we conclude:

Corollary 3.3. *The 6-dimensional simply-connected nilpotent Lie groups H_8, H_{16} and H_{17} corresponding to the Lie algebras $\mathfrak{h}_8 = (0,0,0,0,0,12)$, $\mathfrak{h}_{16} = (0,0,0,12,14,24)$ and $\mathfrak{h}_{17} = (0,0,0,0,12,15)$ have (locally) balanced* SU(3)*-structures.*

It is worthy to remark that H_8 and H_{16} have invariant complex structures, but none of them admit invariant compatible balanced metrics [28]. On the other hand, H_{17} has no invariant complex structures.

Finally, recall that according to [23] if M is a 6-manifold with a family $(F(s), \Psi_+(s), \Psi_-(s))$ of half-flat structures, $s \in I = (a,b)$, satisfying the evolution equations

$$\partial_s \Psi_+ = dF, \qquad F \wedge \partial_s F = -d\Psi_-,$$

then the product manifold $M \times I$ has a Riemannian metric whose holonomy is contained in G_2; in fact, the form $\varphi = F(s) \wedge ds + \Psi_+(s)$ defining the G_2-structure is parallel. Therefore, the balanced SU(3)-structures (10), (11) and (12) can be lifted to a metric with holonomy in G_2.

4. Holonomy of Bismut connection of balanced Hermitian metrics on solvmanifolds

In this section we provide some examples of compact Hermitian manifolds, endowed with an SU(n)-structure whose associated metric is balanced and such that the corresponding Bismut connection has holonomy equal to SU(n), for $n = 3, 4$.

Let (M, J, g) be a Hermitian manifold and F be the Kähler form of the Hermitian structure (J, g). Denote by ∇^{LC} the Levi Civita connection of the metric g. Then the Bismut connection ∇^B of (M, J, F, g) is characterized by the following formula

$$g(\nabla^B_X Y, Z) = g(\nabla^{LC}_X Y, Z) + \frac{1}{2} T(X, Y, Z), \quad \forall X, Y, Z \in \Gamma(M, TM),$$

where the torsion form T is given by

$$T(X, Y, Z) = JdF(X, Y, Z).$$

We will need to compute the curvature of ∇^B. In order to do this, we will use the Cartan structure equations,

$$\begin{cases} de^i + \displaystyle\sum_{j=1}^{2n} \omega^i_j \wedge e^j = \tau^i, & i = 1, \dots, 2n \\ \omega^i_j + \omega^j_i = 0, & i, j = 1, \dots, 2n, \end{cases} \tag{13}$$

$$\begin{cases} d\omega^i_j + \displaystyle\sum_{r=1}^{2n} \omega^i_r \wedge \omega^r_j = \Omega^i_j, & i, j = 1, \dots, 2n \\ \Omega^i_j + \Omega^j_i = 0, & i, j = 1, \dots, 2n, \end{cases} \tag{14}$$

where $\{e^1, \dots, e^{2n}\}$ is an orthonormal coframe, ω^i_j are the connection 1-forms, τ^i are the torsion 2-forms and Ω^i_j are the curvature 2-forms.
If $\{e_1, \dots, e_{2n}\}$ denotes the dual frame of $\{e^1, \dots, e^{2n}\}$, then

$$\tau^i = \sum_{j<k=1}^{2n} T_{ijk}\, e^j \wedge e^k, \quad i = 1, \dots, 2n,$$

where $T_{ijk} = T(e_i, e_j, e_k)$.

4.1. **Dimension six.** There are two 3-dimensional complex-parallelizable (non-abelian) solvable Lie groups [26]. The complex structure equations are given by:

(I) $d\varphi^1 = d\varphi^2 = 0,\ d\varphi^3 = \varphi^{12}$;

(II) $d\varphi^1 = 0,\ d\varphi^2 = \varphi^{12},\ d\varphi^3 = -\varphi^{13}$.

Here (I) is the Lie group underlying the *Iwasawa manifold*. In the following we show that they can be endowed with a balanced SU(3)-structure and determine the holonomy of the corresponding Bismut connection.

Theorem 4.1. *Any 3-dimensional complex-parallelizable (non-abelian) solvable Lie group has a balanced* SU(3)*-structure such that the holonomy of its Bismut connection is equal to* SU(3)*.*

Proof. First we show that for the standard Hemitian balanced structure (J_0, g_0) on the Lie group (I) the holonomy of its Bismut connection equal to SU(3).

