A Decomposition of a Linear Model

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Abstract

In this paper we consider a general linear model in a continuous time. We propose a decomposition of the process which helps us to understand the structure of the model. Moreover, the sufficiency of the BLUE estimator of the expectation of the process can be characterized in terms of the Gaussian character of a component of the decomposition.

Key words: linear model, decomposition, Gaussian distribution, sufficiency

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1 Introduction

The present paper investigates a linear model in continuous time. The objective is to give a decomposition of the process which clarifies its structure. This decomposition has as a starting point the study of the characterization of the Gaussian distribution in terms of the BLUE estimator sufficiency (see Bischoff et al. [1,2] for the discrete case and Ibarrola and Pérez-Palomares [3,4] for the continuous time).

In the literature, there are several decompositions of a classical linear model $Y = X\beta + \epsilon$, where Y is an observable random vector, X is the known design matrix, β is a vector of unknown parameters, and ϵ is a random vector with zero expectation and a known (up to a constant) dispersion matrix. The most classical decomposition consists of writing the vector Y as the sum of the BLUE estimator plus the vector of residuals. When the covariance matrix of ϵ is singular, then there exist linear combinations of the parameters which are completely known when the vector Y is observed. In Nordstrom [5] and in Paige [6] we can find decompositions in which the deterministic part is recovered from the model.

In particular, in Nordstrom [5] the decomposition is derived from an orthogonal partition of R^n and it essentially consists of the following terms:

$$Y = Y_1 + Y_2 + Y_3, \tag{1}$$

where Y_1 is a deterministic component, Y_2 is the BLUE estimator for its expectation and Y_3 is a residual part, i.e. its distribution does not depend on the parameter β. In Paige [6] the authors present a new derivation of the generalized singular value decomposition for matrices and they obtain a decomposition of the linear model as an application. There are three important components in its decomposition: one is a deterministic part, another is a random component which serves to estimate the parameter β, and finally there is a part which does not give information on the parameters.

In this paper, we present a decomposition of a linear model in continuous time, which can be applied to a discrete time linear model, obtaining a different decomposition to those given in the literature. As we will see, if we apply to a discrete model the decomposition developed in this paper, we obtain:

$$Y = Y_1^* + Y_2^* + Y_3^* + Y_4^*,$$

where Y_1^* is a deterministic component equals to Y_1 in (1); Y_2^* is the BLUE for its expectation; Y_3^* is a vector which contains information on the parameter and on the distribution of certain estimators (under some hypotheses); and Y_4^* is a residual part. Moreover, we can give a further decomposition of Y_3^* in two parts, $Y_3^* = Y_3^{*(1} + Y_3^{*(2}$, where $Y_3^{*(1}$ is the BLUE for its expectation and $Y_3^{*(2}$ is a residual term. If we compare our decomposition with (1), we see that $Y_2 = Y_2^* + Y_3^{*(1}$ and $Y_3 = Y_4^* + Y_3^{*(2}$. Therefore, the difference in our decomposition is that we factorize the BLUE term Y_2 into two estimators Y_2^* and $Y_3^{*(1}$ where the last one can give information on the distribution of some linear estimators. The same happens with the residual part, that is, we can give the distribution of $Y_3^{*(2}$ in certain situations.

The structure of the paper is the following: in this section we recall some definitions used throughout the paper (most of them are introduced in Ibarrola and Pérez-Palomares [4]). In Section 2 we introduce each component of the decomposition and, finally, in Section 3 we study the structure and the properties of each component.

