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On the accuracy of de Casteljau-type algorithms and Bernstein representations ☆

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Dedicated to the memory of Paul de Casteljau

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ABSTRACT

This paper summarizes interesting results on systematic backward and forward error analyses performed for corner cutting algorithms providing evaluation of univariate and multivariate functions defined in terms of Bernstein and Bernstein related bases. Relevant results on the conditioning of the bases are also recalled. Finally, the paper surveys important advances, lately obtained, for the design of algorithms adapted to the structure of totally positive matrices, allowing the resolution of interpolation and approximation problems with Bernstein-type bases achieving computations to high relative accuracy.

1. Introduction

Polynomial Bernstein bases (see Farouki, 2012) allow us the definition of Bézier curves and surfaces, which can be used to approximate curves or surfaces to a high degree of accuracy. Bernstein bases have optimal shape preserving (Carnicer and Peña, 1994b) and stability (Farouki and Goodman, 1996) properties and so, are widely used in the field of Computer Aided Geometric Design (CAGD). In the resolution of elliptic and hyperbolic partial differential equations with Galerkin and collocation methods, Bernstein bases have also been considered (cf. (Bhatti and Bracken, 2012; Bhatta and Bhatti, 2006)). Furthermore, Bézier curves can represent the most probable reaction path in a high dimensional configuration space and so, are also useful for the modeling of chemical reactions (cf. Bellucci and Trout, 2014). The de Casteljau algorithm is a well known corner cutting algorithm for the efficient and stable evaluation of polynomials in Bézier form (cf. Mainar and Peña, 1999). On the other hand, the evaluation of rational Bézier curves and surfaces in a floating point arithmetic is an important task in the field of Geometric Modeling (cf. Hoschek and Lasser, 1993; Delgado and Peña, 2010, 2011, 2013) and in other fields.

In Approximation Theory, stability and shape preserving properties as well as properties of evaluation algorithms associated to the Bernstein basis of bivariate polynomials defined on a triangle have also been widely analyzed. For instance, in

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Floater and Peña (2000), Floater and Peña focused on the monotonicity preservation property showing that, for quadratic polynomials, this basis is the unique (up to trivial extensions of it) satisfying a strong monotonicity preservation property. In Delgado and Peña (2007a), axial monotonicity preservation of rational Bernstein bases defined on both rectangular and triangular patches was studied.

In Farouki and Rajan (1987), the inherent numerical stability under the influence of imprecise computer arithmetic or perturbed input data of Bernstein-Bézier methods is described. Let us recall that one practical aspect to be considered in the evaluation of a function f of a vector space of functions is the stability with respect to perturbations of the coefficients. This fundamental concern depends on the basis that has been used for the representation of the functions of the space. In fact, the condition number of a basis allows us to represent how sensitive a value $f(x)$ is to any perturbations of a given maximal relative magnitude in the corresponding coefficients of f. It is well known that, for polynomial evaluation, Bernstein bases on compact intervals are optimally stable among all bases formed by nonnegative polynomials (cf. Farouki and Goodman, 1996). This optimal stability is closely related with a minimality with respect to an order relation among bases of nonnegative functions which has been considered in (Farouki and Goodman, 1996; Carnicer and Peña, 1994a).

Fundamental problems in interpolation and approximation require linear algebra computations with collocation, wronskian and gramian matrices of Bernstein and Bernstein related bases. The conditioning of the explicit conversion between Bernstein and monomial polynomial bases exponentially increases with the polynomial degree (cf. Farouki, 1991) and consequently, it is convenient to avoid the conversion to the power bases when performing numerical computations with these matrices. However, these matrices may become ill-conditioned as their dimension increases and then, standard routines implementing best traditional numerical methods do not obtain accurate results when computing their eigenvalues, their singular values, their inverse matrices or the solution of linear systems of equations. For this reason, during the last years, the design and analysis of algorithms to obtain calculations to high relative accuracy, whose relative errors will be of the order of the machine precision regardless of the dimension or the conditioning, has attracted the interest of many researchers (cf. Demmel and Koev, 2005; Koev, 2005; Mainar et al., 2020a, 2022a,b,c; Marco and Martínez, 2019).

In order to make this paper as self-contained as possible, Section 2 recalls basic concepts and results related to Bernstein bases, Bézier representations, error analysis and high relative accuracy. Section 3 summarizes systematic forward and backward error analyses with respect to the evaluations of functions defined by means of Bernstein-type bases. Results on the conditioning of these bases are collected in Section 4. Finally, Section 5 focuses on the representation of matrices for the interpolation and approximation with Bernstein-type bases, providing factorizations to derive algebraic computations to high relative accuracy.

2. Notations and basic aspects and concepts

We start by recalling some basic concepts and results. Given a sequence of functions $(u_0,\dots,u_n)$ on $I=[a,b]$, and a sequence of points $(C_0,\dots,C_n)$ in $\mathbb{R}^k$, we may define a parametric curve

$$\gamma(t)=\sum_{i=0}^{n} C_i u_i(t), \quad t\in[a,b].$$

The points C_i, $i=0,\dots,n$, are usually called *control points*. The polygonal arc $C_0\cdots C_n$ with vertices $C_0,\dots,C_n$ is usually called the *control polygon* of the curve γ. In CAGD, it is usually required that the functions u_i, $i=0,\dots,n$, are continuous, nonnegative and $\sum_{i=0}^{n} u_i(t)=1$ for all $t\in I$ that is, the system $U=(u_0,\dots,u_n)$ is *normalized*. A normalized system of nonnegative functions is usually called a *blending* system. A required property for curve design is the *convex hull property*, which is satisfied whenever the parametric curves always lie in the convex hull of their control polygons with respect to a given basis. The convex hull property holds if and only if the system of functions is blending.

A matrix is *totally positive (TP)* when all its minors are nonnegative and *strictly totally positive* (STP) when all its minors are positive (see Ando, 1987; Pinkus, 2010). In the literature, totally positive and strictly totally positive matrices are also known as totally nonnegative and totally positive matrices, respectively (see Fallat and Johnson, 2011; Koev, 2007).

Given a system of functions $u=(u_0,\dots,u_n)$ defined on $I\subseteq\mathbb{R}$, the *collocation matrix* of u at $t_0<\cdots<t_m$ in I is given by

$$M\begin{pmatrix} u_0,\dots,u_n \\ t_0,\dots,t_m \end{pmatrix} := (u_j(t_i))_{0\le i\le m;0\le j\le n}.$$

A basis u is TP if all its collocation matrices are TP. If u is normalized totally positive (NTP), then the curve

$$\gamma(t)=\sum_{i=0}^{n} P_i u_i(t), \quad t\in I,$$

inherits many shape properties of the control polygon $P_0\cdots P_n$ (cf. Carnicer and Peña, 1993).

