NON-KAEHLER HETEROTIC STRING COMPACTIFICATIONS WITH NON-ZERO FLUXES AND CONSTANT DILATON

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ABSTRACT. We construct new explicit compact supersymmetric valid solutions with non-zero field strength, non-flat instanton and constant dilaton to the heterotic equations of motion in dimension six. We present balanced Hermitian structures on compact nilmanifolds in dimension six satisfying the heterotic supersymmetry equations with non-zero flux, non-flat instanton and constant dilaton which obey the three-form Bianchi identity with curvature term taken with respect to either the Levi-Civita, the (+)-connection or the Chern connection. Among them, all our solutions with respect to the (+)-connection on the compact nilmanifold M_3 satisfy the heterotic equations of motion.

CONTENTS

1. INTRODUCTION. FIELD AND KILLING-SPINOR EQUATIONS

The bosonic fields of the ten-dimensional supergravity which arises as low energy effective theory of the heterotic string are the spacetime metric g, the NS three-form field strength H, the dilaton ϕ and the gauge connection A with curvature F^A. In this paper, we consider the bosonic geometry to be of the form $R^{1,9-d} \times M^d$, where the bosonic fields are non-trivial only on M^d, $d \leq 8$. We consider the two connections

$$\nabla^{\pm} = \nabla^g \pm \frac{1}{2}H,$$

where ∇^g is the Levi-Civita connection of the Riemannian metric g. Both connections preserve the metric, $\nabla^{\pm}g = 0$ and have totally skew-symmetric torsion $\pm H$, respectively.

The Green-Schwarz anomaly cancellation mechanism requires that the three-form Bianchi identity receives an α' correction of the form

$$dH = \frac{\alpha'}{4}8\pi^2(p_1(M^p) - p_1(E)) = \frac{\alpha'}{4}\Big(Tr(R \wedge R) - Tr(F^A \wedge F^A)\Big), \quad (1.1)$$

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where $p_1(M^p), p_1(E)$ are the first Pontrjagin forms of M^p with respect to a connection ∇ with curvature R and the vector bundle E with connection A, respectively.

A class of heterotic-string backgrounds for which the Bianchi identity of the three-form H receives a correction of type (1.1) are those with (2,0) world-volume supersymmetry. Such models were considered in [31]. The target-space geometry of (2,0)-supersymmetric sigma models has been extensively investigated in [31, 39, 28]. Recently, there is revived interest in these models [21, 9, 22, 23, 24] as string backgrounds and in connection to heterotic-string compactifications with fluxes [8, 2, 3, 4, 36, 19, 20, 5].

In writing (1.1) there is a subtlety to the choice of connection ∇ on M^p since anomalies can be cancelled independently of the choice [29]. Different connections correspond to different regularization schemes in the two-dimensional worldsheet non-linear sigma model. Hence the background fields given for the particular choice of ∇ must be related to those for a different choice by a field redefinition [38]. Connections on M^d proposed to investigate the anomaly cancellation (1.1) are ∇^g [39, 23], ∇^+ [9] and very recently [14], ∇^- [29, 6, 8, 24, 32, 35], Chern connection ∇^c when $d = 6$ [39, 36, 19, 20, 5].

A heterotic geometry will preserve supersymmetry if and only if, in 10 dimensions, there exists at least one Majorana-Weyl spinor ϵ such that the supersymmetry variations of the fermionic fields vanish, i.e. the following Killing-spinor equations hold [39]

$$\begin{aligned}
&\delta_\lambda = \nabla_m \epsilon = \left(\nabla^g_m + \frac{1}{4} H_{mnp}\Gamma^{np}\right)\epsilon = \nabla^+\epsilon = 0\\
&\delta_\Psi = \left(\Gamma^m \partial_m \phi - \frac{1}{12} H_{mnp}\Gamma^{mnp}\right)\epsilon = (d\phi - \frac{1}{2}H)\cdot\epsilon = 0 \qquad (1.2)\\
&\delta_\xi = F^A_{mn}\Gamma^{mn}\epsilon = F^A\cdot\epsilon = 0,
\end{aligned}$$

where λ, Ψ, ξ are the gravitino, the dilatino and the gaugino, fields, respectively and $\cdot$ means Clifford action of forms on spinors.

The bosonic part of the ten-dimensional supergravity action in the string frame is ([6], $R = R^-$)

$$S = \frac{1}{2k^2}\int d^{10}x\sqrt{-g}e^{-2\phi}\Big[Scal^g + 4(\nabla^g\phi)^2 - \frac{1}{2}|H|^2 - \frac{\alpha'}{4}\Big(Tr|F^A|^2) - Tr|R|^2\Big)\Big]. \qquad (1.3)$$

The string frame field equations (the equations of motion induced from the action (1.3)) of the heterotic string up to two-loops [30] in sigma model perturbation theory are (we use the notations in [24])

$$\begin{aligned}
&Ric^g_{ij} - \frac{1}{4}H_{imn}H^{mn}_j + 2\nabla^g_i\nabla^g_j\phi - \frac{\alpha'}{4}\Big[(F^A)_{imab}(F^A)^{mab}_j - R_{imnq}R^{mnq}_j\Big] = 0;\\
&\nabla^g_i(e^{-2\phi}H^i_{jk}) = 0; \qquad (1.4)\\
&\nabla^+_i(e^{-2\phi}(F^A)^i_j) = 0.
\end{aligned}$$

The field equation of the dilaton ϕ is implied from the first two equations above.

The first compact torsional solutions for the heterotic/type I string were obtained via duality from M-theory compactifications on K3 $\times$ K3 proposed in [13]. The metric was first written down on the orientifold limit in [13] and such backgrounds have since been studied (see [2, 3] and references therein). The metric and the H-flux are derived by applying a chain of supergravity dualities and the resulting geometry in the heterotic theory is a $\mathbb{T}^2$ bundle over a K3.

A compact example solving (1.2) and (1.1) with nonzero field strength, constant dilaton and taking $R = R^+$, is constructed in [9] on the Iwasawa nilmanifold which is a $\mathbb{T}^2$ bundle over $\mathbb{T}^4$. However, it has been pointed out in [23] that this example is not a valid solution due to a sign error in the torsional equation derived from the first two equations in (1.2) which leads to the opposite sign in the left hand side of (1.1). Compact example of a balanced 6-manifold with constant dilaton non-trivial warped factor and torsion generated by the Chern-Simons term only is presented very recently in [14].

Compact examples in dimension six solving (1.2) and (1.1) with non-zero flux H and non-constant dilaton were constructed by Li and Yau [36] for U(4) and U(5) principal bundles taking $R = R^c$-the curvature of the Chern connection in (1.1). Non-Kaehler compact solutions of (1.2) and (1.1) on some torus bundles over Calabi-Yau 4-manifold (K3 surfaces or complex torus) provided in [25] are presented by Fu and Yau [19, 20] using the Chern connection in (1.1). It is confirmed in [5] that the examples of torus bundles over the complex torus can not be solutions to (1.2) and (1.1) taking with respect to the curvature of the Chern connection $R = R^c$ with $\alpha' > 0$ while some torus bundles over K3 surfaces are valid solutions.

It is known [15, 22] ([24] for dimension 6) that the equations of motion of type I supergravity are automatically satisfied with $R = 0$ if one imposes, in addition to the preserving supersymmetry equations (1.2), the three-form Bianchi identity (1.1) taking with respect to a flat connection on TM, $R = 0$.

According to no-go (vanishing) theorems (a consequence of the equations of motion [17, 15]; a consequence of the supersymmetry [34, 33] for SU(n)-case and [23] for the general case) there are no compact solutions with non-zero flux and non-constant dilaton satisfying simultaneously the supersymmetry equations (1.2) and the three-form Bianchi identity (1.1) with $Tr(R \wedge R) = 0$.

However, in the presence of a curvature term R the solution of the supersymmetry equations (1.2) and the anomaly cancellation condition (1.1) obey the second and the third equations of motion but do not always satisfy the Einstein equations of motion (the first equation in (1.4)). If R is an SU(3)-instanton then (1.2) and (1.1) imply (1.4). This can be seen from the considerations in the Appendix of [22]. We give a quadratic expression for R which is necessary and sufficient condition in order that (1.2) and (1.1) imply (1.4) based on the properties of the special geometric structure induced from the first two equations in (1.2). More precisely, we prove in Section 2.2 the following

Theorem 1.1. *Let (M, J, g, F^A, R) be a conformally balanced Hermitian manifold with Kähler form F which solves the heterotic Killing spinor equations (1.2) and the anomaly cancellation (1.1).*

a): *The Einstein equations of motion (the first equation in (1.4)) are a consequence of the heterotic Killing spinor equations (1.2) and the anomaly cancellation (1.1) if and only if the next identity holds*

$$\frac{1}{2}\Big[R_{msab}R_{pqab} + R_{mpab}R_{qsab} + R_{mqab}R_{spab}\Big]F^{pq}J^s_n = R_{mpqr}R^{pqr}_n. \tag{1.5}$$

- *If R is $(1,1)$-form, $J^s_m J^p_n R_{spab} = R_{mnab}$ then (1.5) is equivalent to*

$$R_{mjab}R_{klab}F^{kl} = 0. \tag{1.6}$$

  In particular, the Einstein equations of motion with respect to either the Chern connection or the (–)-connection are a consequence of the heterotic Killing spinor equations (1.2) and the anomaly cancellation (1.1) if and only if (1.6) holds.
- *If R is an SU(3)-instanton then (1.5) holds.*

b): *If R^- is an SU(3)-instanton, $Hol(\nabla^+) \subset \mathfrak{su}(3)$ and the manifold is compact then the flux H vanishes, the dilaton is constant and the manifold is a Calabi-Yau space.*

As a consequence of Theorem 1.1, considering solutions involving Chern connection, one may study stability of the tangent bundle.