From now on, let $(Z_t, t \in [0,T])$, $T > 0$, be a stochastic process with distribution P_0 in $(\mathbb{R}^{[0,T]}, \mathcal{F}_T)$ where $\mathcal{F}_T$ is the $\sigma-$algebra generated by $Z_t, t \in [0,T]$. Let E_0 be the mathematical expectation with respect to P_0. Suppose that $E_0[Z_t] = 0$, $t \in [0,T]$ and $E_0[Z_s Z_t] = B(s,t)$, $s,t \in [0,T]$ is a known continuous function in $[0,T]\times[0,T]$. For each $\theta \in \mathbb{R}^p$, we denote by P_θ the distribution of the process $(X_t, t \in [0,T])$ which is defined as $X_t = A(t)\theta + Z_t$, $t \in [0,T]$, where $A(t)'$ is a vector in $\mathbb{R}^p$, with known continuous components in $[0,T]$. Let μ be the normal distribution on $(\mathbb{R}^p, \mathcal{B}(\mathbb{R}^p))$ with zero mean and covariance matrix I. $\overline{P}$ denotes the measure defined on $(\mathbb{R}^{[0,T]}, \mathcal{F}_T)$ as

$$\overline{P}(A) = \int_{\mathbb{R}^p} P_\theta(A) d\mu(\theta), \quad A \in \mathcal{F}_T.$$

The mathematical expectation with respect to $\overline{P}$ will be denoted by $\overline{E}$ and with respect to P_θ by E_θ. Thus, the process $(X_t, t \in [0,T])$ is an element of $L^2(\mathbb{R}^{[0,T]}, \mathcal{F}_T, \overline{P})$ with

$$\overline{E}[X_t] = 0, \qquad \overline{E}[X_s X_t] = B(s,t) + A(s)A(t)', \quad s,t \in [0,T]. \tag{2}$$

We denote by $\overline{\mathcal{L}}(X_t, t \in [0,T])$ the closure in $L^2(\mathbb{R}^{[0,T]}, \mathcal{F}_T, \overline{P})$ of the set of finite linear combinations of type $\sum_{i=1}^n c_i X_{t_i}$, $c_i \in \mathbb{R}$, $t_i \in [0,T]$. $\overline{\mathcal{L}}(X_t, t \in [0,T])$ is a Hilbert space with the inner product $< Y, Z >= \overline{E}[YZ]$. We are interested in estimators constructed by the observed paths of the process $(X_t, t \in [0,T])$ in a linear way, i.e. we are interested in estimators belonging to the class $\overline{\mathcal{L}}(X_t, t \in [0,T])$. We refer to this class of estimators as linear estimators.

An estimable linear combination is a linear combination of θ which can be unbiasedly estimated by elements of $\overline{\mathcal{L}}(X_t, t \in [0,T])$. We say that a linear estimator is the BLUE for an estimable linear combination of θ if it is of minimum variance among all linear estimators unbiased for the linear combination. Let $(\theta_r, r \in K)$ be a family of elements in $\overline{\mathcal{L}}(X_t, t \in [0,T])$, where K is a compact subset of $\mathbb{R}$. If K is not a finite set, we shall suppose that $(\theta_r, r \in K)$ is continuous in square mean sense. We denote by $\overline{\mathcal{L}}(\theta_r, r \in K)$ the closure in $L^2(\mathbb{R}^{[0,T]}, \mathcal{F}_T, \overline{P})$ of the linear combinations of $(\theta_r, r \in K)$. Then $(\theta_r, r \in K)$ is linearly sufficient if the BLUE of each estimable linear combination belongs to $\overline{\mathcal{L}}(\theta_r, r \in K)$.

Let $\hat{\theta} = (\hat{\theta}_1, \cdots, \hat{\theta}_p)'$ be the estimator verifying

$$A(t) =< X_t, \hat{\theta}' >= \overline{E}[X_t \hat{\theta}'], \quad t \in [0,T], \tag{3}$$

which always exists (see for example Ibarrola and Pérez-Palomares [4]). Therefore, we can write

$$E_\theta[Y] = \overline{E}[Y\hat{\theta}']\theta, \quad Y \in \overline{\mathcal{L}}(X_t, t \in [0,T]), \tag{4}$$

and, from (2),

$$\overline{E}[YZ] = E_0[YZ] + \overline{E}[Y\hat{\theta}']\overline{E}[\hat{\theta}Z], \quad Y, Z \in \overline{\mathcal{L}}(X_t, t \in [0,T]). \tag{5}$$