The polynomial basis most used in CAGD is the Bernstein basis. Let us denote by $\mathbf{P}^n(I)$ the space of polynomials of degree not greater than n on the variable t defined on an interval $I\subseteq\mathbb{R}$. The *Bernstein basis* of $\mathbf{P}^n([0,1])$ is

$$b_n = (b_0^n, \dots, b_n^n), \quad b_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}, \quad t \in [0,1], \quad i = 0, \dots, n. \tag{1}$$

Given a triangle T with vertices P_1, P_2, P_3 and $P \in \mathbb{R}^s$, the barycentric coordinates $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ corresponding to P are uniquely defined by

$$\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 = P, \quad \lambda_1 + \lambda_2 + \lambda_3 = 1.$$

Let $\mathbf{\Pi}^n(T)$ be the space of polynomials of total degree n defined on T. The Bernstein basis of $\mathbf{\Pi}^n(T)$ is formed by

$$B_{i,j,k}^n(\lambda_1, \lambda_2, \lambda_3) = \frac{n!}{i!j!k!} \lambda_1^i \lambda_2^j \lambda_3^k, \quad i + j + k = n. \tag{2}$$

When considering the solution of a problem that has been solved in a floating point arithmetic, a forward error analysis measures how far the computed solution is from the exact solution. Nevertheless, one can try to find the initial data for which the computed solution is the exact solution of the problem. Backward error analysis measures how far these data are from the original data and consider rounding errors as perturbations in the initial data of the problem.

Let us introduce some standard notation in error analysis. Given $a \in \mathbb{R}$, the computed element in a floating point arithmetic will be denoted by either $fl(a)$ or by $\hat{a}$. Given $k \in \mathbb{N}_0$ such that $k\varepsilon < 1$, where ε is the unit roundoff, let us define

$$\gamma_k := \frac{k\varepsilon}{1 - k\varepsilon} = k\varepsilon + \mathcal{O}(\varepsilon^2).$$

Error bounds are usually expressed in terms of quantities γ_k.

An algorithm for the resolution of an algebraic problem is performed to *high relative accuracy* in floating point arithmetic if the relative errors in the computations have the order of the unit round-off (or machine precision), without being affected by the dimension or the conventional conditionings of the problem. It is well known that algorithms to high relative accuracy are those avoiding subtractive cancellations, that is, only requiring the following arithmetics operations: products, quotients, and additions of numbers of the same sign (see page 52 in Demmel et al., 1999). Moreover, if the floating-point arithmetic is well implemented, the subtraction of initial data can also be done without losing high relative accuracy (see page 53 in Demmel et al., 1999 and Demmel et al., 2008).

Computations to high relative accuracy with TP matrices can be achieved by means of a proper representation of the matrices in terms of bidiagonal factorizations, which is in turn closely related to their Neville elimination (cf. Gasca and Peña, 1992, 1994, 1996). As a consequence of Theorem 4.2 and the arguments of p. 116 of Gasca and Peña (1996), we know that a nonsingular TP matrix $A \in \mathbb{R}^{(n+1)\times(n+1)}$ admits a decomposition of the form

$$A = F_n F_{n-1} \cdots F_1 D G_1 G_2 \cdots G_n, \tag{3}$$

where $F_i \in \mathbb{R}^{(n+1)\times(n+1)}$ (resp. $G_i \in \mathbb{R}^{(n+1)\times(n+1)}$) is the TP, lower (resp. upper) triangular bidiagonal matrix given by

$$F_i = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & m_{i+1,1} & 1 & & \\ & & & \ddots & \ddots & \\ & & & & m_{n+1,n+1-i} & 1 \end{pmatrix}, \quad G_i^T = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & \widetilde{m}_{i+1,1} & 1 & & \\ & & & \ddots & \ddots & \\ & & & & \widetilde{m}_{n+1,n+1-i} & 1 \end{pmatrix}, \tag{4}$$

and D is the diagonal matrix whose diagonal elements are the diagonal pivots, $p_{i,i} > 0$, $i = 1, \dots, n+1$, of the Neville elimination of A. The elements $m_{i,j}$ (resp. $\widetilde{m}_{j,i}$) of F_i (resp. G_i) are the multipliers of the Neville elimination of A (resp. A^T). Under the following conditions

$$m_{i,j} = 0 \quad \Rightarrow \quad m_{h,j} = 0, \ \forall h > i \quad \text{and} \quad \widetilde{m}_{i,j} = 0 \quad \Rightarrow \quad \widetilde{m}_{i,k} = 0, \ \forall k > j, \tag{5}$$

the factorization (3) is unique and can be represented by means of a matrix $\mathcal{BD}(A) = (b_{i,j})_{1 \le i,j \le n+1}$, such that

$$b_{i,j} := \begin{cases} m_{i,j}, & \text{if } i > j, \\ p_{i,i}, & \text{if } i = j, \\ \widetilde{m}_{j,i}, & \text{if } i < j. \end{cases} \tag{6}$$

3. Error analysis of evaluation algorithms generalizing de Casteljau algorithm

3.1. Evaluation of univariate functions

The de Casteljau algorithm is the usual algorithm for evaluating polynomial curves in Bézier form $\gamma(t) = \sum_{i=0}^n P_i b_i^n(t)$ at $t \in [0,1]$, where $b_n = (b_0^n, \dots, b_n^n)$ is the Bernstein basis given by (1). The de Casteljau algorithm is a corner cutting algorithm since each of their steps consists of a convex combination. This type of algorithms has a nice geometric interpretation and good stability properties. Let us describe this algorithm with more detail.

Let $p(t)=\sum_{i=0}^n c_i b_i^n(t)\in \mathbf{P}^n([0,1])$ and $t\in[0,1]$. Denoting $p_i^{(0)}=c_i$, for $i=0,\dots,n$, and calculating

$$p_i^{(r)}=(1-t)\,p_i^{(r-1)}+t\,p_{i+1}^{(r-1)},\quad i=0,\dots,n-r,\quad r=1,\dots,n, \tag{7}$$

we have $p_0^{(n)}=p(t)$.

In Proposition 3.1 of Mainar and Peña (1999), Mainar and Peña carried out a backward error analysis of corner cutting algorithms for the evaluation of univariate functions. In particular, bounds for the absolute forward errors of these algorithms were provided in Corollary 3.2 of Mainar and Peña (1999). These error bounds are computed separately from the evaluation of the curve, that is, they are a priori bounds. However, in practical applications it is more useful to obtain a bound of the forward error at the same time as the value $p(t)$ is computed. So, in order to get this goal, a running error analysis of corner cutting algorithms for the evaluation of univariate functions was performed in Section 4 of Mainar and Peña (1999). For the sake of completeness, the following theorem summarizes these results for the particular case of the de Casteljau algorithm for the evaluation of polynomials in Bézier form:

Theorem 1. *Let $p(t)=\sum_{i=0}^n c_i b_i^n(t)\in \mathbf{P}^n([0,1])$ represented in terms of the Bernstein basis of degree n in (1), and let us consider the de Casteljau algorithm (7) for the evaluation of $p(t)$. The computed value $\hat{p}(t):=fl(p(t))$ satisfies:*

a) *Backward error: $\hat{p}(t)=\sum_{i=0}^n \bar{c}_i b_i^n(t)$ where*

$$\frac{\left|\bar{c}_i-c_i\right|}{|c_i|}\le \gamma_{2n},\quad \text{for all } i=0,\dots,n.$$

b) *Forward error:*

$$\left|p(t)-\hat{p}(t)\right|\le \gamma_{2n}\sum_{i=0}^n |c_i|\, b_i^n(t), \tag{8}$$

and then

$$\left|p(t)-\hat{p}(t)\right|\le \gamma_{2n}\max_{0\le i\le n}|c_i|\,.$$

c) *Running error: defining $M_i^{(0)}=\frac{1}{2}|c_i|$, for $i=0,\dots,n$, and*

$$M_i^{(r)}=(1-t)M_i^{(r-1)}+tM_{i+1}^{(r-1)}+\left|\hat{p}_i^{(r)}\right|,\quad i=0,\dots,n-r,\quad r=1,\dots,n,$$

we have

$$\left|p(t)-\hat{p}(t)\right|\le \varepsilon\big(2M_0^{(n)}-\left|\hat{p}(t)\right|\big).$$

Let us notice that similar forward error bounds do not hold for other evaluation algorithms, such as the Horner algorithm, which do not consist of convex combinations. Although the complexity of the Horner evaluation method is $O(n)$ instead of $O(n^2)$, the Horner forward error bound in formula (5.3) of Higham (2002) is bigger than the corresponding formula (8) for the forward error of de Casteljau algorithm. This fact can be explained by introducing a condition number for evaluation.