Let us consider the real basis $\{e^1, \dots, e^6\}$ given by

$$\varphi^1 = e^1 + i\,e^2, \quad \varphi^2 = e^3 + i\,e^4, \quad \varphi^3 = e^5 + i\,e^6.$$

Then, the real structure equations are

$$de^1 = de^2 = de^3 = de^4 = 0, \quad de^5 = e^{13} - e^{24}, \quad de^6 = e^{14} + e^{23}.$$

Let J_0 be the complex structure given by

$$J_0e^1 = -e^2, \quad J_0e^2 = e^1, \quad J_0e^3 = -e^4, \quad J_0e^4 = e^3, \quad J_0e^5 = -e^6, \quad J_0e^6 = e^5.$$

The fundamental form F associated with the J_0-Hermitian metric $g_0 = \sum_{i=1}^6 e^i \otimes e^i$ is given by

$$F = e^{12} + e^{34} + e^{56}.$$

Since $dF = e^{136} - e^{145} - e^{235} - e^{246}$, we get that g_0 is balanced and the torsion $T = J_0 dF$ is expressed as

$$T = -e^{135} - e^{146} - e^{236} + e^{245}.$$

From (13), the non-zero connection 1-forms of the Bismut connection ∇^B are

$$\omega^1_5 = \omega^2_6 = -e^3, \qquad \omega^1_6 = -\omega^2_5 = -e^4, \qquad \omega^3_5 = \omega^4_6 = e^1, \qquad \omega^3_6 = -\omega^4_5 = e^2.$$

Now, using (14), the (linearly independent) curvature forms of ∇^B are

$$\Omega^1_2 = 2e^{34}, \quad \Omega^1_3 = -e^{13} - e^{24}, \quad \Omega^2_3 = e^{14} - e^{23}, \quad \Omega^3_4 = 2e^{12}.$$

Computing the covariant derivatives of the curvature, we obtain 4 new linearly independent forms:

$$\nabla^B_{E_1}(\Omega^1_2) = -2(e^{36} - e^{45}), \qquad \nabla^B_{E_2}(\Omega^1_2) = 2(e^{35} + e^{46}),$$

$$\nabla^B_{E_3}(\Omega^3_4) = 2(e^{16} - e^{25}), \qquad \nabla^B_{E_4}(\Omega^3_4) = -2(e^{15} + e^{26}).$$

Therefore, $\mathrm{Hol}(\nabla^B) = \mathrm{SU}(3)$.

Finally, if we take $\Psi = (e^1 + i\,e^2) \wedge (e^3 + i\,e^4) \wedge (e^5 + i\,e^6)$, then $d\Psi_+ = d\Psi_- = 0$, i.e. (F, Ψ_+, Ψ_-) is a balanced SU(3)-structure.

For case (II) we consider the real basis $\{e^1, \dots, e^6\}$ given by

$$\varphi^1 = e^1 + i\,e^2, \quad \varphi^2 = e^3 + i\,e^4, \quad \varphi^3 = e^5 + i\,e^6.$$

In terms of this basis, the structure equations are

$$de^1 = de^2 = 0, \qquad de^3 = e^{13} - e^{24}, \qquad de^4 = e^{14} + e^{23},$$
$$de^5 = -e^{15} + e^{26}, \qquad de^6 = -e^{16} - e^{25}.$$

We consider the Hermitian structure (J_0, g_0) given in case (I). The fundamental form F associated is $F = e^{12} + e^{34} + e^{56}$ and $dF = 2(e^{134} - e^{156})$. We get that g_0 is balanced and the torsion T is given by

$$T = -2(e^{234} - e^{256}).$$

There are 8 linearly independent curvature forms for ∇^B, namely

$$\Omega^1_3 = -e^{13} - e^{24}, \quad \Omega^1_4 = -e^{14} + e^{23}, \quad \Omega^1_5 = -e^{15} - e^{26}, \quad \Omega^1_6 = -e^{16} + e^{25},$$
$$\Omega^3_4 = -2e^{34}, \quad \Omega^3_5 = e^{35} + e^{46}, \quad \Omega^3_6 = e^{36} - e^{45}, \quad \Omega^5_6 = -2e^{56}.$$

Therefore, the holonomy of ∇^B is equal to SU(3).