For a linear estimator Y unbiased for 0 we have from (4) that $\overline{E}[Y\hat{\theta}'] = 0$ and using (5) we see that $E_0[Y\hat{\theta}'] = 0$. This means that $\hat{\theta}$ is uncorrelated with all linear estimators unbiased for 0, so $\hat{\theta}$ is the BLUE for its expectation. Finally, denoting by $\Sigma = \overline{E}[\hat{\theta}\hat{\theta}']$, it holds that $A(t)\Sigma^- \hat{\theta}$ is the BLUE for $A(t)\theta$, $t \in [0,T]$, where Σ^- denotes a generalized inverse of Σ. We shall also use the following notation: $C = E_0[\hat{\theta}\hat{\theta}']$ and $S(t) = E_0[X_t\hat{\theta}']$. Observe that $C = \Sigma(I - \Sigma)$ and $S(t) = A(t)(I - \Sigma)$.

2 Decomposition of the linear model

First, we observe that it is possible to find deterministic elements in $\overline{\mathcal{L}}(X_t, t \in [0,T])$ since we have not imposed restrictions on the covariance function B. The first decomposition that we give consists simply in recovering the main deterministic component.

We define

$$B^{\perp} = \{Z \in \overline{\mathcal{L}}(X_t, t \in [0,T]) : E_0[ZX_t] = 0, t \in [0,T]\}.$$

$B^{\perp}$ is the class of all deterministic linear estimators. Then, we consider for each $t \in [0,T]$ the orthogonal projection of X_t onto $B^{\perp}$ with respect to the inner product $< Y, Z >= \overline{E}[YZ]$, let c_t be this projection. Obviously we can write

$$X_t = c_t + X_t^*, \quad t \in [0,T],$$

with $X_t^* = X_t - c_t$. The process $(X_t^*, t \in [0,T])$ satisfies that $\overline{\mathcal{L}}(X_t^*, t \in [0,T])$ does not contain deterministic linear estimators.

Note 1. If equality (3) is satisfied with E_0, i.e. if there exists a linear estimator $\hat{\theta}^*$ such that

$$A(t) = E_0[X_t \hat{\theta}^{*\prime}], \quad t \in [0,T], \tag{6}$$

then for a $Z \in \overline{\mathcal{L}}(X_t, t \in [0,T])$, we have from (3) and (6)

$$\overline{E}[Z\hat{\theta}'] = E_0[Z\hat{\theta}^{*\prime}].$$

Thus, using (4), $E_\theta[Z] = E_0[Z\hat{\theta}^{*\prime}]\theta$, which means that each $Z \in \overline{\mathcal{L}}(X_t, t \in [0,T])$ with variance zero, it has zero expectation with respect to P_θ, and then $B^{\perp}$ is a trivial set.

Moreover, in this case it is not necessary to consider the measure $\overline{P}$ because each convergent element, in square mean sense with respect to P_0, is convergent with respect to P_θ for all θ (and with respect to $\overline{P}$). Thus, $\overline{\mathcal{L}}(X_t, t \in [0,T])$ would be the set of the linear combinations $\sum_{k=1}^{n} a_k X_{t_k}$ and their limits in square mean sense with respect to P_0.

Lemma 1. A representation of the deterministic linear estimator $(c_t, t \in [0,T])$ is given by

$$c_t = A(t)\Sigma^-(I - CC^+)\hat{\theta}, \quad t \in [0,T],$$

where C^+ is the Moore-Penrose inverse of C.

Proof. First, we see that $c_t \in B^\perp$ because $(I - CC^+)E_0[\hat{\theta}\hat{\theta}'] = (I - CC^+)C = 0$, so $(I - CC^+)\hat{\theta} = 0$, P_0-a.s. showing that $c_t \in B^\perp$. We have to check that $X_t - c_t$ is orthogonal to $B^\perp$ (with respect to the inner product given by $\overline{P}$). Let $Z \in B^\perp$, then from (5), $\overline{E}[\hat{\theta}Z] = \overline{E}[\hat{\theta}\hat{\theta}']\overline{E}[\hat{\theta}Z]$, that is

