

COMPACT SUPERSYMMETRIC SOLUTIONS OF THE HETEROTIC EQUATIONS OF MOTION IN DIMENSIONS 7 AND 8

MARISA FERNÁNDEZ, STEFAN IVANOV, LUIS UGARTE, AND RAQUEL VILLACAMPA

ABSTRACT. We construct explicit compact solutions with non-zero field strength, non-flat instanton and constant dilaton to the heterotic string equations in dimensions seven and eight. We present a quadratic condition on the curvature which is necessary and sufficient the heterotic supersymmetry and the anomaly cancellation to imply the heterotic equations of motion in dimensions seven and eight. We show that some of our examples are compact supersymmetric solutions of the heterotic equations of motion in dimensions seven and eight.

CONTENTS

1. Introduction	1
2. General properties of G_2 and $Spin(7)$ structures	4
3. The supersymmetry equations	6
4. Heterotic supersymmetry and equations of motion	7
4.1. Heterotic supersymmetric equations of motion with constant dilaton	9
5. The Lie group setup	9
5.1. Explicit solutions in dimension 7	10
5.2. Explicit solutions in dimension 8	14
6. Geometric models	17
6.1. T^3 -bundles over a Calabi-Yau surface	18
6.2. S^1 -bundles over a manifold with a balanced $SU(3)$ -structure	20
6.3. S^1 -bundles over a cocalibrated G_2 -manifold of pure type	20
References	22

1. INTRODUCTION

The bosonic fields of the ten-dimensional supergravity which arises as low energy effective theory of the heterotic string are the spacetime metric g , the NS three-form field strength H , the dilaton ϕ and the gauge connection A with curvature F^A . The bosonic geometry considered in this paper is of the form $R^{1,9-d} \times M^d$ where the bosonic fields are non-trivial only on M^d , $d \leq 8$. We consider the two connections

$$\nabla^\pm = \nabla^g \pm \frac{1}{2}H,$$

where ∇^g is the Levi-Civita connection of the Riemannian metric g . Both connections preserve the metric, $\nabla^\pm g = 0$, and have totally skew-symmetric torsion $\pm H$, respectively.

The Green-Schwarz anomaly cancellation mechanism requires that the three-form Bianchi identity receives an α' correction of the form

$$(1.1) \quad dH = \frac{\alpha'}{4}(p_1(M^P) - p_1(E)) = 2\pi^2\alpha' \left(Tr(R \wedge R) - Tr(F^A \wedge F^A) \right),$$

where $p_1(M^P), p_1(E)$ are the first Pontrjagin forms of M^P with respect to a connection ∇ with curvature R , and that of the vector bundle E with connection A , respectively.

A class of heterotic-string backgrounds for which the Bianchi identity of the three-form H receives a correction of type (1.1) are those with (2,0) world-volume supersymmetry. Such models were considered in [54]. The target-space geometry of (2,0)-supersymmetric sigma models has been extensively investigated in [54, 73, 51]. Recently, there is revived interest in these models [37, 13, 38, 39, 41] as string backgrounds and in connection to heterotic-string compactifications with fluxes [12, 1, 2, 3, 66, 34, 35, 4].

In writing (1.1) there is a subtlety to the choice of connection ∇ on M^P since anomalies can be cancelled independently of the choice [52]. Different connections correspond to different regularization schemes in the two-dimensional worldsheet non-linear sigma model. Hence the background fields given for the particular choice of ∇ must be related to those for a different choice by a field redefinition [70]. Connections on M^P proposed to investigate the anomaly cancellation (1.1) are ∇^g [73, 39], ∇^+ [13], ∇^- [5, 12, 41, 55], Chern connection ∇^c when $d = 6$ [73, 66, 34, 35, 4].

An heterotic geometry will preserve supersymmetry if and only if, in 10 dimensions, there exists at least one Majorana-Weyl spinor ϵ such that the supersymmetry variations of the fermionic fields vanish, i.e. the following Killing-spinor equations hold [73]

$$(1.2) \quad \begin{aligned} \delta_\lambda &= \nabla_m \epsilon = \left(\nabla_m^g + \frac{1}{4} H_{mnp} \Gamma^{np} \right) \epsilon = \nabla^+ \epsilon = 0, \\ \delta_\Psi &= \left(\Gamma^m \partial_m \phi - \frac{1}{12} H_{mnp} \Gamma^{mnp} \right) \epsilon = \left(d\phi - \frac{1}{2} H \right) \cdot \epsilon = 0, \\ \delta_\xi &= F_{mn}^A \Gamma^{mn} \epsilon = F^A \cdot \epsilon = 0, \end{aligned}$$

where λ, Ψ, ξ are the gravitino, the dilatino and the gaugino fields, respectively, and \cdot means Clifford action of forms on spinors.

The bosonic part of the ten-dimensional supergravity action in the string frame is [5]

$$(1.3) \quad S = \frac{1}{2k^2} \int d^{10}x \sqrt{-g} e^{-2\phi} \left[Scal^g + 4(\nabla^g \phi)^2 - \frac{1}{2} |H|^2 - \frac{\alpha'}{4} \left(Tr |F^A|^2 - Tr |R|^2 \right) \right].$$

The string frame field equations (the equations of motion induced from the action (1.3)) of the heterotic string up to two-loops [53] in sigma model perturbation theory are (we use the notations in [41])

$$(1.4) \quad \begin{aligned} Ric_{ij}^g - \frac{1}{4} H_{imn} H_j^{mn} + 2\nabla_i^g \nabla_j^g \phi - \frac{\alpha'}{4} \left[(F^A)_{imab} (F^A)_j^{mab} - R_{imnq} R_j^{mnq} \right] &= 0, \\ \nabla_i^g (e^{-2\phi} H_{jk}^i) &= 0, \\ \nabla_i^+ (e^{-2\phi} (F^A)_j^i) &= 0. \end{aligned}$$

The field equation of the dilaton ϕ is implied from the first two equations above.

We search for a solution to lowest nontrivial order in α' of the equations of motion in dimensions seven and eight that follows from the bosonic action which also preserves at least one supersymmetry.

It is known [19, 38] ([41] for dimension 6) that the equations of motion of type I supergravity (1.4) with $R = 0$ are automatically satisfied if one imposes, in addition to the preserving supersymmetry equations (1.2), the three-form Bianchi identity (1.1) taken with respect to a flat connection ($R = 0$) on TM .

A lot of effort had been done in dimension six and compact torsional solutions for the heterotic/type I string are known to exist [18, 1, 2, 13, 39, 66, 34, 35, 4, 17, 24].

In dimension five compact supersymmetric solutions to the heterotic equations of motion with non-zero fluxes and constant dilaton have been constructed recently in [25].

In dimensions 7 and 8 the only known heterotic/type I solutions with non-zero fluxes to the equations of motion preserving at least one supersymmetry (satisfying (1.2) and (1.1) without the curvature term, $R = 0$) are those constructed in [21, 36, 50] for dimension 8, and those presented in [47] for dimension 7. All these solutions are non-compact and conformal to a flat space. Non-compact solutions to (1.2) and (1.1) in dimensions 7 and 8 are presented also in [55].

The main goal of this paper is to construct explicit compact supersymmetric valid solutions with non-zero field strength, non-flat instanton and constant dilaton to the heterotic equations of motion (1.4) in dimensions 7 and 8.

According to no-go (vanishing) theorems (a consequence of the equations of motion [28, 19]; a consequence of the supersymmetry [59, 58] for $SU(n)$ -case and [39] for the general case) there are no compact solutions with non-zero flux and non-constant dilaton satisfying simultaneously the supersymmetry equations (1.2) and the three-form Bianchi identity (1.1) if one takes flat connection on TM , more precisely a connection with zero first Pontrjagin 4-form, $Tr(R \wedge R) = 0$.

In the compact case one necessarily has to have a non-zero term $Tr(R \wedge R)$. However, under the presence of a non-zero curvature 4-form $Tr(R \wedge R)$ a solution of the supersymmetry equations (1.2) and the anomaly cancellation condition (1.1) obeys the second and the third equations of motion (the second and the third equations in (1.4)) but does not always satisfy the Einstein equation of motion (the first equation in (1.4)). We give in Theorem 4.1 a quadratic expression for R which is necessary and sufficient condition in order that (1.2) and (1.1) imply (1.4) in dimensions 7 and 8 based on the properties of the special geometric structure induced from the first two equations in (1.2). (A similar condition in dimensions five and six we presented in [25, 24], respectively.) In particular, if R is a G_2 -instanton (resp. $Spin(7)$ -instanton) the supersymmetry equations together with the anomaly cancellation imply the equations of motion. The latter can also be seen following the considerations in the Appendix of [38].

In this article we present compact nilmanifolds in dimensions seven and eight satisfying the heterotic supersymmetry equations (1.2) with non-zero flux H , non-flat instanton and constant dilaton obeying the three-form Bianchi identity (1.1) with curvature term $R = R^+$ which is of instanton type. According to Theorem 4.1 these nilmanifolds are compact supersymmetric solutions of the heterotic equations of motion (1.4) in dimensions 7 and 8. The solutions in dimension 7 are constructed on the 7-dimensional generalized Heisenberg nilmanifold, which is a circle bundle over a 6-torus with curvature inside the Lie algebra $su(3)$. The 8-dimensional compact solutions can be described as a circle bundle over the product of a 2-torus by the total space of a circle bundle over a 4-torus, or alternatively as the total space of a circle bundle with curvature inside the Lie algebra g_2 over a 7-manifold which is a circle bundle over a 6-torus (see Section 5 for details). Based on the examples we present in Section 5 as well as on constructions proposed in [39], we outline in the last section a more general construction of compact manifolds solving the first two equations in (1.2) with non-constant dilaton depending on reduced number of variables.

Our solutions seem to be the first explicit compact valid supersymmetric heterotic solutions with non-zero flux, non-flat instanton and constant dilaton in dimensions 7 and 8 satisfying the equations of motion (1.4).

Our conventions: We raise and lower the indices with the metric and use the summation convention on repeated indices. For example,

$$B_{ijk}C^{ijk} = B_i^{jk}C_{jk}^i = B_{ijk}C_{ijk} \sum_{i,j,k=1}^d B_{ijk}C_{ijk}.$$

The connection 1-forms σ_{ji} of a metric connection $\nabla, \nabla g = 0$, with respect to a local basis $\{E_1, \dots, E_d\}$ are given by

$$\sigma_{ji}(E_k) = g(\nabla_{E_k} E_j, E_i),$$

since we write $\nabla_X E_j = \sigma_j^s(X) E_s$.

The curvature 2-forms Ω_j^i of ∇ are given in terms of the connection 1-forms σ_j^i by

$$(1.5) \quad \Omega_j^i = d\sigma_j^i + \sigma_k^i \wedge \sigma_j^k, \quad \Omega_{ji} = d\sigma_{ji} + \sigma_{ki} \wedge \sigma_{jk}, \quad R_{ijk}^l = \Omega_k^l(E_i, E_j), \quad R_{ijkl} = R_{ijk}^s g_{ls},$$

and the first Pontrjagin class is represented by the 4-form

$$p_1(\nabla) = \frac{1}{8\pi^2} \sum_{1 \leq i < j \leq d} \Omega_j^i \wedge \Omega_j^i.$$

2. GENERAL PROPERTIES OF G_2 AND $Spin(7)$ STRUCTURES

We recall the basic properties of the geometric structures induced from the gravitino and dilatino Killing spinor equations (the first two equations in (1.2)) in dimensions 7 and 8.

G_2 -structures in $d = 7$. Endow \mathbb{R}^7 with its standard orientation and inner product. Let $\{E_1, \dots, E_7\}$ be an oriented orthonormal basis and $\{e^1, \dots, e^7\}$ its dual basis. Consider the three-form Θ on \mathbb{R}^7 given by

$$(2.1) \quad \Theta = e^{127} - e^{236} + e^{347} + e^{567} - e^{146} - e^{245} + e^{135}.$$

The subgroup of $GL(7, \mathbb{R})$ fixing Θ is the exceptional Lie group G_2 . It is a compact, connected, simply-connected, simple Lie subgroup of $SO(7)$ of dimension 14 [7]. The Lie algebra is denoted by \mathfrak{g}_2 , and it is isomorphic to the two-forms satisfying 7 linear equations, namely $\mathfrak{g}_2 \cong \Lambda_{14}^2(\mathbb{R}^7) = \{\beta \in \Lambda^2(\mathbb{R}^7) \mid *(\beta \wedge \Theta) = -\beta\}$. The 3-form Θ corresponds to a real spinor ϵ and therefore, G_2 can be identified as the isotropy group of a non-trivial real spinor.

The Hodge star operator supplies the 4-form $*\Theta$ given by

$$(2.2) \quad *\Theta = e^{3456} + e^{1457} + e^{1256} + e^{1234} + e^{2357} + e^{1367} - e^{2467}.$$

The space $\Lambda_{14}^2(\mathbb{R}^7)$ can also be described as the subspace of 2-forms β which annihilate $*\Theta$, i.e. $\beta \wedge *\Theta = 0$. A 7-dimensional Riemannian manifold M is called a G_2 -manifold if its structure group reduces to the exceptional Lie group G_2 . The existence of a G_2 -structure is equivalent to the existence of a global non-degenerate three-form which can be locally written as (2.1). The 3-form Θ is called *the fundamental form* of the G_2 -manifold [6]. From the purely topological point of view, a 7-dimensional paracompact manifold is a G_2 -manifold if and only if it is an oriented spin manifold [65]. We will say that the pair (M, Θ) is a G_2 -manifold with G_2 -structure (determined by) Θ .