Again, taking $\Psi = (e^1 + i\, e^2) \wedge (e^3 + i\, e^4) \wedge (e^5 + i\, e^6)$, the triplet (F, Ψ_+, Ψ_-) is a balanced SU(3)-structure. □

Next, we study the holonomy of the Bismut connection for balanced Hermitian metrics on 6-nilmanifolds. First of all, we recall [28] that if a nilmanifold $M = \Gamma\backslash G$ admits a balanced Hermitian metric (not necessarily invariant), then the Lie algebra $\mathfrak{g}$ of G is isomorphic to $\mathfrak{h}_1, \ldots, \mathfrak{h}_6$ or $\mathfrak{h}^-_{19}$, where $\mathfrak{h}_1 = (0,0,0,0,0,0)$ is the abelian Lie algebra and

$$\begin{aligned} \mathfrak{h}_2 &= (0,0,0,0,12,34), & \mathfrak{h}_5 &= (0,0,0,0,13+42,14+23),\\ \mathfrak{h}_3 &= (0,0,0,0,0,12+34), & \mathfrak{h}_6 &= (0,0,0,0,12,13),\\ \mathfrak{h}_4 &= (0,0,0,0,12,14+23), & \mathfrak{h}^-_{19} &= (0,0,0,12,23,14-35). \end{aligned}$$

Observe that $\mathfrak{h}_5$ is the Lie algebra underlying the Iwasawa manifold considered in Theorem 4.1. The Lie algebra $\mathfrak{h}^-_{19}$ is the unique 6-dimensional 3-step nilpotent Lie algebra admitting balanced structures.

On the other hand, in [11, Corollary 6.2, Example 6.3] an explicit balanced Hermitian metric on $\mathfrak{h}_3, \mathfrak{h}_4, \mathfrak{h}_5, \mathfrak{h}_6$ is given such that the holonomy of the corresponding Bismut connection is equal to SU(3) for $\mathfrak{h}_4, \mathfrak{h}_5$ and $\mathfrak{h}_6$. For $\mathfrak{h}_3$ they show that there is a reduction of the holonomy group, whereas it remains to study the Lie algebra $\mathfrak{h}_2$ because the particular coefficients given at the end of the proof of [11, Corollary 6.2] correspond to $\mathfrak{h}_6$.

Next we prove that $\mathfrak{h}_2$ and $\mathfrak{h}^-_{19}$ also admit a balanced SU(3)-structure such that the holonomy of its Bismut connection equals SU(3). That is, we prove the following

Theorem 4.2. *Let M be a 6-dimensional nilmanifold admitting invariant balanced Hermitian structures. If the first Betti number of M is ≤ 4, then there is a balanced* SU(3)*-structure such that the holonomy of its Bismut connection is equal to* SU(3).

Proof. A 6-dimensional nilmanifold with $b_1(M) \leq 4$ admitting invariant balanced Hermitian structures (J, g) has underlying Lie algebra isomorphic to $\mathfrak{h}_2$, $\mathfrak{h}_4$, $\mathfrak{h}_5$, $\mathfrak{h}_6$ or $\mathfrak{h}^-_{19}$.

On $\mathfrak{h}_2$, let us consider the structure equations

$$de^1 = de^2 = de^3 = de^4 = 0, \quad de^5 = e^{13} - e^{24}, \quad de^6 = -2e^{12} + e^{14} + e^{23} + 2e^{34}, \tag{15}$$

Firstly, we notice that the structure equations (15) correspond to the Lie algebra $\mathfrak{h}_2$. To see this it is sufficient to express the equations with respect to the new basis

$$f^1 = -2\,e^2 + \sqrt{3}\,e^3 + e^4, \quad f^2 = e^1 - \sqrt{3}\,e^2 + 2\,e^3, \quad f^3 = 2\,e^2 + \sqrt{3}\,e^3 - e^4,$$
$$f^4 = e^1 + \sqrt{3}\,e^2 + 2\,e^3, \quad f^5 = -\sqrt{3}\,e^5 - e^6, \quad f^6 = -\sqrt{3}\,e^5 + e^6.$$

