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3D LATTICE FLOWER CONSTELLATIONS USING NECKLACES

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A new approach in satellite constellation design is presented in this paper, taking as a base the 3D Lattice Flower Constellation Theory and introducing the necklace problem in its formulation. This creates a further generalization of the Flower Constellation Theory, increasing the possibilities of constellation distribution and maintaining the characteristic symmetries of the original theory in the design.

INTRODUCTION

The space industry has experienced great advances in the last decades due to the number of possibilities and benefits that the space environment brings. Satellites orbiting the Earth have a very advantageous position, they are able to observe vast regions of the Earth in a small amount of time. This advantage can be improved even further with the use of satellite constellations, allowing to study several areas of the surface at the same time.

Satellite constellations are groups of satellites that work cooperatively to achieve a common mission. Satellite constellations allow to optimize the performance of the system, reducing the costs of the mission. However, the study of several satellites at the same time, and more importantly, the relations that appear in the internal structure of the constellation, increases the complexity of the problem to solve, but also expands the possibilities in the design.

In the last decades, several satellite constellation design methodologies have appeared such as the Walker Constellations[1] for circular orbits or the design of Draim[2] for elliptic orbits. In 2004, the Flower Constellation Theory[3–5] was presented, including in its formulation circular and elliptic orbits. The theory was later improved by the 2D Lattice[6] and 3D Lattice[7] theories which simplified the formulation and made the configuration independent of any reference frame.

In the 2D and 3D Lattice Flower Constellation theories, the configuration of the constellation presents symmetries, a property with many advantages in missions such as global coverage or global positioning. Afterwards, realizing that the amount of different configurations of a constellation for a certain number of satellites could be increased in the formulation, the concept of necklaces[8] was introduced for the 2D Lattice Flower Constellation theory. The theory of necklaces is based on the idea of generating a fictitious constellation with more satellites than required and then, selecting a subset of satellites from the fictitious constellation taking into account that the property of symmetry has to be maintained.

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The solution of the necklace problem (as well as the Flower Constellation Theory) is related with Number Theory which implies working with integer numbers in the distribution of the orbital parameters of the constellation. This leads to interesting properties that are not presented with the use of real numbers.

The aim of this paper is to extend the use of necklaces to the 3D Lattice Flower Constellation Theory, maintaining the properties of symmetry of the configuration and including in its formulation all the previous Flower Constellations.

PRELIMINARIES

In this section, a short introduction of the 3D Lattice Theory and the necklace problem is shown, in order to present the base of the problem treated in this paper and as a way to summarize the previous Flower Constellation Theory.

The 3D Lattice Flower Constellation Theory

The 3D Lattice Flower Constellation Theory is a satellite constellation design methodology in which the satellites are distributed in several inertial orbits, where each satellite has a different value of its mean anomaly and argument of perigee. Furthermore, the satellites of the constellation have the same semi-major axis, eccentricity and inclination. This design allows to generate constellations whose satellites present circular or elliptic orbits. The most important property of this constellation design is that the satellites are distributed generating a symmetric configuration in the lattice that is maintained over time.

As it can be seen in Avendaño et al.,[7] a 3D Lattice Flower Constellation can be described by the use of the Hermite Normal Form. The Hermite Normal Form is composed by six integers, three in the diagonal of the matrix and the other three in the inferior part of the matrix. The integers in the diagonal are the number of orbital planes of the constellation (N_o), the number of different argument of perigees in each orbital plane (N_w), and the number of satellites in each orbit (N_{so}). The other three parameters are the configuration numbers ($N_{c_1}, N_{c_2}, N_{c_3}$) defined as follows: $N_{c_1} \in [0, N_o - 1]$, $N_{c_2} \in [0, N_w - 1]$ and $N_{c_3} \in [0, N_o - 1]$.