The fundamental form of a G_2 -manifold determines a Riemannian metric *implicitly* through $g_{ij} = \frac{1}{6} \sum_{kl} \Theta_{ikl} \Theta_{jkl}$ [43]. This is referred to as the metric induced by Θ .

In [23], Fernández and Gray divide G_2 -manifolds into 16 classes according to how the covariant derivative of the fundamental three-form behaves with respect to its decomposition into G_2

irreducible components (see also [14, 37]). If the fundamental form is parallel with respect to the Levi-Civita connection, $\nabla^g \Theta = 0$, then the Riemannian holonomy group is contained in G_2 . In this case the induced metric on the G_2 -manifold is Ricci-flat, a fact first observed by Bonan [6]. It was shown by Gray [43] (see also [23, 7, 71]) that a G_2 -manifold is parallel precisely when the fundamental form is harmonic, i.e. $d\Theta = d * \Theta = 0$. The first examples of complete parallel G_2 -manifolds were constructed by Bryant and Salamon [9, 40]. Compact examples of parallel G_2 -manifolds were obtained first by Joyce [60, 61, 62] and recently by Kovalev [64].

The Lee form θ^7 is defined by [11]

$$(2.3) \quad \theta^7 = -\frac{1}{3} * (*d\Theta \wedge \Theta) = \frac{1}{3} * (*d * \Theta \wedge * \Theta).$$

If the Lee form vanishes, $\theta^7 = 0$, then the G_2 -structure is said to be *balanced*. If the Lee form is closed, $d\theta^7 = 0$, then the G_2 -structure is locally conformally equivalent to a balanced one [32]. If the G_2 -structure satisfies the condition $d * \Theta = \theta^7 \wedge * \Theta$ then it is called *integrable* and an analog of the Dolbeault cohomology is investigated in [27]. A *cocalibrated* G_2 -structure is a balanced G_2 -structure which is also integrable.

Spin(7)-structures in $d = 8$. Consider \mathbb{R}^8 endowed with an orientation and its standard inner product. Let $\{E_1, \dots, E_8\}$ be an oriented orthonormal basis and $\{e^1, \dots, e^8\}$ its dual basis. Consider the 4-form Φ on \mathbb{R}^8 given by

$$(2.4) \quad \begin{aligned} \Phi = & e^{1238} - e^{1347} + e^{1458} + e^{1678} - e^{1257} - e^{1356} + e^{1246} \\ & + e^{4567} + e^{2568} + e^{2367} + e^{2345} + e^{3468} + e^{2478} - e^{3578}. \end{aligned}$$

The 4-form Φ is self-dual $*\Phi = \Phi$ and the 8-form $\Phi \wedge \Phi$ coincides with the volume form of \mathbb{R}^8 . The subgroup of $GL(8, \mathbb{R})$ which fixes Φ is isomorphic to the double covering $Spin(7)$ of $SO(7)$ [49]. Moreover, $Spin(7)$ is a compact simply-connected Lie group of dimension 21 [7]. The Lie algebra of $Spin(7)$ is denoted by $\mathfrak{spin}(7)$ and it is isomorphic to the two-forms satisfying 7 linear equations, namely $\mathfrak{spin}(7) \cong \{\beta \in \Lambda^2(\mathbb{R}^8) \mid *(\beta \wedge \Phi) = -\beta\}$. The 4-form Φ corresponds to a real spinor ϕ and therefore, $Spin(7)$ can be identified as the isotropy group of a non-trivial real spinor.

A *Spin(7)-structure* on an 8-manifold M is by definition a reduction of the structure group of the tangent bundle to $Spin(7)$; we shall also say that M is a *Spin(7)-manifold*. This can be described geometrically by saying that there exists a nowhere vanishing global differential 4-form Φ on M which can be locally written as (2.4). The 4-form Φ is called the *fundamental form* of the *Spin(7)-manifold* M [6].

The fundamental form of a *Spin(7)-manifold* determines a Riemannian metric *implicitly* through $g_{ij} = \frac{1}{24} \sum_{klm} \Phi_{iklm} \Phi_{jklm}$ [43]. This is referred to as the metric induced by Φ .

In general, not every 8-dimensional Riemannian spin manifold M admits a *Spin(7)-structure*. We explain the precise condition [65]. Denote by $p_2(M)$, $\mathbb{X}(M)$, $\mathbb{X}(S_{\pm})$ the second Pontrjagin class, the Euler characteristic of M and the Euler characteristic of the positive and the negative spinor bundles, respectively. It is well-known [65] that a spin 8-manifold admits a *Spin(7)-structure* if and only if $\mathbb{X}(S_+) = 0$ or $\mathbb{X}(S_-) = 0$. The latter conditions are equivalent to $p_1^2(M) - 4p_2(M) + 8\mathbb{X}(M) = 0$, for an appropriate choice of the orientation [65].

Let us recall that a *Spin(7)-manifold* (M, g, Φ) is said to be parallel (torsion-free [61]) if the holonomy of the metric $Hol(g)$ is a subgroup of $Spin(7)$. This is equivalent to saying that the fundamental form Φ is parallel with respect to the Levi-Civita connection ∇^g of the metric g . Moreover, $Hol(g) \subset Spin(7)$ if and only if $d\Phi = 0$ [22] (see also [7, 71]) and any parallel *Spin(7)-manifold* is Ricci flat [6]. The first known explicit example of complete parallel *Spin(7)-manifold* with $Hol(g) = Spin(7)$ was constructed by Bryant and Salamon [9, 40]. The first compact examples of parallel *Spin(7)-manifolds* with $Hol(g) = Spin(7)$ were constructed by Joyce [60, 61].

There are four classes of $Spin(7)$ -manifolds according to the Fernández classification [22] obtained as irreducible representations of $Spin(7)$ of the space $\nabla^g\Phi$.

The Lee form θ^8 is defined by [10]

$$(2.5) \quad \theta^8 - \frac{1}{7} * (*d\Phi \wedge \Phi) = \frac{1}{7} * (\delta\Phi \wedge \Phi).$$

The four classes of Fernández classification can be described in terms of the Lee form as follows [10]: $W_0 : d\Phi = 0$; $W_1 : \theta^8 = 0$; $W_2 : d\Phi = \theta^8 \wedge \Phi$; $W : W = W_1 \oplus W_2$.

A $Spin(7)$ -structure of the class W_1 (i.e. $Spin(7)$ -structure with zero Lee form) is called a *balanced* $Spin(7)$ -structure. If the Lee form is closed, $d\theta^8 = 0$, then the $Spin(7)$ -structure is locally conformally equivalent to a balanced one [57]. It is shown in [10] that the Lee form of a $Spin(7)$ -structure in the class W_2 is closed and therefore such a manifold is locally conformally equivalent to a parallel $Spin(7)$ -manifold. The compact spaces with closed but not exact Lee form (i.e. the structure is not globally conformally parallel) have different topology than the parallel ones [57].

Coeffective cohomology and coeffective numbers of Riemannian manifolds with $Spin(7)$ -structure are studied in [74].

3. THE SUPERSYMMETRY EQUATIONS

Geometrically, the vanishing of the gravitino variation is equivalent to the existence of a non-trivial real spinor parallel with respect to the metric connection ∇^+ with totally skew-symmetric torsion $T = H$. The presence of ∇^+ -parallel spinor leads to restriction of the holonomy group $Hol(\nabla^+)$ of the torsion connection ∇^+ . Namely, $Hol(\nabla^+)$ has to be contained in $SU(3), d = 6$ [73, 59, 58, 39, 48, 13, 1, 2], the exceptional group $G_2, d = 7$ [31, 37, 39, 32], the Lie group $Spin(7), d = 8$ [37, 57, 39]. A detailed analysis of the induced geometries is carried out in [39] and all possible geometries (including non compact stabilizers) are investigated in [44, 46, 45, 68].

Dimension $d = 7$. The precise conditions to have a solution to the gravitino Killing spinor equation in dimension 7 were found in [31]. Namely, there exists a non-trivial parallel spinor with respect to a G_2 -connection with torsion 3-form T if and only if there exists an integrable G_2 -structure (Θ, g) , i.e. $d*\Theta = \theta^7 \wedge *\Theta$. In this case, the torsion connection ∇^+ is unique and the torsion 3-form T is given by

$$H = T = \frac{1}{6}(d\Theta, *\Theta)\Theta - *d\Theta + *(\theta^7 \wedge \Theta).$$

The Riemannian scalar curvature is [32] ([8] for the general case) $s^g = \frac{1}{18}(d\Theta, *\Theta) + \|\theta^7\|^2 - \frac{1}{12}\|T\|^2 + 3\delta\theta^7$.

The necessary conditions to have a solution to the system of dilatino and gravitino Killing spinor equations were derived in [37, 31, 32], and the sufficiency was proved in [31, 32]. The general existence result [31, 32] states that there exists a non-trivial solution to both dilatino and gravitino Killing spinor equations in dimension 7 if and only if there exists a G_2 -structure (Θ, g) satisfying the equations

$$(3.1) \quad d*\Theta = \theta^7 \wedge *\Theta, \quad d\Theta \wedge \Theta = 0, \quad \theta^7 = 2d\phi.$$

Consequently, the torsion 3-form (the flux H) is given by

$$(3.2) \quad H = T = -*d\Theta + 2*(d\phi \wedge \Theta).$$

The Riemannian scalar curvature satisfies $s^g = 8\|d\phi\|^2 - \frac{1}{12}\|T\|^2 + 6\delta d\phi$.

The equations (3.1) hold exactly when the G_2 -structure $(\bar{\Theta} = e^{-\frac{3}{2}\phi}\Theta, \bar{g} = e^{-\phi}g)$ obeys the equations

$$d\bar{*}\bar{\Theta} = d\bar{\Theta} \wedge \bar{\Theta} = 0,$$

i.e. it is cocalibrated of pure type.

Dimension $d = 8$. It is shown in [57] that the gravitino Killing spinor equation always has a solution in dimension 8. Namely, any $Spin(7)$ -structure admits a unique $Spin(7)$ -connection with totally skew-symmetric torsion T satisfying

$$T = *d\Phi - \frac{7}{6} * (\theta^8 \wedge \Phi).$$

(In fact, the converse is also true, namely if there are no obstructions to exist a solution to the gravitino Killing spinor equation then dimension is 8 and the structure is $Spin(7)$ [29, 67].)

The necessary conditions to have a solution to the system of dilatino and gravitino Killing spinor equations were derived in [37, 57], and the sufficiency was proved in [57]. The general existence result [57] states that there exists a non-trivial solution to both dilatino and gravitino Killing spinor equations in dimension 8 if and only if there exists a $Spin(7)$ -structure (Φ, g) with an exact Lee form which is equivalent to the statement that the $Spin(7)$ -structure is conformally balanced, i.e. the $Spin(7)$ structure $(\bar{\Phi} = e^{-\frac{12}{7}\phi}\Phi, \bar{g} = e^{-\frac{6}{7}\phi}g)$ satisfies $\bar{*}d\bar{\Phi} \wedge \bar{\Phi} = 0$.

The torsion 3-form (the flux H) and the Lee form are given by

$$(3.3) \quad H = T = *d\Phi - 2*(d\phi \wedge \Phi), \quad \theta^8 = \frac{12}{7}d\phi.$$

The Riemannian scalar curvature satisfies $s^g = 8\|d\phi\|^2 - \frac{1}{12}\|T\|^2 + 6\delta d\phi$.

In addition to these equations, the vanishing of the gaugino variation requires the 2-form F^A to be of instanton type [16, 73, 50, 69, 20, 39]:

Case $d = 7$: a G_2 -instanton, i.e. the gauge field A is a G_2 -connection and its curvature 2-form $F^A \in \mathfrak{g}_2$. The latter can be expressed in any of the following two equivalent ways

$$(3.4) \quad F_{mn}^A \Theta^{mn}{}_p = 0 \quad \Leftrightarrow \quad F_{mn}^A = -\frac{1}{2}F_{pq}^A (*\Theta)^{pq}{}_{mn};$$

Case $d = 8$: an $Spin(7)$ -instanton, i.e. the gauge field A is a $Spin(7)$ -connection and its curvature 2-form $F^A \in \mathfrak{spin}(7)$. The latter is equivalent to

$$(3.5) \quad F_{mn}^A = -\frac{1}{2}F_{pq}^A \Phi^{pq}{}_{mn}.$$

4. HETEROTIC SUPERSYMMETRY AND EQUATIONS OF MOTION

It is known [19, 38] ([41] for dimension 6) that the equations of motion of type I supergravity (1.4) with $R = 0$ are automatically satisfied if one imposes, in addition to the preserving supersymmetry equations (1.2), the three-form Bianchi identity (1.1) taken with respect to a flat connection on $TM, R = 0$. However, the no-go theorem [28, 19, 59, 58, 39] states that if even $Tr(R \wedge R) = 0$ there are no compact solutions with non-zero flux H and non-constant dilaton.

In the presence of a curvature term $Tr(R \wedge R) \neq 0$, a solution of the supersymmetry equations (1.2) and the anomaly cancellation condition (1.1) obeys the second and the third equations in (1.4) but does not always satisfy the Einstein equation of motion (the first equation in (1.4)). However if the curvature R is of instanton type (1.2) and (1.1) imply (1.4). This can be seen following the considerations in the Appendix of [38]. We shall give below an independent proof for the Einstein equation of motion (the first equation in (1.4)) based on the properties of the special geometric structure induced from the first two equations in (1.2).