Now, the fundamental form F associated to the Hermitian structure (J_0, g_0) is given by $F = e^{12} + e^{34} + e^{56}$. Since

$$dF = e^{136} - e^{246} + 2e^{125} - e^{145} - e^{235} - 2e^{345}\,,$$

we get that g_0 is balanced and the torsion T is given by

$$T = -2e^{126} - e^{135} - e^{146} - e^{236} + e^{245} + 2e^{346}.$$

The following 8 curvature forms for the Bismut connection

$$\Omega^1_2 = -2(2e^{12} - e^{14} - e^{23} - 3e^{34}), \quad \Omega^1_3 = -(e^{13} + e^{24}),$$
$$\Omega^1_4 = -(e^{14} - e^{23}), \quad \Omega^1_5 = -2e^{46},$$
$$\Omega^1_6 = -2e^{36}, \quad \Omega^3_4 = 2(3e^{12} - e^{14} - e^{23} - 2e^{34}),$$
$$\Omega^3_5 = -2e^{26}, \quad \Omega^3_6 = 2e^{16}.$$

are linearly independent, therefore, $\mathrm{Hol}(\nabla^B) = \mathrm{SU}(3)$.

Now, if we consider the $(3,0)$-form $\Psi = (e^1 + i\,e^2) \wedge (e^3 + i\,e^4) \wedge (e^5 + i\,e^6)$, then (F, Ψ_+, Ψ_-) is a balanced SU(3)-structure.

On $\mathfrak{h}^-_{19}$, we consider the following structure equations:

$$d\,e^1 = d\,e^2 = d\,e^3 = 0, \quad d\,e^4 = e^{12}, \quad d\,e^5 = e^{23}, \quad d\,e^6 = e^{14} - e^{35}.$$

We define the complex structure J given by

$$Je^1 = e^3, \quad Je^2 = e^6, \quad Je^3 = -e^1, \quad Je^4 = -e^5, \quad Je^5 = e^4, \quad Je^6 = -e^2.$$

The fundamental form F associated to the J-Hermitian metric $g = \sum_{i=1}^6 e^i \otimes e^i$ is given by $F = -e^{13} - e^{26} + e^{45}$. Since $dF = -e^{124} + e^{125} - e^{234} - e^{235}$, we get that g is balanced and the torsion T is given by

$$T = e^{146} - e^{156} - e^{346} - e^{356}.$$

Also in this case, $\mathrm{Hol}(\nabla^B)$=SU(3) because there are 8 linearly independent curvature forms for the Bismut connection:

$$\Omega^1_2 = -\tfrac{1}{4}(3e^{12} + 2e^{16} + e^{36}), \quad \Omega^1_3 = \tfrac{1}{2}(e^{26} + e^{45}),$$
$$\Omega^1_4 = -\tfrac{3}{4}(e^{14} - e^{35}), \quad \Omega^1_5 = \tfrac{1}{4}(2e^{14} - e^{15} - e^{34} - 2e^{35}),$$
$$\Omega^1_6 = -\tfrac{1}{4}(e^{16} + 3e^{23} - 2e^{36}), \quad \Omega^2_4 = \tfrac{1}{4}(e^{24} - 2e^{46} - e^{56}),$$
$$\Omega^2_5 = \tfrac{1}{4}(e^{25} + e^{46} - 2e^{56}), \quad \Omega^2_6 = \tfrac{1}{2}(e^{13} - e^{45}).$$

Moreover, if $\Psi = (e^1 - i\,e^3) \wedge (e^2 - i\,e^6) \wedge (e^4 + i\,e^5)$, then (F, Ψ_+, Ψ_-) is a balanced SU(3)-structure. $\square$

To finish this section, we show an example of a 6-dimensional compact solvmanifold with a balanced SU(3)-structure such that the holonomy of its Bismut connection is equal to SU(3).
Let $\mathfrak{g}$ be the solvable Lie algebra whose structure equations are

$$de^1 = 0,\ \ de^2 = 0,\ \ de^3 = e^{13},\ \ de^4 = -e^{14},\ \ de^5 = e^{15},\ \ de^6 = -e^{16}\,.$$