Remark 1. By Theorem 1 b) we have that

$$\left|p(t)-\hat{p}(t)\right|\le \gamma_{2n}\, C_{b_n}(p,t),$$

where $C_{b_n}(p,t):=\sum_{i=0}^n \left|c_i\, b_i^n(t)\right|$ is called a *condition number* for the evaluation of $p(t)$ with the basis b_n (see Farouki and Goodman, 1996; Farouki and Rajan, 1988 and Section 4). In fact, the formula for the forward error satisfies the classical relation:

$$\text{forward error} \le \text{backward error} \times \text{condition number}.$$

For de Casteljau and Horner algorithms, the term corresponding to the backward error in the previous formula is the same (γ_{2n}). However, it is well known (cf. [5]) that the condition number corresponding to the Bernstein basis is always lower than or equal to the condition number corresponding to the monomial basis (used for the polynomial representation when evaluating with the Horner algorithm). In fact, Bernstein bases are optimally stable for polynomial evaluation among all bases formed by nonnegative polynomials, as it is going to be explained with more detail in Section 4.

Let us observe that Theorem 1 provides bounds for absolute errors. However, the relative errors are more significant. In Delgado and Peña (2009a), Delgado and Peña provided a sufficient condition depending on $p(t)$ and an absolute error bound εK for obtaining relative error bounds. The following result regards the forward and running relative error bounds for the evaluation of polynomials represented in the Bernstein basis, obtained in Delgado and Peña (2009a).

Theorem 2. *Let $p(t) = \sum_{i=0}^n c_i b_i^n(t) \in \mathbf{P}^n([0,1])$ represented in terms of the Bernstein basis of degree n in (1), and let us consider the de Casteljau algorithm (7) for the evaluation of $p(t)$. The computed value $\hat{p}(x) := fl(p(x))$ satisfies:*

a) *Forward relative error: if $\left|\hat{p}(t)\right| > \gamma_{2n}\, C_{b_n}(p,t)$ then*

$$\frac{\left|p(t)-\hat{p}(t)\right|}{|p(t)|} \leq \gamma_{2n} \frac{\sum_{i=0}^n |c_i|\; b_i^n(t)}{\left|\hat{p}(t)\right|}.$$

b) *Running relative error: if $\left|\hat{p}(t)\right| > \varepsilon(2M_0^{(n)} - \left|\hat{p}(t)\right|)$ then*

$$\frac{\left|p(t)-\hat{p}(t)\right|}{|p(t)|} \leq \varepsilon \frac{(2M_0^{(n)} - \left|\hat{p}(t)\right|)}{\left|\hat{p}(t)\right|}.$$

3.2. Evaluation of multivariate functions

In Delgado and Peña (2008), both backward and forward error analyses of corner cutting algorithms for the evaluation of tensor product surfaces were performed. It was proved that these algorithms are backward stable. Now, let us recall this error analysis for the particular case of the generalization of the de Casteljau algorithm for the evaluation of tensor product Bézier functions

$$F(x,y) = \sum_{i=0}^m \sum_{j=0}^n f_{i,j} b_i^m(x) b_j^n(y), \quad (x,y) \in [0,1]\times[0,1], \quad f_{i,j} \in \mathbb{R},\ \forall i,j. \tag{9}$$

In order to evaluate $F(x,y)$ with the de Casteljau-type algorithm, we denote $f_{i,j}^{(0,0)} := f_{i,j}$ for all $i = 0,\dots,m$ and $j = 0,\dots,n$. Calculating the coefficients

$$f_{i,j}^{(0,r)} = (1-y)\, f_{i,j}^{(0,r-1)} + y\, f_{i,j+1}^{(0,r-1)}, \quad i = 0,\dots,m, \quad r = 1,\dots,n, \quad j = 0,\dots,n-r,$$

and then, computing

$$f_{i,0}^{(r,n)} = (1-x)\, f_{i,0}^{(r-1,n)} + x\, f_{i+1,0}^{(r-1,n)}, \quad r = 1,\dots,m, \quad i = 0,\dots,m-r,$$

the following value is obtained $f_{0,0}^{(m,n)} = F(x,y)$.

In Theorem 5 of Delgado and Peña (2008) a forward error analysis of corner cutting algorithms for tensor product functions was presented and the corresponding running error analysis was derived in Theorem 8. The following theorem summarizes these results, for the particular case of de Casteljau-type algorithm for tensor product Bézier functions:

Theorem 3. *Let F be a tensor product Bézier function given by (9) and let us consider the de Casteljau-type algorithm for the evaluation of $F(x,y)$. The computed value $\hat{F}(x,y) = fl(F(x,y))$ satisfies:*

a) *Backward error: $\hat{F}(x,y) = \sum_{i=0}^m \sum_{j=0}^n \overline{f}_{i,j} b_i^m(x) b_j^n(y)$ where*

$$\frac{\left|\overline{f}_{i,j} - f_{i,j}\right|}{\left|f_{i,j}\right|} \leq \gamma_{2(m+n)}, \quad \textit{for all } i = 0,\dots,m \textit{ and } j = 0,\dots,n.$$

b) *Forward error:*

$$\left|F(x,y) - \hat{F}(x,y)\right| \leq \gamma_{2(m+n)} \sum_{i=0}^m \sum_{j=0}^n \left|f_{i,j}\right|\; b_i^m(x) b_j^n(y) \leq \gamma_{2(m+n)} \max_{i,j} \left|f_{i,j}\right|.$$

c) *Running error:*

$$\left|F(x,y) - \hat{F}(x,y)\right| \leq \varepsilon\, \pi_{0,0}^{(m,n)},$$

where $\pi_{0,0}^{(m,n)}$ is computed recursively, using data computed while applying the de Casteljau-type algorithm (see Theorem 8 of Delgado and Peña, 2008 for more details).

The numerical experimentation described in Delgado and Peña (2008) illustrates the better stability properties of the de Casteljau algorithm over the extension of the Horner algorithm for the evaluation of tensor product polynomial surfaces.

In Delgado and Peña (2010), forward and running error analyses of the commonly used algorithm for the evaluation of rational Bézier functions were performed. But this algorithm uses the projection operator in its last step and so it is not a corner cutting algorithm. In Delgado and Peña (2007b), Delgado and Peña presented a corner cutting algorithm for the evaluation of rational Bézier functions. As a novelty, this algorithm avoided using the projection operator for the evaluation.