The main goal of this paper is to construct explicit compact valid solutions with non-zero field strength, non-flat instanton and constant dilaton to the heterotic equations of motion (1.4) in dimension six. We present compact nilmanifolds in dimension six satisfying the heterotic supersymmetry equations (1.2) with non-zero flux $H \neq 0$, non-flat instanton $F^A \neq 0$ and constant dilaton obeying the three-form Bianchi identity (1.1) with curvature term $R = R^g$, $R = R^+$ or $R = R^c$. Some of them are torus bundles over the complex torus but this does not violate the non-existence result in [5] since we use different curvature term (R^g or R^+) in (1.1). In particular, we present a valid solution on the Iwasawa manifold but with respect to a non-standard complex structure. We find compact valid solutions to (1.2) with non-zero flux, non-flat instanton and constant dilaton satisfying anomaly cancellation condition (1.1) using the curvature R^c of the Chern connection on an S^1 bundle over a 5-manifold which is a $\mathbb{T}^2$ bundle over $\mathbb{T}^3$. All manifolds do not admit any Kaehler metric and seem to be the first explicit compact valid supersymmetric heterotic solutions to (1.2) and (1.1) with non-zero flux, non-flat instanton and constant dilaton in dimension six.

However, because of Theorem 1.1, the Einstein equations of motion (the first equation in (1.4)) is not satisfied in most cases. Only the solutions constructed in Theorem 5.1 b), Theorem 5.2 b) on the compact nilmanifold $M_3 = \Gamma\backslash H(2,1) \times S^1$, where $H(2,1)$ is the 5-dimensional Heisenberg group and Γ is a lattice, solve in addition the heterotic equations of motion (1.4) with non-zero fluxes and constant dilaton. It seems that these are the first compact supersymmetric solutions to the heterotic equations of motion with non-zero flux $H \neq 0$, non-flat instanton $F^A \neq 0$ and constant dilaton in dimension six.

Our convention for the curvature is given in Section 3.

2. The supersymmetry equations in dimension 6

Necessary and sufficient conditions to have a solution to the system of gravitino and dilatino equations (the first two equations in (1.2)) in dimension 6 were derived by Strominger in [39] involving the notion of SU(n)-structure and then studied by many authors [21, 22, 23, 9, 8, 32, 2, 3, 24, 36, 19, 20, 5].

2.1. **SU(3)-structures in $d = 6$.** Let (M, J, g) be an almost Hermitian 6-manifold with Riemannian metric g and almost complex structure J, i.e. (J, g) define an $U(3)$-structure. The Nijenhuis tensor N, the Kaehler form F and the Lee form θ are defined by

$$N(\cdot,\cdot) = [J\cdot, J\cdot] - [\cdot,\cdot] - J[J\cdot,\cdot] - J[\cdot, J\cdot], \quad F(\cdot,\cdot) = g(\cdot, J\cdot), \quad \theta(\cdot) = \delta F(J\cdot),$$

respectively, where $*$ is the Hodge operator and δ is the co-differential, $\delta = - * d *$.

An SU(3)-structure is determined by an additional non-degenerate (3,0)-form $\Psi = \Psi^+ + i\,\Psi^-$, or equivalently by a non-trivial spinor. The subgroup of $SO(6)$ fixing the forms F and Ψ simultaneously is SU(3). The Lie algebra of SU(3) is denoted $\mathfrak{su}(3)$.

The failure of the holonomy group of the Levi-Civita connection to reduce to SU(3) can be measured by the intrinsic torsion τ, which is identified with $\nabla^g F$ or $\nabla^g J$ and can be decomposed into five classes [10], $\tau \in W_1 \oplus \dots \oplus W_5$. The intrinsic torsion of an $U(n)$-structure belongs to the first four components described by Gray-Hervella [27]. The five components of an SU(3)-structure are first described by Chiossi-Salamon [10] (for interpretation in physics see [9]) and are determined by $dF, d\Psi^+, d\Psi^-$ as well as by dF and N. The Hermitian manifolds belong to $W_3 \oplus W_4$. In the paper we are interested in the class W_3 of balanced Hermitian manifolds [37] characterized by the conditions $N = 0, \theta = 0$ or, equivalently, $N = 0$, $d * F = 0$.

Necessary conditions to solve the gravitino equation (the first equation in(1.2)) are given in [18]. The presence of a parallel spinor in dimension 6 leads firstly to the reduction to $U(3)$, i.e. the existence of an almost Hermitian structure, secondly to the existence of a linear connection preserving the almost Hermitian structure with torsion 3-form and thirdly to the reduction of the holonomy group of the torsion connection to SU(3), i.e. its Ricci 2-form has to be identically zero. It is shown in [18] that there exists a unique linear connection preserving an almost Hermitian structure having totally skew-symmetric torsion if and only if the Nijenhuis tensor is a 3-form, i.e. the intrinsic torsion $\tau \in W_1 \oplus W_3 \oplus W_4$. The torsion connection ∇^+ with torsion T is determined by

$$\nabla^+ = \nabla^g + \frac{1}{2}T, \qquad T = JdF + N = -dF(J\cdot, J\cdot, J\cdot) + N.$$

Necessary and sufficient conditions to solve the gravitino equation (the first equation in (1.2)) in dimension 6 are given in [32]. Namely, there exists a unique linear connection with torsion 3-form which preserves the almost Hermitian structure whose holonomy is contained in SU(3) if and only if the first Chern class vanishes, $c_1(M, J) = 0$ and the SU(3)-structure $(M, g, F, \Psi^+, \Psi^-)$ satisfies the differential equations [32]

$$d\Psi^+ = \theta \wedge \Psi^+ - \frac{1}{4}(N, \Psi^+) * F, \qquad d\Psi^- = \theta \wedge \Psi^- - \frac{1}{4}(N, \Psi^-) * F. \tag{2.1}$$

The torsion T is given by $T = - * dF + *(\theta \wedge F) + \frac{1}{4}(N, \Psi^+)\Psi^+ + \frac{1}{4}(N, \Psi^-)\Psi^-$.

Necessary and sufficient conditions to solve the gravitino and dilatino equations (the first two equations in (1.2)) are presented in [39]. The dilatino equation forces the almost complex structure to be integrable ($N = 0$) and the Lee form to be closed (for applications in physics the Lee form has to be exact) determined by the dilaton due to $\theta = 2d\phi$ [39]. The three-form field strength is given by $H = T = -dF(J\cdot, J\cdot, J\cdot) = - * dF + *(2d\phi \wedge F)$. Solutions with constant dilaton are those with zero Lee form, $dF^{n-1} = 0$, i.e. balanced Hermitian manifolds.

When the almost complex structure is integrable, $N = 0$, the torsion connection ∇^+ is also known as the Bismut connection (we shall call it Bismut-Strominger (B-S) connection) and was used by Bismut to prove a local index theorem for the Dolbeault operator on non-Kaehler Hermitian manifolds [7]. This formula was recently applied in string theory [3]. Vanishing theorems for the Dolbeault cohomology on compact non-Kaehler Hermitian manifolds were found in terms of the B-S connection [1, 33, 34].

In addition to these equations, the vanishing of the gaugino variation (the third equation in (1.2)) requires the non-zero 2-form F^A to be of instanton type ([12, 39, 23]). A Donaldson-Uhlenbeck-Yau SU(3)-instanton i.e. the gauge field A is a connection on a holomorphic vector bundle with curvature 2-form $F^A \in \mathfrak{su}(3)$. The SU(3)-instanton condition can be written in local holomorphic coordinates in the form [12, 39]

$$F^A_{\alpha\beta} = F^A_{\bar\alpha\bar\beta} = 0, \qquad F^A_{\alpha\bar\beta}F^{\alpha\bar\beta} = 0.$$

2.2. **Proof of Theorem 1.1.** A consequence of the gravitino and dilatino equations (the first two equations in (1.2)) is the expression of the Ricci tensor $Ric^+_{mn} = R^+_{imnj}g^{ij}$ of the (+)-connection established in [34], Proposition 3.1:

$$Ric^+_{mn} = -2\nabla^+_m d\phi_n - \frac{1}{4}dT_{mspq}J^s_nF^{pq} = -2\nabla^g_m d\phi_n + d\phi_s T^s_{mn} - \frac{1}{4}dT_{mspq}J^s_nF^{pq}. \tag{2.2}$$

The four-form $dT = dJdF$ is a (2,2)-form with respect to the complex structure J. Therefore, the last term in (2.2) is symmetric.

