Design of Flower Constellations Using Necklaces

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This paper introduces a new approach in the design of optimal satellite constellations comprising few satellites and characterized by symmetric distribution. Typical examples are missions requiring persistent observation of sites or regions for intelligence or science. To minimize the number of satellites, the mathematical necklace theory is applied to the two-dimensional Lattice Flower Constellation framework. The necklace theory identifies in the phasing space all satellite subsets characterized by symmetric distributions. Mathematically, these subsets are parameterized by necklaces, identifying the actual satellite locations in the first orbital plane and a shifting parameter governing phasing between subsequent orbital planes. This article specifically targets the emerging importance of small satellite formations and constellations.

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I. INTRODUCTION

Constellations of satellites have been used for a variety of space missions (e.g., global navigation systems, communications, Earth observation, reconnaissance, etc.), and the improvement and design of new constellations are a current topic aimed to reduce the cost of missions as much as possible.

Symmetry plays a key role in satellite constellation design. However, the concept of symmetry can be seen from different perspectives. One of these is associated with global coverage, wherein symmetry has spatial meaning associated with the uniform distribution of satellites. When the satellites all belong to the same relative trajectory, as in regional or persistent observation missions, then symmetry has a temporal meaning, in that the time interval between the passage of two subsequent satellites must be constant. Spatial and time symmetries can be merged into symmetry in phasing space—the (Ω, M)-space—where spatial and time distributions are interconnected. This paper describes a study on how to find symmetries in (Ω, M)-space for solutions that are characterized by time and spatial regularities.

General theories in the design of satellite constellations include the classic Walker constellations [1] and the more recent Flower Constellations (FCs) [2]. The philosophic difference between Walker constellations and the original FCs is the reference frame selected for building symmetric distributions of satellites. Whereas Walker chose the inertial reference frame, a generic rotating reference frame was selected in the theory of FCs.

The original theory of FCs, first presented in [2] and then expanded in detail in [3, 4], was substantially improved with the two-dimensional lattice theory (2D-LFC) [5], making the theory independent of any reference frame (inertial or rotating) and making the theory work with minimal parameterization. More recently, the three-dimensional (3D) lattice theory [6] extended the 2D-LFC theory to account for the J_2 effect due to the Earth's oblateness.

The evolution of the Flower Constellation theories is interesting for many reasons: first, the deep connection with the mathematical tools and properties of number theory (Chinese remainder theorem, theory of lattices, Hermite normal form, etc.); second, the level of description using minimal parameterization, a property useful for ensuring inclusion of all possible symmetric solutions; and third, the important practical reason for including the J_2 effect; that is, to allow constellation designers to use any inclination when selecting elliptical orbits [6]. From a mathematical point of view, the theory appears to have reached the final level of maturity, whereas from a practical point of view, the following question arises: Since most of these Lattice constellations involve an impractically high number of satellites to obtain full symmetry, is it possible to select a subset of them and still obtain a symmetric phasing distribution?

This paper provides a positive answer to this question and gives a method to compute all these symmetric subsets.

The location of all the satellites of a FC corresponds to a lattice in (Ω, M)-space that can be regarded as a 3D torus (both axes, M and Ω, are modulo 2π) [7, 8]. Throughout this paper, symmetry is invariant from rigid translations in (Ω, M)-space.

In this (Ω, M)-space, the initial orbit plane is made with N_{so} admissible locations (available for the 2D-LFC satellites), and these locations can be seen as a necklace of N_{so} empty pearls. An actual number of satellites (N_{rso}, actual pearls) less than the number of empty pearls can be distributed in the empty pearls necklace. The purpose is to find the proper necklaces and associated suitable shifting parameters (to duplicate and shift the initial necklace in the following orbit planes) to obtain the same initial necklace when we reach the last orbit plane.

By solving the above problem, we are able to design optimal satellite constellations made of few satellites, while keeping the design parameter space as big as the computer can tolerate. To solve this problem, basic number theory knowledge is required. Due to the mathematical complexity, to best explain the proposed methodology, a final flowchart is provided to clarify how the algorithms should be used to generate the necklaces and the associated shifting parameters during the design process.

II. FLOWER CONSTELLATION BACKGROUND

A. The Evolution of the Theory of Flower Constellations

A Flower Constellation, as defined in [2–4], is a set of N_s satellites following the same (closed) trajectory with respect to a rotating reference frame (e.g., fixed to the Earth). This condition implies that:

1) The orbital period T_p of each satellite is a rational multiple of the rotating frame period T_d. That is, $N_pT_p = N_dT_d$ for some positive (coprime) integers N_d and N_p.
2) The orbital parameters a, e, i and ω are the same for all satellites.
3) The mean anomaly M_i and the right ascension of the ascending node Ω_i of each satellite satisfy

$$N_p\,\Omega_i \equiv -N_d\,M_i \bmod (2\pi).$$

Item 1 guarantees that the trajectory in the rotating frame is closed. Items 2 and 3 are necessary and sufficient conditions to have all the satellites on the same trajectory (a complete proof of this fact is given in [7, 8]).

Usually, when designing a Flower Constellation, the compatibility (or resonant) parameters N_d and N_p are decided first, which immediately determines the orbital period T_p and, therefore, the semimajor axis a. Then, the orbital parameters e, i, and ω are selected, and finally, the angles Ω_i and M_i, starting from $\Omega_1 = M_1 = 0$, are computed by the recursive sequence

$$\Omega_{i+1} \equiv \Omega_i + 2\pi\frac{F_n}{F_d} \quad \text{and}$$

$$M_{i+1} \equiv M_i - 2\pi\frac{N_pF_n + F_dF_h(i)}{F_dN_d},$$

where F_n and F_d are two coprime positive integers and $F_h(i)$ is any sequence of numbers chosen in the set $\{1, 2, \ldots, N_d\}$. It is easy to show that this procedure always produces pairs (Ω_i, M_i) consistent with the equation $N_p\Omega_i \equiv -N_dM_i \bmod(2\pi)$. For simplicity, the parameter F_h will be considered constant. Currently a FC is specified by six integer parameters (N_d, N_p, F_d, F_n, F_h, N_s), as well as the continuous parameters e, i, ω. This is the approach followed so far in all the papers on Flower Constellations, as well as in the simulation and visualization software FCVAT [9].

It has been shown in [7, Thm 1], that the number of satellites in a Flower Constellation designed under this procedure cannot exceed N_dF_d/G satellites, where $G = \gcd(N_d, N_pF_n + F_dF_h)$. A constellation with the maximum number of satellites allowed by this theorem is called either a secondary path (as in [4]) or a Harmonic Flower Constellation (HFC) (as in [7]). The location of HFC satellites in (Ω, M)-space is determined [7, Thm 2] by three invariants: the number of inertial orbits F_d, the number of satellites per orbit $N_{so} = N_d/G$, and the configuration number $N_c \in [1, F_d)$ given by the formula

$$N_c = E_n\,\frac{N_p\,F_n + F_d\,F_h}{G} \mod (F_d), \tag{1}$$

where E_n and E_d are any integers such that $E_nF_n + E_dF_d = 1$. The numbers F_d, N_{so}, and N_c are always coprime.