Given a matrix $(c_{i,j})_{0\le i\le m;0\le j\le n}$ of real numbers and a matrix $(w_{i,j})_{0\le i\le m;0\le j\le n}$ of positive weights, let us consider a rational Bézier function

$$F(x,y) := \sum_{i=0}^{m}\sum_{j=0}^{n} c_{i,j}\frac{w_{i,j}\,b_i^m(x)\,b_j^n(y)}{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{i,j}\,b_i^m(x)\,b_j^n(y)}, \quad (x,y)\in[0,1]\times[0,1]. \tag{10}$$

In order to evaluate this function, let us denote $f_{i,j}^{(0,0)} := c_{i,j}$ and $w_{i,j}^{(0,0)} := w_{i,j}$, for all $i = 0,\dots,m$ and $j = 0,\dots,n$. First, we obtain the coefficients $f_{i,j}^{(0,r)}$ as

$$\begin{aligned} w_{i,j}^{(0,r)} &= (1-y)\,w_{i,j}^{(0,r-1)} + y\,w_{i,j+1}^{(0,r-1)},\\ f_{i,j}^{(0,r)} &= (1-y)\,\frac{w_{i,j}^{(0,r-1)}}{w_{i,j}^{(0r)}}\,f_{i,j}^{(0,r-1)} + y\,\frac{w_{i,j+1}^{(0,r-1)}}{w_{i,j}^{(0,r)}}\,f_{i,j+1}^{(0,r-1)},\ i=0,\dots,m, r=1,\dots,n,\ j=0,\dots,n-r. \end{aligned}$$

Then, we compute the coefficients $f_{i,0}^{(r,n)}$ as

$$\begin{aligned} w_{i,0}^{(r,n)} &= (1-x)\,w_{i,0}^{(r-1,n)} + x\,w_{i+1,0}^{(r-1,n)},\\ f_{i,0}^{(r,n)} &= (1-x)\,\frac{w_{i,0}^{(r-1,n)}}{w_{i,0}^{(r,n)}}\,f_{i,0}^{(r-1,n)} + x\,\frac{w_{i+1,0}^{(r-1,n)}}{w_{i,0}^{(r,n)}}\,f_{i+1,0}^{(r-1,n)}, \quad r=1,\dots,m, \quad i=0,\dots,m-r. \end{aligned}$$

Finally, $f_{0,0}^{(m,n)} = F(x,y)$ is obtained.

This algorithm was called *corner cutting rational surface algorithm* in Delgado and Peña (2007b) and it can be considered a de Casteljau-type algorithm for rational Bézier surfaces.

In Theorem 3.1 of Delgado and Peña (2007b), a forward error analysis of the corner cutting rational surface algorithm was carried out. In Theorem 5 of Delgado and Peña (2011), Delgado and Peña provided a running error analysis for the corner cutting rational surface algorithm. These results are summarized in the following theorem.

Theorem 4. *Let us consider the rational function F given by (10) and let $\hat F(x,y)$ be the computed value with floating point arithmetic by the corner cutting rational surface algorithm. Then*

a) *Forward error bound:*

$$\left|F(x,y)-\hat F(x,y)\right| \le \gamma_k \sum_{i=0}^{m}\sum_{j=0}^{n}\left|c_{i,j}\right|\frac{w_{i,j}\,b_i^m(x)\,b_j^n(y)}{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{i,j}\,b_i^m(x)\,b_j^n(y)} \le \gamma_k \max_{i,j}\left|c_{i,j}\right|,$$

where $k = (3m^2+5m)+(3n^2+5n)+6mn$.

b) *Running error bound:*

$$\left|F(x,y)-\hat F(x,y)\right| \le \varepsilon\pi_{0,0}^{(m,n)},$$

where $\pi_{0,0}^{(m,n)}$ is computed recursively using data computed while evaluating $F(x,y)$ by using the corner cutting rational surface algorithm (see Theorem 5 of Delgado and Peña, 2011 for more details).

Remark 2. By a) of the previous theorem, we have that

$$\left|F(x,y)-\hat F(x,y)\right| \le \gamma_{2n}\, S_{rb_{m,n}}(F(x,y)),$$

where $S_{rb_{m,n}}(F(x,y)) := \sum_{i=0}^{m}\sum_{i=0}^{n}\left|c_{i,j}\,rb_{i,j}^{m,n}(x)\right|$ is called a *condition number* for the evaluation of $F(x,y)$ with the basis

$$rb_{m,n} = (rb_{i,j}^{m,n}(x,y))_{0\le i\le m;0\le j\le n}, \quad rb_{i,j}^{m,n}(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{w_{i,j}\,b_i^m(x)\,b_j^n(y)}{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{i,j}\,b_i^m(x)\,b_j^n(y)},$$

for $i=0,\dots,m,\ j=0,\dots,n$.

Triangular Bézier patches are an alternative to tensor product patches, providing in some senses more flexibility in the design than the latter. In geometric modeling, bivariate polynomials defined on a triangle are usually stored in a Bernstein-Bézier form and evaluated by the de Casteljau algorithm. The *Bernstein-Bézier representation* of $p \in \mathbf{\Pi}^n(T)$ is

$$p(\lambda_1, \lambda_2, \lambda_3) = \sum_{i=0}^{n} \sum_{j=0}^{i} b_{n-i,i-j,j} B^n_{n-i,i-j,j}(\lambda_1, \lambda_2, \lambda_3), \tag{11}$$

where $B^n_{i,j,k}(\lambda_1, \lambda_2, \lambda_3)$, $i + j + k = n$, are the Bernstein polynomials on the triangle T (see (2)).

Starting with $b^{(0)}_{i,j,k} := b_{i,j,k}$, for all $i + j + k = n$, the de Casteljau algorithm computes

$$b^{(k)}_{n-i-k,i-j,j} = \lambda_1\, b^{(k-1)}_{n-i-k+1,i-j,j} + \lambda_2\, b^{(k-1)}_{n-i-k,i-j+1,j} + \lambda_3\, b^{(k-1)}_{n-i-k,i-j,j+1}, \tag{12}$$

for $j = 0, \dots, i$, $i = 0, \dots, n-k$, $k = 1, \dots, n$. So that $b^{(n)}_{0,0,0} = p(\lambda_1, \lambda_2, \lambda_3)$.

Let us observe that each step of the algorithm is a convex combination of values computed at the previous iteration, such as it happened with univariate corner cutting algorithms (cf. Mainar and Peña, 1999). This property implies some good stability properties of the de Casteljau algorithm (12).

In Mainar and Peña (2006a), a forward error analysis for the evaluation of bivariate polynomials defined on a triangle through the de Casteljau algorithm was performed. Some obtained results are sumarized in the following theorem.