A consequence of the gravitino and dilatino Killing spinor equations is an expression of the Ricci tensor $Ric_{mn}^+ = R_{imnj}^+ g^{ij}$ of the (+)-connection, and therefore an expression of the Ricci tensor Ric^g of the Levi-Civita connection, in terms of the suitable trace of the torsion three-form $T = H$ (the Lee form) and the exterior derivative of the torsion form $dT = dH$ (see [31] in dimension 7 and [57] in dimension 8). We outline an unified proof for dimensions 7 and 8.

Indeed, the two Ricci tensors are connected by (see e.g. [31])

$$(4.1) \quad Ric_{mn}^g = Ric_{mn}^+ + \frac{1}{4} T_{mpq} T_n^{pq} - \frac{1}{2} \nabla_s^+ T_{mn}^s, \quad Ric_{mn}^+ - Ric_{nm}^+ = \nabla_s^+ T_{mn}^s = \nabla_s^g T_{mn}^s,$$

$$(4.2) \quad Ric_{mn}^g = \frac{1}{2} (Ric_{mn}^+ + Ric_{nm}^+) + \frac{1}{4} T_{mpq} T_n^{pq}.$$

Denote by Ψ the 4-form $- * \Theta$ in dimension 7 or the $Spin(7)$ -form $-\Phi$ in dimension 8. Since $Hol(\nabla^+) \subset \{\mathfrak{g}_2, \mathfrak{spin}(7)\}$, we have the next sequence of identities

$$(4.3) \quad 2Ric_{mn}^+ = R_{mjkl}^+ \Psi_{jklm} = \frac{1}{3} (R_{mjkl}^+ + R_{mklj}^+ + R_{mljk}^+) \Psi_{jklm}.$$

We apply the following identity established in [31]

$$(4.4) \quad R_{jklm}^+ + R_{kljm}^+ + R_{ljkm}^+ - R_{mjkl}^+ - R_{mklj}^+ - R_{mljk}^+ \frac{3}{2} dT_{jklm} - T_{jks} T_{lms} - T_{kls} T_{jms} - T_{ljs} T_{kms}.$$

The first Bianchi identity for ∇^+ reads (see e.g. [31])

$$(4.5) \quad R_{jklm}^+ + R_{kljm}^+ + R_{ljkm}^+ = dT_{jklm} - T_{jks} T_{lms} - T_{kls} T_{jms} - T_{ljs} T_{kms} + \nabla_m^+ T_{jkl}.$$

Now, (4.5), (4.4) and (4.3) yield

$$(4.6) \quad Ric_{mn}^+ = \frac{1}{12} dT_{mjkl} \Psi_{jklm} + \frac{1}{6} \nabla_m^+ T_{jkl} \Psi_{jklm}.$$

Using the special expression of the torsion (3.2) and (3.3) for dimension 7 and 8, respectively, the equation (4.6) takes the form

$$(4.7) \quad Ric_{mn}^+ = \frac{1}{12} dT_{mjkl} \Psi_{jklm} - 2 \nabla_m^+ d\phi_n \frac{1}{12} dT_{mjkl} \Psi_{jklm} - 2 \nabla_m^g d\phi_n + d\phi_s T_{mn}^s.$$

Substitute (4.7) into (4.2), insert the result into the first equation of (1.4) and use the anomaly cancellation (1.1) to conclude

Theorem 4.1. *The Einstein equation of motion (the first equation in (1.4)) in dimensions 7 and 8 is a consequence of the heterotic Killing spinor equations (1.2) and the anomaly cancellation (1.1) if and only if the next identity holds*

$$(4.8) \quad \frac{1}{6} \left[R_{mjab} R_{klab} + R_{mkab} R_{ljab} + R_{mlab} R_{jkab} \right] \Psi_{jklm} = R_{mpqr} R_n^{pqr},$$

where the 4-form Ψ is equal to $- * \Theta$ in dimension 7 and to the $Spin(7)$ -form $-\Phi$ in dimension 8.

In particular, if R is an instanton then (4.8) holds.

It is shown in [56] that the curvature of R^+ satisfies the identity $R_{ijkl}^+ = R_{klij}^+$ if and only if $\nabla_i^+ T_{jkl}$ is a 4-form. Now Theorem 4.1 yields

Corollary 4.2. *Suppose the torsion 3-form is ∇^+ -parallel, $\nabla_i^+ T_{jkl} = 0$. The equations of motion (1.4) with respect to the curvature R^+ of the (+)-connection are consequences of the heterotic Killing spinor equations (1.2) and the anomaly cancellation (1.1).*

Manifolds with parallel torsion 3-form are studied in detail in dimension 6 [72] and dimension 7 [30].

4.1. Heterotic supersymmetric equations of motion with constant dilaton. In the case when the dilaton is constant we arrive to the following problems:

Dimension 7. We look for a compact G_2 -manifold (M, Θ) which satisfies the following conditions

- a). Gravitino and dilatino Killing spinor equations with constant dilaton: Search for a cocalibrated G_2 -manifold of pure type, i.e. $d * \Theta = d\Theta \wedge \Theta = 0$.
- b). Gaugino equation: look for a vector bundle E of rank r over M equipped with a G_2 -instanton, i.e. a connection A with curvature 2-form Ω^A satisfying

$$(4.9) \quad (\Omega^A)_{E_i, E_j}(E_k, E_l)(* \Theta)(E_m, E_n, E_k, E_l) = -2(\Omega^A)_{E_i, E_j}(E_m, E_n),$$

where $\{E_1, \dots, E_7\}$ is a G_2 -adapted basis on M .

- c). Anomaly cancellation condition:

$$(4.10) \quad dH = dT = -d * d\Theta = 2\pi^2 \alpha' (p_1(M) - p_1(A)), \quad \alpha' > 0.$$

- d). The first Pontrjagin form $p_1(M)$ satisfies equation (4.8).

Dimension 8. We look for a compact $Spin(7)$ -manifold (M, Φ) satisfying the following conditions

- a). Gravitino and dilatino Killing spinor equations with constant dilaton: (M, Φ) is balanced, i.e. $*d\Phi \wedge \Phi = 0$.
- b). Gaugino equation: look for a vector bundle E of rank r over M equipped with an $Spin(7)$ -instanton, i.e. a connection A with curvature 2-form Ω^A satisfying

$$(4.11) \quad (\Omega^A)_{E_i, E_j}(E_k, E_l)(\Phi)(E_m, E_n, E_k, E_l) = -2(\Omega^A)_{E_i, E_j}(E_m, E_n),$$

where $\{E_1, \dots, E_8\}$ is a $Spin(7)$ -adapted basis on M .

- c). Anomaly cancellation condition:

$$(4.12) \quad dH = dT = d * d\Phi = 2\pi^2 \alpha' (p_1(M) - p_1(A)), \quad \alpha' > 0.$$

- d). The first Pontrjagin form $p_1(M)$ satisfies equation (4.8).

5. THE LIE GROUP SETUP

Let us suppose that g is a left-invariant Riemannian metric on a Lie group G of dimension m , and let $\{e^1, \dots, e^m\}$ be an orthonormal basis of left-invariant 1-forms, so that $g = e^1 \otimes e^1 + \dots + e^m \otimes e^m$. Let

$$de^k = \sum_{1 \leq i < j \leq m} a_{ij}^k e^i \wedge e^j, \quad k = 1, \dots, m$$

be the structure equations in the basis $\{e^k\}_{k=1}^m$.

Let us denote by $\{E_1, \dots, E_m\}$ the dual basis. Since $de^k(E_i, E_j) = -e^k([E_i, E_j])$, the Levi-Civita connection 1-forms $(\sigma^g)_j^i$ are

$$(5.1) \quad (\sigma^g)_j^i(E_k) = -\frac{1}{2}(g(E_i, [E_j, E_k]) - g(E_k, [E_i, E_j]) + g(E_j, [E_k, E_i])) = \frac{1}{2}(a_{jk}^i - a_{ij}^k + a_{ki}^j).$$

The connection 1-forms $(\sigma^+)_j^i$ for the torsion connection ∇^+ are given by

$$(5.2) \quad (\sigma^+)_j^i(E_k) = (\sigma^g)_j^i(E_k) - \frac{1}{2}T_j^i(E_k), \quad T_j^i(E_k) = T(E_i, E_j, E_k).$$

We shall focus on 7 and 8-dimensional nilmanifolds $M = \Gamma \backslash G$ endowed with an invariant special structure.

5.1. Explicit solutions in dimension 7. We consider cocalibrated G_2 -structures of pure type. From (3.2) we have that the torsion 3-form in this case is given by

$$(5.3) \quad \nabla^+ = \nabla^g + \frac{1}{2}T, \quad H = T = - * d\Theta.$$

Starting from a balanced $SU(3)$ -structure (F, Ψ_+, Ψ_-) on a manifold M^6 it is easy to see that the G_2 -structure given by $\Theta = F \wedge e^7 + \Psi_+$ on the product $M^7 = M^6 \times S^1$ is cocalibrated of pure type, where e^7 denotes the standard 1-form on the circle S^1 . Moreover, following the argument given in [55, Theorem 4.6] we conclude that the natural extension of an $SU(3)$ -instanton on M^6 gives rise to a G_2 -instanton on M^7 , and if the torsion connection of the $SU(3)$ -structure satisfies the modified Bianchi identity then the corresponding ∇^+ given in (5.3) also satisfies (4.10). We can apply this to the compact 6-dimensional explicit solutions given in [24] to get compact solutions in dimension 7:

Corollary 5.1. *Let (M^6, F, Ψ_+, Ψ_-) be a compact balanced $SU(3)$ -nilmanifold with an $SU(3)$ -instanton solving the modified Bianchi identity for $\nabla = \nabla^+$ or ∇^g . Then, the G_2 -manifold $M^7 = M^6 \times S^1$ with the structure $\Theta = F \wedge e^7 + \Psi_+$, the G_2 -instanton obtained as an extension of the $SU(3)$ -instanton and ∇ being the Levi-Civita connection ∇^g or the torsion connection ∇^+ given in (5.3), provides a compact valid solution to the supersymmetry equations in dimension 7.*

Our next goal is to find more compact G_2 -solutions to the supersymmetry equations with non-zero flux and constant dilaton on non-trivial extensions of the balanced Hermitian structures on the Lie algebra \mathfrak{h}_3 given in [24]. We also provide a new solution to the equations of motion based on the 7-dimensional generalized Heisenberg compact nilmanifold.

Seven-dimensional extensions of \mathfrak{h}_3 : For any $t \neq 0$, the structure equations

$$(5.4) \quad \begin{cases} de^1 = de^2 = de^3 = de^4 = de^5 = 0, \\ de^6 = -2t e^{12} + 2t e^{34}, \end{cases}$$

correspond to the nilpotent Lie algebra $\mathfrak{h}_3 = (0, 0, 0, 0, 0, 12 + 34)$. As it is shown in [24], the $SU(3)$ -structure given by

$$F = e^{12} + e^{34} + e^{56}, \quad \Psi = \Psi_+ + i\Psi_- = (e^1 + i e^2) \wedge (e^3 + i e^4) \wedge (e^5 + i e^6),$$

is balanced for all the values of the parameter t . Consider any nilpotent 7-dimensional extension $\mathfrak{h}^7 = \mathfrak{h}_3 \oplus \langle E_7 \rangle$ such that the G_2 -structure

$$(5.5) \quad \Theta = F \wedge e^7 + \Psi_+$$

is cocalibrated of pure type on \mathfrak{h}^7 , where e^7 denotes the dual of E_7 . Using (2.1) and (2.2) it is easy to check that $de^7 = c_0(e^{12} - e^{34}) + c_1(e^{13} + e^{24}) + c_2(e^{14} - e^{23})$, where $c_0, c_1, c_2 \in \mathbb{R}$. Since $t \neq 0$ in (5.4), we can consider $c_0 = 0$. Therefore, the nilpotent Lie algebra \mathfrak{h}^7 must be given by the structure equations

$$(5.6) \quad \begin{cases} de^1 = de^2 = de^3 = de^4 = de^5 = 0, \\ de^6 = -2t(e^{12} - e^{34}), \\ de^7 = c_1(e^{13} + e^{24}) + c_2(e^{14} - e^{23}), \end{cases}$$

where $c_1, c_2 \in \mathbb{R}$. Moreover, a direct calculation shows that the torsion is given by

$$T = - * d\Theta = -2t(e^{12} - e^{34})e^6 + c_1(e^{13} + e^{24})e^7 + c_2(e^{14} - e^{23})e^7.$$

Hence, $dT = -2(4t^2 + c_1^2 + c_2^2)e^{1234}$. It is easy to prove that T is parallel with respect to the torsion connection ∇^+ if and only if $c_1 = c_2 = 0$, which corresponds to the situation described in Theorem 5.1.