On the other hand, the Ricci tensors of ∇^g and ∇^+ are connected by (see e.g. [18])

$$Ric^g_{mn} = Ric^+_{mn} + \frac{1}{4}T_{mpq}T^{pq}_n - \frac{1}{2}\nabla^+_s T^s_{mn}, \qquad Ric^+_{mn} - Ric^+_{nm} = \nabla^+_s T^s_{mn} = \nabla^g_s T^s_{mn}, \tag{2.3}$$

$$Ric^g_{mn} = \frac{1}{2}(Ric^+_{mn} + Ric^+_{nm}) + \frac{1}{4}T_{mpq}T^{pq}_n. \tag{2.4}$$

Substitute (2.2) into (2.4), insert the result into the first equation of (1.4) and use the anomaly cancellation (1.1) to conclude (1.5). If R is a (1,1)-form then (1.6) is a consequence of (1.5). It is well known that the curvature of the Chern connection R^c is always a (1,1)-form. When $Hol(\nabla^+) \subset \mathfrak{su}(3)$ the curvature R^- of the (–)-connection is also an (1,1)-form. This follows from the well known identity

$$dT_{ijkl} = 2R^+_{ijkl} - 2R^-_{klij} \tag{2.5}$$

and the fact that dT is a (2,2)-form. This completes the proof of a).

The proof of b) is essentially contained in [34, 33]. Indeed, if $Hol(\nabla^+) \subset \mathfrak{su}(3)$ and R^- is an SU(3)-instanton, (2.5) yields $dT_{ispq}F^{pq} = 0$, i.e. the manifold is almost strong in the terminology of [34]. Then Corollary 4.2 a) in [34] asserts that there are no holomorphic (3,0) forms which contradicts the result in [39] except $T = d\phi = 0$. This completes the proof of Theorem 1.1.

2.3. **Heterotic supersymmetry with constant dilaton.** We look for a compact Hermitian 6-manifold (M, J, g) which satisfies the following conditions

a). Gravitino equation (the first equation in (1.2)): $Hol(\nabla^+) \subset \mathfrak{su}(3)$, i.e.

$$\sum_{i=1}^{6}(\Omega^+)^{E_i}_{JE_i} = 0, \tag{2.6}$$

where $\{E_1, \dots, E_6\}$ is an orthonormal basis on M.

b). Dilatino equation (the second equation in (1.2)) with constant dilaton: the Lee form $\theta = 2d\phi = 0$, i.e. (M, J, g) is a balanced manifold.

c). Gaugino equation (the third equation in (1.2)): look for a Hermitian vector bundle E of rank r over M equiped with an SU(3)-instanton, i.e. a connection A with curvature 2-form Ω^A satisfying

$$(\Omega^A)^i_j(JE_k, JE_l) = (\Omega^A)^i_j(E_k, E_l), \qquad \sum_{k=1}^{6}(\Omega^A)^i_j(E_k, JE_k) = 0. \tag{2.7}$$

d). Anomaly cancellation condition:

$$dH = dT = \frac{\alpha'}{4}8\pi^2\Big(p_1(M) - p_1(A)\Big), \qquad \alpha' > 0. \tag{2.8}$$

3. General preliminaries

For a linear connection ∇, the connection 1-forms ω^i_j with respect to a fixed basis $E_1, \dots, E_6$ are

$$\omega^i_j(E_k) = g(\nabla_{E_k} E_j, E_i)$$

since we write $\nabla_X E_j = \omega^1_j(X)\, E_1 + \cdots + \omega^6_j(X)\, E_6$.

The curvature 2-forms Ω^i_j of ∇ are given in terms of the connection 1-forms ω^i_j by

$$\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j, \quad \Omega_{ji} = d\omega_{ji} + \omega_{ki} \wedge \omega_{jk}, \quad R^l_{ijk} = \Omega^l_k(E_i, E_j), \quad R_{ijkl} = R^s_{ijk} g_{ls}.$$

and the first Pontrjagin class is represented by the 4-form

$$p_1(\nabla) = \frac{1}{8\pi^2} \sum_{1 \le i < j \le 6} \Omega^i_j \wedge \Omega^i_j.$$

Let (M, J, g) be a 6-dimensional Hermitian manifold. Consider the connections with torsion $\nabla^{\pm}$ given by $\nabla^{\pm} = \nabla^g \pm \frac{1}{2} T$ with torsion T given by

$$T = JdF = - * dF. \tag{3.1}$$

Notice that ∇^+ is precisely the B-S connection of the Hermitian structure.

The Chern connection ∇^c is defined by,

$$\nabla^c = \nabla^g + \frac{1}{2} C, \qquad C(.,.,.) = dF(J.,.,.).$$

Observe that the tensor field C satisfies that $C(X, \cdot, \cdot) = (JX \lrcorner dF)(\cdot, \cdot)$ is a 2-form on M.

Let us suppose that (J, g) is a left invariant Hermitian structure on a 6-dimensional Lie group M and let $\{e^1, \dots, e^6\}$ be an orthonormal basis of left invariant 1-forms, that is, $g = e^1 \otimes e^1 + \cdots + e^6 \otimes e^6$. Let

$$d\, e^k = \sum_{1 \le i < j \le 6} a^k_{ij}\, e^i \wedge e^j, \qquad k = 1, \dots, 6,$$

be the structure equations in the basis $\{e^k\}$.

Let us denote by $\{E_1, \dots, E_6\}$ the dual basis. Since $de^k(E_i, E_j) = -e^k([E_i, E_j])$, we have that the Levi-Civita connection 1-forms $(\omega^g)^i_j$ are

$$(\omega^g)^i_j(E_k) = -\frac{1}{2}(g(E_i, [E_j, E_k]) - g(E_k, [E_i, E_j]) + g(E_j, [E_k, E_i])) = \frac{1}{2}(a^i_{jk} - a^k_{ij} + a^j_{ki}). \tag{3.2}$$

The connection 1-forms $(\omega^{\pm})^i_j$ for the connections with torsion $\nabla^{\pm}$ are given by

$$(\omega^{\pm})^i_j(E_k) = (\omega^g)^i_j(E_k) + \frac{1}{2}(T^{\mp})^i_j(E_k), \qquad (T^{\pm})^i_j(E_k) = T^{\pm}(E_i, E_j, E_k) = \mp dF(JE_i, JE_j, JE_k). \tag{3.3}$$

The connection 1-forms $(\omega^c)^i_j$ for the Chern connection ∇^c are determined by

$$(\omega^c)^i_j(E_k) = (\omega^g)^i_j(E_k) + \frac{1}{2} C^i_j(E_k), \qquad C^i_j(E_k) = dF(JE_k, E_i, E_j). \tag{3.4}$$

We shall focus on six-dimensional nilmanifolds $M = \Gamma \backslash G$ endowed with an invariant (integrable almost) complex structure J. According to Proposition 6.1 in [16], for invariant Hermitian metrics on compact nilmanifolds the balanced condition is equivalent to $Hol(\nabla^+) \subset \mathfrak{su}(3)$. The equivalence of the conditions a) and b) in subsection 2.3 can also be derived from (2.1) and the fact, established in [16], that for any invariant Hermitian structure on a nilmanifold the (3,0)-form $\Psi = \Psi^+ + i\, \Psi^-$ is closed.

3.1. Six-dimensional balanced Hermitian nilmanifolds. Next we review the main results given in [40] concerning balanced J-Hermitian metrics on M in order to apply them to the construction of solutions to the equations (2.6)-(2.8) above. First of all, if (M, J) admits a balanced J-Hermitian metric (not necessarily invariant) then the Lie algebra $\mathfrak{g}$ of G is isomorphic to $\mathfrak{h}_1, \ldots, \mathfrak{h}_6$ or $\mathfrak{h}_{19}^-$, where $\mathfrak{h}_1 = (0,0,0,0,0,0)$ is the abelian Lie algebra and

$$\begin{aligned} \mathfrak{h}_2 &= (0,0,0,0,12,34), & \mathfrak{h}_5 &= (0,0,0,0,13+42,14+23),\\ \mathfrak{h}_3 &= (0,0,0,0,0,12+34), & \mathfrak{h}_6 &= (0,0,0,0,12,13),\\ \mathfrak{h}_4 &= (0,0,0,0,12,14+23), & \mathfrak{h}_{19}^- &= (0,0,0,12,23,14-35). \end{aligned}$$

Here $\mathfrak{h}_5$ is the Lie algebra underlying the Iwasawa manifold. For the canonical complex structure J_0 on $\mathfrak{h}_5$ there exists a complex basis $\{\omega^j\}_{j=1}^3$ of 1-forms of type (1,0) satisfying $d\omega^1 = d\omega^2 = 0$ and $d\omega^3 = \omega^{12}$.