Theorem 5. *Let us consider $p \in \mathbf{\Pi}^n(T)$ in its Bernstein-Bézier form (11) and let $\hat{p}(\lambda_1, \lambda_2, \lambda_3)$ be the computed value with floating point arithmetic through the de Casteljau algorithm (12). Then*

a) *Forward error bound:*

$$|p(\lambda_1, \lambda_2, \lambda_3) - \hat{p}(\lambda_1, \lambda_2, \lambda_3)| \le \gamma_{3n} \sum_{i=0}^{n} \sum_{j=0}^{i} |b_{n-i,i-j,j}| B^n_{n-i,i-j,j}(\lambda_1, \lambda_2, \lambda_3) \le \gamma_{3n} \max_{i+j+k=n} \left| b_{i,j,k} \right| .$$

b) *Running error bound:*

$$\left| p(\lambda_1, \lambda_2, \lambda_3) - \hat{p}(\lambda_1, \lambda_2, \lambda_3) \right| \le \varepsilon \left(3M^{(d)}_{0,0,0} - 2 \left| \hat{b}^{(d)}_{0,0,0} \right| \right),$$

where $M^{(d)}_{0,0,0}$ and $\hat{b}^{(d)}_{0,0,0}$ can be obtained while running the de Casteljau algorithm (see pages 101-102 of Mainar and Peña, 2006a for more details).

In addition to the de Casteljau algorithm, Schumaker and Volk proposed in Schumaker and Volk (1986) a multivariate Horner-type algorithm (called VS algorithm) with lower computational cost.

In Mainar and Peña (2006a), the de Casteljau algorithm for the evaluation of n-degree bivariate polynomials was also compared to the VS algorithm. A posteriori error bound for the VS algorithm was derived and it was also shown that the forward error bounds for both evaluation algorithms only differ in the constants γ_{3n} (for the de Casteljau algorithm) and γ_{4n} (for the Volk and Schumaker algorithm).

The analysis in Mainar and Peña (2006a) for the evaluation of polynomials defined on triangles was generalized in Mainar and Peña (2006b) for the evaluation of multivariate polynomials of d variables in Bernstein–Bézier form. A forward error analysis for the corresponding de Casteljau and VS algorithm was performed. The results in this paper show that both algorithms present similar nice stability properties in the trivariate case and, when increasing the number of variables, the VS algorithm presents better properties than the de Casteljau algorithm.

In Delgado and Peña (2013), it was presented and analyzed a generalization of the usual de Casteljau-type algorithm for the evaluation of rational Bernstein–Bézier surfaces defined on triangular patches for surfaces with any number of barycentric coordinates. In particular, a forward error analysis of the generalized algorithm was carried out. Here, for the sake of simplicity, let us recall the triangular case.

Given a sequence of positive weights $w_{i,j,k}$, $i + j + k = n$, let us consider the rational triangular Bernstein basis for three barycentric coordinates given by

$$rb_n = (rb^n_{i,j,k})_{i+j+k=n}, \quad rb^n_{i,j,k}(\lambda_1, \lambda_2, \lambda_3) = \frac{w_{i,j,k} B^n_{i,j,k}(\lambda_1, \lambda_2, \lambda_3)}{\sum_{i+j+k=n} w_{i,j,k} B^n_{i,j,k}(\lambda_1, \lambda_2, \lambda_3)}, \quad i + j + k = n,$$

where $B^n_{i,j,k}$ are the Bernstein polynomials on the triangle T (see (2)). Given a sequence of real numbers $c_{i,j,k}$, $i + j + k = n$, we define a rational Bézier triangle function as

$$B(\lambda_1, \lambda_2, \lambda_3) = \sum_{i+j+k=n} c_{i,j,k}\, rb^n_{i,j,k}(\lambda_1, \lambda_2, \lambda_3). \tag{13}$$

Rational Bézier functions on triangles can be evaluated by the rational triangular de Casteljau algorithm. In order to evaluate $B(\lambda_1, \lambda_2, \lambda_3)$ with this algorithm, let us denote by e_k the multi-index such that the k-th index is 1 and the remaining indices are 0 and, $c_\alpha^{(0)} := c_\alpha$ $w_\alpha^{(0)} := w_\alpha$ for all multi-index α such that $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 = n$. Then, computing

$$w_{\alpha_r}^{(r)} = \sum_{k=1}^{3} \lambda_k w_{\alpha_r+e_k}^{(r-1)}, \quad c_{\alpha_r}^{(r)} = \frac{\sum_{k=1}^{3} \lambda_k w_{\alpha_r+e_k}^{(r-1)} c_{\alpha_r+e_k}^{(r-1)}}{w_{\alpha_r}^{(r)}}, \tag{14}$$

for $r = 1, \dots, n$ and $|\alpha_r| = n - r$. So, $c_{0,0,0}^{(n)} = B(\lambda_1, \lambda_2, \lambda_3)$ is obtained.

The following result provides a forward error bound for the rational triangular de Casteljau algorithm (Theorem 2 of Delgado and Peña, 2013 for $s = 2$).

Theorem 6. *Let B be the rational triangular function (13), and let $\hat{B}(\lambda_1, \lambda_2, \lambda_3)$ be the computed value with floating point arithmetic by the rational triangular de Casteljau algorithm (14). Then*

$$\left| B(\lambda_1, \lambda_2, \lambda_3) - \hat{B}(\lambda_1, \lambda_2, \lambda_3) \right| \le \left((3n^2 + 5n)u + \mathcal{O}(u^2) \right) \sum_{i_1+i_2+i_3=n} |c_{i_1,i_2,i_3}| \, r b_{i,j,k}^n(\lambda_1, \lambda_2, \lambda_3).$$

The extension of the previous evaluation algorithm for surfaces with any number of barycentric coordinates was called in Delgado and Peña (2013) the rational triangular de Casteljau-Kahan algorithm since it uses Kahan-summation, which allows us to obtain a bound independent of the number of barycentric coordinates. For the error analysis of this algorithm see Theorem 2, for $s \ge 3$, of Delgado and Peña (2013). Finally, let us note that on the numerical stability of linear barycentric rational interpolation, interested readers can see Berrut and Trefethen (2004); Fuda et al. (2022).

4. Optimal stability of the bases for the evaluation

Given a basis $u = (u_0, \dots, u_n)$ of a vector space $\mathcal{U}$ of real functions defined on a subset S of $\mathbb{R}^m$ ($m \ge 1$) and a function $f \in \mathcal{U}$, we can write $f(x) = \sum_{i=0}^n c_i u_i(x)$ for all $x \in S$, where $c_i \in \mathbb{R}$ for all $i = 0, \dots, n$. The stability of the basis u with respect to the evaluation at a point is measured by the function $C_u : \mathcal{U} \times I \to \mathbb{R}_+$ given by

$$C_u(f, x) := \sum_{i=0}^{n} |c_i u_i(x)|. \tag{15}$$

This condition number (15) was introduced in Farouki and Rajan (1988) for polynomial evaluation and has been used in the previous section. Later, in Lyche and Peña (2004), a corresponding relative condition number was defined:

$$cond(u; f, x) = \frac{\sum_{i=0}^n |c_i u_i(x)|}{\|f\|_\infty} = \frac{\sum_{i=0}^n |c_i u_i(x)|}{\left\| \sum_{i=0}^n c_i u_i \right\|_\infty}.$$

Given two bases, u and v, of nonnegative functions, let A be the matrix of change of basis such that $v = uA$. The following result (see Lemma 3.1 of Lyche and Peña, 2004) compares the conditioning of the bases by means of the nonnegativity of A.