From (1.5), (5.1) and (5.2), it follows that the non-zero curvature forms $(\Omega^+)^i_j$ of the torsion connection ∇^+ are

$$\begin{aligned} (\Omega^+)^1_2 &= -(\Omega^+)^3_4 = -4t^2(e^{12} - e^{34}), \\ (\Omega^+)^1_3 &= (\Omega^+)^2_4 = -c_1^2(e^{13} + e^{24}) - c_1c_2(e^{14} - e^{23}) + 4tc_2e^{67}, \\ (\Omega^+)^1_4 &= -(\Omega^+)^2_3 = -c_1c_2(e^{13} + e^{24}) - c_2^2(e^{14} - e^{23}) - 4tc_1e^{67}, \end{aligned}$$

which implies that the first Pontrjagin form of ∇^+ is

$$p_1(\nabla^+) = -\frac{1}{2\pi^2} (16t^4 + (c_1^2 + c_2^2)^2) e^{1234}.$$

Let us consider $(c_1, c_2) \in \mathbb{Q}^2 - \{(0, 0)\}$. The well-known Malcev theorem asserts that the simply-connected nilpotent Lie group H^7 corresponding to the Lie algebra \mathfrak{h}^7 has a lattice Γ of maximal rank. We denote by M^7 the compact nilmanifold $\Gamma \backslash H^7$.

Lemma 5.2. *Let A_λ be the linear connection preserving the metric on M^7 defined by the connection 1-forms*

$$(\sigma^{A_\lambda})^i_j = \lambda e^7, \quad (\sigma^{A_\lambda})^6_7 = e^1 + e^2 + e^3 + e^4 + e^5 + \lambda e^6 + \lambda e^7,$$

for $(i, j) = (1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)$, where $\lambda \in \mathbb{R}$. Then, A_λ is a G_2 -instanton with respect to the structure (5.5), and

$$p_1(A_\lambda) = -\frac{\lambda^2}{4\pi^2} (4t^2 + 11(c_1^2 + c_2^2)) e^{1234}.$$

Proof. A direct calculation shows that the non-zero curvature forms $(\Omega^{A_\lambda})^i_j$ of the connection A_λ are given by:

$$(\Omega^{A_\lambda})^i_j = \lambda c_1(e^{13} + e^{24}) + \lambda c_2(e^{14} - e^{23}), \quad (\Omega^{A_\lambda})^6_7 = -2\lambda t(e^{12} - e^{34}) + \lambda c_1(e^{13} + e^{24}) + \lambda c_2(e^{14} - e^{23}).$$

for $(i, j) = (1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)$. On the other hand, since the Lie algebra of G_2 can be identified with the subspace of 2-forms which annihilate $*\Theta$ and $(e^{12} - e^{34}) \wedge *\Theta = (e^{13} + e^{24}) \wedge *\Theta = (e^{14} - e^{23}) \wedge *\Theta = 0$, the connection A_λ is a G_2 -instanton. \square

As a consequence we get the following compact 7-dimensional solutions.

Theorem 5.3. *Let A_λ be the G_2 -instanton on M^7 given above. If $\lambda^2 < \min\{8t^2, 2(c_1^2 + c_2^2)/11\}$, then*

$$dT = 2\pi^2 \alpha' (p_1(\nabla^+) - p_1(A_\lambda)),$$

with $\alpha' > 0$ and $(M^7, \Theta, \nabla^+, A_\lambda)$ is a compact solution to the heterotic Killing-spinor equations (1.2) satisfying the anomaly cancellation condition (1.1).

Proof. Notice that $p_1(\nabla^+) - p_1(A_\lambda) = \frac{1}{4\pi^2} [4t^2(\lambda^2 - 8t^2) + (c_1^2 + c_2^2)(11\lambda^2 - 2(c_1^2 + c_2^2))] e^{1234}$. Therefore, if $\lambda^2 < \min\{8t^2, 2(c_1^2 + c_2^2)/11\}$, then $p_1(\nabla^+) - p_1(A_\lambda)$ is a negative multiple of e^{1234} . Since $dT = -2(4t^2 + c_1^2 + c_2^2)e^{1234}$, the result follows. \square

Remark 5.4. *The first Pontrjagin form of the Levi-Civita connection is given by*

$$p_1(\nabla^g) = -\frac{1}{16\pi^2} [3(4t^2 - c_1^2 - c_2^2)^2 + 16t^2(c_1^2 + c_2^2)] e^{1234},$$

so there is $\lambda \neq 0$ sufficiently small such that $dT = 2\pi^2 \alpha' (p_1(\nabla^g) - p_1(A_\lambda))$, with $\alpha' > 0$.

The 7-dimensional generalized Heisenberg group: Next we construct a 7-dimensional compact solution to the equations of motion which is not an extension of the 6-dimensional nilmanifolds given in [24]. Let $H(3, 1)$ be the 7-dimensional generalized Heisenberg group, i.e. the nilpotent Lie group consisting of the matrices of real numbers of the form

$$H(3, 1) = \left\{ \left(\begin{array}{ccccc} 1 & x_1 & x_2 & x_3 & z \\ 0 & 1 & 0 & 0 & y_1 \\ 0 & 0 & 1 & 0 & y_2 \\ 0 & 0 & 0 & 1 & y_3 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right) \mid x_i, y_i, z \in \mathbb{R}, 1 \leq i \leq 3 \right\}.$$

We consider the basis for the left-invariant 1-forms on $H(3, 1)$ given by

$$e^1 = \frac{1}{a} dx_1, \quad e^2 = dy_1, \quad e^3 = \frac{1}{b} dx_2, \quad e^4 = dy_2, \quad e^5 = \frac{1}{c} dx_3, \quad e^6 = dy_3, \quad e^7 = x_1 dy_1 + x_2 dy_2 + x_3 dy_3 - dz,$$

where $a, b, c \in \mathbb{R} - \{0\}$, so that the structure equations become

$$(5.7) \quad \begin{cases} de^1 = de^2 = de^3 = de^4 = de^5 = de^6 = 0, \\ de^7 = a e^{12} + b e^{34} + c e^{56}. \end{cases}$$

Lemma 5.5. *The G_2 -structure given by*

$$\Theta = (e^{12} + e^{34} + e^{56}) \wedge e^7 + e^{135} - e^{146} - e^{236} - e^{245}$$

is cocalibrated for each $a, b, c \in \mathbb{R} - \{0\}$. Moreover, Θ is of pure type if and only if $c = -a - b$ or, equivalently, $de^7 \in \mathfrak{su}(3)$.

Proof. A direct simple calculation shows that $d * \Theta = 0$ and $\Theta \wedge d\Theta = 2(a + b + c)e^{1234567}$. \square

From now on, let us consider $c = -a - b \neq 0$ in equations (5.7). The torsion 3-form for the cocalibrated G_2 -structure of pure type is given by

$$T = - * d\Theta = (de^7) \wedge e^7 = a e^{127} + b e^{347} - (a + b)e^{567}.$$

Hence

$$(5.8) \quad dT = 2ab e^{1234} - 2a(a + b)e^{1256} - 2b(a + b)e^{3456}.$$

Moreover, it is forward to check that T is parallel with respect to the torsion connection ∇^+ , i.e.

Lemma 5.6. *For any $a, b \in \mathbb{R} - \{0\}$ such that $b \neq -a$, we have $\nabla^+ T = 0$.*

On the other hand, by (1.5), (5.1) and (5.2) we have that the non-zero curvature forms $(\Omega^+)^i_j$ of the torsion connection ∇^+ are

$$(5.9) \quad \begin{aligned} (\Omega^+)^1_2 &= -a (a e^{12} + b e^{34} - (a + b)e^{56}), \\ (\Omega^+)^3_4 &= -b (a e^{12} + b e^{34} - (a + b)e^{56}), \\ (\Omega^+)^5_6 &= -(\Omega^+)^1_2 - (\Omega^+)^3_4 = (a + b) (a e^{12} + b e^{34} - (a + b)e^{56}). \end{aligned}$$

Let $\Gamma(3, 1)$ denote the subgroup of matrices of $H(3, 1)$ with integer entries and consider the compact nilmanifold $N(3, 1) = \Gamma(3, 1) \backslash H(3, 1)$. We can describe $N(3, 1)$ as a principal circle bundle over a 6-torus

$$S^1 \hookrightarrow N(3, 1) \rightarrow \mathbb{T}^6,$$

whose connection 1-form $\eta = e^7$ has curvature $d\eta = a(e^{12} - e^{56}) + b(e^{34} - e^{56})$ in $\mathfrak{su}(3)$.

Next we show a 3-parametric family of G_2 -instantons on the nilmanifold $N(3, 1)$.

Proposition 5.7. *Let $A_{\lambda,\mu,\tau}$ be the linear connection on $N(3,1)$ defined by the connection 1-forms $(\sigma^{A_{\lambda,\mu,\tau}})_2^1 = -(\sigma^{A_{\lambda,\mu,\tau}})_1^2 = \lambda e^7$, $(\sigma^{A_{\lambda,\mu,\tau}})_4^3 = -(\sigma^{A_{\lambda,\mu,\tau}})_3^4 = \mu e^7$, $(\sigma^{A_{\lambda,\mu,\tau}})_6^5 = -(\sigma^{A_{\lambda,\mu,\tau}})_5^6 = \tau e^7$, and $(\sigma^{A_{\lambda,\mu,\tau}})_j^i = 0$ for the remaining (i,j) , where $\lambda, \mu, \tau \in \mathbb{R}$. Then, $A_{\lambda,\mu,\tau}$ is a G_2 -instanton with respect to the cocalibrated G_2 -structure of pure type given in Lemma 5.5 for any a, b , $A_{\lambda,\mu,\tau}$ preserves the metric, and its first Pontrjagin form is given by*

$$p_1(A_{\lambda,\mu,\tau}) = \frac{\lambda^2 + \mu^2 + \tau^2}{4\pi^2} (ab e^{1234} - a(a+b)e^{1256} - b(a+b)e^{3456}).$$

Proof. A direct calculation shows that the non-zero curvature forms $(\Omega^{A_{\lambda,\mu,\tau}})_j^i$ of the connection $A_{\lambda,\mu,\tau}$ are:

$$\begin{aligned} (\Omega^{A_{\lambda,\mu,\tau}})_2^1 &= \lambda (a e^{12} + b e^{34} - (a+b)e^{56}), \\ (\Omega^{A_{\lambda,\mu,\tau}})_4^3 &= \mu (a e^{12} + b e^{34} - (a+b)e^{56}), \\ (\Omega^{A_{\lambda,\mu,\tau}})_6^5 &= \tau (a e^{12} + b e^{34} - (a+b)e^{56}). \end{aligned}$$

On the other hand, the Lie algebra of G_2 can be identified with the subspace of 2-forms which annihilate $*\Theta$. Since $(a e^{12} + b e^{34} - (a+b)e^{56}) \wedge *\Theta = 0$, the connection $A_{\lambda,\mu,\tau}$ is a G_2 -instanton. \square

The next result gives explicit compact valid solutions on $N(3,1)$ to the heterotic supersymmetry equations with non-zero flux and constant dilaton satisfying the anomaly cancellation condition which also solve the equations of motion (1.4) due to Lemma 5.6 and Theorem 4.1.

Theorem 5.8. *Let $N(3,1)$ be the compact cocalibrated G_2 -nilmanifold of pure type, ∇^+ be the torsion connection and $A_{\lambda,\mu,\tau}$ the G_2 -instanton given in Proposition 5.7. If $(\lambda, \mu, \tau) \neq (0, 0, 0)$ are small enough so that $\lambda^2 + \mu^2 + \tau^2 < 2(a^2 + ab + b^2)$, then*

$$dT = 2\pi^2 \alpha' (p_1(\nabla^+) - p_1(A_{\lambda,\mu,\tau})),$$

where $\alpha' = 4(2(a^2 + ab + b^2) - \lambda^2 - \mu^2 - \tau^2)^{-1} > 0$.

Therefore, the manifold $(N(3,1), \Theta, \nabla^+, A_{\lambda,\mu,\tau})$ is a compact solution to the supersymmetry equations (1.2) obeying the anomaly cancellation (1.1) and solving the equations of motion (1.4).

The Riemannian metric is locally given by

$$g = \frac{1}{a^2} dx_1^2 + dy_1^2 + \frac{1}{b^2} dx_2^2 + dy_2^2 + \frac{1}{a^2 + b^2} dx_3^2 + dy_3^2 + (x_1 dy_1 + x_2 dy_2 + x_3 dy_3 - dz)^2.$$

Proof. The non-zero curvature forms of the torsion connection $\nabla_{a,b}^+$ are given by (5.9), which implies that its first Pontrjagin form is

$$p_1(\nabla^+) = \frac{a^2 + ab + b^2}{2\pi^2} (ab e^{1234} - a(a+b)e^{1256} - b(a+b)e^{3456}).$$

Now the proof follows directly from (5.8) and Proposition 5.7. The final assertion in the theorem follows from Lemma 5.6 and Theorem 4.1. \square

Remark 5.9. *The first Pontrjagin form of the Levi-Civita connection is given by*

$$\begin{aligned} p_1(\nabla^g) &= \frac{1}{32\pi^2} [ab(5a^2 + 4ab + 5b^2)e^{1234} - a(a+b)(6a^2 + 6ab + 5b^2)e^{1256} \\ &\quad - b(a+b)(5a^2 + 6ab + 6b^2)e^{3456}]. \end{aligned}$$

It is easy to see that there is no solution to the heterotic supersymmetry equations for $\nabla = \nabla^g$ using the instantons of Lemma 5.7.