Let G be the simply-connected Lie group whose Lie algebra is $\mathfrak{g}$. It can be easily checked that, for any $X \in \mathfrak{g}$, ad_X has real eigenvalues, i.e. $\mathfrak{g}$ is completely solvable. Thus G has a uniform discrete subgroup Γ, such that $M = \Gamma\backslash G$ is a 6-dimensional compact solvmanifold.
We define an (integrable) complex structure J on M by setting

$$Je^1 = -e^2,\ \ Je^2 = e^1,\ \ Je^3 = -e^5,\ \ Je^4 = -e^6,\ \ Je^5 = e^3,\ \ Je^6 = e^4,$$

The fundamental form F associated with the J-Hermitian metric $g = \sum_{i=1}^6 e^i \otimes e^i$ is given by $F = e^{12} + e^{35} + e^{46}$. Hence $dF = 2\left(e^{135} - e^{146}\right)$ and g is balanced. The torsion form of the Bismut connection is given by

$$T = 2\left(e^{246} - e^{235}\right)\,.$$

The following curvature forms are linearly independent:

$$\Omega^1_2 = 2\left(e^{35} + e^{46}\right), \quad \Omega^1_3 = -e^{13} - e^{25}, \quad \Omega^1_4 = -e^{14} - e^{26}, \quad \Omega^1_5 = -e^{15} + e^{23},$$
$$\Omega^1_6 = -e^{16} + e^{24}, \quad \Omega^3_4 = e^{34} + e^{56}, \quad \Omega^3_5 = -2e^{35}, \quad \Omega^3_6 = e^{36} + e^{45}.$$

Consequently, $\mathrm{Hol}(\nabla^B) = \mathrm{SU}(3)$.

Taking $\Psi = (e^1 + i\,e^2) \wedge (e^3 + i\,e^5) \wedge (e^4 + i\,e^6)$ we obtain that (F, Ψ_+, Ψ_-) is a balanced SU(3)-structure.

Finally, we observe that the manifold M has no Kähler structures. Indeed, in view of the Main Theorem in [21], a compact solvmanifold of completely solvable type has a Kähler structure if and only if it is a complex torus.

4.2. **Balanced Hermitian metrics on compact 8-solvmanifolds.** Here we present examples of compact balanced Hermitian 8-manifolds with an SU(4)-structure such that the holonomy of its Bismut connection is equal to SU(4). Furthermore, according to [26], all the examples are holomorphic parallelizable. We will describe carefully only one example. In the other cases the computations are similar.

Example 4.3. Consider the complex structure equations

$$d\varphi^1 = 0\,, \quad d\varphi^2 = \varphi^{12}\,, \quad d\varphi^3 = -\varphi^{13}\,, \quad d\varphi^4 = -\varphi^{23}\,.$$

Let $\{e^1, \ldots, e^8\}$ be the basis given by $\varphi^j = e^{2j-1} + i\,e^{2j}$, $j = 1, \ldots, 4$. The corresponding real equations are

$$de^1 = 0,\ \ de^2 = 0,\ \ de^3 = e^{13} - e^{24}, \qquad de^4 = e^{14} + e^{23},$$
$$de^5 = -e^{15} + e^{26},\ \ de^6 = -e^{16} - e^{25}, \qquad de^7 = -e^{35} + e^{46},\ \ de^8 = -e^{36} - e^{45}\,.$$

Let $\mathfrak{g}$ be the real 8-dimensional Lie algebra whose dual is spanned by $\{e^1, \ldots, e^8\}$, and let G be the simply-connected Lie group whose Lie algebra is $\mathfrak{g}$. It is immediate to see that $\mathfrak{g}$ is a 3-step solvable but not completely solvable Lie algebra. In fact, $\mathfrak{g}' = \langle e_3, e_4, e_5, e_6, e_7, e_8\rangle\,, \mathfrak{g}'' = \langle e_7, e_8\rangle\,, \mathfrak{g}''' = \{0\}$. However, by [26] it turns out that G has a uniform discrete subgroup Γ, such that $M = \Gamma\backslash G$ is a compact solvmanifold of dimension 8. Let J_0 be the natural complex structure given by

$$J_0e^1 = -e^2, \qquad J_0e^2 = e^1, \qquad J_0e^3 = -e^4, \qquad J_0e^4 = e^3,$$
$$J_0e^5 = -e^6, \qquad J_0e^6 = e^5, \qquad J_0e^7 = -e^8, \qquad J_0e^8 = e^7.$$