$$(I - \Sigma)\overline{E}[\hat{\theta}Z] = 0. \tag{7}$$

Thus, again from (5), $\overline{E}[(X_t - c_t)Z] = \overline{E}[(X_t - c_t)\hat{\theta}']\overline{E}[\hat{\theta}Z] = A(t)(I - \Sigma^-(I - CC^+)\Sigma)\overline{E}[\hat{\theta}Z] = A(t)(\Sigma^- CC^+\Sigma)\overline{E}[\hat{\theta}Z]$. The Moore-Penrose inverse C^+ satisfies $(CC^+)' = CC^+$. On the other hand $C\Sigma = \Sigma(I - \Sigma)\Sigma = \Sigma C$. With these properties and (7) we conclude

$$\overline{E}[(X_t - c_t)Z] = A(t)(\Sigma^-(C^+)'\Sigma C)\overline{E}[\hat{\theta}Z] = A(t)(\Sigma^-(C^+)'\Sigma\Sigma)(I - \Sigma)\overline{E}[\hat{\theta}Z] = 0,$$

for all $Z \in B^\perp$. This shows the lemma.

We shall now give some properties of the process $(X_t^*, t \in [0,T])$.

Lemma 2. Let $a(t) := \overline{E}[X_t^*\hat{\theta}']$, $t \in [0,T]$. Then, there exists an estimator

$\hat{\theta}^* \in \overline{\mathcal{L}}(X_t^*, t \in [0,T])$ such that

$$a(t) = E_0[X_t^* \hat{\theta}^{*\prime}], \quad t \in [0,T]. \tag{8}$$

Proof. From Lemma 1, $a(t) = A(t)\Sigma^- CC^+\Sigma = A(t)(I-\Sigma)C^+\Sigma = S(t)C^+\Sigma = E_0[X_t\hat{\theta}' C^+\Sigma]$. Now, $X_t = X_t^*, P_0$-a.s. and we take $\widetilde{\theta} \in \overline{\mathcal{L}}(X_t^*, t \in [0,T])$ with $\hat{\theta} = \widetilde{\theta}$, P_0-a.s. (for example the projection of $\hat{\theta}$ onto this space). Finally, defining $\hat{\theta}^* = \Sigma(C^+)'\widetilde{\theta}$, we prove Lemma 2.

Note 2. (i) Lemma 2 implies that,

$$E_\theta[Z] = E_0[Z\hat{\theta}^{*\prime}]\theta, \quad Z \in \overline{\mathcal{L}}(X_t^*, t \in [0,T]). \tag{9}$$

(ii) According to Note 1, the class $\overline{\mathcal{L}}(X_t^*, t \in [0,T])$ coincides with the limits of $\sum_{k=1}^n a_k X_{t_k}^*$ in square mean sense with respect to P_0. Moreover, an equality P_0-a.s. among elements of $\overline{\mathcal{L}}(X_t^*, t \in [0,T])$ is satisfied $\overline{P}$-a.s. (and therefore P_θ-a.s. for all θ).

To obtain our decomposition, we consider the eigenvalues λ_k and the eigenfunctions $e_k(t)$, $t \in [0,T]$, of the covariance function $B(t,s)$, which verify that

$$B(t,s) = \sum_{k=1}^\infty \lambda_k e_k(t) e_k(s), \quad t,s \in [0,T],$$

where

$$\lambda_k > 0, \qquad \lambda_k e_k(t) = \int_0^T B(t,s)e_k(s)ds, \quad t \in [0,T]$$

and $\int_0^T e_k(t)e_j(t)dt = \delta_{kj}$, with $\delta_{kj} = 0$ if $k \neq j$ and $\delta_{kj} = 1$ if $k = j$. Define $\overline{X}_k^* = \frac{1}{\sqrt{\lambda_k}} \int_0^T X_t^* e_k(t)dt$, $k = 1, 2, \ldots$ The Karhunen-Loève expansion yields (see for instance Todorovic [7])

$$X_t^* = \sum_{k=1}^\infty \sqrt{\lambda_k} e_k(t) \overline{X}_k^* = \sum_{k=1}^\infty E_0[X_t^* \overline{X}_k^*]\overline{X}_k^*, \quad P_0 - a.s. \tag{10}$$