5.2. Explicit solutions in dimension 8. We consider balanced $Spin(7)$ -structures, i.e. $\theta^8 = 0$. From (3.3) we have that the torsion 3-form in this case is given by

$$(5.10) \quad \nabla^+ = \nabla^g + \frac{1}{2}T, \quad H = T = *^8 d\Phi.$$

Starting from a cocalibrated G_2 -structure of pure type Θ on a 7-manifold M^7 it is easy to see that the $Spin(7)$ -structure given by $\Phi = e^1 \wedge \Theta + *^7 \Theta$ on the product $M^8 = M^7 \times S^1$ is balanced, where e^1 denotes the standard 1-form on the circle S^1 . Moreover, following the argument given in [55, Theorem 5.1] we conclude that the natural extension of a G_2 -instanton on M^7 gives rise to a $Spin(7)$ -instanton on M^8 , and if the torsion connection of the G_2 -structure satisfies the Bianchi identity then the corresponding ∇^+ given in (5.10) also satisfies (4.12). We can apply this to the compact 7-dimensional explicit solutions given in the preceding section to get compact solutions in dimension 8:

Corollary 5.10. *Let (M^7, Θ) be a compact cocalibrated G_2 -nilmanifold of pure type with a G_2 -instanton solving the modified Bianchi identity for $\nabla = \nabla^+$ or ∇^g . Then, the $Spin(7)$ -manifold $M^8 = M^7 \times S^1$ with the structure $\Phi = e^1 \wedge \Theta + *^7 \Theta$, the $Spin(7)$ -instanton obtained as an extension of the G_2 -instanton and ∇ being the Levi-Civita connection ∇^g or the torsion connection ∇^+ given in (5.10), provides a compact valid solution to the supersymmetry equations in dimension 8. In particular, starting with the solutions on the generalized Heisenberg compact nilmanifold $N(3, 1)$ given in Theorem 5.8 one obtains solutions to the equations of motion in dimension 8 for $\nabla = \nabla^+$.*

Next we find more compact $Spin(7)$ -solutions to the supersymmetry equations with non-zero flux and constant dilaton on non-trivial extensions of the cocalibrated G_2 -structures of pure type given on the 7-dimensional generalized Heisenberg group. Moreover, we also provide new 8-dimensional solutions to the equations of motion on some of these non-trivial $Spin(7)$ -extensions.

Non-trivial $Spin(7)$ extension of the 7-dimensional generalized Heisenberg group: Let us consider the 8-dimensional extension of (5.7) given by:

$$(5.11) \quad \begin{cases} de^1 = c(e^{24} + e^{25} - e^{34} + e^{35}), \\ de^2 = de^3 = de^4 = de^5 = de^6 = de^7 = 0, \\ de^8 = ae^{23} + be^{45} - (a+b)e^{67}. \end{cases}$$

These equations correspond to the structure equations of an 8-dimensional nilpotent Lie algebra, which we denote by \mathfrak{h}^8 . Let us consider the $Spin(7)$ -structure defined by (2.4). A direct calculation shows that the torsion is given by

$$T = *d\Phi = ce^{124} + ce^{125} - ce^{134} + ce^{135} + ae^{238} + be^{458} - (a+b)e^{678}.$$

The torsion satisfies $T \wedge \Phi = 0$ and

$$(5.12) \quad dT = 2(ab - 2c^2)e^{2345} - 2a(a+b)e^{2367} - 2b(a+b)e^{4567}.$$

There are some special cases for which T is parallel with respect to the torsion connection, more concretely:

Lemma 5.11. $\nabla^+ T = 0$ if and only if $(a-b)c = 0$.

Using again (1.5), (5.1) and (5.2), the non-zero curvature forms $(\Omega^+)_j^i$ of the torsion connection ∇^+ are given by

$$\begin{aligned} (\Omega^+)_3^2 &= -a^2 e^{23} - ab e^{45} + a(a+b) e^{67}, \\ (\Omega^+)_4^2 &= (\Omega^+)_5^3 = (a-b)c e^{18} - c^2 e^{24} - c^2 e^{25} + c^2 e^{34} - c^2 e^{35}, \\ (\Omega^+)_5^2 &= -(\Omega^+)_4^3 = -(a-b)c e^{18} - c^2 e^{24} - c^2 e^{25} + c^2 e^{34} - c^2 e^{35}, \\ (\Omega^+)_5^4 &= -ab e^{23} - b^2 e^{45} + b(a+b) e^{67}, \\ (\Omega^+)_7^6 &= a(a+b) e^{23} + b(a+b) e^{45} - (a+b)^2 e^{67}, \end{aligned}$$

which implies that the first Pontrjagin form $p_1(\nabla^+)$ is given by

$$\begin{aligned} 2\pi^2 p_1(\nabla^+) &= (ab(a^2 + ab + b^2) - 4c^4) e^{2345} - a(a+b)(a^2 + ab + b^2) e^{2367} \\ &\quad - b(a+b)(a^2 + ab + b^2) e^{4567}. \end{aligned}$$

Let us denote by H^8 the simply-connected nilpotent Lie group corresponding to the Lie algebra \mathfrak{h}^8 . From the explicit description of the Lie group $H(3, 1)$ and from (2.4), it follows that the left-invariant metric g on H^8 determined by the $Spin(7)$ -structure Φ can be expressed globally as

$$\begin{aligned} (5.13) \quad g &= (dw + \frac{c}{b}(\frac{x_1}{a} - y_1)dx_2 + c(\frac{x_1}{a} + y_1)dy_2)^2 + (\frac{1}{a}dx_1)^2 + (dy_1)^2 + (\frac{1}{b}dx_2)^2 \\ &\quad + (dy_2)^2 + (\frac{-1}{a+b}dx_3)^2 + (dy_3)^2 + (x_1dy_1 + x_2dy_2 + x_3dy_3 - dz)^2, \end{aligned}$$

where $(w, x_1, y_1, x_2, y_2, x_3, y_3, z)$ denote the (global) coordinates of H^8 , and the w -coordinate of the left translation $L_{(w^0, x_1^0, y_1^0, x_2^0, y_2^0, x_3^0, y_3^0, z^0)}$ by an element $(w^0, x_1^0, y_1^0, x_2^0, y_2^0, x_3^0, y_3^0, z^0)$ of H^8 is given by

$$w \circ L_{(w^0, x_1^0, y_1^0, x_2^0, y_2^0, x_3^0, y_3^0, z^0)} = w - \frac{c}{b} \left(\frac{x_1^0}{a} - y_1^0 \right) x_2 - c \left(\frac{x_1^0}{a} + y_1^0 \right) y_2 + w^0.$$

Notice that the remaining coordinates of $L_{(w^0, x_1^0, y_1^0, x_2^0, y_2^0, x_3^0, y_3^0, z^0)}$ come easily from the matrix description of $H(3, 1)$.

Let Γ be a lattice of maximal rank of H^8 and denote by M^8 the compact nilmanifold $\Gamma \backslash H^8$. Clearly, M^8 can be described as a circle bundle over the compact 7-manifold $N(3, 1)$ (defined by (5.7))

$$S^1 \hookrightarrow M^8 \rightarrow N(3, 1),$$

with connection 1-form $\eta = e^1$ such that the curvature form $d\eta = c(e^{24} + e^{25} - e^{34} + e^{35}) \in \mathfrak{g}_2$.

Alternatively, the manifold M^8 may be viewed as the total space of a circle bundle over the product of a 2-torus by a 5-manifold M^5 , which is also the total space of a principal circle bundle over a 4-torus, i.e. $S^1 \hookrightarrow M^5 \rightarrow \mathbb{T}^4$. In fact, let $\{e^2, \dots, e^5\}$ be a basis for the closed 1-forms on \mathbb{T}^4 . Then, M^5 is the circle bundle over \mathbb{T}^4 with connection 1-form $\eta = e^1$ such that the curvature form is $d\eta = c(e^{24} + e^{25} - e^{34} + e^{35})$. Now, let e^6 and e^7 be a basis for the closed 1-forms on \mathbb{T}^2 . Take the product manifold $M^5 \times \mathbb{T}^2$. Then, M^8 is the circle bundle over $M^5 \times \mathbb{T}^2$

$$S^1 \hookrightarrow M^8 \rightarrow M^5 \times \mathbb{T}^2,$$

with connection form $\nu = e^8$ such that $d\nu = a e^{23} + b e^{45} - (a+b) e^{67}$.

Proposition 5.12. *For each $\lambda, \mu \in \mathbb{R}$, let $A_{\lambda, \mu}$ be the linear connection on M^8 defined by the connection forms:*

$$\begin{aligned} (\sigma^{A_{\lambda, \mu}})_3^2 &= -(\sigma^{A_{\lambda, \mu}})_2^3 = (\sigma^{A_{\lambda, \mu}})_5^4 = -(\sigma^{A_{\lambda, \mu}})_4^5 = \lambda e^8, \\ (\sigma^{A_{\lambda, \mu}})_4^2 &= (\sigma^{A_{\lambda, \mu}})_5^2 = (\sigma^{A_{\lambda, \mu}})_3^5 = (\sigma^{A_{\lambda, \mu}})_3^4 = -\mu e^1, \\ (\sigma^{A_{\lambda, \mu}})_4^3 &= (\sigma^{A_{\lambda, \mu}})_2^4 = (\sigma^{A_{\lambda, \mu}})_5^2 = (\sigma^{A_{\lambda, \mu}})_3^5 = \mu e^1, \\ (\sigma^{A_{\lambda, \mu}})_7^6 &= -(\sigma^{A_{\lambda, \mu}})_6^7 = -2\lambda e^8, \end{aligned}$$

and $(\sigma^{A_{\lambda, \mu}})_j^i = 0$ for the remaining (i, j) . Then, $A_{\lambda, \mu}$ is a $Spin(7)$ -instanton with respect to the $Spin(7)$ -structure (2.4) for any a, b, c , $A_{\lambda, \mu}$ preserves the metric, and its first Pontrjagin form is given by

$$2\pi^2 p_1(A_{\lambda, \mu}) = (3ab\lambda^2 - 4c^2\mu^2)e^{2345} - 3a(a+b)\lambda^2 e^{2367} - 3b(a+b)\lambda^2 e^{4567}.$$

Proof. The non-zero curvature forms $(\Omega^{A_{\lambda, \mu}})_j^i$ of the connection $A_{\lambda, \mu}$ are:

$$\begin{aligned} (\Omega^{A_{\lambda, \mu}})_3^2 &= (\Omega^{A_{\lambda, \mu}})_5^4 = \lambda (a e^{23} + b e^{45} - (a+b)e^{67}), \\ (\Omega^{A_{\lambda, \mu}})_4^2 &= (\Omega^{A_{\lambda, \mu}})_5^2 = -(\Omega^{A_{\lambda, \mu}})_4^3 = (\Omega^{A_{\lambda, \mu}})_5^3 = -\mu c(e^{24} + e^{25} - e^{34} + e^{35}), \\ (\Omega^{A_{\lambda, \mu}})_7^6 &= -(\Omega^{A_{\lambda, \mu}})_3^2 - (\Omega^{A_{\lambda, \mu}})_5^4 = -2\lambda (a e^{23} + b e^{45} - (a+b)e^{67}). \end{aligned}$$

Since the Lie algebra of $Spin(7)$ can be identified with the subspace Λ_{21}^2 of 2-forms β such that $\ast(\beta \wedge \Phi) = -\beta$, and since $a e^{23} + b e^{45} - (a+b)e^{67}$, $e^{24} + e^{25} - e^{34} + e^{35} \in \Lambda_{21}^2$ the connection $A_{\lambda, \mu}$ is a $Spin(7)$ -instanton for any λ, μ . \square

Theorem 5.13. *Let (M^8, Φ) be the compact balanced $Spin(7)$ -nilmanifold, ∇^+ be the torsion connection and $A_{\lambda, \mu}$ the $Spin(7)$ -instanton given in Proposition 5.7. If $(\lambda, \mu) \neq (0, 0)$ satisfy $3\lambda^2 < a^2 + ab + b^2$ and $3\lambda^2 - 2\mu^2 = a^2 + ab + b^2 - 2c^2$, then*

$$dT = 2\pi^2 \alpha' (p_1(\nabla^+) - p_1(A_{\lambda, \mu})),$$

where $\alpha' = 2(a^2 + ab + b^2 - 3\lambda^2)^{-1} > 0$.

Therefore, the manifold $(M^8, \Phi, \nabla^+, A_{\lambda, \mu})$ is a compact solution to the supersymmetry equations (1.2) satisfying the anomaly cancellation (1.1).

If $a = b$ then the manifold $(M^8, \Phi, \nabla^+, A_{\lambda, \mu})$ with $(\lambda, \mu) \neq (0, 0)$ satisfying

$$\lambda^2 < a^2, \quad 3\lambda^2 - 2\mu^2 = 3a^2 - 2c^2$$

is a compact supersymmetric solution to the heterotic equations of motion (1.4) in dimension 8.

The Riemannian metric is locally given by (5.13) with $a = b$.