B. 2D Lattice Flower Constellations

The 2D-LFC [5] can be described by five integer parameters and three continuous parameters. The integer parameters can be broken into two sets, the first describing the phasing of the satellites and the second describing the orbital period (or semimajor axis). The first set is $\{N_o, N_{so}, N_c\}$ where N_o is the number of orbital planes, N_{so} is the number of satellites per orbit, and N_c is a phasing parameter. The second set is $\{N_p, N_d\}$, which satisfies the compatibility equation

$$N_p\,T_p = N_d\,T_d, \tag{2}$$

where T_p is the Keplerian orbital period and T_d is the period of the rotating reference frame (e.g., the sidereal period of Earth's rotation). This definition enforces the repeating space-track requirement.

The phasing parameters define the right ascension of the ascending node (Ω) and initial mean anomaly (M) as

$$\Omega_{ij} = \frac{2\pi}{N_o}(i-1) \quad \text{and} \quad M_{ij} = \frac{2\pi}{N_{so}}(j-1) - \frac{N_c\Omega_{ij}}{N_{so}}. \tag{3}$$

These equations can be rewritten in matrix notation as

$$\begin{bmatrix} N_o & 0 \\ N_c & N_{so} \end{bmatrix} \begin{Bmatrix} \Omega_{ij} \\ M_{ij} \end{Bmatrix} = 2\pi \begin{Bmatrix} i-1 \\ j-1 \end{Bmatrix}, \tag{4}$$

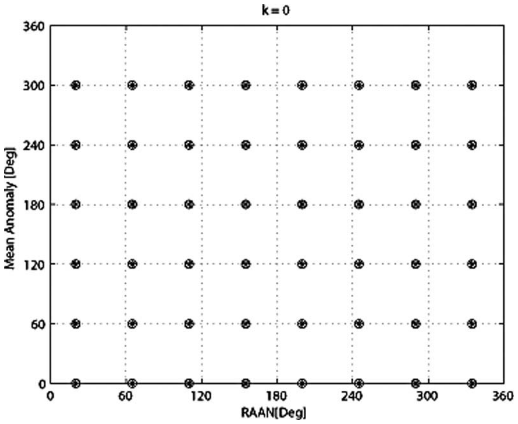

Fig. 1. (Ω, M)-space of 2D-LFC with $N_c = 0$.

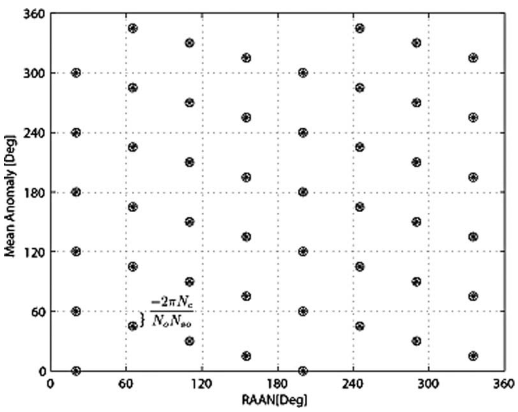

Fig. 2. (Ω, M)-space of 2D-LFC with $N_c = 2$.

where $i = 1, \ldots, N_o$; $j = 1, \ldots, N_{so}$; and $N_c \in [1, N_o]$. Satellite (i, j) is the jth satellite on the ith orbital plane. The remaining parameters required to define the constellation are continuous parameters that are the same for all orbits in the constellation: the inclination angle, eccentricity, and argument of periapsis. Note that since the 2D-LFC separates the satellite phasing from the orbit size, nonrepeating space tracks can be used without affecting the uniformity of the satellite distribution.

Since all satellites of a 2D-LFC have the same orbital parameters a, e, i, and ω, the constellation is then completely defined when the satellite phasing, provided by the (Ω, M)-space, is given. Fig. 1 shows the distribution of satellites in the LFC with $N_{so} = 6$, $N_o = 8$, and $N_c = 0$, obtained by solving (4). To show how the value of N_c influences the distribution of satellites, Fig. 2 shows the distribution of satellites in the 2D-LFC with $N_{so} = 6$, $N_o = 8$, and $N_c = 2$, obtained by solving (4).

Each point in (Ω, M)-space identifies one satellite of the constellation. Usually, the mission budget limits the number of constellation satellites to an upper assigned value, say $N_{s\,\max}$. The number of satellites in the constellation, which can be computed as the determinant of the 2×2 matrix of (4), satisfies $N_s = N_o N_{so} \leq N_{s\,\max}$. On the other hand, N_o defines the number of orbital planes, a number that is proportional to the number of distinct launches needed to deploy the entire constellation, which is also strongly constrained by the mission budget. The remaining parameter, the configuration number N_c, remains the only (integer) variable to play with. Because of the limited possible values for N_c (they are actually N_o values, only), there are not many different potential configurations. This is a strong limitation in the design process. To overcome this limitation, the following idea is proposed and analyzed in this article.

Instead of directly searching for a 2D-LFC made with a given number of satellites, we introduce a fictitious satellite constellation with a much larger number of satellites, and then we extract our constellation as a subset of the larger one. Because we would like to preserve all the nice properties of LFCs, we are automatically led to the following problem: find all the subsets of N_{rs} real satellites, selected from the fictitious constellation made of $N_s \gg N_{rs}$ total satellites, such that the satellite distribution in (Ω, M)-space is symmetric in both M and Ω axes. Here, symmetry should be understood in the following sense: the satellites in each orbit have the same exact pattern of mean anomalies, and orbit planes are uniformly distributed in space.

Finding all these subsets will be a high-payoff effort because the benefits of the necklace theory applied to 2D-LFC will be outstanding: new optimal solutions will be found with an assigned minimum number of satellites in a solution space whose dimension is only limited by the available computational capability.

III. THE NECKLACE PROBLEM

Consider a set of N_{rso} satellites that can be arranged in N_{so} available locations (with $N_{so} \geq N_{rso}$) in a given orbit. This set of satellites forms a "necklace" that is rotating along the orbit and comes back to the original setup in an orbital period. If the satellite locations are defined in terms of mean anomaly, then the satellite necklace structure moves rigidly in the mean anomaly space. The question we answer here is: How many and which are all these necklaces?

A. The Necklace Theory

In general, the necklace problem is a combinatorial problem that answers the following question: How many different arrangements of n pearls in a circular loop are there, assuming that each pearl comes in one of k different colors? Two arrangements that differ only by a rotation of the loop are consider to be identical. The mathematical solution to this problem (see [10]) is a simple application of Burnside's counting theorem and is summarized by the following formula:

$$N_k(n) = \frac{1}{n} \sum_{d|n} \varphi(d) k^{n/d},$$

where the sum is taken over all the divisors d of n, and $\varphi(d)$ is called Euler's totient function of d, an arithmetic function that counts the number of positive integers less

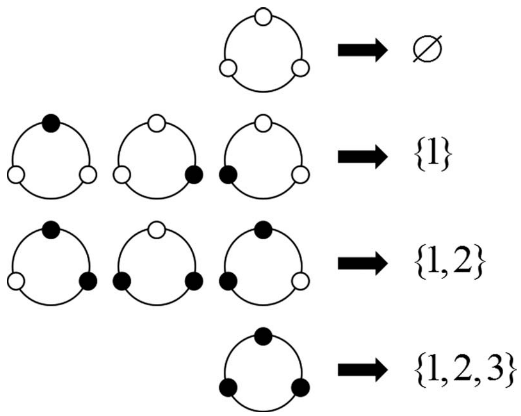

Fig. 3. Unlabeled necklaces with three pearls and two colors.