Lemma 7. *Let $\mathcal{U}$ be a finite dimensional vector space of functions defined on a subset S of $\mathbb{R}^m$. Let u and v be bases of nonnegative functions of $\mathcal{U}$. Then*

$$cond(u; f, x) \le cond(v; f, x), \quad \forall f \in \mathcal{U}, \quad \forall x \in S, \tag{16}$$

if and only if the matrix A such that $v = uA$ is nonnegative.

For the space $\mathbf{P}^n([0, 1])$, Theorem 3 of Farouki and Goodman (1996) leads to the optimality of the Bernstein basis.

Theorem 8. *Let $b = (b_0^n, \dots, b_n^n)$ be the Bernstein basis in (1). Then there does not exist (up to reordering and positive scaling) another basis $u = (u_0, \dots, u_n)$ of nonnegative functions in $\mathbf{P}^n([0, 1])$ such that $cond(u; p, t) \le cond(b; p, t)$, for all $t \in [0, 1]$ and $p \in \mathbf{P}^n([0, 1])$.*

This result guarantees that the Bernstein basis is best conditioned among all nonnegative bases of the corresponding space of polynomials in the sense that there is not a better conditioned nonnegative basis. Similar optimality results to the previous one are satisfied, for instance, by the B-spline basis (Peña, 1997) or by the bases of spaces of real multivariate functions, such as the tensor product B-spline basis and the triangular Bernstein basis (see Lyche and Peña, 2004), as well as the corresponding rational bases (see Delgado and Peña, 2007b, Delgado and Peña, 2013). Moreover, we have found examples of bases used in CAGD that are always worse conditioned than the corresponding optimal basis. For instance, in Delgado et al. (2023a) it has been proved that the q-Bernstein basis of triangular polynomials is worse conditioned than the Bernstein basis of triangular polynomials.

Very general sources of examples of optimally stable bases in the previous sense for spaces of univariate functions were found in Peña (2002) and Peña (2006), and are closely related to the concept of B-basis. This last concept, in turn, is related to the concept of optimal shape preserving basis in CAGD, which we are now going to recall.

Given u, v two NTP bases of $\mathcal{U}$, then u has better shape preserving properties than v (in the sense that the control polygon of a curve with respect to u is closer in shape to the curve than the control polygon with respect to v) if the matrix A such that $v = uA$ is (stochastic) TP. Since a stochastic matrix is a nonnegative square matrix whose entries of each row sum 1, observe that, in the previous claim, the matrix A is necessarily stochastic because the bases u, v are both normalized and formed by nonnegative functions. Recall that a normalized B-basis of a space is an NTP basis $b = (b_0, \dots, b_n)$ of the space such that any other NTP basis v is of the form $v = (b_0, \dots, b_n)K$, where K is a (stochastic) TP matrix. In other words, a normalized B-basis generates all NTP bases of the space by means of stochastic TP matrices. Therefore, a normalized B-basis has optimal shape preserving properties. Besides, by Theorem 4.2 of Carnicer and Peña (1994b), a space with an NTP basis has always a unique normalized B-basis b. For the space of polynomials of degree less than or equal to n on $[a, b]$, the Bernstein basis is the normalized B-basis (see Carnicer and Peña, 1993) and, for the corresponding space of polynomial splines, the B-spline basis is the normalized B-basis (see Theorem 4.6 of Carnicer and Peña, 1994b).

There are also some optimal conditioning properties of the collocation matrices of normalized B-bases. Let us start by recalling the case of the Bernstein basis of polynomials studied in Delgado and Peña (2009b).

Given a nonsingular matrix $A = (a_{ij})_{1\le i,j\le n}$, the classical condition number is

$$\kappa_\infty(A) := \|A\|_\infty \, \|A^{-1}\|_\infty.$$

Moreover, denoting by $|A|$ the matrix whose (i, j)-entry is $|a_{ij}|$, the Skeel condition number is

$$\mathrm{Cond}(A) := \|\, |A^{-1}|\,|A|\, \|_\infty.$$

The following result corresponds to Theorem 2.1 of Delgado and Peña (2009b). It shows that the collocation matrices of the Bernstein basis are the best conditioned among all the corresponding collocation matrices of NTP bases of $\mathbf{P}^n([0, 1])$, and a similar result using the Skeel condition number of the transposes of the collocation matrices.

Theorem 9. *Let $(b_0^n, \dots, b_n^n)$ be the Bernstein basis of $\mathbf{P}^n([0, 1])$ in (1), let $(v_0, \dots, v_n)$ be another NTP basis of $\mathbf{P}^n([0, 1])$. For a sequence $0 \le t_0 < \cdots < t_n \le 1$, let $V := M\begin{pmatrix} v_0,\dots,v_n \\ t_0,\dots,t_n \end{pmatrix}$ and $B := M\begin{pmatrix} b_0^n,\dots,b_n^n \\ t_0,\dots,t_n \end{pmatrix}$. Then:*

$$\kappa_\infty(B) \le \kappa_\infty(V), \quad \mathrm{Cond}(B^T) \le \mathrm{Cond}(V^T).$$

In Delgado and Peña (2020), the previous result was extended to the collocation matrices of any normalized B-basis and, in Delgado et al. (2021), to any collocation matrix of the tensor product of normalized B-bases.

The previous results concern with the conditioning of TP matrices associated to NTP bases. In the next section, we show some subclasses of TP matrices associated with CAGD representations for which many algebraic computations can be performed with high relative accuracy (HRA).

5. Computations to high relative accuracy associated to Bernstein representations

When considering Lagrange interpolation problems, the polynomial interpolant of a function f, at a given sequence of nodes satisfying $0 < t_0 < \cdots < t_n < 1$, can be writen in terms of the n-degree Bernstein basis (1) as follows,

$$p(t) = \sum_{i=0}^{n} a_i b_i^n(t), \quad t \in [0, 1].$$

The vector of the coefficients $(a_0, \dots, a_n)^T$ is the solution of the linear system of equations

$$A_n(a_0, \dots, a_n)^T = (f(t_0), \dots, f(t_n))^T, \tag{17}$$

and the coefficient matrix,

$$A_n := M\begin{pmatrix} b_0^n,\dots,b_n^n \\ t_0,\dots,t_n \end{pmatrix} = \left(\binom{n}{j-1} t_{i-1}^{j-1} (1 - t_{i-1})^{n-j+1} \right)_{1\le i,j\le n+1}, \tag{18}$$

is called *Bernstein-Vandermonde matrix*. Proposition 3.1 of Marco and Martínez (2007) proves that

$$\det A_n = \prod_{k=0}^{n} \binom{n}{k} \prod_{0\le i<j\le n} (t_j - t_i). \tag{19}$$

Let us observe that $\det A_n^{-1}$ goes to zero very quickly as the degree increases and then, most numerical algorithms would conclude that A_n^{-1} is a singular matrix.

Taking into account the partition of unity property of the Bernstein basis (1), one can easily derive that $\|A_n\|_\infty = 1$, which implies large values of $\|A_n^{-1}\|_\infty$. As a consequence, Bernstein-Vandermonde matrices are ill conditioned and we can predict that the solution of the linear system (17) is very sensitive to small errors in the evaluation of the function to be interpolated (cf. Higham, 2002). Moreover, standard algorithms for the resolution of Bernstein-Vandermonde linear systems may suffer from subtractive cancellations and the accuracy of the solutions is lost when the dimension of the system increases.