Proof. The proof follows directly from (5.12), the expression of the first Pontrjagin form of ∇^+ and Proposition 5.12. The final assertion in the theorem follows from Lemma 5.11 and Theorem 4.1. \square

Remark 5.14. *There are also solutions on M^8 to the supersymmetry equations taking ∇ as the Levi-Civita connection ∇^g . For example, if $a = b = c = 1$ in (5.11) then a direct computation shows that the first Pontrjagin form of ∇^g is given by*

$$16\pi^2 p_1(\nabla^g) = -5 e^{2345} - 19 e^{2367} - 19 e^{4567}.$$

From Proposition 5.12 for $a = b = c = 1$ we get

$$2\pi^2 p_1(A_{\lambda, \mu}) = (3\lambda^2 - 4\mu^2)e^{2345} - 6\lambda^2 e^{2367} - 6\lambda^2 e^{4567}.$$

Since $dT = -2e^{2345} - 4e^{2367} - 4e^{4567}$, if we choose the $Spin(7)$ -instanton $A_{\lambda,\mu}$ such that $48\lambda^2 < 19$ and $64\mu^2 = 96\lambda^2 - 9$, then

$$dT = 2\pi^2 \alpha' (p_1(\nabla^g) - p_1(A_{\lambda,\mu})),$$

where $\alpha' = 32(19 - 48\lambda^2)^{-1} > 0$.

6. GEOMETRIC MODELS

The structure of the examples that we have presented as well as constructions proposed in [39] suggest a more general construction. In this section we describe how to derive compact solutions to the system of gravitino and dilatino Killing spinor equations (the first two equations in (1.2)) in dimensions seven and eight starting with a solution of these equations in low dimensions. The construction is a \mathbb{T}^k -bundle with curvature of instanton type over a compact low dimensional solution. The benefit of this construction is the obtained reduction of the dilaton variables, i.e. the non-constant dilaton depend on reduced number of variables.

First we recall the dimensions 5 and 6.

D=5 The gravitino and dilatino Killing spinor equations in dimension 5 define a reduction of the structure group $SO(5)$ to $SU(2)$ which is described in terms of differential forms by Conti and Salamon in [15] as follows: an $SU(2)$ -structure on a 5-dimensional manifold M is the quadruplet $(\eta, \omega_1, \omega_2, \omega_3)$, where η is a 1-form with a dual vector field ξ and $\omega_i, i = 1, 2, 3$, are 2-forms on M satisfying

$$\omega_i \wedge \omega_j = \delta_{ij} v, \quad v \wedge \eta \neq 0,$$

for some 4-form v , and $X \lrcorner \omega_1 = Y \lrcorner \omega_2 \Rightarrow \omega_3(X, Y) \geq 0$, where \lrcorner denotes the interior multiplication.

Let $\mathbb{H} = \text{Ker}\eta$. The 2-forms $\omega_i, i = 1, 2, 3$, can be chosen to form a basis of the \mathbb{H} -self-dual 2-forms [15], i.e. $*_{\mathbb{H}}\omega_i = \omega_i$, where $*_{\mathbb{H}}$ denotes the Hodge operator on the 4-dimensional distribution \mathbb{H} .

Based on analysis done in [31, 33] it is shown in [25] that

The first two equations in (1.2) admit a solution in dimension five exactly when there exists a five dimensional manifold M endowed with an $SU(2)$ -structure $(\eta, \omega_1, \omega_2, \omega_3)$ satisfying the structure equations:

$$(6.1) \quad d\omega_i = 2df \wedge \omega_i, \quad *_{\mathbb{H}}d\eta = -d\eta$$

where f is a smooth function which does not depend on ξ , $df(\xi) = 0$.

The flux H is given by $H = T = \eta \wedge d\eta - 2*_4df$ and the dilaton ϕ is equal to $\phi = f + \text{cons.}$

Therefore, if the dilaton is constant then the structure equations are

$$(6.2) \quad d\omega_i = 0, \quad *_{\mathbb{H}}d\eta = -d\eta$$

and the flux H is given by $H = T = \eta \wedge d\eta$.

If the $SU(2)$ structure is regular, i.e. the orbit space $N = M/\xi$ is a smooth manifold then M is an S^1 -bundle over a Calabi-Yau 4-fold (flat torus or $K3$ surface) with \mathbb{H} -anti-self-dual curvature form equal to $d\eta$. The metric has the form

$$g = e^{2f} g_{cy} + \eta \otimes \eta,$$

where g_{cy} is the metric on the Calabi-Yau base and f is a smooth function on the base.

We do not know whether there exist non-regular $SU(2)$ -structures (the integral curves of ξ are not closed) on a compact 5-manifold.

D=6 The gravitino and dilatino Killing spinor equations in dimension 6 define a reduction of the structure group $SO(6)$ to $SU(3)$ which is described in terms of forms by Chiossi and Salamon in [14] as follows: an $SU(3)$ -structure is $(F, \Psi = \Psi^+ + \sqrt{-1}\Psi^-)$ with Kähler form F and complex volume form Ψ which satisfy the compatibility relations

$$F \wedge \Psi^\pm = 0, \quad \Psi^+ \wedge \Psi^- = \frac{2}{3} F \wedge F \wedge F.$$

The necessary and sufficient condition for the existence of solutions to the first two equations in (1.2) in dimension 6 were derived by Strominger [73], namely the manifold should be complex conformally balanced manifold with non-vanishing holomorphic volume form Ψ satisfying additional condition. In terms of the five torsion classes described in [14], the Strominger condition is interpreted in [13] as follows (see [55] for a slightly different expression):

$$2F \lrcorner dF + \Psi^+ \lrcorner d\Psi^+ = 0.$$

If the dilaton is constant then the Strominger conditions read

$$(6.3) \quad dF \wedge F = d\Psi^+ = d\Psi^- = 0.$$

Examples of the latter via evolution equations were presented recently in [26].

A very promising geometric model in dimension 6 was proposed in [42] to be a certain \mathbb{T}^2 -bundle over a Calabi-Yau surface (see [42] and references therein). Starting with an $SU(2)$ -structure $(\eta, \omega_1, \omega_2, \omega_3)$ on (a regular) 5-manifold M satisfying (6.2) one considers an S^1 -bundle over M with curvature an exact \mathbb{H} -anti-self-dual 2 form, $d\alpha$ and the $SU(3)$ -structure $(F, \Psi = \Psi^+ + \sqrt{-1}\Psi^-)$ defined by

$$(6.4) \quad F = \omega_1 + \eta \wedge \alpha; \quad \Psi^+ = \omega_2 \wedge \eta - \omega_3 \wedge \alpha; \quad \Psi^- = \omega_3 \wedge \eta + \omega_2 \wedge \alpha.$$

Using (6.2) and the fact that $d\alpha$ is \mathbb{H} -anti-self-dual it can be shown following Goldstein and Prokushkin [42] that (6.3) hold as a consequence of (6.4). When M is regular, i.e. it is an S^1 -bundle over a Calabi-Yau 4-manifold one gets a holomorphic \mathbb{T}^2 -bundle over a Calabi-Yau surface with anti-self-dual integral curvature 2-forms which solves the first two equations in (1.2) with constant dilaton [42]. It also follows from considerations in [42] that if the starting $SU(2)$ -structure solves the equations with non-constant dilaton, i.e. (6.1) hold, then the $SU(3)$ -structure on the circle bundle also solves the first two Killing spinor equations with non-constant dilaton in dimension 6. The \mathbb{T}^2 -bundle over a K3 surface construction was used in [66, 34, 35, 4] to produce the first compact examples in dimension 6 solving the heterotic supersymmetry equations (1.2) with non-zero flux and non-constant dilaton together with the anomaly cancellation (1.1) with respect to the Chern connection.

6.1. \mathbb{T}^3 -bundles over a Calabi-Yau surface. The structure of the example Γ/H^7 , where H^7 is the nilpotent Lie group defined by (5.6), is generalized in the following

Theorem 6.1. *Let Γ_i , $1 \leq i \leq 3$, be three closed anti-self-dual 2-forms on a Calabi-Yau surface M^4 , which represent integral cohomology classes. Denote by ω_1 and by $\omega_2 + \sqrt{-1}\omega_3$ the (closed) Kähler form and the holomorphic volume form on M^4 , respectively. Then, there is a compact 7-dimensional manifold $M^{1,1,1}$, which is the total space of a \mathbb{T}^3 -bundle over M^4 , and it has a G_2 -structure*

$$(6.5) \quad \Theta = \omega_1 \wedge \eta_1 + \omega_2 \wedge \eta_2 - \omega_3 \wedge \eta_3 + \eta_1 \wedge \eta_2 \wedge \eta_3,$$

solving the first two Killing spinor equations in (1.2) with constant dilaton in dimension 7, where η_i , $1 \leq i \leq 3$, is a 1-form on $M^{1,1,1}$ such that $d\eta_i = \Gamma_i$, $1 \leq i \leq 3$.

For any smooth function f on M^4 , the G_2 -structure on $M^{1,1,1}$ given by

$$(6.6) \quad \Theta_f = e^{2f} \left[\omega_1 \wedge \eta_1 + \omega_2 \wedge \eta_2 - \omega_3 \wedge \eta_3 \right] + \eta_1 \wedge \eta_2 \wedge \eta_3$$

solves the first two Killing spinor equations in (1.2) with non-constant dilaton $\phi = 2f$ (in dimension 7). The metric has the form

$$g_f = e^{2f} g_{cy} + \eta_1 \otimes \eta_1 + \eta_2 \otimes \eta_2 + \eta_3 \otimes \eta_3.$$

Proof. Since $[\Gamma_i]$, $1 \leq i \leq 3$, define integral cohomology classes on M^4 , the well-known result of Kobayashi [63] implies that there exists a circle bundle $S^1 \hookrightarrow M^5 \rightarrow M^4$, with connection 1-form η_1 on M^5 whose curvature form is $d\eta_1 = \Gamma_1$. (From now on, we write with the same symbol the 2-form Γ_i on M^4 and its lifting to M^5 via the projection $M^5 \rightarrow M^4$.) Because Γ_i ($i = 2, 3$) defines an integral cohomology class on M^5 , there exists a principal circle bundle $S^1 \hookrightarrow M^6 \rightarrow M^5$ corresponding to $[\Gamma_2]$ and a connection 1-form η_2 on M^6 such that Γ_2 is the curvature form of η_2 . Using again the result of Kobayashi, there exists a principal circle bundle $S^1 \hookrightarrow M^{1,1,1} \rightarrow M^6$ with connection 1-form η_3 such that $d\eta_3 = \Gamma_3$ since Γ_3 defines an integral cohomology class on M^6 . The actions of S^1 on each one of the manifolds M^5 , M^6 and $M^{1,1,1}$ define an action of the 3-torus on $M^{1,1,1}$ doing $M^{1,1,1}$ a \mathbb{T}^3 -bundle over M^4 .

We have to show that (6.6) implies (3.1). We calculate using (6.6) that

$$\begin{aligned} * \Theta_f &= e^{2f} \left[\omega_1 \wedge \eta_2 \wedge \eta_3 + \omega_2 \wedge \eta_3 \wedge \eta_1 - \omega_3 \wedge \eta_1 \wedge \eta_2 + \frac{e^{2f}}{2} \omega_1 \wedge \omega_1 \right]; \\ d\Theta_f &= 2df \wedge \Theta_f - 2df \wedge \eta_1 \wedge \eta_2 \wedge \eta_3 + d\eta_1 \wedge \eta_2 \wedge \eta_3 - \eta_1 \wedge d\eta_2 \wedge \eta_3 + \eta_1 \wedge \eta_2 \wedge d\eta_3. \end{aligned}$$

From the last two equalities we derive

$$d * \Theta_f = 2df \wedge * \Theta_f, \quad d\Theta_f \wedge \Theta_f = 0,$$

where we have used the equalities $d\omega_i = 0$, $\omega_i \wedge d\eta_j = 0$ ($i, j = 1, 2, 3$) since $d\eta_j = \Gamma_j$ are anti-self-dual 2-forms on M^4 , and $df \wedge \omega_i \wedge \omega_i = 0$ as a 5-form on a four-dimensional Calabi-Yau manifold. \square

Notice that in the previous theorem, if we start with a 4-torus, we have essentially 3 possibilities:

- 1) Only one of the three 2-forms Γ_i is independent. In this case, we get (5.6) with $c_1 = c_2 = 0$. The resulting compact nilmanifold satisfies the equations of motion.
- 2) Two of the three 2-forms Γ_i are independent. Then, we get (5.6) with $(c_1, c_2) \neq (0, 0)$. The resulting compact nilmanifold satisfies the supersymmetric equations but not the equations of motion.
- 3) The three 2-forms Γ_i are independent. In this case, essentially we get the quaternionic Heisenberg nilmanifold. We did not get any instanton satisfying the supersymmetric equations, but at least the first 2 Killing spinor equations are satisfied as the previous theorem asserts.

Remark 6.2. Clearly the conclusions of the above theorem are valid also if we start with a compact non-regular M^5 with an $SU(2)$ -structure satisfying (6.1). In this case, we take two anti-self-dual 2-forms Γ_2 and Γ_3 on M^5 , and we consider $M^{1,1,1}$ the principal circle bundle over M^6 corresponding to $[\Gamma_3]$, which in turn is a principal circle bundle over M^5 corresponding to $[\Gamma_2]$. Now, $M^{1,1,1}$ is a \mathbb{T}^2 -bundle over M^5 , and the G_2 -structure defined by (6.5) solves the first two Killing spinor equations.

Suppose that M has a G_2 -structure defined by a 3-form Θ . Let us recall that a 3-dimensional submanifold X of M is called *associative*, with respect to Θ , if the restriction to X of Θ coincides with the Riemannian volume form on X induced by the G_2 -metric determined by Θ . (Here we

do not assume that Θ is closed.) We don't know whether $M^{1,1,1}$ has a G_2 -structure, defined by a 3-form Θ , such the fibers are associative with respect to Θ .