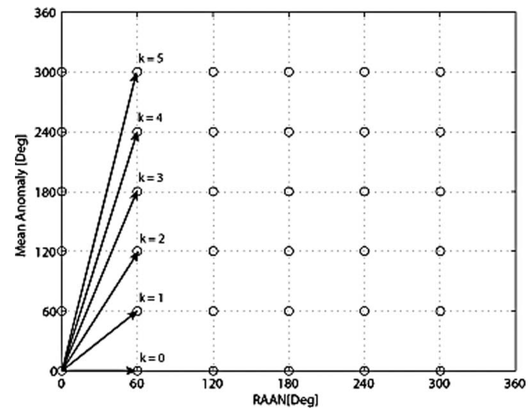

Fig. 4. Shifting determines location of satellites in constellation.

than or equal to d that are coprime to d. For example a simple computation shows that $\varphi(1) = \varphi(2) = 1$, $\varphi(3) = \varphi(4) = 2$, $\varphi(5) = 4$, $\varphi(6) = 2$, $\varphi(7) = 6$, and so on. In our physical example $k = 2$, and these two "colors" represent the presence and the absence of a satellite in the various admissible locations. Therefore, the total number of satellite necklaces is

$$N_2(n) = \frac{1}{n}\sum_{d|n} \varphi(d)\, 2^{n/d}. \tag{5}$$

Mathematically, a necklace will be represented as a subset $\mathcal{G} \subseteq \{1, \ldots, n\}$. Since we only consider unlabeled necklaces, the two subsets $\mathcal{G}$ and $\mathcal{G}'$ that differ by an additive constant are considered identical:

$$\mathcal{G} = \mathcal{G}' \;\Leftrightarrow\; \exists s : \; \mathcal{G} \equiv \mathcal{G}' + s \bmod (n).$$

The set of all possible unlabeled necklaces with n pearls and two colors will be identified by $K(n)$. Fig. 3 shows all possible unlabeled necklaces using three pearls of two colors, that is, the elements of $K(3)$. Notice that in Fig. 3, the configurations $\{1, 2\}$, $\{2, 3\}$, and $\{1, 3\}$ are all represented with the set $\{1, 2\}$ because it is possible to obtain $\{1, 3\}$ and $\{2, 3\}$ from $\{1, 2\}$ by performing a suitable rotation. Similarly, the configurations $\{1\}$, $\{2\}$, and $\{3\}$ are all equivalent to $\{1\}$. Therefore $K(3)$ contains only four elements: $\emptyset$, $\{1\}$, $\{1, 2\}$, and $\{1, 2, 3\}$.

Algorithm 1 (provided in the Appendix in pseudocode), computes all possible necklaces involving a total of N_{so} pearls, of which N_{rso} are black and $N_{so} - N_{rso}$ are white. To obtain all possible necklaces with N_{so} pearls, it is necessary to call the algorithm with $N_{rso} = 0, \ldots, N_{so}$.

B. Symmetries of the Necklaces

Let $\mathcal{G}$ be a necklace such that $\mathcal{G} \in K(n)$. We say that $\mathcal{G}$ has a symmetry of length r if $\mathcal{G}$ and $\mathcal{G} + r$ coincide modulo n.

As an example, consider the necklace $\mathcal{G} = \{1, 3, 5, 7\} \in K(8)$. What symmetries does it have?

- $r = 2$ is a symmetry, since $\mathcal{G} + 2 = \{3, 5, 7, 9\}$ is equivalent to $\mathcal{G}$ modulo 8.
- $r = 4$ and $r = 6$ are also symmetries, since $\{5, 7, 9, 11\}$ and $\{7, 9, 11, 13\}$ reduce to $\{1, 3, 5, 7\}$ modulo 8
- $r = 1$ is not a symmetry, since $\{2, 4, 6, 8\}$ and $\{1, 3, 5, 7\}$ do not coincide modulo 8.

From the example, it is easy to see that if r is a symmetry of a necklace, then any multiple of r is also a symmetry. This remark motivates our following definition: for each necklace $\mathcal{G} \in K(n)$, the symmetry number of $\mathcal{G}$, denoted $\mathrm{Sym}(\mathcal{G})$, is the shortest of the symmetries of $\mathcal{G}$. Note that $\mathrm{Sym}(\mathcal{G})$ always divides n.

$$\mathrm{Sym}(\mathcal{G}) = \min\{1 \le r \le n : \mathcal{G} + r \equiv \mathcal{G} \bmod (n)\} \tag{6}$$

Algorithm 2 (provided in the Appendix) can be used to find all the symmetries and the symmetry number of a given necklace.

IV. NECKLACES AND 2D LATTICE FLOWER CONSTELLATIONS

To generate the necklaces, the following idea is adopted: consider a standard 2D-LFC (with parameters N_{so}, N_o, and N_c), and instead of placing all satellites in admissible locations, as provided by (3), a subset (necklace) of admissible locations $\mathcal{G} \subseteq \{1, 2, \ldots, N_{so}\}$ is selected for actual satellites in the first orbital plane. This configuration is then duplicated for each subsequent orbital plane using a constant shifting parameter (an integer $k \in \{0, \ldots, N_{so} - 1\}$). The subset $\mathcal{G}$ can be any necklace. Once $\mathcal{G}$ and the shifting parameters are given, the constellation is automatically determined. Fig. 4 shows the various positions of a satellite in the second orbital plane with respect the first one in the first orbital plane as a function of the shifting parameter k.

To perform a correct and unique shifting between subsequent orbital planes, two problems must be taken into consideration:

1) *Consistency Problem:* Because of the modular nature of the Ω parameter, the shifting has to be chosen in such a way that the group of satellites (necklace) in the orbit with $\Omega = 0$ coincides with the group of satellites (necklace) in the orbit with $\Omega = 2\pi$. This problem is discussed in detail in the next subsection.

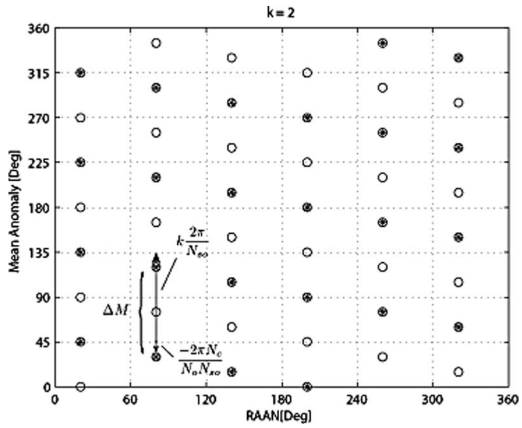

Fig. 5. Amount ΔM in (Ω, M)-space.

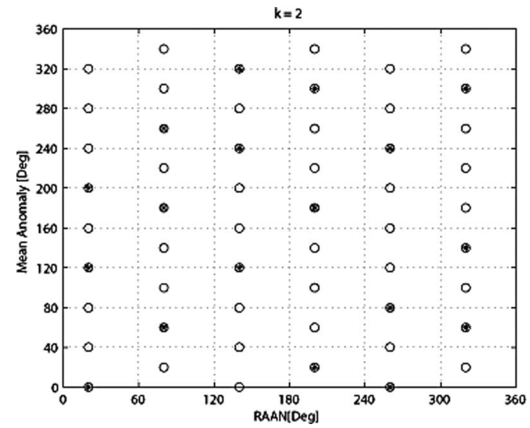

Fig. 6. NFC generated by admissible pair.