The expression that summarizes the distribution of the satellites in a 3D Lattice Flower Constellation is:

$$\begin{bmatrix} N_o & 0 & 0 \\ N_{c_3} & N_w & 0 \\ N_{c_1} & N_{c_2} & N_{so} \end{bmatrix} \begin{pmatrix} \Delta\Omega_{ijk} \\ \Delta\omega_{ijk} \\ \Delta M_{ijk} \end{pmatrix} = 2\pi \begin{pmatrix} i-1 \\ k-1 \\ j-1 \end{pmatrix}; \tag{1}$$

where $\Delta\Omega_{ijk}$ is distribution in the right ascension of the ascending node of the constellation, $\Delta\omega_{ijk}$ is the distribution of argument of perigee, and ΔM_{ijk} is the distribution of the mean anomaly with respect a reference satellite of the constellation with orbital elements $\{\Omega_{000}, \omega_{000}, M_{000}\}$. Moreover, the list (i, j, k) represents the position of a satellite in the orbital plane i, with the argument of perigee k and the mean anomaly j. Note also that the values of Ω_{ijk}, ω_{ijk} and M_{ijk} represent three angles and thus are defined in the range $[0, 2\pi]$.

A distribution as the one shown in Equation (1) can be represented as a set of points that are situated over the surface of a three dimensional torus in a four dimensional space (a representation that is non practical from a graphical point of view). However, the same distribution can also be represented by three different two dimensional torus in a three dimensional space.

As an example of that, a constellation with parameters: $N_o = 7$, $N_w = 5$, $N_{so} = 10$, $N_{c_1} = 5$, $N_{c_2} = 4$ and $N_{c_3} = 6$ is generated. Using Equation (1), the distribution of the satellites is obtained, where the constellation is made by $N_o N_w N_{so} = 350$ satellites. The torus representation of this constellation can be seen in Figure 1 where it is clearly shown that the points are situated generating closed lines in the torus, the lattice of the constellation.

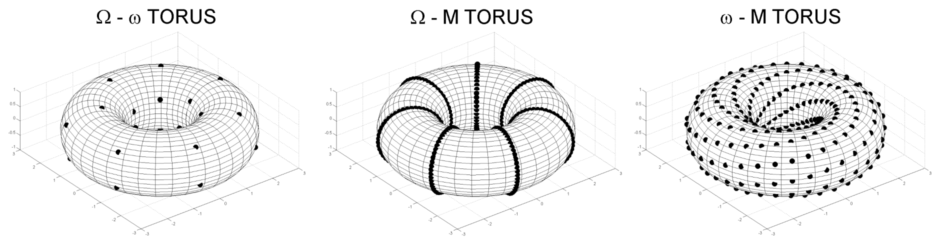

Figure 1. Torus representation of the constellation distribution.

Another useful representation of this distribution can be seen in Figure 2, where the (Ω, ω, M)-space for this particular configuration is shown. As it can be seen, the satellites are distributed in several planes in this space and these planes are not parallel with respect to the axis. This is caused by the configuration numbers (N_{c_1}, N_{c_2} and N_{c_3}) which produce this effect in the distribution. As it will be seen later, this property has deep implications in the development of the necklace theory.

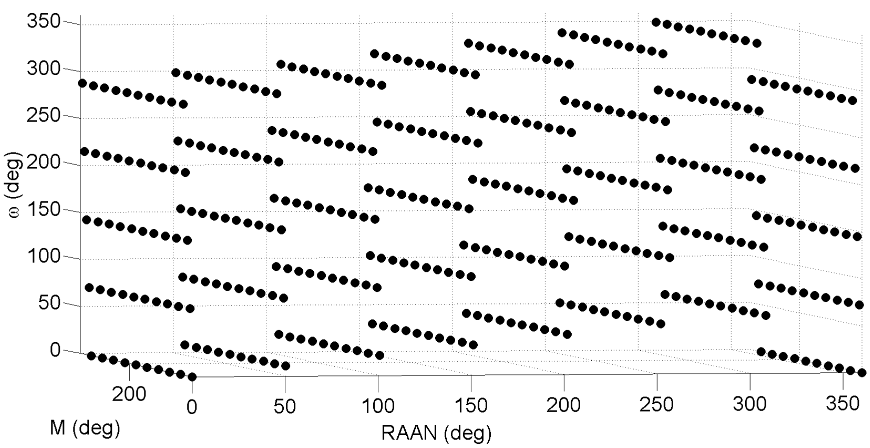

Figure 2. (Ω, ω, M)-space representation of the constellation.