In [42], it is proved that certain non-trivial \mathbb{T}^2 -bundles M over a Calabi-Yau surface have a natural complex structure not admitting Kähler metric. The key of their proof is that the fibers are complex submanifolds of M . For the previous construction of \mathbb{T}^3 -bundles $M^{1,1,1}$ over a Calabi-Yau surface we have

Lemma 6.3. *In the conditions of Theorem 6.1, suppose that one of the integral cohomology classes represented by Γ_i is non-trivial on M^4 . Let Θ be a 3-form defining a G_2 -structure on $M^{1,1,1}$, such that there is a fibre \mathbb{T}^3 which is associative with respect to Θ . Then Θ is not closed. Therefore, the G_2 -structure on $M^{1,1,1}$ is non-parallel.*

Proof. We know that one of the circle bundles considered in the construction of $M^{1,1,1}$ is non-trivial since one of the forms Γ_i defines a non-zero cohomology class on M^4 . Then, one can check that the homology class in $H_3(M^{1,1,1}, \mathbb{R})$ defined by the fibres is trivial. Therefore, if some \mathbb{T}^3 fibre is associative, then Θ cannot be closed. Otherwise, there is a well-defined cohomology class $[\Theta]$ in $H^3(M^{1,1,1}, \mathbb{R})$ and it evaluates on $[\mathbb{T}^3]$ to give a positive number, i.e. the volume of \mathbb{T}^3 , which is a contradiction with the triviality of $[\mathbb{T}^3]$. \square

6.2. \mathbb{S}^1 -bundles over a manifold with a balanced $SU(3)$ -structure. Next result generalizes the structure of the example $N(3, 1)$ defined by (5.7).

Theorem 6.4. *Let M^6 be a compact complex 6-manifold solving the first two Killing spinor equations with constant dilaton in dimension 6, i.e. there exists an $SU(3)$ -structure (F, Ψ^+, Ψ^-) satisfying (6.3). Let Γ be a closed integral 2-form which is an $SU(3)$ -instanton, $\Gamma \in su(3)$, i.e. $\Gamma_{\alpha\beta} = \Gamma_{\bar{\alpha}\bar{\beta}} = \Gamma_{\alpha\bar{\beta}} F^{\alpha\bar{\beta}} = 0$ in local holomorphic coordinates. Then, there is a principal circle bundle $\pi : M^7 \rightarrow M^6$ with a connection 1-form η such that $\Gamma = d\eta$ is the curvature of η and the G_2 -structure*

$$(6.7) \quad \Theta = F \wedge \eta + \Psi^+, \quad *\Theta = \frac{1}{2}F \wedge F + \Psi^- \wedge \eta$$

solves the first two Killing spinor equations in (1.2) with constant dilaton.

Proof. The exterior derivative of (6.7), with the help of (6.3), yields

$$d*\Theta = \frac{1}{2}d(F \wedge F) + d\Psi^- \wedge \eta - \Psi^- \wedge d\eta = 0,$$

and

$$d\Theta \wedge \Theta = F^2 \wedge d\eta \wedge \eta + (F \wedge \eta + \Psi^+) \wedge d\Psi^+ - dF \wedge \Psi^+ \wedge \eta = 0,$$

because of the algebraic facts $\Psi^- \wedge d\eta = 0$, $F^2 \wedge d\eta = 0$ since $d\eta \in su(3)$, and because $dF \wedge \Psi^+ = 0$ on a complex manifold (see e.g. [14]). Hence, (3.1) holds with $\theta^7 = 0$.

The existence of a principal circle bundle in the conditions above follows again from [63]. \square

6.3. \mathbb{S}^1 -bundles over a cocalibrated G_2 -manifold of pure type. We describe a more general situation inspired by the structure of the example Γ/H^8 defined by (5.11) and by considerations in [39].

Theorem 6.5. *Let M^7 be a compact G_2 -manifold solving the first two equations of (1.2) with constant dilaton in dimension 7, i.e. there exists a G_2 -structure Θ satisfying $d*\Theta = d\Theta \wedge \Theta = 0$. Let f be a smooth function on M^7 , and let Γ_4 be a closed integral 2-form on M^7 which is a G_2 -instanton, $\Gamma_4 \in g_2$, i.e. it satisfies (3.4). Then, we have*

- i) There is a principal circle bundle $\pi : M^8 \longrightarrow M^7$ corresponding to $[\Gamma_4]$ and a connection 1-form η_4 on M^8 whose curvature form is Γ_4 , such that the $Spin(7)$ -structure

$$(6.8) \quad \Phi_f = e^{3f} \Theta \wedge \eta_4 + e^{4f} *_7 \Theta,$$

solves the first two Killing spinor equations in (1.2) with non-constant dilaton $\phi = 2f$ in dimension 8, where $*_7$ denotes the Hodge star operator on M^7 . The $Spin(7)$ -metric has the form

$$g_f = e^{2f} g_7 + \eta_4 \otimes \eta_4.$$

- ii) If M^7 is a circle bundle over a compact 6-manifold (M^6, F, Ψ^+, Ψ^-) as in Theorem 6.4, f is a smooth function on M^6 and the form Γ_4 of the part i) is such that $\Gamma_4 \in su(3)$, then there is a compact 8-dimensional manifold $M^{1,1}$ with a free structure preserving \mathbb{T}^2 -action and a fibration $\pi : M^{1,1}/\mathbb{T}^2 \cong M^6$ with the $Spin(7)$ -structure

$$(6.9) \quad \Phi_f = e^{3f} \left[F \wedge \eta + \Psi^+ \right] \wedge \eta_4 + e^{4f} \left[\frac{1}{2} F \wedge F + \Psi_- \wedge \eta \right],$$

solving the first two Killing spinor equations in (1.2) with non-constant dilaton $\phi = 2f$ in dimension 8, where η is the connection 1-form on the circle bundle over M^6 corresponding to $[\Gamma]$. The metric has the form

$$g_f = e^{2f} (g_6 + \eta \otimes \eta) + \eta_4 \otimes \eta_4.$$

Proof. To prove i) first we show that the Lee form $7\theta_f^8 = - * (*d\Phi \wedge \Phi)$ is an exact 1-form. The exterior derivative of (6.8) yields

$$d\Phi_f = 3e^{3f} df \wedge \Theta \wedge \eta_4 + e^{3f} d\Theta \wedge \eta_4 + 4e^{4f} df \wedge *_7 \Theta - e^{3f} \Theta \wedge d\eta_4.$$

The latter leads to

$$*d\Phi_f = -3e^{4f} *_7 (df \wedge \Theta) + e^{4f} *_7 d\Theta + 4e^{3f} *_7 (df \wedge *_7 \Theta) \wedge \eta_4 + 2e^{4f} d\eta_4 \wedge \eta_4,$$

where we have used the well-known fact that $*_7(\Theta \wedge d\eta_4) = -2d\eta_4$ since $d\eta \in g_2$.

Consequently, we claim

$$(6.10) \quad \begin{aligned} *d\Phi_f \wedge \Phi_f &= -3e^{7f} *_7 (df \wedge \Theta) \wedge \Theta \wedge \eta_4 + 4e^{7f} *_7 (df \wedge *_7 \Theta) \wedge *_7 \Theta \wedge \eta_4 + \\ &3e^{8f} *_7 (df \wedge \Theta) \wedge *_7 \Theta + e^{7f} *_7 d\Theta \wedge \Theta \wedge \eta_4 + e^{8f} *_7 d\Theta \wedge *_7 \Theta + 2e^{8f} *_7 \Theta \wedge d\eta_4 \wedge \eta_4 \\ &= 24e^{7f} *_7 df \wedge \eta_4. \end{aligned}$$

Indeed, the second line in (6.10) gives no contribution since the first term vanishes because it is a general algebraic identity valid on any G_2 -manifold, the second term is zero due to the second equality in (2.3), the third term is zero because of the following chain of equalities

$$*_7 d\Theta \wedge *_7 \Theta = g(*_7 d\Theta, \Theta) vol_{.7} = g(d\Theta, *_7 \Theta) vol_{.7} = d\Theta \wedge \Theta = 0$$

and the fourth term is zero because $*_7 \Theta \wedge d\eta_4 = 0$ since $d\eta_4 \in g_2$.

The terms in the first line are subject to the following well-known algebraic G_2 -identities:

$$*_7 (df \wedge \Theta) \wedge \Theta = -4 *_7 df, \quad *_7 (df \wedge *_7 \Theta) \wedge *_7 \Theta = 3 *_7 df.$$

Hence, we obtain from (2.5) and (6.10) that $\theta_f^8 = \frac{24}{7} df$, i.e. the Lee form is an exact form which completes the proof of i). The existence of the principal circle bundle $S^1 \hookrightarrow M^8 \rightarrow M^7$ in the conditions above follows from the result of Kobayashi [63].

Now, let us suppose that Γ and Γ_4 are closed integral 2-forms on M^6 , such that Γ and $\Gamma_4 \in su(3)$. Let M^7 be the principal circle bundle over M^6 corresponding to $[\Gamma]$ as in Theorem 6.4. Since $[\Gamma_4]$ defines an integral cohomology class on M^7 , Kobayashi theorem implies that there exists a principal circle bundle $S^1 \hookrightarrow M^{1,1} \rightarrow M^7$ corresponding to $[\Gamma_4]$ and a connection 1-form η_4 whose

curvature is Γ_4 . The actions of S^1 on each one of the manifolds M^7 and $M^{1,1}$ define an action of the 2-torus on $M^{1,1}$ and $M^{1,1}$ can be considered a \mathbb{T}^2 -bundle over M^6 . Substituting (6.7) in (6.8), and using Theorem 6.4 and the part i), we conclude ii). \square

Remark 6.6. *In Theorem 6.5, if M^7 is a \mathbb{T}^2 -bundle over a compact non-regular M^5 as in Remark 6.2, such that M^5 has an $SU(2)$ -structure $(\eta_1, \omega_1, \omega_2, \omega_3)$ satisfying (6.1), and there exist three closed anti-self-dual 2-forms Γ_2, Γ_3 and Γ_4 on M^5 representing integral cohomology classes, then the S^1 -bundle over M^7 , constructed in Theorem 6.5, is a \mathbb{T}^3 -bundle over M^5 with $Spin(7)$ -structure*

$$\Phi_f = e^{3f} \Theta_f \wedge \eta_4 + * \Theta_f,$$

solving the first two equations in (1.2) with non-constant dilaton, where the G_2 -form Θ_f on M^7 is given by (6.6). The $Spin(7)$ -metric is

$$g_f = e^{2f} (g_5 + \eta_2 \otimes \eta_2 + \eta_3 \otimes \eta_3) + \eta_4 \otimes \eta_4,$$

where f and g_5 denote a smooth function and the metric on M^5 , respectively.

Moreover, we must notice that in Theorem 6.5, if M^7 is a \mathbb{T}^3 -bundle over a Calabi-Yau surface as in Theorem 6.1, and the form Γ_4 considered in Theorem 6.5 is such that $\Gamma_4 \in su(2)$, i.e. anti-self-dual 2-form on M^4 , then the S^1 -bundle constructed in Theorem 6.5 is a \mathbb{T}^4 -bundle over the Calabi-Yau M^4 with a $Spin(7)$ -structure given by

$$\Phi = \Theta_f \wedge \eta_4 + * \Theta_f,$$

which solves the first two equations in (1.2) with non-constant dilaton, where the G_2 -form Θ_f is given by (6.6). The metric is given by

$$g_f = e^{2f} g_{cy} + \eta_1 \otimes \eta_1 + \eta_2 \otimes \eta_2 + \eta_3 \otimes \eta_3 + \eta_4 \otimes \eta_4.$$

Suppose that one of the integral cohomology classes represented by Γ_i is non-trivial on M^4 . Let Φ be a 4-form defining a $Spin(7)$ -structure on the total space of the S^1 -bundle over M^7 , such that there is a fibre \mathbb{T}^4 which is associative with respect to Φ . Then we conclude that Φ is not closed similarly as in the proof of Lemma 6.3. Therefore, the $Spin(7)$ -structure on the total space of the S^1 -bundle over M^7 is non-parallel.

Acknowledgments. We would like to thank Alexei Kovalev for useful discussions about associative submanifolds. This work has been partially supported through grant MEC (Spain) MTM 2005-08757-C04-02 and under project "Ingenio Mathematica (i-MATH)" No. CSD2006-00032 (Consolider Ingenio 2010). The second author is partially supported by the Contract 154/2008 with the University of Sofia 'St.Kl.Ohridski'. S.I. is a Senior Associate to the Abdus Salam ICTP, Trieste and the final stage of the research was done during his stay in the ICTP, Fall 2008.

REFERENCES

- [1] K. Becker, M. Becker, K. Dasgupta, P.S. Green, *Compactifications of Heterotic Theory on Non-Kähler Complex Manifolds: I*, JHEP 0304 (2003) 007.
- [2] K. Becker, M. Becker, K. Dasgupta, P.S. Green, E. Sharpe, *Compactifications of Heterotic Strings on Non-Kähler Complex Manifolds: II*, Nucl. Phys. **B 678** (2004), 19–100.
- [3] K. Becker, M. Becker, K. Dasgupta, S. Prokushkin, *Properties of heterotic vacua from superpotentials*, Nucl. Phys. **B 666** (2003), 144–174.
- [4] K. Becker, M. Becker, J-X. Fu, L-S. Tseng, S-T. Yau, *Anomaly Cancellation and Smooth Non-Kähler Solutions in Heterotic String Theory*, Nucl.Phys. **B 751** (2006) 108–128.
- [5] E. A. Bergshoeff, M. de Roo, *The quartic effective action of the heterotic string and supersymmetry*, Nucl. Phys. **B 328** (1989), 439–468.