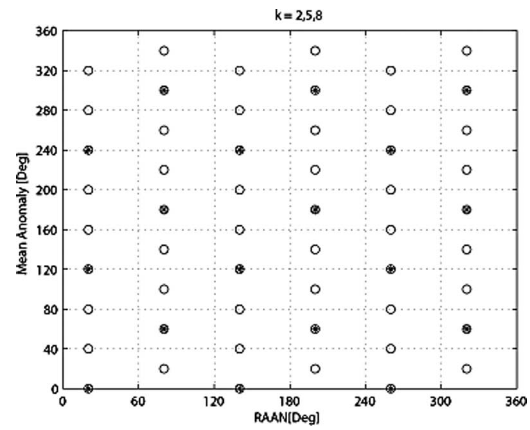

Fig. 7. Different values of k can generate same configuration.

2) *Minimality Problem:* Sometimes, for the same $\mathcal{G}$, two values of the shifting parameter generate the same distribution of satellites in (Ω, M)-space. This is solved by simply taking $0 \leq k \leq \text{Sym}(\mathcal{G}) - 1$.

Satellite constellations obtained from the above procedure are called necklace Flower Constellations (NFCs).

A. ΔM Shifting Between Subsequent Orbital Planes

According to (4), the first satellite ($j = 1$) in the first orbit ($i = 1$) is chosen (without loss of generality) as $M_{11} = 0$ and $\Omega_{11} = 0$. Taking into account (3), the mean anomaly of our satellite in the next orbit will be:

$$M_{21} = -\frac{2\pi N_c}{N_o N_{so}}. \tag{7}$$

Then, the amount ΔM, called ΔM-shifting, between subsequent orbits will be:

$$\Delta M = -\frac{2\pi N_c}{N_o N_{so}} + k\frac{2\pi}{N_{so}}. \tag{8}$$

This means that the mean anomalies of the satellites in the second orbit can be obtained by adding ΔM to the mean anomalies of the satellites of the first orbit. Similarly, the mean anomalies on the third orbit are the mean anomalies on the second plus ΔM, and so on.

After an orbital period, the mean anomaly of the satellite will increase by

$$N_o \Delta M = N_o \left(-\frac{2\pi N_c}{N_o N_{so}} + k\frac{2\pi}{N_{so}} \right) = \frac{2\pi}{N_{so}}(kN_o - N_c). \tag{9}$$

Fig. 5 shows the meaning of the value ΔM in a (Ω, M)-space of a NFC $\mathcal{G} = \{1, 3, 5, 7\}$ with $N_{so} = 8$, $N_o = 6$, and $N_c = 2$.

B. Admissible Pair $(\mathcal{G}, k)$

If $\mathcal{G}$ is a necklace such that $\mathcal{G} \in K(N_{so})$ and is paired with a shifting parameter $k \in \{0, \ldots, N_{so} - 1\}$, then the pair $(\mathcal{G}, k)$ is called admissible if the distribution of satellites in the initial orbit is invariant by the adding $N_o \Delta M$ to the mean anomaly of each satellite:

$$\frac{2\pi}{N_{so}}\mathcal{G} + \frac{2\pi}{N_{so}}(kN_o - N_c) \equiv \frac{2\pi}{N_{so}}\mathcal{G} \bmod (2\pi) \tag{10}$$

The logic behind this equation is the following: the term $\frac{2\pi}{N_{so}}\mathcal{G}$ represents the mean anomalies of the satellites in the first orbital plane, the second term represents the shifting in mean anomaly that will affect all satellites because of shifting between the first and the last orbit. Finally, the right-hand side represents the mean anomalies of the satellites in the last plus one orbit that must coincide with the initial mean anomalies up to some integer multiple of 2π. Multiplying (10) by $\frac{N_{so}}{2\pi}$ and using the definition of symmetry number, the condition translates to

$$\text{Sym}(\mathcal{G}) | kN_o - N_c, \tag{11}$$

reading $\text{Sym}(\mathcal{G})$ divides $(kN_o - N_c)$. Equation (11) represents the solution to the consistency problem; that is, it provides the values of the shifting parameters (k) that are all admissible to create NFCs. Again, these values of k are such that the initial necklace in orbital plane $\Omega = 0$ is the same when shifted N_o times by the mean anomaly variation given in (8).

Figs. 6 and 7 show two examples of 2D-LFCs generated by an admissible pair $(\mathcal{G}, k)$. In both cases, the design parameters were $N_{so} = 9$, $N_o = 6$, and $N_c = 3$. The

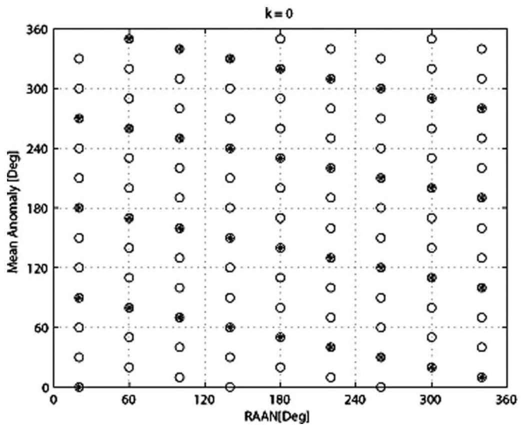

Fig. 8. NFC with $k = 0$.

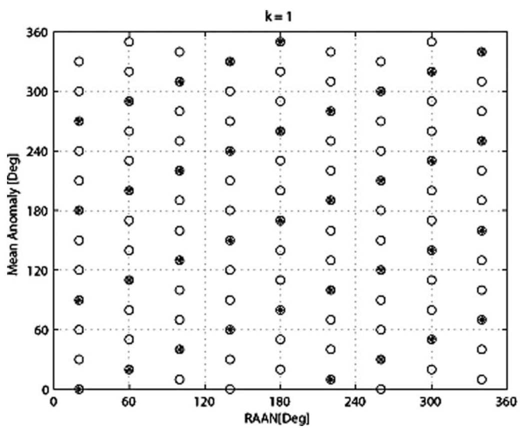

Fig. 9. NFC with $k = 1$.

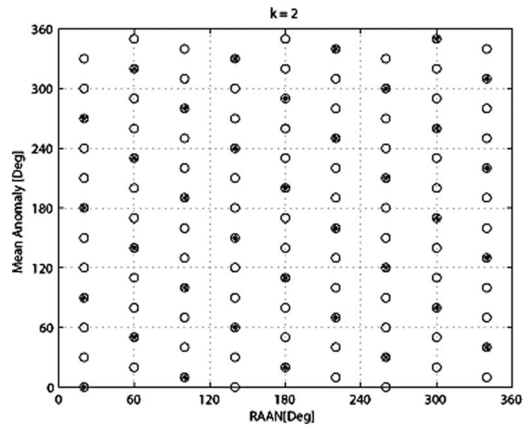

Fig. 10. NFC with $k = 2$.

necklace in Fig. 6 is $\mathcal{G} = \{1, 4, 6\}$ with symmetry number $\mathrm{Sym}(\mathcal{G}) = 9$ and shifting parameter $k = 2$. The consistency condition is satisfied since $9 \mid 2{\cdot}6 - 3$, so the pair $(\{1, 4, 6\}, 2)$ is admissible. This can be seen in Fig. 6 as follows: shifting the three satellites of the last orbit (the one with $\Omega = 320°$) with $\Delta M = 60°$ as given by (8) for $k = 2$ reproduces exactly the configuration in the first orbit (the one with $\Omega = 20°$). In Fig. 7, the necklace is $\mathcal{G} = \{1, 4, 7\}$, which has symmetry number $\mathrm{Sym}(\mathcal{G}) = 3$ and shifting parameter $k = 2$. Again, the consistency condition is satisfied: $3 \mid 2{\cdot}6 - 3$.