Taking into account (8), the estimator $\hat{\theta}^*$ generates the BLUE estimators for the class $\overline{\mathcal{L}}(X_t^*, t \in [0,T])$ and thus we define, as in Ibarrola and Pérez-Palomares [4], the subset $S^\perp = \{Z \in \overline{\mathcal{L}}(X_t^*, t \in [0,T])$ with $E_0[Z\hat{\theta}^{*\prime}] = 0\}$ or equivalently, from (9), $S^\perp = \{Z \in \overline{\mathcal{L}}(X_t^*, t \in [0,T])$ unbiased for $0\}$. We will now define the index sets

$$T_{S^\perp} = \{k \in \mathbb{N} : \overline{X}_k^* \in S^\perp\}$$

and

$$T_S = \{k \in \mathbb{N} : \overline{X}_k^* = \hat{\theta}^{*\prime} c, P_0 - a.s. \text{ for some } c \in \mathbb{R}^p\}.$$

We have the following

Lemma 3. $T_S = \{k \in \mathbb{N} : e_k(t) = a(t)c, \text{ for some } c \in \mathbb{R}^p\}$.

Proof. First, by definition of $\overline{X}_k^*$, $E_0[X_t^*\overline{X}_k^*] = \sqrt{\lambda}_k e_k(t)$. Thus, $\overline{X}_k^* = \hat{\theta}^{*\prime} c \Leftrightarrow E_0[X_t^*\overline{X}_k^*] = E_0[X_t^*\hat{\theta}^{*\prime}]c \Leftrightarrow \sqrt{\lambda}_k e_k(t) = a(t)c$, where the last equivalence follows from (8). Lemma 3 is proved.

Using (10) we can factorize the process $(X_t^*, t \in [0,T])$ as

$$X_t^* = Y_t + W_t + G_t, \tag{11}$$

with $Y_t := \sum_{k \in T_S} E_0[X_t^*\overline{X}_k^*]\overline{X}_k^*$,

$W_t := \sum_{k \in T_{S\perp}} E_0[X_t^*\overline{X}_k^*]\overline{X}_k^*$,

$G_t := \sum_{k \in (T_S \cup T_{S^\perp})^c} E_0[X_t^*\overline{X}_k^*]\overline{X}_k^*$,

where $(T_S \cup T_{S^\perp})^c$ denotes the complementary set of $(T_S \cup T_{S^\perp})$.

Note 3. From (10) it follows that

$$\overline{\mathcal{L}}(X_t^*, t \in [0,T]) = \overline{\mathcal{L}}(\overline{X}_k^*, k = 1, 2, \dots)$$

and

$$\overline{\mathcal{L}}(\overline{X}_k^*, k=1,2,\dots) = \overline{\mathcal{L}}(\overline{X}_k^*, k \in T_S) \oplus \overline{\mathcal{L}}(\overline{X}_k^*, k \in T_{S^\perp}) \oplus \overline{\mathcal{L}}(\overline{X}_k^*, k \in (T_S \cup T_{S^\perp})^c),$$

where $\oplus$ means the orthogonal direct sum with respect to P_0. Thus the components Y_t, W_t and G_t can be seen as the orthogonal projections (with respect to P_0) of X_t^* onto each space $\overline{\mathcal{L}}(\overline{X}_k^*, k \in T_S)$, $\overline{\mathcal{L}}(\overline{X}_k^*, k \in T_{S^\perp})$ and $\overline{\mathcal{L}}(\overline{X}_k^*, k \in (T_S \cup T_{S^\perp})^c$, respectively. From definition, it is obvious that for $k \in T_S$, $\overline{X}_k^* = \frac{1}{\sqrt{\lambda_k}} \int_0^T Y_t e_k(t) dt$; for $k \in T_{S^\perp}$, $\overline{X}_k^* = \frac{1}{\sqrt{\lambda_k}} \int_0^T W_t e_k(t) dt$ and for $k \in (T_S \cup T_{S^\perp})^c$, $\overline{X}_k^* = \frac{1}{\sqrt{\lambda_k}} \int_0^T G_t e_k(t) dt$.