The fundamental form F associated with the J_0-Hermitian metric $g_0 = \sum_{i=1}^8 e^i \otimes e^i$ is given by $F = e^{12} + e^{34} + e^{56} + e^{78}$ and, consequently,

$$dF = 2\left(e^{134} - e^{156}\right) - e^{358} + e^{468} + e^{367} + e^{457}\,.$$

It is straightforward to see that g_0 is balanced and the torsion of the Bismut connection is

$$T = 2\left(e^{256} - e^{234}\right) - e^{467} + e^{357} + e^{458} + e^{368}\,.$$

By a direct computation, we get only 6 linearly independent curvature forms, given by

$$\Omega^1_2 = 2(e^{34} + e^{56})\,, \quad \Omega^1_3 = -e^{24} - e^{13}\,, \quad \Omega^1_4 = e^{23} - e^{14}\,,$$
$$\Omega^1_5 = -e^{26} - e^{15}\,, \quad \Omega^1_6 = e^{25} - e^{16}\,, \quad \Omega^3_4 = 2(-e^{34} + e^{56}).$$

Computing the covariant derivatives of the curvature forms, we obtain that

$$\begin{aligned}
&\nabla^B_{e_3}\Omega^1_2 = 2\left(-e^{14} + e^{23} - e^{67} + e^{58}\right), && \nabla^B_{e_4}\Omega^1_2 = 2\left(e^{13} + e^{24} - e^{68} - e^{57}\right),\\
&\nabla^B_{e_5}\Omega^1_2 = 2\left(e^{47} - e^{38} + e^{16} - e^{25}\right), && \nabla^B_{e_6}\Omega^1_2 = 2\left(e^{48} + e^{37} - e^{26} - e^{15}\right),\\
&\nabla^B_{e_6}\Omega^1_4 = -e^{35} - e^{28} - e^{46} - e^{17}, && \nabla^B_{e_4}\Omega^1_5 = -e^{18} + e^{27} + e^{36} - e^{45},\\
&\nabla^B_{e_3}\Omega^1_6 = e^{45} + e^{27} - e^{36} - e^{18}, && \nabla^B_{e_4}\Omega^1_6 = -e^{35} + e^{28} - e^{46} + e^{17},\\
&\nabla^B_{e_5}\Omega^1_6 = 2(e^{56} - e^{12})\,,
\end{aligned}$$

are linearly independent and so, Hol (∇^B)=SU(4).

Finally, taking $\Psi = (e^1 + i\,e^2) \wedge (e^3 + i\,e^4) \wedge (e^5 + i\,e^6) \wedge (e^7 + i\,e^8)$, we obtain that (F, Ψ_+, Ψ_-) is a balanced SU(4)-structure.

For the holomorphic parallelizable solvmanifolds corresponding to the Lie algebras defined respectively by the following structure equations,

$$\begin{array}{lllll}
d\varphi^1 = 0\,, & d\varphi^2 = 0\,, & d\varphi^3 = -\varphi^{12}\,, & d\varphi^4 = -2\varphi^{13}\,, & \\
d\varphi^1 = 0\,, & d\varphi^2 = \varphi^{12}\,, & d\varphi^3 = c\varphi^{13}\,, & d\varphi^4 = -(1+c)\varphi^{14} & c(1+c) \neq 0\,,\\
d\varphi^1 = 0\,, & d\varphi^2 = \varphi^{12}\,, & d\varphi^3 = -2\varphi^{13}\,, & d\varphi^4 = \varphi^{14} - \varphi^{12}\,. &
\end{array}$$

the computations of the holonomy of the Bismut connection are similar and therefore they are omitted.

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References

[1] L.C. de Andrés, L.A. Cordero, M. Fernández, J.J. Mencía, Examples of four-dimensional compact locally conformal Kähler manifolds, *Geom. Dedicata* **29** (1989), 227–232.

[2] K. Becker, M. Becker, K. Dasgupta, P.S. Green, Compactifications of heterotic theory on non-Kähler complex manifolds: I, *JHEP* **04** (2003), 007, 60pp.

[3] K. Becker, M. Becker, K. Dasgupta, P.S. Green, E. Sharpe, Compactifications of heterotic strings on non-Kähler complex manifolds: II, *Nuclear Phys. B* **678** (2004), 19–100.

[4] J.M. Bismut, A local index theorem of non-Kähler manifolds, *Math. Ann.* **284** (1989), 681–699.