Since the Lie algebras $\mathfrak{h}_2, \ldots, \mathfrak{h}_6$ are 2-step nilpotent, for any complex structure J ($\neq J_0$ for $\mathfrak{h}_5$) there is a basis $\{\omega^j\}_{j=1}^3$ of (1,0)-forms such that

$$d\omega^1 = d\omega^2 = 0, \qquad d\omega^3 = \rho\,\omega^{12} + \omega^{1\bar{1}} + B\,\omega^{1\bar{2}} + D\,\omega^{2\bar{2}}, \tag{3.5}$$

where $B, D \in \mathbb{C}$, and $\rho = 0, 1$. In particular, J is a nilpotent complex structure on $\mathfrak{h}_2, \ldots, \mathfrak{h}_6$ in the sense [11]. Recall that a complex structure J on a $2n$-dimensional nilpotent Lie algebra $\mathfrak{g}$ is called *nilpotent* if there is a basis $\{\omega^j\}_{j=1}^n$ of (1,0)-forms satisfying $d\omega^1 = 0$ and

$$d\omega^j \in \bigwedge^2(\omega^1, \ldots, \omega^{j-1}, \omega^{\bar{1}}, \ldots, \omega^{\overline{j-1}}),$$

for $j = 2, \cdots, n$.

Any complex structure on the Lie algebra $\mathfrak{h}_{19}^-$ is not nilpotent and there is a (1,0)-basis $\{\omega^j\}_{j=1}^3$ satisfying

$$d\omega^1 = 0, \quad d\omega^2 = E\,\omega^{13} + \omega^{1\bar{3}}, \quad d\omega^3 = C\,\omega^{1\bar{1}} + ia\,\omega^{1\bar{2}} - ia\bar{E}\,\omega^{2\bar{1}}, \tag{3.6}$$

where $E \in \mathbb{C}$ with $|E| = 1$, $\bar{C} = CE$ and $a \in \mathbb{R} - \{0\}$.

Now, the fundamental form F of any invariant J-Hermitian structure is given in terms of the basis $\{\omega^j\}_{j=1}^3$ by

$$2\,F = i(r^2\omega^{1\bar{1}} + s^2\omega^{2\bar{2}} + t^2\omega^{3\bar{3}}) + u\,\omega^{1\bar{2}} - \bar{u}\,\omega^{2\bar{1}} + v\,\omega^{2\bar{3}} - \bar{v}\,\omega^{3\bar{2}} + z\,\omega^{1\bar{3}} - \bar{z}\,\omega^{3\bar{1}}, \tag{3.7}$$

where $r, s, t \in \mathbb{R} - \{0\}$ and $u, v, z \in \mathbb{C}$ must satisfy those restrictions coming from the positive definiteness of the associated metric $g(X, Y) = -F(X, JY)$. The following result gives necessary and sufficient conditions, in terms of the different coefficients involved, in order the Hermitian structure to be balanced.

Proposition 3.1. [40] *In the notation above, we have:*

(*i*) *If J is a nonnilpotent complex structure defined by* (3.6)*, then (J, F) is balanced if and only if*

$$z = -iuv/s^2 \qquad \text{and} \qquad Cs^2 + a\bar{E}u + a\bar{u} = 0.$$

(*ii*) *If J is a nilpotent complex structure defined by* (3.5)*, then (J, F) is balanced if and only if*

$$s^2t^2 - |v|^2 + D(r^2t^2 - |z|^2) = B(it^2\bar{u} - v\bar{z}).$$

4. The Iwasawa manifold revisited

Apart from the abelian Lie algebra, $\mathfrak{h}_5$ is the only 6-dimensional nilpotent Lie algebra which can be given a *complex* Lie algebra structure. The corresponding complex parallelizable nilmanifold is the well-known Iwasawa manifold. This manifold is studied in [9]; however, as it is pointed out in the introduction, this example is not a valid solution due to a sign error in the torsional equation. More general, we show in Remark 4.1 that there are no valid solutions on the Iwasawa manifold with respect to the standard complex structure and any invariant compatible Hermitian metric, a fact which leads us to study general complex nilmanifolds in the subsequent sections.

The standard complex structure J_0 on $\mathfrak{h}_5$ is defined by the following complex structure equations:

$$d\omega^1 = d\omega^2 = 0, \qquad d\omega^3 = \omega^{12}.$$

For any $t \neq 0$, let us consider F given by

$$F = \frac{i}{2}(\omega^{1\bar{1}} + \omega^{2\bar{2}} + t^2\,\omega^{3\bar{3}}).$$

It is easy to see that the Hermitian structure (J_0, F) is balanced for any value of the parameter.

Notice that the Iwasawa manifold is a $\mathbb{T}^2$ bundle over $\mathbb{T}^4$, where the parameter t scales the fiber.

From a real point of view, let us consider the real basis of 1-forms $\{e^1, \ldots, e^6\}$ given by

$$e^1 + i\,e^2 = \omega^1, \quad e^3 + i\,e^4 = \omega^2, \quad e^5 + i\,e^6 = t\,\omega^3.$$

Now, in terms of this basis, we have that the structure equations are

$$\begin{cases} de^1 = de^2 = de^3 = de^4 = 0, \\ de^5 = t\,e^{13} - t\,e^{24}, \\ de^6 = t\,e^{14} + t\,e^{23}, \end{cases} \tag{4.1}$$

the complex structure J_0 is given by $J_0 e^1 = -e^2, J_0 e^3 = -e^4, J_0 e^5 = -e^6$, the J_0-Hermitian metric $g = e^1 \otimes e^1 + \cdots + e^6 \otimes e^6$ has the associated fundamental form $F = e^{12} + e^{34} + e^{56}$. The structure equations (4.1) give $dF = t\,e^{136} - t\,e^{145} - t\,e^{235} - t\,e^{246}$. Apply (3.1) to verify that the torsion T of ∇^+ satisfies

$$T = -t\,e^{135} - t\,e^{146} - t\,e^{236} + t\,e^{245}, \qquad dT = -4t^2 e^{1234}.$$

All the curvature forms $(\Omega^c)^i_j$ of the Chern connection vanish. In view of (3.2) and (3.3), the non-zero curvature forms $(\Omega^g)^i_j$ and $(\Omega^{\pm})^i_j$ for the Levi-Civita connection and the connections $\nabla^{\pm}$ are given by:

$$\begin{aligned}
&(\Omega^g)^1_2 = \tfrac{t^2}{2}(e^{34} - e^{56}), \quad (\Omega^g)^1_3 = -\tfrac{t^2}{4}(3e^{13} - e^{24}), \quad (\Omega^g)^1_4 = -\tfrac{t^2}{4}(3e^{14} + e^{23}),\\
&(\Omega^g)^1_5 = -(\Omega^g)^2_6 = \tfrac{t^2}{4}(e^{15} - e^{26}), \quad (\Omega^g)^1_6 = (\Omega^g)^2_5 = \tfrac{t^2}{4}(e^{16} + e^{25}), \quad (\Omega^g)^2_3 = -\tfrac{t^2}{4}(e^{14} + 3e^{23}),\\
&(\Omega^g)^2_4 = \tfrac{t^2}{4}(e^{13} - 3e^{24}), \quad (\Omega^g)^3_4 = \tfrac{t^2}{2}(e^{12} - e^{56}), \quad (\Omega^g)^3_5 = -(\Omega^g)^4_6 = \tfrac{t^2}{4}(e^{35} - e^{46}),\\
&(\Omega^g)^3_6 = (\Omega^g)^4_5 = \tfrac{t^2}{4}(e^{36} + e^{45}), \quad (\Omega^g)^5_6 = -\tfrac{t^2}{2}(e^{12} + e^{34});
\end{aligned}$$

$$\begin{aligned}
&(\Omega^+)^1_2 = 2t^2 e^{34}, \quad (\Omega^+)^1_3 = (\Omega^+)^2_4 = -t^2(e^{13} + e^{24}), \quad (\Omega^+)^1_4 = -(\Omega^+)^2_3 = -t^2(e^{14} - e^{23}),\\
&(\Omega^+)^3_4 = 2t^2 e^{12}, \quad (\Omega^+)^5_6 = -2t^2(e^{12} + e^{34});
\end{aligned}$$

$$(\Omega^-)^1_2 = (\Omega^-)^3_4 = -2t^2 e^{56}, \quad (\Omega^-)^1_3 = -(\Omega^-)^2_4 = -t^2(e^{13} - e^{24}), \quad (\Omega^-)^1_4 = (\Omega^-)^2_3 = -t^2(e^{14} + e^{23}).$$