In order to abbreviate the notation, we define

$\overline{\mathcal{L}}_Y := \overline{\mathcal{L}}(\overline{X}_k^*, k \in T_S) = \overline{\mathcal{L}}(Y_t, t \in [0,T])$,

$\overline{\mathcal{L}}_W := \overline{\mathcal{L}}(\overline{X}_k^*, k \in T_{S^\perp}) = \overline{\mathcal{L}}(W_t, t \in [0,T])$ and

$\overline{\mathcal{L}}_G := \overline{\mathcal{L}}(\overline{X}_k^*, k \in (T_S \cup T_{S^\perp})^c) = \overline{\mathcal{L}}(G_t, t \in [0,T])$.

3 Structure of the decomposition

In this section we will see the role of each process in our decomposition.

Lemma 4. (i) $(W_t, t \in [0,T])$ is a process with null expectation and therefore its distribution is independent of the parameter θ.

(ii) If $Z \in \overline{\mathcal{L}}(X_t^*, t \in [0,T])$ is the BLUE for its expectation, then $Z \in \overline{\mathcal{L}}_Y \oplus \overline{\mathcal{L}}_G$.

Proof. (i) It follows from the definition of W_t and from the fact that for $k \in T_{S^\perp}$, $E_\theta[\overline{X}_k^*] = 0$.

(ii) For each $Z \in \overline{\mathcal{L}}(X_t^*, t \in [0,T])$ we can write, using the decomposition in (11),

$$Z = Z_Y + Z_W + Z_G,$$

where Z_Y, Z_W and Z_G are the components in the spaces $\overline{\mathcal{L}}_Y$, $\overline{\mathcal{L}}_W$ and $\overline{\mathcal{L}}_G$, respectively. If Z is the BLUE for its expectation, then Z and Z_W are uncorrelated since Z_W is unbiased for 0. Then, $0 = E_0[ZZ_W] = E_0[Z_Y Z_W] + E_0[Z_W Z_W] + E_0[Z_G Z_W] = E_0[Z_W Z_W]$, showing that $Z_W = 0$, P_0-a.s. and $Z = Z_Y + Z_G$. It concludes the proof of Lemma 4.

Theorem 1. (i) $E_\theta[Y_t] = a(t)R\theta$, $t \in [0,T]$ where R is a $p \times p$ idempotent matrix.

(ii) $(Y_t, t \in [0,T])$ is the BLUE for $(a(t)R\theta, t \in [0,T])$. Thus, if $M\theta$ is an estimable linear combination with $\overline{\mathcal{L}}(X_t^*, t \in [0,T])$, then the BLUE of $MR\theta$ belongs to the class $\overline{\mathcal{L}}_Y$. All the elements in $\overline{\mathcal{L}}_Y$ are BLUE estimators for their expectations.

Proof. (i) Since $E_0[X_t^* \overline{X}_k^*] = \sqrt{\lambda_k} e_k(t)$,

$$E_\theta[Y_t] = \textstyle\sum_{k \in T_S} E_0[X_t^* \overline{X}_k^*] E_\theta[\overline{X}_k^*] = \sum_{k \in T_S} e_k(t) \int_0^T e_k(t) E_\theta[X_t^*] dt =$$

$$= \textstyle\sum_{k \in T_S} e_k(t) \int_0^T e_k(t) a(t)\theta dt.$$

Using Lemma 3, for each $k \in T_S$, there is a $c_k \in \mathbb{R}^p$ such that $e_k(t) = a(t)c_k$, and then

$$E_\theta[Y_t] = a(t)(\sum_{k \in T_S} c_k c_k')(\int_0^T a(t)' a(t) dt)\theta = a(t)R\theta,$$

with

$$R = \sum_{k \in T_S} (c_k c_k')(\int_0^T a(t)' a(t) dt). \tag{12}$$

On the other hand,

$$Rc_j = \left(\sum_{k\in T_S} c_k c_k'\right) \int_0^T a(t)'a(t)c_j dt = \left(\sum_{k\in T_S} c_k c_k'\right) \int_0^T a(t)'e_j(t)dt =$$

$$= \sum_{k\in T_S} c_k \int_0^T e_k(t)'e_j(t)dt = c_j,$$

which proves that R is a idempotent matrix.