As we mentioned before, the minimality problem is solved by restricting the range of values of k to the interval $[0, \mathrm{Sym}(\mathcal{G}) - 1]$. It is clear that $(\mathcal{G}, k)$ and $(\mathcal{G}, k')$ will generate the same constellation if, and only if, $k - k'$ is an integer multiple of $\mathrm{Sym}(\mathcal{G})$. This is impossible for two values in the proposed interval. Fig. 7 shows an example of this situation: in this 2D-LFC ($N_{so} = 9$, $N_o = 6$, and $N_c = 3$), the necklace $\mathcal{G} = \{1, 4, 7\}$, which has $\mathrm{Sym}(\mathcal{G}) = 3$, generates the same configuration for $k = 2$, $k = 5$, and $k = 8$.

The above leads to the main result: each NFC corresponds with one (and only one) pair $(\mathcal{G}, k)$ with $\mathcal{G} \in K(N_{so})$, $0 \leq k \leq \mathrm{Sym}(\mathcal{G}) - 1$, and $\mathrm{Sym}(\mathcal{G}) \mid kN_o - N_c$.

Figs. 8–10 show the only three possible NFCs (according to our main result) induced by the necklace $\mathcal{G} = \{1, 4, 7, 10\} \in K(12)$, which has symmetry number $\mathrm{Sym}(\mathcal{G}) = 3$. The underlying 2D-LFC has parameters $N_{so} = 12$, $N_o = 9$, and $N_c = 3$, so the three possible values of $k \in \{0, 1, 2\}$ are admissible.

C. The Diophantine Equation for the Shifting Parameter

The admissibility condition for a pair $(\mathcal{G}, k)$, motivates us to study the Diophantine equation $d \mid ak - b$ (11), where a, b, and d are given (positive) integers, and the unknown k takes integer values in the range $[0, d - 1]$. All the solutions can be obtained by trial and error (since there are finitely many possibilities for k), but we would like a more efficient procedure.

The number of solutions of this Diophantine equation, denoted by $Y(d, a, b)$, is exactly

$$Y(d, a, b) = \begin{cases} 0 & \text{if } \gcd(d, a) \nmid b \\ \gcd(d, a) & \text{otherwise.} \end{cases} \tag{12}$$

Equation (12) can be proved as follows. Independent of the value of k, the product ak is always divisible by $\gcd(d, a)$, so when $\gcd(d, a) \nmid b$, reading $\gcd(d, a)$ does not divide b, it is impossible to have $\gcd(d, a) \mid ak - b$; therefore, we will never have $d \mid ak - b$. In the case where $\gcd(d, a) \mid b$, we can divide a, b, and d by $\gcd(d, a)$ and reduce the problem to the equation $d' \mid a'k - b'$, where $a' = a/\gcd(d, a)$, $b' = b/\gcd(d, a)$, and $d' = d/\gcd(d, a)$. This problem has only one solution in the interval $[0, d' - 1]$, since a' and d' have no common factor and, therefore, has $d/d' = \gcd(d, a)$ solutions in $[0, d - 1]$.

An efficient algorithm computing all the solutions of equation $d \mid ak - b$ is given in the Appendix (see Algorithm 3).

V. THE TOTAL NUMBER OF NECKLACE FLOWER CONSTELLATIONS

To implement the necklace theory in an optimization process successfully, it is important to have an algorithm providing all the necklaces that can be obtained from a 2D-LFC with parameters N_{so}, N_o, and N_c. However, before listing all these necklaces, it is important to know how many they are. The total number of necklaces, here denoted by $W(N_{so}, N_o, N_c)$, is exactly the number of admissible pairs; that is,

$$\begin{aligned} &W(N_{so}, N_o, N_c) \\ &\quad = \#\{(\mathcal{G}, k) : \mathcal{G} \in K(N_{so}),\ 0 \le k \le \mathrm{Sym}(\mathcal{G}) - 1,\ kN_o \\ &\quad \equiv N_c \bmod (\mathrm{Sym}(\mathcal{G}))\}. \end{aligned} \tag{13}$$

Let $X(d)$ be the number of necklaces with symmetry number d; then (13) can be rewritten as

$$W(N_{so}, N_o, N_c) = \sum_{d|N_{so}} X(d) Y(d, N_o, N_c). \tag{14}$$

It should be natural to adopt the notation $X(d, N_{so})$ rather than $X(d)$, since we are considering necklaces in $K(N_{so})$. However, the number of necklaces with symmetry number d in $K(N_{so})$ has a one-to-one correspondence with the number of necklaces in $K(d)$ with symmetry number d. This shows that $X(d, N_{so})$ does not depend on N_{so}, as long as $d \mid N_{so}$. For practical purposes we can define $X(d) = X(d, d)$, that is, the number of necklaces in $K(d)$ with symmetry number d. A simple corollary of this discussion is the formula

$$\sum_{d|n} X(d) = N_2(n), \tag{15}$$

where $N_2(n)$ is given in (5). Equation (15) is derived from the fact that $X(d) = X(d, n)$ for any $d \mid n$, and that any necklace in $K(n)$ has a symmetry number that divides n.

A simple way to compute $W(N_{so}, N_o, N_c)$ is explained as follows. Consider two positive integers n and m. Denote $(n : m^\infty)$ the integer obtained by removing from n all the prime factors corresponding to the primes that appear in m. For instance, $(120 : 70^\infty) = 3$, since $120 = 2^3 \cdot 3 \cdot 5$ and the primes 2 and 5 appear in $70 = 2 \cdot 5 \cdot 7$. Using this definition and assuming $\gcd(N_{so}, N_o, N_c) = 1$, then

$$W(N_{so}, N_o, N_c) = N_2(N_{so} : N_o^\infty) \tag{16}$$

regardless of the value of N_c. Equation (16) can be proven as follows. First, (14) is used to compute the value of $W(N_{so}, N_o, N_c)$. In (14) the sum ranges over all divisors d of N_{so}. However, if the divisor d has a common factor with N_o, then it cannot have any common factor with N_c by our assumption $\gcd(N_{so}, N_o, N_c) = 1$; therefore, $Y(d, N_o, N_c) = 0$ according to (12). This means that it is enough to consider divisors of $(N_{so} : N_o^\infty)$. For any of these divisors, we have $Y(d, N_o, N_c) = 1$, since $\gcd(d, N_o) = 1$. All together this means that

$$W(N_{so}, N_o, N_c) = \sum_{d|(N_{so}:N_o^\infty)} X(d),$$

which is equal to $N_2(N_{so} : N_o^\infty)$ by (15), thus proving (16).