Necklace Theory

The necklace problem is a combinatorial problem which answers how many different arrangements of n pearls in a circular loop can be produced assuming that each pearl comes in one of k different colors. In the case of study, there are just two colors $k = 2$, representing an empty position or a satellite in the constellation. Thus, we can define a necklace as the subset of points selected from a set of available positions, that is, a necklace $\mathcal{G}$ is a subset of a ring of integers $\mathbb{Z}_n$:

$$\mathcal{G} \subseteq \mathbb{Z}_n = \{1, ..., n\} . \quad (2)$$

In this definition, two arrangements are considered to be identical if they only differ by a rotation inside the loop, that is:

$$\mathcal{G}_1 = \mathcal{G}_2 \Longleftrightarrow \exists s : \mathcal{G}_1 = \mathcal{G}_2 + s \mod (n), \tag{3}$$

where s is an integer that belongs to the ring $\mathbb{Z}_n$. In addition, another important concept to introduce is the symmetry of a necklace ($Sym(\mathcal{G})$), defined as the shortest value of r such that:

$$Sym(\mathcal{G}) = \min \{1 \leq r \leq n : \mathcal{G} + r \equiv \mathcal{G} \mod (n)\}, \tag{4}$$

where $Sym(\mathcal{G})$ is the number of times that the configuration can be shifted before obtaining the initial configuration.

INTRODUCING NECKLACES IN THE MEAN ANOMALY

Equation (1) defines the distribution of a 3D Lattice Flower Constellation. This distribution has the particularity of presenting a symmetric configuration in the lattice of the constellation with respect to all its variables, the right ascension of the ascending node, the argument of perigee and the mean anomaly. The objective now is to introduce the concept of necklaces in the formulation, but preserving the symmetries of the initial configuration.

In order to introduce the necklaces, first, an expansion of Equation (1) must be done. The resulting expression is the following:

$$\begin{aligned}
\Delta\Omega_{ijk} &= \frac{2\pi}{N_o}(i-1), \\
\Delta\omega_{ijk} &= \frac{2\pi}{N_w}(k-1) - \frac{2\pi}{N_w}\frac{N_{c3}}{N_o}(i-1), \\
\Delta M_{ijk} &= \frac{2\pi}{N_{so}}(j-1) - \frac{2\pi}{N_{so}}\frac{N_{c2}}{N_w}(k-1) - \frac{2\pi}{N_{so}}\left(\frac{N_{c_1}}{N_o} - \frac{N_{c2}}{N_w}\frac{N_{c3}}{N_o}\right)(i-1),
\end{aligned} \tag{5}$$

where this configuration corresponds to a fictitious constellation that is used to define the available positions in which the real satellites of the constellation are positioned.

From Equation (5), it can be observed that the value of $\Delta\omega_{ijk}$ is different for $i = 1$ and $i = N_o+1$, and thus, moving in $i \in [1, N_o + 1]$ does not close the configuration in the torus for a particular value of k. In the 3D Lattice formulation this has no effect because all the positions are filled and thus, the configuration is complete. However, with the use of necklaces, this effect has to be taken into account in order to generate symmetric configurations. The same consideration has to be made in the expression of the mean anomaly, but for $i \in [1, N_o + 1]$ and $k \in [1, N_w + 1]$.

As done in Casanova et al.,[8] necklaces are defined in the mean anomaly, but, as a consequence of working in a three dimensional space of configuration, two parameters of shifting have to be established, one related to the movement of the mean anomaly with the right ascension of the ascending node ($S_{M\Omega}$), and the other related to the movement of the mean anomaly with the argument of perigee ($S_{M\omega}$).