(ii) is immediate because Lemma 3 establishes that, for each $k \in T_S$, $\overline{X}_k^*$ is the BLUE for its expectation, which yields that Y_t is the BLUE for its expectation. Theorem 1 is proved.

Theorem 2. (i) $E_\theta[G_t] = a(t)(I - R)\theta$, $t \in [0, T]$ where R is defined in (12).

(ii) Let $M\theta$ be an estimable linear combination with $\overline{\mathcal{L}}(X_t^*, t \in [0, T])$, then the BLUE of $M(I - R)\theta$ belongs to the class $\overline{\mathcal{L}}_G$.

Proof. (i) is obvious from (11), Theorem 1(i) and Lemma 4(i). (ii) From Theorem 1(i) and part (i) of the present theorem, we have for each $U \in \overline{\mathcal{L}}_Y$, $E_0[U\hat{\theta}^{*\prime}] = cR$ for some vector c, and for each $U \in \overline{\mathcal{L}}_G$, $E_0[U\hat{\theta}^{*\prime}] = c(I - R)$, for a c. Then, let Z be the BLUE for $M(I - R)$ with $M\theta$ an estimable linear combination. From Lemma 4(ii) $Z = Z_Y + Z_G$, where Z_Y and Z_G are the corresponding elements in $\overline{\mathcal{L}}_Y$ and $\overline{\mathcal{L}}_G$, respectively. Then, $M(I - R) = E_0[Z\hat{\theta}^{*\prime}] = E_0[Z_Y\hat{\theta}^{*\prime}] + E_0[Z_G\hat{\theta}^{*\prime}] = c_1R + c_2(I - R)$. Multiplying by R and using the fact that it is idempotent, it holds that $0 = c_1R = E_0[Z_Y\hat{\theta}^{*\prime}]$. But, Z_Y is the BLUE for its expectation, so $Z_Y = 0$, a.s. showing that $Z = Z_G \in \overline{\mathcal{L}}_G$. Theorem 2 is proved.

Now we shall see that the distribution of the component $(G_t, t \in [0, T])$ can be characterized.

First, we factorize the estimator $\hat{\theta}$ as follows:

$$\hat{\theta} = \hat{\theta}_D + \hat{\theta}^* = \hat{\theta}_D + \hat{\theta}^*_Y + \hat{\theta}^*_G, \tag{13}$$

where $\hat{\theta}_D$ is the deterministic part of $\hat{\theta}$ and $\hat{\theta}^*_Y$ and $\hat{\theta}^*_G$ are the components of $\hat{\theta}^*$ in $\overline{\mathcal{L}}_Y$ and $\overline{\mathcal{L}}_G$, respectively. Denoting by $\sigma(A)$ the minimum σ-field generated by A, it is easy to see that in the model $(G_t, \sigma(G_t, t \in [0,T]), P_\theta)$, $\hat{\theta}^*_G$ generates the BLUE estimators of all estimable functions. Thus, we have the following lemma.

Lemma 5. If $\overline{X}^*_j$, $\overline{X}^*_{j+1} \ldots$, $j \in (T_S \cup T_{S^\perp})^c$ are independent random variables, then the process $(G_t, t \in [0,T])$ has a normal distribution if and only if $\hat{\theta}^*_G$ is a sufficient estimator in the model $(G_t, \sigma(G_t, t \in [0,T]), P_\theta)$. In this case, for each estimable linear combination $M\theta$ (in $\overline{\mathcal{L}}(X^*_t, t \in [0,T])$), the BLUE estimator of $M(I-R)\theta$ has a normal distribution.

Proof. We can apply Corollary 1 in Ibarrola and Pérez-Palomares [4], because $(G_t, t \in [0,T])$ verifies the hypotheses of this result.