From (16) two particular cases of independent interest are derived. In case 1,

$$\text{If} \quad \gcd(N_{so}, N_o) = 1 \quad \text{then} \quad W(N_{so}, N_o, N_c) = N_2(N_{so}). \tag{17}$$

In fact, when N_{so} and N_o have no common factors, then $(N_{so} : N_o^\infty) = N_{so}$ because there are no primes to remove from N_{so}. Equation (17) then immediately follows from (16). In case 2,

$$\begin{aligned} &\text{If} \quad N_{so} | N_o \quad \text{and} \quad \gcd(N_c, N_{so}) = 1, \\ &\quad \text{then} \quad W(N_{so}, N_o, N_c) = 2. \end{aligned} \tag{18}$$

In fact, the assumption $N_{so} \mid N_o$, implies that all the primes in N_{so} appear in N_o; consequently, $(N_{so} : N_o^\infty) = 1$. Using (16), we obtain $W(N_{so}, N_o, N_c) = N_2(1) = 2$.

Equation (16) is particularly useful with HFCs that are 2D-LFCs with the additional constraint $\gcd(N_{so}, N_o, N_c) = 1$ (see [5]). For the general case of $W(N_{so}, N_o, N_c) \neq 1$, the following two results constitute positive steps toward the general solution.

First, a formula for $X(d)$ is provided. For any positive integer d, we have

$$X(d) = \frac{1}{d} \sum_{e|d} \mu(e)\, 2^{d/e}, \tag{19}$$

where μ is the multiplicative Möbius function. The Möbius function is defined as follows: $\mu(n)$ is zero when the factorization of n contains a prime number to a power greater than 1, and is equal to $(-1)^r$ when n is the product of r different primes.

The proof of (19) can be done by inverting (15) using the Möbius inversion formula [11, Thm 2.9]

$$X(d) = \sum_{e|d} \mu\left(\frac{d}{e}\right) N_2(e) = \sum_{e|d} \sum_{f|e} \mu\left(\frac{d}{e}\right) \frac{\varphi(f)}{e} 2^{e/f}.$$

Setting $r = e/f$ and changing the order of summation, we obtain

$$X(d) = \sum_{r|d} \frac{2^r}{r} \sum_{f|(d/r)} \mu\left(\frac{d}{rf}\right) \frac{\varphi(f)}{f}.$$

Finally, the theorem of multiplicative arithmetic functions [11, Thm 2.14, Thm 2.15] shows that the second sum reduces to $\mu(d/r)/(d/r)$; therefore,

$$X(d) = \sum_{r|d} \frac{2^r}{r} \frac{\mu(d/r)}{d/r} = \frac{1}{d} \sum_{r|d} \mu\left(\frac{d}{r}\right) 2^r,$$

thus proving (19).

For the cases not included in (16) or in any of its corollaries, the following formula for $W(N_{so}, N_o, N_c)$ is provided:

$$\text{If} \quad N_{so}|N_o \quad \text{and} \quad N_c = 0 \quad \text{then} \quad W(N_{so}, N_o, N_c) = 2^{N_{so}}. \tag{20}$$

To derive (20) the following observation should be noted: for any divisor d of N_{so}, we have $Y(d, No, 0) = d$, since d also divides N_o. Therefore, using (14) and (19), we can write

$$W(N_{so}, N_o, N_c) = \sum_{d|N_{so}} X(d)\, d = \sum_{d|N_{so}} \sum_{e|d} \mu(e) 2^{d/e}.$$

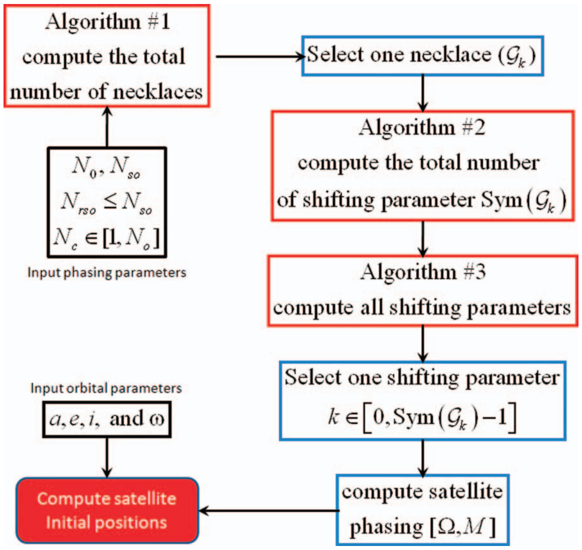

Fig. 11. Program flowchart.

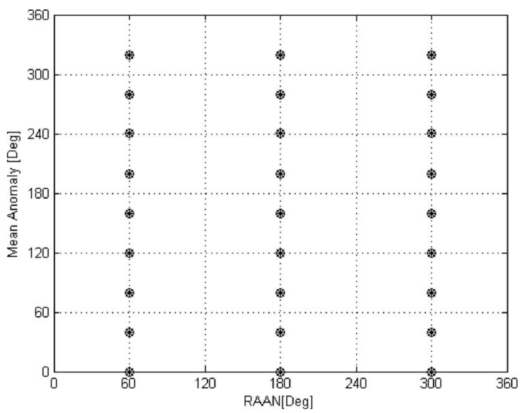

Fig. 12. 2D-LFC with $N_c = 0$.

Now, by setting $d = ek$ and changing the order of summation, the previous equation reduces to

$$W(N_{so}, N_o, N_c) = \sum_{k|N_{so}} \sum_{e|(N_{so}/k)} \mu(e)\, 2^k.$$

Note that the sum $\Sigma_{e\,|\,r}\, \mu(e)$ is equal to 1 when $r = 1$ and equal to 0 in the other cases. In particular, the sum above (the one depending on e) will vanish unless $k = N_{so}$, but this shows $W(N_{so}, N_o, Nc) = 2^{N_{so}}$ as claimed.

To summarize, the design of a NFC is as follows: given the number of admissible locations per orbit N_{so} and the real number of satellites per orbit N_{rso}, is possible determine all the different configurations to distribute the satellites in the first orbit of our constellation using the Necklace Theory and Algorithm 1. After that, an initial necklace is selected, and it is possible to compute its symmetry number using Algorithm 2. Finally, Algorithm 3 is used with parameters $A = N_o$, $B = N_c$, and $C = \mathrm{Sym}(\mathcal{G})$ to compute the values of the shifting k that give different NFCs.

The flowchart given in Fig. 11 summarizes the design procedure described above. This flowchart can be read as follows. For input, we have the mission parameters N_o and

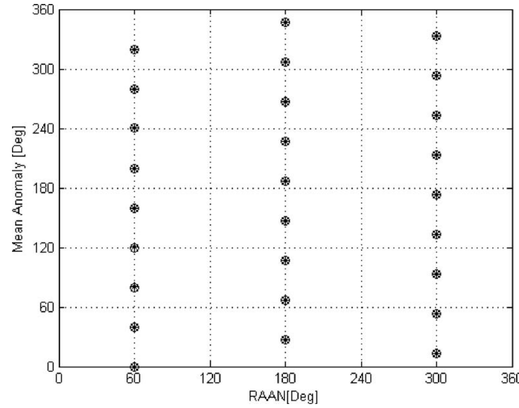

Fig. 13. 2D-LFC with $N_c = 1$.