Let $\mathcal{G}_M$ be a necklace comprised of N_{rso} elements and such that $\mathcal{G}_M \subseteq \mathbb{Z}_{N_{fso}}$. This represents N_{rso} satellites taken from a set of N_{fso} available positions. The necklace in the mean anomaly $\mathcal{G}_M$ is represented as a vector of dimension N_{rso}:

$$\mathcal{G}_M = \{\mathcal{G}_M(1), \dots, \mathcal{G}_M(j^*), \dots, \mathcal{G}_M(N_{rso})\}, \tag{6}$$

with

$$1 \leq \mathcal{G}_M(1) \leq \cdots \leq \mathcal{G}_M(j^*) \leq \cdots \leq \mathcal{G}_M(N_{rso}) \leq N_{fso}, \tag{7}$$

where the index j^* represents an integer modulo N_{rso}, that is, $j^* + N_{rso}$ is the same index as j^*. This allows to define an application (T1) that points to the positions occupied by the necklace from the available positions:

$$\begin{aligned} \text{T1}: \quad \mathbb{Z}_{N_{rso}} &\longrightarrow \mathbb{Z}_{N_{fso}} \\ j^* &\longmapsto \mathcal{G}_M(j^*). \end{aligned} \tag{8}$$

Thus, it makes sense to refer to $\mathcal{G}_M(j^*)$, where the integer parameter $j^* \in \{1, \ldots, N_{rso}\}$ represents the movement inside the necklace defined. In addition, and for simplicity of notation, we denote mod$(a, b) = a \mod (b)$. Thus, due to the modular arithmetic inside the necklace:

$$\mathcal{G}_M(j^*) = \mathcal{G}_M(\text{mod}(j^* + N_{rso}, N_{rso})), \tag{9}$$

which corresponds to a complete loop in the available positions in the mean anomaly. It is important to note that this rotation is equivalent to a movement in the admissible locations defined by:

$$j = j + N_{fso} \mod (N_{fso}), \tag{10}$$

as both represent the same movement of the necklace, one using the parametrization of the necklace and the other using the parametrization of the fictitious constellation.

Now, the effect of the possible movement respect to the right ascension of the ascending node and the argument of perigee is introduced in the formulation by the use of the two shifting parameters ($S_{M\Omega}$ and $S_{M\omega}$). That way, an application (T2) is defined, which relates the ring in the necklace and the ring in the available positions:

$$\begin{aligned} \text{T2}: \quad \mathbb{Z}_{N_o} \times \mathbb{Z}_{N_{rso}} \times \mathbb{Z}_{N_\omega} &\longrightarrow \mathbb{Z}_{N_o} \times \mathbb{Z}_{N_{fso}} \times \mathbb{Z}_{N_\omega} \\ (i, j^*, k) &\longmapsto (i, j, k), \end{aligned} \tag{11}$$

where the integer j is described as:

$$j = \mathcal{G}_M(j^*) + S_{M\omega}(k-1) + S_{M\Omega}(i-1). \tag{12}$$

In order to agree with the formulation of Equation (5), one unit is subtracted from the former expression, leading to:

$$j - 1 = \mathcal{G}_M(j^*) - 1 + S_{M\omega}(k-1) + S_{M\Omega}(i-1), \tag{13}$$

and applying the definition of symmetry of the necklace (Equation (4)), the arithmetic nature of the necklace $\mathcal{G}_M$ is introduced in the formulation, resulting in:

$$j - 1 = \text{mod}\left(\mathcal{G}_M(j^*) - 1 + S_{M\omega}(k-1) + S_{M\Omega}(i-1), Sym(\mathcal{G}_M)\right). \tag{14}$$

This expression represents all possible regular movements that the necklace can perform over the available positions defined by j. Introducing this expression in Equation (5), we obtain:

$$\begin{aligned} \Delta M_{ij^*k} &= \frac{2\pi}{N_{fso}} \Bigg[\text{mod}\left(\mathcal{G}_M(j^*) - 1 + S_{M\omega}(k-1) + S_{M\Omega}(i-1), Sym(\mathcal{G}_M)\right) - \\ &- \frac{N_{c_2}}{N_\omega}(k-1) - \left(\frac{N_{c_1}}{N_o} - \frac{N_{c_2}}{N_\omega}\frac{N_{c_3}}{N_o}\right)(i-1)\Bigg], \end{aligned} \tag{15}$$

which allows to generate all possible configurations of necklaces in the mean anomaly. However, we are interested in the symmetric solutions, thus, the possible values of the shifting parameters that allows the condition of symmetry have to be found.