Theorem 3. Suppose that $(\overline{X}^*_k, k \in T_S)$, $(\overline{X}^*_k, k \in T_{S^\perp})$, $\overline{X}^*_j$, $\overline{X}^*_{j+1} \ldots$, $j \in (T_S \cup T_{S^\perp})^c$ are independent random variables. Then, the estimator $\hat{\theta}$ is a sufficient estimator in the complete model, i.e. in the model $(X_t, \sigma(X_t, t \in [0,T]), P_\theta)$ if and only if $(G_t, t \in [0,T])$ is a Gaussian process.

Proof. For the right implication it is enough to use Theorem 2 of Ibarrola and Pérez-Palomares [4]. For the left implication, first note that $\hat{\theta}$ is a sufficient estimator in the model $(X_t, \sigma(X_t, t \in [0,T]), P_\theta)$ if and only if $\hat{\theta}^*$ is a sufficient estimator in the model $(X^*_t, \sigma(X^*_t, t \in [0,T]), P_\theta)$. Thus, we shall prove the last assertion. In order to do this, we observe that

$$\sigma(X_t^*, t \in [0,T]) = \sigma(\sigma(Y_t, t \in [0,T]) \cup \sigma(G_t, t \in [0,T]) \cup \sigma(W_t, t \in [0,T])).$$

Now, let $A \in \sigma(Y_t, t \in [0,T])$, $B \in \sigma(G_t, t \in [0,T])$ and $C \in \sigma(W_t, t \in [0,T])$. By definition Y_t is measurable with respect to $\sigma(\hat{\theta}^*)$ and from hypothesis W_t is independent of $(G_t, \hat{\theta}^*)$, so

$$E_\theta[1_A 1_B 1_C | \sigma(\hat{\theta}^*)] = 1_A P_0(C) E_\theta[1_B | \sigma(\hat{\theta}^*)]. \tag{14}$$

On the other hand, we use the decomposition of $\hat{\theta}^*$ as in (13) and we have $\sigma(\hat{\theta}^*) = \sigma(\sigma(\hat{\theta}_Y^*) \cup \sigma(\hat{\theta}_G^*))$. Thus from the independence between $(\sigma(G_t, t \in [0,T])$ and $\sigma(\hat{\theta}_Y^*)$ we obtain

$$E_\theta[1_B | \sigma(\hat{\theta}^*)] = E_\theta[1_B | \sigma(\hat{\theta}_G^*)].$$

Using Lemma 5, we can assure the sufficiency of $\hat{\theta}_G^*$ for $\sigma(G_t, t \in [0,T])$, so taking into account (14) we conclude that for $A \in \sigma(Y_t, t \in [0,T])$, $B \in \sigma(G_t, t \in [0,T])$ and $C \in \sigma(W_t, t \in [0,T])$ there exists a version of $E_\theta[1_A 1_B 1_C | \sigma(\hat{\theta}^*)]$ independent of θ. Finally, using a standard argument we obtain the sufficiency of $\hat{\theta}$.

Corollary 1. Suppose that $(\overline{X}_k^*, k \in T_S)$, $(\overline{X}_k^*, k \in T_{S^\perp})$, $\overline{X}_j^*$, $\overline{X}_{j+1}^* \ldots$, $j \in (T_S \cup T_{S^\perp})^c$ are independent random variables. Then, the estimator $\hat{\theta}$ is a sufficient estimator in the complete model, i.e in the model $(X_t, \sigma(X_t, t \in [0,T]), P_\theta)$ if and only if $\hat{\theta}_G^*$ is a sufficient estimator in $(G_t, \sigma(G_t, t \in [0,T]), P_\theta)$.

Note 5. As we mentioned in the introduction, we can factorize the process G_t to obtain, finally, the following decomposition,

$$X_t = c_t + Y_t + W_t + G_t^{(B)} + G_t^{(R)},$$

where c_t is a deterministic estimator, Y_t and $G_t^{(B)}$ are the BLUE estimators for $a(t)R\theta$ and $a(t)(I-R)\theta$, respectively and, finally, W_t and $G_t^{(R)}$ are residual components. Moreover, if the hypotheses of Lemma 5 are satisfied, $G_t^{(B)}$ and $G_t^{(R)}$ have a normal distribution.

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