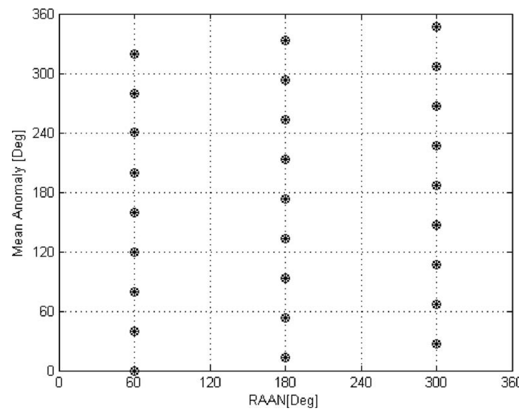

Fig. 14. 2D-LFC with $\mathrm{N}_c = 2$.

N_{rso} indicating the number of orbit planes and the number of satellites per orbit plane and an arbitrary number of potential locations (per orbit plane) to locate our satellites $N_{so} \geq N_{rso}$. Now, by choosing a value for the configuration number N_c in the interval $[1, N_o]$, we can compute all possible necklaces of the N_{rso} satellites in the N_{so} potential locations using Algorithm 1. Now, by selecting one of these necklaces, say $\mathcal{G}_k$, we can compute $\mathrm{Sym}(\mathcal{G}_k)$ using Algorithm 2 and all the $\mathrm{Sym}(\mathcal{G}_k)$ possible values of the shifting parameter $k \in [0,\ \mathrm{Sym}(\mathcal{G})-1]$ using Algorithm 3, where the algorithm parameters are $A = N_o$, $B = N_c$, and $C = \mathrm{Sym}(\mathcal{G}_k)$. At this point, using the selected necklace and shifting parameter, we compute the phasing of all the $N_{rs} = N_{rso}N_o$ satellites in (Ω, M)-space as shown in Fig. 4. The location of the satellites in the first orbit is given by the necklace $\mathcal{G}$, and the location of the satellites in subsequent orbits is controlled by the shifting parameter k. Finally, we optimize the common orbital parameters $(a, e, i,$ and $\omega)$ to minimize the mission cost function.

Note that the formulas we obtained for the total number of NFCs can be used in practice to select values of N_c (given N_{so} and N_o) that produce the largest number of different patterns. This is useful, because the more configurations there are, the more design possibilities we have.

A 27-satellite constellation is designed to illustrate a practical example of usage. The number of orbital planes

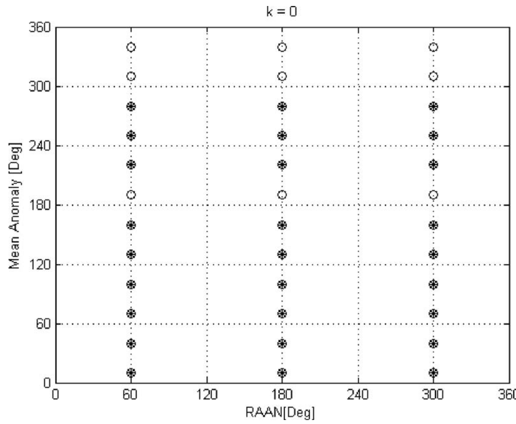

Fig. 15. NFC with $N_c = 0$ and $k = 0$.

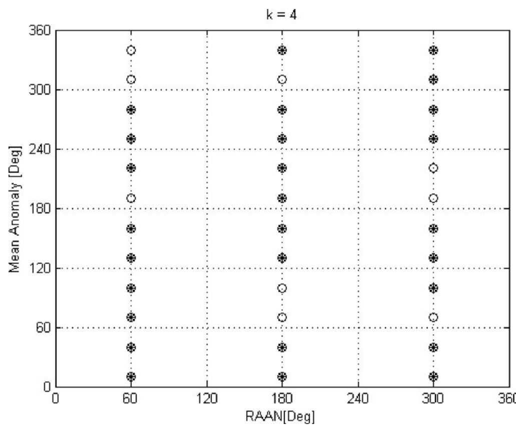

Fig. 16. NFC with $N_c = 0$ and $k = 4$.

is three, $N_o = 3$. By using the 2D-LFC theory, the remaining parameters must be $N_{so} = 9$ and $N_c \in \{0, 1, 2\}$. Consequently, we have three unique design possibilities illustrated in Figs. 12–14.

However, by using the NFC theory, more designs are possible. Consider a NFC with parameters $N_o = 3$, $N_{so} = 12$, $N_{rso} = 9$, and $N_c \in \{0, 1, 2\}$. First, as the theory states, the first orbit of the constellation is given by a necklace. In particular, there are 19 different necklaces to associate with the first orbit. These are:

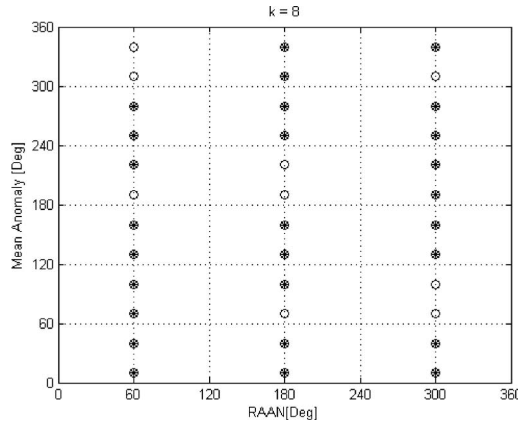

Fig. 17. NFC with $N_c = 0$ and $k = 8$.

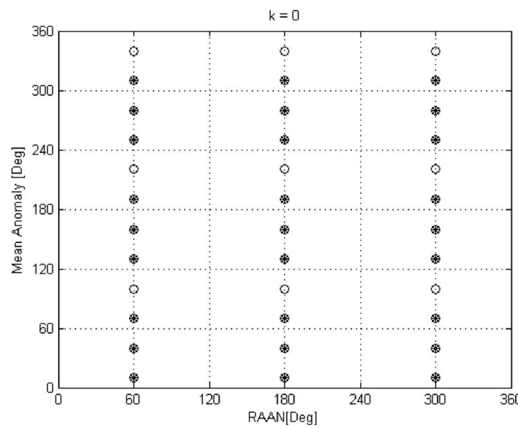

Fig. 18. NFC with $N_c = 0$ and $k = 0$.

Only two particular cases are analyzed. The necklace $\mathcal{G}_4$ has symmetry number $\mathrm{Sym}(\mathcal{G}_4) = 12$. When $N_c = 0$ the consistency condition—see (11)—implies that the shifting parameter must be $k \in \{0, 4, 8\}$, whereas for the other values for $N_c \in \{1, 2\}$, no values for the shifting parameter satisfy the consistency condition. By using necklace $\mathcal{G}_4$ we have three new designs for the constellation illustrated in Figs. 15–17.