Symmetry in the 3D Lattice

Once a particular lattice $\mathcal{G}_M$ is selected, the configuration of the Flower Constellation has to present a symmetrical behavior with respect the argument of perigee, the right ascension of the ascending node and the mean anomaly. These symmetries are calculated separately in the following subsections to present the process in a clearer way.

Symmetry with respect to the mean anomaly The conditions that are required to be imposed in order to have symmetry in the distribution with respect to the variations in the mean anomaly are:

$$\begin{aligned} \Delta\Omega_{ij^*k} &= \Delta\Omega_{i(j^*+N_{rso})k}, \\ \Delta\omega_{ij^*k} &= \Delta\omega_{i(j^*+N_{rso})k}, \\ \Delta M_{ij^*k} &= \Delta M_{i(j^*+N_{rso})k}. \end{aligned} \tag{16}$$

where the right ascension of the ascending node and the argument of perigee fulfill automatically these relations as they are not subjected to the variations in the mean anomaly. On the other hand, the relation in the mean anomaly is also fulfilled as:

$$j = j + N_{fso} \mod (N_{fso}), \tag{17}$$

is equivalent to:

$$j^* = j^* + N_{rso} \mod (N_{rso}), \tag{18}$$

due to the modular arithmetic that controls the problem in both rings (the necklace and the available positions).

Symmetry with respect to the argument of perigee In this subsection, a symmetry of the configuration in the argument of perigee is imposed. Thus, Equation (15) has to fulfill the following conditions:

$$\begin{aligned} \Delta\Omega_{ij^*k} &= \Delta\Omega_{ij^*(k+N_\omega)}, \\ \Delta\omega_{ij^*k} &= \Delta\omega_{ij^*(k+N_\omega)}, \\ \Delta M_{ij^*k} &= \Delta M_{ij^*(k+N_\omega)}. \end{aligned} \tag{19}$$

Following this conditions, the expressions for $\Delta\Omega_{ij^*k}$ and $\Delta\omega_{ij^*k}$ are symmetric due to Equation (5). However, the parameters of shifting have to present particular values in order to make the expression in the mean anomaly symmetric. Applying the condition of symmetry to Equation (15):

$$\begin{aligned} \Delta M_{ij^*k} &= \frac{2\pi}{N_{fso}}\Bigg[\text{mod}\left(\mathcal{G}_M(j^*) - 1 + S_{M\omega}(k-1) + S_{M\Omega}(i-1), Sym(\mathcal{G}_M)\right) - \\ &- \frac{N_{c_2}}{N_\omega}(k-1) - \left(\frac{N_{c_1}}{N_o} - \frac{N_{c_2}}{N_\omega}\frac{N_{c_3}}{N_o}\right)(i-1)\Bigg], \\ \Delta M_{ij^*(k+N_\omega)} &= \frac{2\pi}{N_{fso}}\Bigg[\text{mod}\left(\mathcal{G}_M(j^*) - 1 + S_{M\omega}(k+N_\omega-1) + S_{M\Omega}(i-1), Sym(\mathcal{G}_M)\right) - \\ &- \frac{N_{c_2}}{N_\omega}(k+N_\omega-1) - \left(\frac{N_{c_1}}{N_o} - \frac{N_{c_2}}{N_\omega}\frac{N_{c_3}}{N_o}\right)(i-1)\Bigg], \end{aligned} \tag{20}$$

and equating the previous expressions we obtain:

$$\begin{aligned}\mathrm{mod}\,(\mathcal{G}_M(j^*) - 1 + S_{M\omega}(k + N_\omega - 1) + S_{M\Omega}(i-1), Sym(\mathcal{G}_M)) - \\ -\mathrm{mod}\,(\mathcal{G}_M(j^*) - 1 + S_{M\omega}(k-1) + S_{M\Omega}(i-1), Sym(\mathcal{G}_M)) - N_{c_2} = 0.\end{aligned} \tag{21}$$