Clearly $Hol(\nabla^+) \subset \mathfrak{su}(3)$ and the Pontrjagin classes of the four connections are then represented by

$$p_1(\nabla^g) = \frac{t^4}{4\pi^2} e^{1234}, \quad p_1(\nabla^+) = 0, \quad p_1(\nabla^-) = \frac{t^4}{\pi^2} e^{1234}, \quad p_1(\nabla^c) = 0. \tag{4.2}$$

4.1. **Cardoso et al. abelian instanton.** Cardoso et al. consider in [9] an abelian field strength configuration with (1,1)-form

$$\mathfrak{F} = if\,dz_1 \wedge d\bar{z}_1 - if\,dz_2 \wedge d\bar{z}_2 + \mathrm{e}^{i\gamma}\sqrt{\frac{1}{4} - f^2}\,dz_1 \wedge d\bar{z}_2 - \mathrm{e}^{-i\gamma}\sqrt{\frac{1}{4} - f^2}\,dz_2 \wedge d\bar{z}_1,$$

where the function f satisfies

$$i\partial_{z_2} f + \partial_{z_1}\left(\mathrm{e}^{-i\gamma}\sqrt{\frac{1}{4} - f^2}\right) = 0, \qquad i\partial_{z_1} f + \partial_{z_2}\left(\mathrm{e}^{i\gamma}\sqrt{\frac{1}{4} - f^2}\right) = 0.$$

Under these conditions one gets

$$Tr\ F^A \wedge F^A = \mathfrak{F} \wedge \mathfrak{F} = -\frac{1}{2} dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2.$$

Here dz_1 and dz_2 denote the (2,0)-forms at the level of the Lie group, which descend to the forms ω^1 and ω^2 on the compact nilmanifold. Therefore, on the Iwasawa manifold we have

$$Tr\ F^A \wedge F^A = \frac{1}{2}\,\omega^{1\bar{1}} \wedge \omega^{2\bar{2}} = -2\,e^{1234}.$$

Now, taking A as one of these abelian instantons we have that

$$dT = -4t^2 e^{1234} = -16\pi^2 t^2 (p_1(\nabla^+) - p_1(A)), \tag{4.3}$$

which is not a valid solution for any t (see [23] for details). Moreover, the whole space of complex structures compatible with the canonical metric obtained when $t = 1$ in (4.1) is studied in [9] where the authors proved that the behavior is the same as in (4.3).

Remark 4.1. *It is not difficult to prove that any J_0-Hermitian invariant metric g is equivalent to one in the 1-parameter family given above. Since $dT = -4t^2 e^{1234}$, in view of* (4.2) *there is no way to find a satisfactory solution with* (J_0, g) *as the underlying Hermitian structure. Indeed, it is proved in* [5] *that torus bundles over the complex torus can not be solutions to* (1.2) *and* (1.1) *taken with respect to the curvature of the Chern connection $R = R^c$ with $\alpha' > 0$. Since $p_1(\nabla^c) = 0$ and $dT = -4t^2 e^{1234}$ we conclude from* (1.1) *that $p_1(A)$ cannot be a positive multiple of e^{1234} for any* SU(3)*-instanton A on the Iwasawa manifold which is a torus bundle over the complex torus. Hence,* (1.1) *cannot be satisfied for any $\alpha' > 0$ neither for $R = R^g$ nor for $R = R^{\pm}$ because of* (4.2).

Therefore, in order to find solutions we need to consider other compact nilmanifolds or metrics and/or complex structures different from the canonical ones on the nilmanifold underlying the Iwasawa manifold. In the following sections we show many explicit solutions.

5. A family of balanced Hermitian structures on the Lie algebra $\mathfrak{h}_3$

In this section we construct explicit solutions on compact nilmanifold corresponding to the Lie algebra $\mathfrak{h}_3$. First we recall [40] that, up to equivalence, there exist two complex structures $J^{\pm}$ on $\mathfrak{h}_3$, namely

$$J^{\pm}: \quad d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{1\bar{1}} \pm \omega^{2\bar{2}},$$

but only J^- admits compatible balanced structures. Notice that the balanced condition for J^- given in Proposition 3.1 (ii) reduces to

$$(r^2 - s^2)t^2 = |z|^2 - |v|^2.$$

For any $t \neq 0$, let us consider the balanced structure F given by

$$F = \frac{i}{2}(\omega^{1\bar{1}} + \omega^{2\bar{2}} + t^2\, \omega^{3\bar{3}}),$$

which corresponds to $r = s = 1$ and $u = v = z = 0$.

From a real point of view, let us consider the basis of 1-forms $\{e^1, \ldots, e^6\}$ given by

$$e^1 + i\, e^2 = \omega^1, \quad e^3 + i\, e^4 = \omega^2, \quad e^5 + i\, e^6 = t\, \omega^3.$$

Now, in terms of this basis, we have the structure equations

$$\begin{cases} de^1 = de^2 = de^3 = de^4 = de^5 = 0, \\ de^6 = -2t\, e^{12} + 2t\, e^{34}, \end{cases} \tag{5.1}$$

and the complex structure $J = J^-$ is given by $Je^1 = -e^2, Je^3 = -e^4, Je^5 = -e^6$. The balanced J-Hermitian metric $g = e^1 \otimes e^1 + \cdots + e^6 \otimes e^6$ has the associated fundamental form $F = e^{12} + e^{34} + e^{56}$. The structure equations (5.1) yield $dF = 2t(e^{12} - e^{34})e^5$. For the torsion T of ∇^+ we calculate using (3.1), (3.2) and (3.3) that

$$T = -2t(e^{12} - e^{34})e^6, \qquad dT = -8t^2 e^{1234}, \qquad \nabla^+ T = 0.$$

A direct calculation applying (3.2) and (3.3) shows that the non-zero curvature forms $(\Omega^g)^i_j$ of the Levi-Civita connection ∇^g are given by

$$(\Omega^g)^1_2 = -t^2(3\, e^{12} - 2\, e^{34}), \quad (\Omega^g)^1_3 = t^2 e^{24}, \quad (\Omega^g)^1_4 = -t^2 e^{23}, \quad (\Omega^g)^1_6 = t^2 e^{16}, \quad (\Omega^g)^2_3 = -t^2 e^{14},$$

$$(\Omega^g)^2_4 = t^2 e^{13}, \quad (\Omega^g)^2_6 = t^2 e^{26}, \quad (\Omega^g)^3_4 = t^2(2\, e^{12} - 3\, e^{34}), \quad (\Omega^g)^3_6 t^2 e^{36}, \quad (\Omega^g)^4_6 = t^2 e^{46},$$

and the non-zero curvature forms $(\Omega^+)^i_j$ of the connection ∇^+ are

$$(\Omega^+)^1_2 = -(\Omega^+)^3_4 = -4t^2(e^{12} - e^{34}). \tag{5.2}$$

Therefore, (2.6) is satisfied and the Pontrjagin classes are represented by

$$p_1(\nabla^g) = \frac{-3t^4}{\pi^2} e^{1234}, \qquad p_1(\nabla^+) = \frac{-8t^4}{\pi^2} e^{1234}.$$

Now, let us consider the new basis $\{f^1, \dots, f^6\}$ given by $f^i = e^i$, for $i = 1, \dots, 5$, and $f^6 = \frac{1}{t} e^6$. In terms of this basis, the structure equations (5.1) become

$$df^1 = df^2 = df^3 = df^4 = df^5 = 0, \qquad df^6 = -2\, f^{12} + 2\, f^{34},$$

and the family (J_t, g_t) of balanced Hermitian SU(3)-structures on $\mathfrak{h}_3$ is given by

$$J_t f^1 = -f^2, \quad J_t f^2 = f^1, \quad J_t f^3 = -f^4, \quad J_t f^4 = f^3, \quad J_t f^5 = -t\, f^6, \quad J_t f^6 = \frac{1}{t} f^5,$$
$$g_t = f^1 \otimes f^1 + \cdots + f^5 \otimes f^5 + t^2 f^6 \otimes f^6, \quad F_t = f^{12} + f^{34} + t\, f^{56}.$$

Let us fix $t' \neq 0$ and denote by $\nabla^+_{t'}$ the connection corresponding to the balanced structure $(J_{t'}, g_{t'})$ in the previous family. It follows from (5.2) that the non-zero curvature forms $(\Omega^+_{t'})^i_j$ of $\nabla^+_{t'}$ are

$$(\Omega^+_{t'})^1_2 = -(\Omega^+_{t'})^3_4 = -4{t'}^2 (f^{12} - f^{34}).$$

Therefore, (2.6) and (2.7) are satisfied and $\nabla^+_{t'}$ is an SU(3)-instanton with respect to any other balanced structure in the family (J_t, g_t).