- [6] E. Bonan, *Sur des variétés riemanniennes à groupe d'holonomie G_2 ou $Spin(7)$* , C. R. Acad. Sci. Paris **262** (1966), 127–129.
- [7] R. Bryant, *Metrics with exceptional holonomy*, Ann. Math. **126** (1987), 525–576.
- [8] R. Bryant, *Some remarks on G_2 -structures*, Proceeding of Gokova Geometry-Topology Conference 2005 edited by S. Akbulut, T. Onder, and R.J. Stern (2006), International Press, 75–109.
- [9] R. Bryant, S. Salamon, *On the construction of some complete metrics with exceptional holonomy*, Duke Math. J. **58** (1989), 829–850.
- [10] F. Cabrera, *On Riemannian manifolds with $Spin(7)$ -structure*, Publ. Math. Debrecen **46** (3-4) (1995), 271–283.
- [11] F. Cabrera, *On Riemannian manifolds with G_2 -structure*, Bolletino UMI A (7) **10** (1996), 98–112.
- [12] G. L. Cardoso, G. Curio, G. Dall'Agata, D. Lüst, *BPS Action and Superpotential for Heterotic String Compactifications with Fluxes*, JHEP 0310 (2003) 004.
- [13] G.L. Cardoso, G. Curio, G. Dall'Agata, D. Lüst, P. Manousselis, G. Zoupanos, *Non-Kähler string backgrounds and their five torsion classes*, Nuclear Phys. **B 652** (2003), 5–34.
- [14] S. Chiossi, S. Salamon, *The intrinsic torsion of $SU(3)$ and G_2 -structures*, Differential Geometry, Valencia 2001, World Sci. Publishing, 2002, pp. 115–133.
- [15] D. Conti, S. Salamon, *Generalized Killing spinors in dimension 5*, Trans. Amer. Math. Soc. **359** (2007), 5319–5343.
- [16] E. Corrigan, C. Devchand, D.B. Fairlie, J. Nuyts, *First-order equations for gauge fields in spaces of dimension greater than four*, Nuclear Phys. **B 214** (1983), no. 3, 452–464.
- [17] K. Dasgupta, H. Firouzjahi, R. Gwyn, *On the warped heterotic axion*, JHEP 0806, 056 (2008).
- [18] K. Dasgupta, G. Rajesh, S. Sethi, *M theory, orientifolds and G-flux*, JHEP **0211**, 006 (2002).
- [19] B. de Wit, D.J. Smit, N.D. Hari Dass, *Residual Supersymmetry Of Compactified $d=10$ Supergravity*, Nucl. Phys. **B 283** (1987), 165–191.
- [20] S.K. Donaldson, R.P. Thomas, *Gauge theory in higher dimensions*, The geometric universe (Oxford, 1996), 31–47, Oxford Univ. Press, Oxford, 1998.
- [21] D.B. Fairlie, J. Nuyts, *Spherically symmetric solutions of gauge theories in eight dimensions*, J. Phys. **A17** (1984) 2867–2872.
- [22] M. Fernández, *A classification of Riemannian manifolds with structure group $Spin(7)$* , Ann. Mat. Pura Appl. (4) **143** (1986), 101–122.
- [23] M. Fernández, A. Gray, *Riemannian manifolds with structure group G_2* , Ann. Mat. Pura Appl. (4) **132** (1982), 19–45.
- [24] M. Fernández, S. Ivanov, L. Ugarte, R. Villacampa, *Non-Kähler heterotic-string compactifications with non-zero fluxes and constant dilaton*, Commun. Math. Phys. **288** (2009), 677–697.
- [25] M. Fernández, S. Ivanov, L. Ugarte, R. Villacampa, *Compact supersymmetric solutions of the heterotic equations of motion in dimension 5*, Nuclear Phys. **B 820** (2009), 483–502.
- [26] M. Fernández, A. Tomassini, L. Ugarte, R. Villacampa, *Balanced hermitian metrics from $SU(2)$ -structures*, J. Math. Phys. **50** (2009), 033507.
- [27] M. Fernández, L. Ugarte, *Dolbeault cohomology for G_2 -manifolds*, Geom. Dedicata **70** (1998), no. 1, 57–86.
- [28] D.Z. Freedman, G.W. Gibbons, P.C. West, *Ten Into Four Won't Go*, Phys. Lett. **B 124** (1983), 491–492.
- [29] Th. Friedrich, *On types of non-integrable geometries*, Rend. Circ. Mat. Palermo (2) Suppl. No. **71** (2003), 99–113.
- [30] Th. Friedrich, *G_2 -Manifolds With Parallel Characteristic Torsion*, Diff. Geom. Appl. **25** (2007), no. 6, 632–648.
- [31] Th. Friedrich, S. Ivanov, *Parallel spinors and connections with skew-symmetric torsion in string theory*, Asian J. Math. **6** (2002), 303–336.
- [32] Th. Friedrich, S. Ivanov, *Killing spinor equations in dimension 7 and geometry of integrable G_2 -manifolds*, J. Geom. Phys. **48** (2003), 1–11.
- [33] Th. Friedrich, S. Ivanov, *Almost contact manifolds, connections with torsion, and parallel spinors*, J. Reine Angew. Math. **559** (2003), 217–236.
- [34] J-X. Fu, S-T. Yau, *Existence of Supersymmetric Hermitian Metrics with Torsion on Non-Kähler Manifolds*, arXiv:hep-th/0509028.
- [35] J-X. Fu, S-T. Yau, *The theory of superstring with flux on non-Kähler manifolds and the complex Monge-Ampère equation*, J. Diff. Geom. **78** (2008), no. 3, 369–428.
- [36] S. Fubini, H. Nicolai, *The octonionic instanton*, Phys. Lett. **B 155** (1985), 369–372.
- [37] J. Gauntlett, N. Kim, D. Martelli, D. Waldram, *Fivebranes wrapped on SLAG three-cycles and related geometry*, JHEP 0111 (2001) 018.
- [38] J.P. Gauntlett, D. Martelli, S. Pakis, D. Waldram, *G-Structures and Wrapped NS5-Branes*, Commun. Math. Phys. **247** (2004), 421–445.

- [39] J. Gauntlett, D. Martelli, D. Waldram, *Superstrings with Intrinsic torsion*, Phys. Rev. **D69** (2004), 086002.
- [40] G.W. Gibbons, D.N. Page, C.N. Pope, *Einstein metrics on S^3 , \mathbb{R}^3 and \mathbb{R}^4 bundles*, Commun. Math. Phys. **127** (1990), 529–553.
- [41] J. Gillard, G. Papadopoulos, D. Tsimpis, *Anomaly, Fluxes and (2,0) Heterotic-String Compactifications*, JHEP **0306** (2003), 035.
- [42] E. Goldstein, S. Prokushkin, *Geometric Model for Complex Non-Kähler Manifolds with $SU(3)$ Structure*, Commun. Math. Phys. **251** (2004), 65–78.
- [43] A. Gray, *Vector cross product on manifolds*, Trans. Am. Math. Soc. **141** (1969), 463–504, Correction **148** (1970), 625.
- [44] U. Gran, P. Lohrmann, G. Papadopoulos, *The spinorial geometry of supersymmetric heterotic string backgrounds*, JHEP **0602** (2006), 063.
- [45] U. Gran, G. Papadopoulos, D. Roest *Supersymmetric heterotic string backgrounds*, Phys. Lett. B **656** (2007), 119–126.
- [46] U. Gran, G. Papadopoulos, D. Roest, P. Sloane *Geometry of all supersymmetric type I backgrounds*, JHEP **0708** (2007), 074.
- [47] M. Günaydin, H. Nicolai, *Seven-dimensional octonionic Yang-Mills instanton and its extension to an heterotic string soliton*, Phys. Lett. B **351** (1995) 169–172.
- [48] J. Gutowski, S. Ivanov, G. Papadopoulos, *Deformations of generalized calibrations and compact non-Kähler manifolds with vanishing first Chern class*, Asian J. Math. **7** (2003), 39–80.
- [49] R. Harvey, H.B. Lawson, *Calibrated geometries*, Acta Math. **148** (1982), 47–157.
- [50] J.A. Harvey, A. Strominger, *Octonionic superstring solitons*, Phys. Review Lett. **66** 5 (1991), 549–552.
- [51] P.S. Howe, G. Papadopoulos, *Ultra-violet behaviour of two-dimensional supersymmetric non-linear σ -models*, Nucl. Phys. **B 289** (1987), 264–276.
- [52] C.M. Hull, *Anomalies, ambiguities and superstrings*, Phys. Lett. **B 167** (1986), 51–55.
- [53] C.M. Hull, P.K. Townsend, *The two-loop β -function for σ -models with torsion*, Phys. Lett. **B 191** (1987), 115–121.
- [54] C.M. Hull, E. Witten, *Supersymmetric sigma models and the Heterotic String*, Phys. Lett. **B 160** (1985), 398–402.
- [55] P. Ivanov, S. Ivanov, *$SU(3)$ -instantons and G_2 , $Spin(7)$ -Heterotic string solitons*, Comm. Math. Phys. **259** (2005), 79–102.
- [56] S. Ivanov, *Geometry of quaternionic Kähler connections with torsion*, J. Geom. Phys. **41** (2002), no. 3, 235–257.
- [57] S. Ivanov, *Connections with torsion, parallel spinors and geometry of $Spin(7)$ manifolds*, Math. Res. Lett. **11** (2004), no. 2-3, 171–186.
- [58] S. Ivanov, G. Papadopoulos, *A no-go theorem for string warped compactifications*, Phys. Lett. **B 497** (2001) 309–316.
- [59] S. Ivanov, G. Papadopoulos, *Vanishing Theorems and String Backgrounds*, Class. Quant. Grav. **18** (2001) 1089–1110.
- [60] D. Joyce, *Compact Riemannian 7-manifolds with holonomy G_2 . I*, J. Diff. Geom. **43** (1996), 291–328.
- [61] ———, *Compact Riemannian 7-manifolds with holonomy G_2 . II*, J. Diff. Geom., **43** (1996), 329–375.
- [62] ———, *Compact Riemannian manifolds with special holonomy*, Oxford University Press, 2000.
- [63] S. Kobayashi, *Principal fibre bundles with the 1-dimensional toroidal group*, Tohoku Math. J. **8** (1956), 29–45.
- [64] A. Kovalev, *Twisted connected sums and special Riemannian holonomy*, J. Reine Angew. Math. **565** (2003), 125–160.
- [65] B. Lawson, M.-L. Michelsohn, *Spin Geometry*, Princeton University Press, 1989.
- [66] J. Li, S.-T. Yau, *The Existence of Supersymmetric String Theory with Torsion*, J. Diff. Geom. **70**, no. 1, (2005), 143–181.
- [67] P.-A. Nagy, *Prolongations of Lie algebras and applications*, arXiv:0712.1398.
- [68] G. Papadopoulos, *Solution of heterotic Killing spinor equations and special geometry*, arXiv:0811.1539 [math.DG].
- [69] R. Reyes Carrión, *A generalization of the notion of instanton*, Diff. Geom. Appl. **8** (1998), no. 1, 1–20.
- [70] A. Sen, *(2,0) supersymmetry and space-time supersymmetry in the heterotic string theory*, Nucl. Phys. **B 278** (1986), 289–308.
- [71] S. Salamon, *Riemannian geometry and holonomy groups*, Pitman Res. Notes Math. Ser., 201 (1989).
- [72] N. Schoemann, *Almost Hermitian Structures with Parallel Torsion*, J. Geom. Phys. **57** (2007), no. 11, 2187–2212.
- [73] A. Strominger, *Superstrings with torsion*, Nucl. Phys. **B 274** (1986) 253–284.

- [74] L. Ugarte, *Coeffective Numbers of Riemannian 8-manifolds with Holonomy in Spin(7)*, Ann. Glob. Anal. Geom. **19** (2001), 35–53.

(Fernández) UNIVERSIDAD DEL PAÍS VASCO, FACULTAD DE CIENCIA Y TECNOLOGÍA, DEPARTAMENTO DE MATEMÁTICAS, APARTADO 644, 48080 BILBAO, SPAIN

E-mail address: `marisa.fernandez@ehu.es`

(Ivanov) UNIVERSITY OF SOFIA "ST. KL. OHRIDSKI", FACULTY OF MATHEMATICS AND INFORMATICS, BLVD. JAMES BOURCHIER 5, 1164 SOFIA, BULGARIA

E-mail address: `ivanovsp@fmi.uni-sofia.bg`

(Ugarte) DEPARTAMENTO DE MATEMÁTICAS - I.U.M.A., UNIVERSIDAD DE ZARAGOZA, CAMPUS PLAZA SAN FRANCISCO, 50009 ZARAGOZA, SPAIN

E-mail address: `ugarte@unizar.es`

(Villacampa) DEPARTAMENTO DE MATEMÁTICAS - I.U.M.A., UNIVERSIDAD DE ZARAGOZA, CAMPUS PLAZA SAN FRANCISCO, 50009 ZARAGOZA, SPAIN

E-mail address: `raquelvg@unizar.es`