The necklace $\mathcal{G}_{19}$ has symmetry number $\mathrm{Sym}(\mathcal{G}_{19}) = 4$. When $N_c = 0$ the consistency condition—see (11)—implies that the shifting parameter must be $k = 0$. When

$$
\begin{aligned}
&\mathcal{G}_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}, && \mathcal{G}_2 = \{1, 2, 3, 4, 5, 6, 7, 8, 10\},\\
&\mathcal{G}_3 = \{1, 2, 3, 4, 5, 6, 7, 9, 10\}, && \mathcal{G}_4 = \{1, 2, 3, 4, 5, 6, 8, 9, 10\},\\
&\mathcal{G}_5 = \{1, 2, 3, 4, 5, 7, 8, 9, 10\}, && \mathcal{G}_6 = \{1, 2, 3, 4, 6, 7, 8, 9, 10\},\\
&\mathcal{G}_7 = \{1, 2, 3, 5, 6, 7, 8, 9, 10\}, && \mathcal{G}_8 = \{1, 2, 4, 5, 6, 7, 8, 9, 10\},\\
&\mathcal{G}_9 = \{1, 3, 4, 5, 6, 7, 8, 9, 10\}, && \mathcal{G}_{10} = \{1, 2, 3, 4, 5, 6, 7, 9, 11\},\\
&\mathcal{G}_{11} = \{1, 2, 3, 4, 5, 6, 8, 9, 11\}, && \mathcal{G}_{12} = \{1, 2, 3, 4, 5, 7, 8, 9, 11\},\\
&\mathcal{G}_{13} = \{1, 2, 3, 4, 6, 7, 8, 9, 11\}, && \mathcal{G}_{14} = \{1, 2, 3, 5, 6, 7, 8, 9, 11\},\\
&\mathcal{G}_{15} = \{1, 2, 4, 5, 6, 7, 8, 9, 11\}, && \mathcal{G}_{16} = \{1, 2, 3, 4, 5, 7, 8, 10, 11\},\\
&\mathcal{G}_{17} = \{1, 2, 3, 4, 6, 7, 8, 10, 11\}, && \mathcal{G}_{18} = \{1, 2, 3, 5, 6, 7, 8, 10, 11\},\\
&\mathcal{G}_{19} = \{1, 2, 3, 5, 6, 7, 9, 10, 11\}.
\end{aligned}
$$

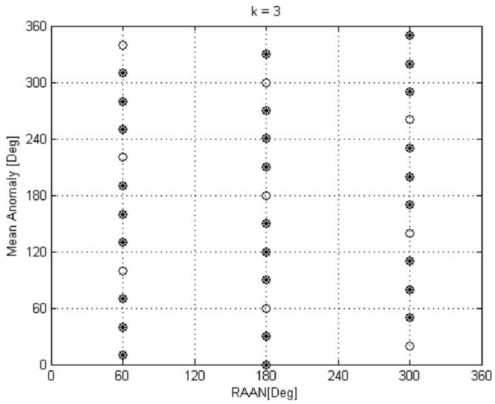

Fig. 19. NFC with $N_c = 1$ and $k = 3$.

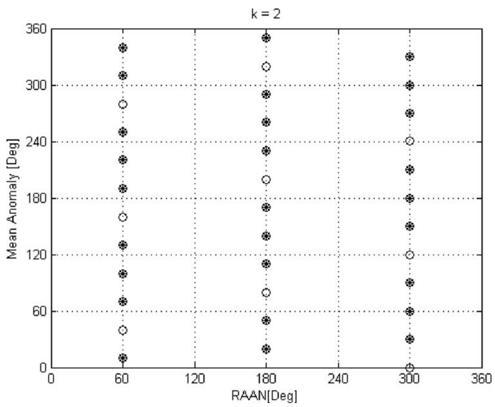

Fig. 20. NFC with $N_c = 2$ and $k = 2$.

$N_c = 1$ the shifting parameter must be $k = 3$; finally, when $N_c = 2$ the shifting parameter must be $k = 2$. Then, with the necklace $\mathcal{G}_{19}$, three different designs are possible for the constellation illustrated in Figs. 18–20.

Only two necklaces have been analyzed. Note that the more necklaces that can be associated with the first orbit, the more design possibilities there are.

VI. CONCLUSIONS

The cost of the missions is one of the most important factors for which to account when building a constellations of satellites. The theory of necklaces allows us to reduce the number of satellites in a Flower Constellation without losing its symmetric character. Throughout the paper we have shown what parameters are needed to define one of these objects (basically, a pair $(\mathcal{G}, k)$ consisting of a necklace and a positive integer) and which constraints have to be imposed on these parameters (a simple Diophantine equation). We have also provided algorithms in pseudocode, ready to be implemented, that enumerate all the possible necklace constellations that can be extracted from a 2D Lattice Flower Constellation.

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APPENDIX: PSEUDOCODE OF THE PROPOSED ALGORITHMS

Algorithm 1: Necklaces of N_{so} admissible locations and N_{rso} satellites per orbit.

Input: Number of admissible locations (N_{so}) and number of satellites per orbit (N_{rso}).
Output: Matrix with necklaces of N_{so} pearls, of which N_{rso} are black.

$\vec{a} = \text{zeros}(1, N_{so})$; $\vec{b} = []$;
Initial call; neckrec(1,1,0,N_{so},N_{rso})
neckrec(t,p,$ones$,N_{so},N_{rso})
if $ones <= N_{rso}$ **then**
  if $t > N_{so}$ **then**
    if mod(N_{so}, p) = 0 **then**
      if sum($\vec{a}$) = N_{rso} **then**
        $\vec{b}(\text{size}(\vec{b}, 1) + 1, :) = \vec{a}(2 : end)$
      end
    end
    else
      $\vec{a}(t + 1) = \vec{a}(t - p + 1)$
      if $\vec{a}(t + 1) > 0$ **then**
        neckrec($t + 1$,p,$ones + 1$,N_{so},N_{rso}
      end
      else
        neckrec($t + 1$,p,$ones$,N_{so},N_{rso})
      end
      for $j \leftarrow \vec{a}(t + 1 - p) + 1$ **to** 1 **do**
        $\vec{a}(t + 1) = j$
        neckrec($t + 1$,t,$ones + 1$,N_{so},N_{rso})
      end
    end
  end
end

Algorithm 2: Symmetries and the symmetry number of a given necklace.

Input: Matrix of necklaces M and number of admissible locations N_{so}.
Output: A matrix S indicating the symmetries and a vector N indicating the number of vectors for each necklace.

$[row, N_{rso}] = \text{size}(M)$
$[nod, d] = \text{divisors}(N_{so})$
$N = \text{zeros}(row, 1)$
$S = -\text{ones}(row, nod)$
for $i \leftarrow 1$ **to** row **do**
  $counter = 0$
  for $j \leftarrow 1$ **to** nod **do**
    $sym = true$;
    A = 1 : d(j)
    for $k \leftarrow 1$ **to** $\frac{N_{so}}{d(j)} - 1$ **do**
      $B = A + k^* d(j)$
      if isequal($M(i, A)$, $M(i, B)$) = 0 **then**
        $sym = false;$
        break;
      end
    end
    if $sym = true$ **then**
      $counter = counter + 1$
      $S(i, counter) = d(j)$
    end
  end
  $N(i) = counter$
end

Algorithm 3: Solutions of the Diophantine equation $kA \equiv B \bmod(C)$

Input: C, A, B.

Output: A vector w with possible values of k such that $C \mid kA - B$.

```
[d, x1, k1] = gcd(–C, A)
w = –ones(1, C + 1);
counter = 0;
if mod(B, d) = 0 then
   for i = – C : C do
      k = (k1*B/d) + (i – 1)*(C/d);
      if (k >= 0 and k < C) then
         counter = counter + 1
         w(counter) = k;
      end
   end
end
w(counter + 1) = – 1;
```

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