Bernstein bases defined on compact intervals are TP and then Bernstein-Vandermonde matrices are TP as well. An explicit expression for the determinants involved in the computation of the pivots of the Neville elimination of the Bernstein-Vandermonde matrix A_n in (18) can be found in page 363 of Marco and Martínez (2013).

Moreover, an algorithm for the fast and accurate computation of the matrix $\mathcal{BD}(A_n)$ in (6), storing the bidiagonal factorization of A_n, is provided. The computation of $\mathcal{BD}(A_n)$ satisfies the non inaccurate cancellation condition and so, its computation to high relative accuracy is guaranteed. Then, using the algorithms in Koev (2007), algebraic problems such as the computation of A_n^{-1}, the determination of the eigenvalues and singular values of A_n, as well as the resolution of linear systems of equations $A_n x = b$, for vectors b whose entries have alternating signs, can be performed to high relative accuracy. All these functions require the matrix $\mathcal{BD}(A_n)$ as input argument.

As for the analysis of the accuracy in the computation of $\mathcal{BD}(A_n) = (b_{i,j})_{1\le i,j\le n+1}$, Theorem 5.1 of Marco and Martínez (2013) shows that the matrix $(\hat{b}_{i,j})_{1\le i,j\le n+1}$ computed to this algorithm in floating point arithmetic satisfies

$$|\hat{b}_{i,j} - b_{i,j}| \le \frac{(4n^2+2n)\varepsilon}{1-(4n^2+2n)\varepsilon} b_{i,j}, \quad 1\le i,j\le n+1,$$

where ε denotes the machine precision. The previous bounds for the errors in the entries, confirm that the algorithm computes $\mathcal{BD}(A_n)$ accurately. In addition, the sensitivity of the bidiagonal factorization with respect to perturbations of t_i, $i = 1,\dots,n+1$, is also analyzed. In this sense, Theorem 6.1 of Marco and Martínez (2013) derives a condition number for Benstein-Vandermonde matrices A_n with respect to perturbations of the nodes.

For negative degrees, Bernstein bases were introduced in Goldman (1999). Given $m \in \mathbb{N}$, the $(n+1)$-dimensional *Bernstein basis of degree* $-m$ is the basis of rational functions defined as

$$(b_0^{-m}, b_1^{-m}, \dots, b_n^{-m}), \quad b_i^{-m}(t) := \binom{-m}{i} t^i (1-t)^{-m-i}, \quad i = 0,\dots,n, \tag{20}$$

with the convention

$$\binom{-m}{i} := \frac{(-m)(-m-1)\cdots(-m-i+1)}{i!} = (-1)^i \binom{m+i-1}{i}, \quad i \in \mathbb{N}\cup\{0\}. \tag{21}$$

Bernstein rational functions in (20) are non-negative, linearly independent, and form a partition of unity over the interval $(-\infty, 0]$ (cf. Goldman, 1999). They generate spaces of rational functions allowing an exact representation of functions, which are analytic in a neighborhood of zero, and uniformly approximate all continuous functions vanishing at minus infinity. These rational bases share many properties of Bernstein polynomials: they satisfy Descartes' Rule of Signs, form a partition of unity, and possess recurrence relations, as well as two-term formulas for differentiation and degree elevation. Moreover, they are known to be TP (see Mainar et al., 2020b). In Mainar et al. (2021) it is shown that the collocation matrix of the Bernstein basis of degree $-m$ in (20),

$$A_{-m} = M\begin{pmatrix} b_0^{-m},\dots,b_n^{-m} \\ t_0,\dots,t_n \end{pmatrix} = \left(\binom{m+j-2}{j-1} (-t_i)^{j-1} (1-t_i)^{-m-j+1} \right)_{1\le i,j\le n+1},$$

can be factorized as in (3), and the entries $m_{i,j}, \widetilde{m}_{i,j}$ and $p_{i,i}$ of F_i, G_i, $i = 1,\dots,n$, and D, respectively, satisfy formulae (13) in page 6 of Mainar et al. (2021). The computation of this bidiagonal factorization can be performed with no inaccurate cancellations and algebraic problems related to these matrices are solved to high relative accuracy.

In the literature, other extensions of Bernstein-Vandermonde matrices are analyzed for several generalizations of Bernstein bases. Unfortunately, these collocation matrices are also badly conditioned and traditional algorithms fail to obtain accurate solutions when solving related algebraic problems.

In Mainar and Peña (2018), the Bernstein basis (1) is generalized by considering two nonnegative continuous functions $f, g : I \to \mathbb{R}$ such that $f(t) \ne 0$, $g(t) \ne 0$ for all $t \in (a,b)$ and f/g is a strictly increasing function on I. The system

$$(u_0^n, \dots, u_n^n), \quad u_i^n(t) := \binom{n}{i} f^i(t) g^{n-i}(t), \quad t \in [a,b], \quad i = 0,\dots,n, \tag{22}$$

is called *fg-Bernstein basis*. For a given sequence of nodes, $a < t_1 < \cdots < t_{n+1} < b$, the fg-Vandermonde matrix at nodes t_i, $i = 1, \ldots, n+1$, is defined as

$$M_n := M \begin{pmatrix} u_0^n, \ldots, u_n^n \\ t_0, \ldots, t_n \end{pmatrix} = \left(\binom{n}{j-1} f^{j-1}(t_i) g^{n-j+1}(t_i) \right)_{1 \le i, j \le n+1}. \tag{23}$$

Theorem 3 of Mainar and Peña (2018) obtains an explicit expression of the bidiagonal factorization (3) of fg-Vandermonde matrices. In addition, an algorithm for its efficient and accurate computation is provided. It is also proved that a sufficient condition to obtain $\mathcal{BD}(M_n)$ to high relative accuracy is that the expressions $f(t_i)g(t_k) - f(t_k)g(t_i)$ for all $k < i$ are computed to high relative accuracy. Algebraic problems with collocation matrices of fg-Bernstein bases using algebraic, trigonometric or hyperbolic polynomials are accurately solved using the provided factorization.

The *q-Bernstein polynomials* of degree n on the interval $[0, 1]$ are defined as

$$q_i^n(t) := \begin{bmatrix} n \\ k \end{bmatrix}_q t^i \prod_{k=0}^{n-i-1} (1 - q^k t), \quad t \in [0, 1], \quad i = 0, \ldots, n, \tag{24}$$

where, for $q > 0$, the q-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$, $k = 0, \ldots, n$, are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! \, [n-k]_q!}$$

and, for any non-negative integer n, the q-factorial $[n]_q!$ is defined by

$$[n]_q! := [n]_q \, [n-1]_q \cdots [1]_q,$$

and the q-integer $[n]_q$ by

$$[n]_q := \begin{cases} 1 + q + \cdots + q^{n-1} = \dfrac{1-q^n}{1-q}, & \text{if } q \neq 1, \\ n, & \text{if } q = 1, \end{cases} \tag{25}$$

(cf. Phillips, 1997; Oruç and Phillips, 2003; Phillips, 2010).