Let $H(2,1)$ denote the 5-dimensional generalized Heisenberg group, and let Γ be a lattice of maximal rank. The nilpotent Lie algebra $\mathfrak{h}_3$ is the Lie algebra underlying the compact nilmanifold $M_3 = \Gamma\backslash H(2,1) \times S^1$.

Theorem 5.1. *In the notation above, for each $t' \neq t$, we consider the* SU(3)*-instanton $\nabla^+_{t'}$. Then we have:*

a) $$dT = \frac{8\pi^2 t^2}{3t^4 - 8{t'}^4} (p_1(\nabla^g_t) - p_1(\nabla^+_{t'})),$$

b) $$dT = \frac{\pi^2 t^2}{t^4 - {t'}^4} (p_1(\nabla^+_t) - p_1(\nabla^+_{t'})).$$

Hence, for any pair (t, t') such that $8{t'}^4 < 3t^4$ we obtain explicit valid solutions to the heterotic supersymmetry equations (1.2) *with non-zero flux $H = T$ and constant dilaton satisfying the three-form Bianchi identity* (1.1) *for the Levi-Civita connection and for the (+)-connection on the compact nilmanifold M_3.*

The compact manifold $(M_3, g, J, A = \nabla^+_{t'}, R(\nabla^+_t))$ described in b) solves the equations of motion (1.4).

Moreover, we can also use the abelian instanton A given in Subsection 4.1 to find more solutions. In fact, we can take dz_1 and dz_2 as (2,0)-forms at the level of the Lie group $H(2,1) \times \mathbb{R}$ which descend to the forms ω^1 and ω^2 on the compact nilmanifold M_3.

Theorem 5.2. *In the notation above and taking A as the abelian* SU(3)*-instanton given in* [9] *we have:*

a) $$dT = \frac{32\pi^2 t^2}{12t^4 - 1} (p_1(\nabla^g_t) - p_1(A)),$$

b) $$dT = \frac{32\pi^2 t^2}{32t^4 - 1} (p_1(\nabla^+_t) - p_1(A)).$$

Thus, for any t such that $12t^4 > 1$ we obtain explicit valid solutions to the heterotic supersymmetry equations (1.2) *with non-zero flux $H = T$ and constant dilaton satisfying the three-form Bianchi identity* (1.1) *for the Levi-Civita connection and for the (+)-connection on the compact nilmanifold M_3.*

The space $(M_3, g, J, A, R(\nabla^+_t))$ described in b) is a compact solution to the equations of motion (1.4).

Remark 5.3. *A direct calculation for ∇^- and for the Chern connection ∇^c shows that*

$$p_1(\nabla^-) = 0, \qquad p_1(\nabla^c) = 0.$$

The nilmanifold M_3 is a torus bundle over a complex torus, therefore we can use the argument given in Remark 4.1 to conclude that the family above cannot provide any solution for the connections ∇^- and ∇^c.

6. Balanced Hermitian structures on the Lie algebras $\mathfrak{h}_2$, $\mathfrak{h}_4$ and $\mathfrak{h}_5$

In this section we construct explicit solutions on compact nilmanifolds corresponding to the Lie algebras $\mathfrak{h}_2$, $\mathfrak{h}_4$ and $\mathfrak{h}_5$.

Let us consider the complex structure equations

$$d\omega^1 = d\omega^2 = 0, \qquad d\omega^3 = \omega^{12} + \omega^{1\bar 1} + b\,\omega^{1\bar 2} - \omega^{2\bar 2},$$

where $b \in \mathbb{R}$. According to [40, Proposition 13], the Lie algebras underlying this 1-parameter family of complex equations are:

$$\mathfrak{h}_2, \text{ for } b \in (-1,1); \qquad \mathfrak{h}_4, \text{ for } b = \pm 1; \qquad \mathfrak{h}_5, \text{ for any } b \text{ such that } b^2 > 1. \tag{6.1}$$

Notice that the latter condition defines a 1-parameter family of complex structures J on the Iwasawa manifold which are not equivalent to the standard J_0.

For any $t \neq 0$, let us consider F given by

$$F = \frac{i}{2}(\omega^{1\bar 1} + \omega^{2\bar 2} + t^2\,\omega^{3\bar 3}).$$

Since $D = -1$, $r = s = 1$ and the coefficients u, v, z in (3.7) vanish, it follows from Proposition 3.1 (ii) that all the Hermitian structures (J, F) are balanced.

Notice that the associated compact nilmanifolds are $\mathbb{T}^2$ bundles over $\mathbb{T}^4$ for any b, whereas the parameter t scales the fiber.

In terms of the real basis of 1-forms $\{e^1, \ldots, e^6\}$ defined by

$$e^1 + i\,e^2 = \omega^1, \quad e^3 + i\,e^4 = \omega^2, \quad e^5 + i\,e^6 = t\,\omega^3,$$

the structure equations are

$$\begin{cases} de^1 = de^2 = de^3 = de^4 = 0, \\ de^5 = t(b+1)e^{13} + t(b-1)e^{24}, \\ de^6 = -2t\,e^{12} - t(b-1)e^{14} + t(b+1)e^{23} + 2t\,e^{34}, \end{cases} \tag{6.2}$$

the complex structure J is given by $Je^1 = -e^2, Je^3 = -e^4, Je^5 = -e^6$, and the balanced J-Hermitian metric $g = e^1 \otimes e^1 + \cdots + e^6 \otimes e^6$ has the associated fundamental form $F = e^{12} + e^{34} + e^{56}$.

Use (6.2) to get $dF = 2t\,e^{125} + t(b+1)e^{136} + t(b-1)e^{145} - t(b+1)e^{235} + t(b-1)e^{246} - 2t\,e^{345}$. Due to (3.1) the torsion T of ∇^+ satisfies

$$\begin{aligned} T &= -2t\,e^{126} + t(b-1)e^{135} - t(b+1)e^{146} + t(b-1)e^{236} + t(b+1)e^{245} + 2t\,e^{346}, \\ dT &= -4t^2(b^2+3)e^{1234}. \end{aligned}$$

A direct calculation using (3.3) gives that the non-zero curvature forms $(\Omega^+)^i_j$ of the connection ∇^+ are:

$$\begin{aligned} (\Omega^+)^1_2 &= -4t^2e^{12} - 2t^2(b-1)e^{14} + 2t^2(b+1)e^{23} + 6t^2e^{34} - 2t^2b^2e^{56}, \\ (\Omega^+)^1_3 = (\Omega^+)^2_4 &= -t^2(b^2+b+1)e^{13} - t^2(b^2-b+1)e^{24}, \\ (\Omega^+)^1_4 = -(\Omega^+)^2_3 &= -2t^2b\,e^{12} - t^2(b^2-b+1)e^{14} + t^2(b^2+b+1)e^{23} + 2t^2b\,e^{34} + 4t^2b\,e^{56}, \\ (\Omega^+)^1_5 = (\Omega^+)^2_6 &= t^2b\,e^{15} + t^2b\,e^{26} - 2t^2e^{46}, \\ (\Omega^+)^1_6 = -(\Omega^+)^2_5 &= -t^2b\,e^{16} + t^2b\,e^{25} + 2t^2e^{36}, \\ (\Omega^+)^3_4 &= 6t^2\,e^{12} + 2t^2(b-1)e^{14} - 2t^2(b+1)e^{23} - 4t^2e^{34} + 2t^2b^2e^{56}, \\ (\Omega^+)^3_5 = (\Omega^+)^4_6 &= -2t^2e^{26} + t^2b\,e^{35} - t^2b\,e^{46}, \\ (\Omega^+)^3_6 = -(\Omega^+)^4_5 &= 2t^2e^{16} + t^2b\,e^{36} + t^2b\,e^{45}, \\ (\Omega^+)^5_6 = -(\Omega^+)^1_2 - (\Omega^+)^3_4 &= -2t^2e^{12} - 2t^2e^{34}. \end{aligned}$$