[5] C. Boyer, K. Galicki, *3-Sasakian manifolds*, Surveys Diff. Geom. **6** (1999), 123–184.

[6] S. Chiossi, S. Salamon, The intrinsic torsion of SU(3) and G_2 structures, In: Gil-Medrano, O., Miquel, V. (eds.), Differential geometry, Valencia, 2001, World Sci. Publishing, River Edge, NJ, 2002, 115–133.

[7] D. Conti, S. Salamon, Generalized Killing spinors in dimension 5, *Trans. AMS* **359** (2007), 5319–5343.

[8] D. Conti, A. Tomassini, Special symplectic six-manifolds, *Q. J. Math.* **58** (2007), 297–311.

[9] M. Fernández, S. Ivanov, L. Ugarte, R. Villacampa, Non-Kähler Heterotic String Compactifications with non-zero fluxes and constant dilaton, *Commun. Math. Phys.* (to appear), arXiv:0804.1648 [math.DG].

[10] A. Fino, G. Grantcharov, Properties of manifolds with skew-symmetric torsion and special holonomy, *Adv. Math.* **189** (2004), 439–450.

[11] A. Fino, M. Parton, S. Salamon, Families of strong KT structures in six dimensions, *Comment. Math. Helv.* **79** (2004), 317–340.

[12] J.-X. Fu, J. Li, S.-T. Yau, Constructing balanced metrics on some families of non-Kähler Calabi-Yau threefolds, arXiv:0809.4748 [math.DG].

[13] J.-X. Fu, L.-S. Tseng, S.-T. Yau, Local heterotic torsional models, arXiv:0806.2392 [hep-th].

[14] J.-X. Fu, S.-T. Yau, The theory of superstring with flux on non-Kähler manifolds and the complex Monge-Ampère equation, *J. Differential Geom.* **78** (2008), 369–428.

[15] J.P. Gauntlett, D. Martelli, D. Waldram, Superstrings with intrinsic torsion, *Phys. Rev. D* **69** (2004), no. 8, 086002, 27 pp.

[16] H. Geiges, Symplectic couples on 4-manifolds, *Duke Math. J.* **85** (1986), 701–711.

[17] E. Goldstein, S. Prokushkin, Geometric model for complex non-Kähler manifolds with SU(3) structure, *Commun. Math. Phys.* **251** (2004), 65–78.

[18] D. Grantcharov, G. Grantcharov, Y.S. Poon, Calabi-Yau connections with torsion on toric bundles, *J. Differential Geom.* **78** (2008), 13–32.

[19] A. Gray, L.M. Hervella, The sixteen classes of almost Hermitian manifolds and their linear invariants, *Ann. Mat. Pura Appl.* (4) **123** (1980), 35–58.

[20] J. Gutowski, S. Ivanov, G. Papadopoulos, Deformations of generalized calibrations and compact non-Kähler manifolds with vanishing first Chern class, *Asian J. Math.* **7** (2003), 39–80.

[21] K. Hasegawa, A note on compact solvmanifolds with Kähler structures, *Osaka J. Math.* **43** (2006), 131-135.

[22] A. Hattori, Spectral sequence in the de Rham cohomology of fibre bundles, *J. Fac. Sci. Univ. Tokyo* **8** (1960), 298–331.

[23] N.J. Hitchin, Stable forms and special metrics. In: Fernández, M., Wolf J. (eds.), Global differential geometry: the mathematical legacy of Alfred Gray (Bilbao, 2000), Contemp. Math. **288**, Amer. Math. Soc., providence, RI, 2001, 70–89.

[24] S. Kobayashi, Principal fibre bundles with the 1-dimensional toroidal group, *Tôhoku Math. J.* (56) **8** (1956), 29–45.

[25] M.L. Michelsohn, On the existence of special metrics in complex geometry, *Acta Math.* **149** (1982), 261–295.

[26] I. Nakamura, Complex parallelisable manifolds and their small deformations, *J. Differential Geom.* **10** (1975), 85–112.

[27] A. Strominger, Superstrings with torsion, *Nuclear Phys. B* **274** (1986), 253–284.

[28] L. Ugarte, Hermitian structures on six dimensional nilmanifolds, *Transform. Groups* **12** (2007), 175–202.

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