For the particular case $q = 1$, the q-Bernstein basis $(q_0^n, \ldots, q_n^n)$ coincides with the Bernstein basis (1). By Corollary 3.3 of Carnicer and Peña (1994b), for any $q \in (0, 1]$, the basis $(q_0^n, \ldots, q_n^n)$ is TP on the interval $[0, 1]$ and STP on $(0, 1)$. Moreover, the partition of unity property of $(q_0^n, \ldots, q_n^n)$ can be deduced from Proposition 5.2 of Goldman and Simeonov (2015a).

In Delgado and Peña (2015), the collocation matrices at ordered sequences of nodes on $(0, 1)$ of $(q_0^n, \ldots, q_n^n)$ are called q-Bernstein–Vandermonde matrices. These matrices are STP and using the bidiagonal factorization for their converse matrix, an algorithm for the resolution of algebraic problems with these matrices to high relative accuracy is achieved.

For a given $h \in \mathbb{R}$, the *h-Bernstein basis* of $\mathbf{P}^n([0, 1])$ is

$$(H_0^n, \ldots, H_n^n), \quad H_i^n(t) := \binom{n}{i} \frac{\prod_{k=0}^{i-1}(t + kh) \prod_{k=0}^{n-i-1}(1 - t + kh)}{\prod_{k=0}^{n-1}(1 + kh)}, \quad k = i, \ldots, n, \tag{26}$$

(cf. Goldman and Simeonov, 2015b).

In Marco et al. (2019), the collocation matrices of $(H_0^n, \ldots, H_n^n)$, at ordered sequences of nodes on $(0, 1)$, are called h-Bernstein–Vandermonde matrices. Their strict total positivity is proved and an algorithm for the fast and accurate computation of the bidiagonal factorization is proposed.

In Delgado et al. (2023b), new bidiagonal decompositions of Vandermonde and related matrices such as the (q-, h-) Bernstein-Vandermonde ones, among others were introduced. These new decompositions can be computed efficiently and to high relative accuracy for TP matrices of the above classes of arbitrary rank. In turn, matrix computations (e.g., eigenvalue computation) can also be performed efficiently and to high relative accuracy for potentially singular TP matrices of Vandermonde-type.

Given real values $d_0, \ldots, d_n$ and $t_0 \in (0, 1)$, the corresponding Taylor polynomial interpolant is $p \in \mathbf{P}^n([0, 1])$ such that $p^{(k)}(t_0) = d_k$, $k = 0, \ldots, n$. The Taylor interpolant can be expressed in terms of the Bernstein basis (1) as

$$p(t) = \sum_{i=0}^{n} c_i b_i^n(t), \quad t \in [0, 1],$$

where $(c_0, \ldots, c_n)^T$ is the solution of the linear system

$$W (c_0, \ldots, c_n)^T = (d_0, \ldots, d_n)^T, \tag{27}$$

and W is the wronskian matrix

$$W = ((b_{j-1}^n)^{(i-1)}(t_0))_{1\le i,j\le n+1},$$

where the i-th derivative of a function f at the value t is denoted by $f^{(i)}(t)$.

In Mainar et al. (2021), wronskian matrices of Bernstein polynomials and other related bases, including Bernstein basis of negative degree or the negative binomial bases are analyzed. It is shown that these matrices are not TP. Nevertheless, they are closely related to TP matrices whose bidiagonal decomposition to high relative accuracy is obtained. Using this factorization, the computation of algebraic problems related to these non totally positive matrices is also achieved to high relative accuracy.

Given $r, l \in \mathbb{N} \cup \{0\}$ and $r + l \le n \in \mathbb{N}$, let $\mathbf{P}_{r,l}^n([0,1]) = \mathrm{span}\{b_r^n, \dots, b_{n-l}^n\}$. Clearly, $\mathbf{P}_{r,l}^n([0,1])$ is formed by all polynomials defined on $[0,1]$, whose degree is at most n and their derivatives of order at most $r-1$ at $t=0$ and of order at most $l-1$ at $t=1$ vanish.

When considering the following inner product

$$\langle f, g\rangle := \int_0^1 t^\alpha (1-t)^\beta f(t)g(t)\,dt, \quad \alpha,\beta > -1, \tag{28}$$

the *constrained dual Bernstein basis* of degree n is the basis $(D_r^{(n,r,l)}, \dots, D_{n-l}^{(n,r,l)})$ of $\mathbf{P}_{r,l}^n([0,1])$ satisfying

$$\langle D_i^{(n,r,l)}, b_j^n\rangle = \delta_{i,j} = \begin{cases} 1, & \text{for } i = j, \\ 0, & \text{for } i \ne j, \end{cases}$$

for $i, j = r, \dots, n-l$ (cf. Lewanowicz and Woźny, 2011).

In Mainar et al. (2022a), it is shown that the matrix $M^{r,l}$ satisfying

$$(b_r^n, \dots, b_{n-l}^n)^T = M^{r,l}(D_r^n, \dots, D_{n-l}^n)^T \tag{29}$$

is the gramian matrix of $(b_r^n, \dots, b_{n-l}^n)$ with respect (28) and can be described as $M^{r,l} = (M_{i,j})_{1\le i.j\le m+1}$, with $m =: n-r-l$, and

$$M_{i,j} := \binom{n}{r+i-1}\binom{n}{r+j-1}\frac{\Gamma(2r+i+j+\alpha-1)\Gamma(2n-2r-i-j+\beta+3)}{\Gamma(2n+\alpha+\beta+2)}. \tag{30}$$

Theorem 2 of Mainar et al. (2022a) proves that $M^{r,l}$ is an STP matrix, providing the multipliers and the diagonal pivots of its Neville elimination and so, an explicit expression of the entries of the bidiagonal factorization (3) of $M^{r,l}$ and its inverse. Furthermore, Corollary 1 shows that whenever $\Gamma(\alpha+1)$, $\Gamma(\beta+1)$ and $\Gamma(\alpha+\beta+2)$ can be evaluated to high relative accuracy, $M^{r,l}$ and its inverse $(M^{r,l})^{-1}$ can also be computed to high relative accuracy. Note that these conditions are satisfied for values $\alpha, \beta \in \mathbb{N} \cup \{0\}$ since, in this case, $\Gamma(\alpha+1) = \alpha!$, $\Gamma(\beta+1) = \beta!$, $\Gamma(\alpha+\beta+2) = (\alpha+\beta+1)!$. In addition, they also hold for $\alpha, \beta \in \{-1/2, 1/2\}$, corresponding to four Chebyshev-type weights, since for any $n \in \mathbb{N}$, $\Gamma(n+1/2) = \frac{(2n)!}{4^n n!}\sqrt{\pi}$.

In Section 4 of Mainar et al. (2022a) the strict total positivity of gramian matrices of Bernstein bases of negative degree is proved and their bidiagonal factorization to derive accurate algorithms is also deduced.

Finally, let us observe that the proposed factorizations can be used in degree reduction methods or in the approximation, in the least-squares sense, of curves by Bézier curves, requiring the inversion of Bernstein mass matrices (30). Other approaches for the inversion of Bernstein mass matrices can be found in Allen and Kirby (2020).

CRediT authorship contribution statement

J. Delgado: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing. **E. Mainar:** Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing. **J.M. Peña:** Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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