Similarly, applying (3.2), we calculate that the non-zero curvature forms $(\Omega^g)^i_j$ of the Levi-Civita connection ∇^g are:

$$
\begin{aligned}
(\Omega^g)^1_2 &= -3t^2e^{12} - \frac{3}{2}t^2(b-1)e^{14} + \frac{3}{2}t^2(b+1)e^{23} - \frac{t^2}{2}(b^2-5)e^{34} - \frac{t^2}{2}(b^2+1)e^{56},\\
(\Omega^g)^1_3 &= -\frac{3}{4}t^2(b+1)^2e^{13} - \frac{t^2}{4}(b^2-5)e^{24},\\
(\Omega^g)^1_4 &= -\frac{3}{2}t^2(b-1)e^{12} - \frac{3}{4}t^2(b-1)^2e^{14} + \frac{t^2}{4}(b^2-5)e^{23} + \frac{3}{2}t^2(b-1)e^{34} + t^2b\,e^{56},\\
(\Omega^g)^1_5 &= \frac{t^2}{4}(b+1)^2e^{15} - \frac{t^2}{4}(b-1)^2e^{26} + \frac{t^2}{2}(b-1)e^{46},\\
(\Omega^g)^1_6 &= \frac{t^2}{4}(b^2-2b+5)e^{16} + \frac{t^2}{4}(b+1)^2e^{25} + t^2e^{36} - \frac{t^2}{2}(b+1)e^{45},\\
(\Omega^g)^2_3 &= \frac{3}{2}t^2(b+1)e^{12} + \frac{t^2}{4}(b^2-5)e^{14} - \frac{3}{4}t^2(b+1)^2e^{23} - \frac{3}{2}t^2(b+1)e^{34} - t^2b\,e^{56},\\
(\Omega^g)^2_4 &= -\frac{t^2}{4}(b^2-5)e^{13} - \frac{3}{4}t^2(b-1)^2e^{24},\\
(\Omega^g)^2_5 &= \frac{t^2}{4}(b+1)^2e^{16} + \frac{t^2}{4}(b-1)^2e^{25} - \frac{t^2}{2}(b+1)e^{36},\\
(\Omega^g)^2_6 &= -\frac{t^2}{4}(b-1)^2e^{15} + \frac{t^2}{4}(b^2+2b+5)e^{26} + \frac{t^2}{2}(b-1)e^{35} - t^2e^{46},\\
(\Omega^g)^3_4 &= -\frac{t^2}{2}(b^2-5)e^{12} + \frac{3}{2}t^2(b-1)e^{14} - \frac{3}{2}t^2(b+1)e^{23} - 3t^2e^{34} + \frac{t^2}{2}(b^2-1)e^{56},\\
(\Omega^g)^3_5 &= \frac{t^2}{2}(b-1)e^{26} + \frac{t^2}{4}(b+1)^2e^{35} + \frac{t^2}{4}(b^2-1)e^{46},\\
(\Omega^g)^3_6 &= t^2e^{16} - \frac{t^2}{2}(b+1)e^{25} + \frac{t^2}{4}(b^2+2b+5)e^{36} - \frac{t^2}{4}(b^2-1)e^{45},\\
(\Omega^g)^4_5 &= -\frac{t^2}{2}(b+1)e^{16} - \frac{t^2}{4}(b^2-1)e^{36} + \frac{t^2}{4}(b-1)^2e^{45},\\
(\Omega^g)^4_6 &= \frac{t^2}{2}(b-1)e^{15} - t^2e^{26} + \frac{t^2}{4}(b^2-1)e^{35} + \frac{t^2}{4}(b^2-2b+5)e^{46},\\
(\Omega^g)^5_6 &= -\frac{t^2}{2}(b^2+1)e^{12} + t^2b\,e^{14} - t^2b\,e^{23} + \frac{t^2}{2}(b^2-1)e^{34}.
\end{aligned}
$$

Hence, $Hol(\nabla^+) \subset \mathfrak{su}(3)$ and the Pontrjagin classes of the connections ∇^g and ∇^+ are represented by

$$p_1(\nabla^g) = -\frac{t^4}{4\pi^2}(b^4+4b^2+11)e^{1234}, \quad p_1(\nabla^+) = -\frac{t^4}{\pi^2}(b^4+5b^2+10)e^{1234}.$$

As we mentioned above, $\mathfrak{h}_5$ is the nilpotent Lie algebra underlying the Iwasawa manifold. Notice that $\mathfrak{h}_2$ is the Lie algebra of $H^3 \times H^3$, where H^3 is the Heisenberg group. Let us denote by M_2, M_4, M_5 any compact nilmanifold whose underlying Lie algebra is isomorphic to $\mathfrak{h}_2$, $\mathfrak{h}_4$ or $\mathfrak{h}_5$, respectively. We can take dz_1 and dz_2 as (2,0)-forms at the level of the associated Lie group which descend to the forms ω^1 and ω^2 on M_2, M_4, M_5, so using again the abelian instanton given in Section 4 we get:

Theorem 6.1. *In the notation above and taking A as the abelian* SU(3)*-instanton given in* [9] *we have:*

$$dT = \frac{16\pi^2t^2(b^2+3)}{t^4(b^4+4b^2+11)-1}(p_1(\nabla^g) - p_1(A)),$$

$$dT = \frac{16\pi^2t^2(b^2+3)}{4t^4(b^4+5b^2+10)-1}(p_1(\nabla^+) - p_1(A)).$$

For any $b \in \mathbb{R}$ we can choose $t \neq 0$ such that

$$t^4(b^4+5b^2+10) > 1/4 \qquad \text{and} \qquad t^4(b^4+4b^2+11) > 1,$$

which, in view of (6.1)*, provides explicit valid solutions to the heterotic supersymmetry equations* (1.2) *with non-zero flux* $H = T$ *and constant dilaton satisfying the three-form Bianchi identity* (1.1) *for the Levi-Civita connection and for the (+)-connection on the compact nilmanifolds* M_2, M_4, M_5*.*

Remark 6.2. *Finally, a direct calculation for* ∇^- *and for the Chern connection* ∇^c *shows that*

$$p_1(\nabla^-) = \frac{t^4}{\pi^2}(b^2+3)e^{1234}, \qquad p_1(\nabla^c) = 0.$$

Since M_2*,* M_4 *and* M_5 *are torus bundles over a complex torus, notice that the same argument as in Remark 4.1 shows that the family above cannot provide any satisfactory solution for the connections* ∇^- *and* ∇^c*.*

7. The space of balanced structures on $\mathfrak{h}_6$

In this section we study the space of balanced Hermitian structures on the nilpotent Lie algebra $\mathfrak{h}_6$.

The complex equations

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{12} - \omega^{2\bar{1}},$$

define a complex structure J on $\mathfrak{h}_6$, and any complex structure on the Lie algebra $\mathfrak{h}_6$ is equivalent to J [40, Corollary 15]. Moreover, it is easy to see that any J-balanced structure F is equivalent to one of the form

$$F = \frac{i}{2}(\omega^{1\bar{1}} + \omega^{2\bar{2}} + t^2\,\omega^{3\bar{3}}),$$

for some $t \neq 0$.

From a real point of view, the whole space of balanced Hermitian structures on $\mathfrak{h}_6$ is described as follows. Let us consider the basis of 1-forms $\{e^1, \ldots, e^6\}$ given by

$$e^1 + i\,e^2 = \omega^1, \quad e^3 + i\,e^4 = \omega^2, \quad e^5 + i\,e^6 = t\,\omega^3.$$

Now, in terms of this basis, we have the structure equations

$$\begin{cases} de^1 = de^2 = de^3 = de^4 = 0, \\ de^5 = 2t\,e^{13}, \\ de^6 = 2t\,e^{14}. \end{cases} \tag{7.1}$$

The complex structure J is given by $Je^1 = -e^2, Je^3 = -e^4, dJe^5 = -e^6$, the J-Hermitian metric $g = e^1 \otimes e^1 + \cdots + e^6 \otimes e^6$ has the associated fundamental form $F = e^{12} + e^{34} + e^{56}$.

The structure equations (7.1) yield $dF = 2t(e^{136} - e^{145})$. Consequently, applying (3.1), we obtain that the torsion T of ∇^+ satisfies

$$T = -2t(e^{236} - e^{245}), \qquad dT = -8t^2 e^{1234}.$$

Using (3.3) we calculate that the non-zero curvature forms $(\Omega^+)^i_j$ for the connection ∇^+ are given by:

$$(\Omega^+)^1_2 = 2t^2(e^{34} + e^{56}), \quad (\Omega^+)^1_3 = (\Omega^+)^2_4 = -t^2(3e^{13} + e^{24}), \quad (\Omega^+)^1_4 = -(\Omega^+)^2_3 = -t^2(3e^{14} - e^{23}),$$

$$(\Omega^+)^1_5 = (\Omega^+)^2_6 = t^2(e^{15} - e^{26}), \quad (\Omega^+)^1_6 = -(\Omega^+)^2_5 = t^2(e^{16} + e^{25}), \quad (\Omega^+)^3_4 = 2t^2(e^{12} - e^{56}),$$

$$(\Omega^+)^3_5 = (\Omega^+)^4_6 = t^2(e^{35} + e^{46}), \quad (\Omega^+)^3_6 = -(\Omega^+)^4_5 = -t^2(e^{36} - e^{45}), \quad (\Omega^+)^5_6 = -2t^2(e^{12} + e^{34}),$$

so (2.6) holds and the first Pontrjagin class is represented by

$$p_1(\nabla^+) = -\frac{2t^4}{\pi^2} e^{1234}.$$

Let us denote by M_6 any compact nilmanifold whose underlying Lie algebra is isomorphic to $\mathfrak{h}_6$. We can take dz_1 and dz_2 as (2,0)-forms at the level of the Lie group corresponding to $\mathfrak{h}_6$ which descend to the forms ω^1 and ω^2 on M_6, so using again the abelian instanton given in Section 4 we get:

Theorem 7.1. *In the notation above and taking A as the abelian* SU(3)*-instanton given in* [9] *we have:*

$$dT = \tfrac{32\pi^2 t^2}{8t^4-1}(p_1(\nabla^+) - p_1(A)).$$

Thus, for any t such that $t^4 > \frac{1}{8}$ we obtain explicit valid solutions to the heterotic supersymmetry equations (1.2) *with non-zero flux $H = T$ and constant dilaton satisfying the three-form Bianchi identity* (1.1) *for the (+)-connection on the compact nilmanifold M_6.*

Remark 7.2. *The Pontrjagin classes of the Levi-Civita connection, ∇^- and the Chern connection are represented by*

$$p_1(\nabla^g) = 0, \qquad p_1(\nabla^-) = \frac{2t^4}{\pi^2}e^{1234}, \qquad p_1(\nabla^c) = 0.$$

Since the nilmanifold M_6 is a torus bundle over a complex torus, the same argument as in Remark 4.1 shows that there is no way to find a satisfactory solution for the connections ∇^g, ∇^- and ∇^c on the whole space of invariant balanced Hermitian structures on M.

8. Balanced structures on the Lie algebra $\mathfrak{h}_{19}^-$

In this section we construct compact valid solutions to (1.2) with non-zero flux and constant dilaton satisfying anomaly cancellation condition (1.1) using the curvature R^c of the Chern connection.

Consider the complex structure equations

$$d\omega^1 = 0, \quad d\omega^2 = \omega^{13} + \omega^{1\bar{3}}, \quad d\omega^3 = i(\omega^{1\bar{2}} - \omega^{2\bar{1}}),$$

which in view of (3.6) correspond to a complex structure J on the 3-step nilpotent Lie algebra $\mathfrak{h}_{19}^-$.

The associated real structure equations are

$$(8.1) \qquad \begin{cases} de^1 = de^2 = de^5 = 0, \\ de^3 = 2e^{15}, \\ de^4 = 2e^{25}, \\ de^6 = 2(e^{13} + e^{24}), \end{cases}$$

and the complex structure J is given by $Je^1 = -e^2, Je^3 = -e^4, Je^5 = -e^6$. The fundamental form F of the J-Hermitian metric $g = e^1 \otimes e^1 + \cdots + e^6 \otimes e^6$ is given by $F = e^{12} + e^{34} + e^{56}$. It follows from Proposition 3.1 (i) that the structure (J, g) is balanced.

The structure equations (8.1) imply $dF = -2(e^{135} + e^{145} - e^{235} + e^{245})$. Apply (3.1) to verify that the torsion T satisfies

$$T = 2(e^{136} + e^{146} - e^{236} + e^{246}) \qquad dT = -8(e^{1234} + e^{1256}).$$

Using (3.2), (3.3) and (3.4) we obtain that the non-zero curvature forms $(\Omega^c)^i_j$ and $(\Omega^+)^i_j$ of the Chern connection and the (+)-connection are given by:

$$\begin{aligned}
&(\Omega^c)^1_2 = -2e^{34} - 2e^{56}, \quad (\Omega^c)^1_3 = (\Omega^c)^2_4 = -e^{13} - e^{24}, \quad (\Omega^c)^1_4 = -(\Omega^c)^2_3 = 2e^{13} + e^{14} - e^{23} + 2e^{24},\\
&(\Omega^c)^1_5 = (\Omega^c)^2_6 = e^{16} - e^{25}, \quad (\Omega^c)^1_6 = (\Omega^c)^2_5 - e^{15} - e^{26}, \quad (\Omega^c)^3_4 = -2e^{12} + 2e^{56},\\
&(\Omega^c)^3_5 = (\Omega^c)^4_6 = -e^{36} + e^{45}, \quad (\Omega^c)^3_6 = -(\Omega^c)^4_5 = e^{35} + e^{46}, \quad (\Omega^c)^5_6 = -(\Omega^c)^1_2 - (\Omega^c)^3_4 = 2e^{12} + 2e^{34}.
\end{aligned}$$

$$\begin{aligned}
&(\Omega^+)^1_2 = -2e^{34} + 2e^{56}, \quad (\Omega^+)^1_3 = (\Omega^+)^2_4 = -3e^{13} - 3e^{24},\\
&(\Omega^+)^1_4 = -(\Omega^+)^2_3 = -2e^{13} - e^{14} + e^{23} - 2e^{24}, \quad (\Omega^+)^1_5 = (\Omega^+)^2_6 = -3e^{15} - 2e^{16} - e^{26},\\
&(\Omega^+)^1_6 = -(\Omega^+)^2_5 = -e^{16} + 3e^{25} + 2e^{26}, \quad (\Omega^+)^3_4 = -2e^{12} - 2e^{56},\\
&(\Omega^+)^3_5 = (\Omega^+)^4_6 = e^{35} + 2e^{36} - e^{46}, \quad (\Omega^+)^3_6 = -(\Omega^+)^4_5 = -e^{36} - e^{45} - 2e^{46},\\
&(\Omega^+)^5_6 = -(\Omega^+)^1_2 - (\Omega^+)^3_4 = 2e^{12} + 2e^{34};
\end{aligned}$$

A direct calculation shows that the Pontrjagin classes are represented by

$$p_1(\nabla^+) = -\frac{2}{\pi^2}(3e^{1234} + e^{1256}), \qquad p_1(\nabla^c) = -\frac{2}{\pi^2}(e^{1234} + e^{1256}).$$

Let M_{19} be a compact nilmanifold corresponding to the Lie algebra $\mathfrak{h}_{19}^-$. From (8.1) we have that M_{19} is an S^1-bundle over a compact 5-nilmanifold N, which is a $\mathbb{T}^2$-bundle over $\mathbb{T}^3$.

Lemma 8.1. *For each $\lambda, \mu \in \mathbb{R}$, let $A_{\lambda,\mu}$ be the $\mathrm{U}(3)$-connection on M_{19} with respect to structure (J, g) defined by the connection forms*

$$(\sigma^{A_{\lambda,\mu}})^2_3 = (\sigma^{A_{\lambda,\mu}})^2_5 = (\sigma^{A_{\lambda,\mu}})^4_5 - \lambda\, e^1 - \mu\, e^6, \qquad (\sigma^{A_{\lambda,\mu}})^i_j = \lambda\, e^1 + \mu\, e^6,$$

for $1 \le i < j \le 6$ such that $(i, j) \ne (2, 3), (2, 5), (4, 5)$. Then, $A_{\lambda,\mu}$ is an $\mathrm{SU}(3)$-instanton and

$$p_1(A_{\lambda,\mu}) = -\frac{15}{\pi^2}\mu^2 e^{1234}.$$

Proof. A direct calculation shows that the curvature forms $(\Omega^{A_{\lambda,\mu}})^i_j$ of the connection $A_{\lambda,\mu}$ are given by

$$(\Omega^{A_{\lambda,\mu}})^2_3 = (\Omega^{A_{\lambda,\mu}})^2_5 = (\Omega^{A_{\lambda,\mu}})^4_5 = -2\mu(e^{13} + e^{24}), \qquad (\Omega^{A_{\lambda,\mu}})^i_j = 2\mu(e^{13} + e^{24}),$$

for $1 \le i < j \le 6$ such that $(i, j) \ne (2, 3), (2, 5), (4, 5)$. Now it is clear that $A_{\lambda,\mu}$ satisfies (2.7). □

Theorem 8.2. *Let $A_{\lambda,\mu}$ be the $\mathrm{SU}(3)$-instanton above.*

(i) *If $\mu^2 = \frac{4}{15}$, then*

$$dT = 4\pi^2(p_1(\nabla^+) - p_1(A_{\lambda,\mu})).$$

(ii) *If $\mu = 0$, then $p_1(A_{\lambda,0}) = 0$ and*

$$dT = 4\pi^2\,(p_1(\nabla^c) - p_1(A_{\lambda,0})).$$

Hence, we obtain explicit valid solutions to the heterotic supersymmetry equations (1.2) with non-zero flux $H = T$ and constant dilaton satisfying the three-form Bianchi identity (1.1) for the Chern connection and the (+)-connection on the compact nilmanifold M_{19}.

Remark 8.3. *During the preparation of the paper we learned that a compact example solving (1.2) with non-zero flux, constant dilaton satisfying (1.1) with respect to a metric connection on the tangent bundle, and trivial instanton ($A = 0$) on M_3 is announced [26].*

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