Then, an expansion of the modulo in $Sym(\mathcal{G}_M)$ is performed, leading to the following expression:

$$A Sym(\mathcal{G}_M) = S_{M\omega} N_\omega - N_{c_2}, \tag{22}$$

where A is an arbitrary integer. Equation (22) shows a relation between the parameter of shifting of the mean anomaly with the argument of perigee ($S_{M\omega}$), with parameters of the Hermite Normal Form and the symmetry of the necklace ($Sym(\mathcal{G}_M)$). Equation (22) can also be expressed in a more compact way:

$$Sym(\mathcal{G}_M) \mid S_{M\omega} N_\omega - N_{c_2}. \tag{23}$$

reading $Sym(\mathcal{G}_M)$ divides $(S_{M\omega} N_\omega - N_{c_2})$. Equation (23) is the first relation in the necklace symmetry problem, which allows to obtain all the possible values of the shifting parameter $S_{M\omega}$ that are admissible to create symmetric configurations. In fact, Equation (23) is a Diophantine equation in the variable $S_{M\omega}$ whose solutions are integer values in the range $[0, Sym(\mathcal{G}_M) - 1]$.

Symmetry with respect to the right ascension of the ascending node In a similar procedure as the one done with the symmetry with the argument of perigee, in this section the condition for the symmetry in the configuration of the constellation with the variation of the right ascension of the ascending node is described. First, as done before, the conditions for symmetry are:

$$\begin{aligned}\Delta\Omega_{ij^*k} &= \Delta\Omega_{(i+N_o)j^*k}, \\ \Delta\omega_{ij^*k} &= \Delta\omega_{(i+N_o)j^*k}, \\ \Delta M_{ij^*k} &= \Delta M_{(i+N_o)j^*k}.\end{aligned} \tag{24}$$

However, in this case, the symmetry with the argument of perigee is not fulfilled automatically, and thus, the condition of symmetry in the argument of perigee has to be also imposed:

$$\Delta\omega_{ij^*k} = \Delta\omega_{(i+N_o)j^*k'}, \tag{25}$$

where it is important to notice that the right side of the equation presents terms in k whilst the right side has terms in k'. This is not an arbitrary decision, in fact it is related on how this formulation is generated. In Equation (15) has been introduced two degrees of freedom that provide the possibility to impose the symmetries. However, in the argument of perigee there is no degree of freedom so, although the symmetry is achieved in ω because the distribution in that variable is complete (there is no necklace and as a consequence the lattice is closed), we require to compute which is the movement that a complete rotation in Ω provokes. Thus, imposing this symmetry to Equation (15), we obtain:

$$\begin{aligned}\Delta\omega_{ij^*k} &= \frac{2\pi}{N_\omega}(k-1) \\ \Delta\omega_{(i+N_o)j^*k'} &= \frac{2\pi}{N_\omega}(k' - 1 - N_{c_3}),\end{aligned} \tag{26}$$

where k' is an integer number different in general from k and that is given by:

$$k' = k + N_{c_3} \mod (N_\omega), \tag{27}$$

where the modular operation appears as a result of the boundaries in the definition of both k and k', in particular, $k, k' \in \{1, \ldots, N_\omega\}$.

Now, imposing the condition of symmetry in the mean anomaly:

$$\begin{aligned}
\Delta M_{ij^*k} &= \frac{2\pi}{N_{fso}}\Bigg[\text{mod}\left(\mathcal{G}_M(j^*) - 1 + S_{M\omega}(k-1) + S_{M\Omega}(i-1), Sym(\mathcal{G}_M)\right) - \\
&- \frac{N_{c_2}}{N_\omega}(k-1) - \left(\frac{N_{c_1}}{N_o} - \frac{N_{c_2}}{N_\omega}\frac{N_{c_3}}{N_o}\right)(i-1)\Bigg], \\
\Delta M_{(i+N_o)j^*k'} &= \frac{2\pi}{N_{fso}}\Bigg[\text{mod}\left(\mathcal{G}_M(j^*) - 1 + S_{M\omega}(k' - 1 + N_{c_3}) + S_{M\Omega}(i + N_o - 1), Sym(\mathcal{G}_M)\right) - \\
&- \frac{N_{c_2}}{N_\omega}\left(k' - 1 + N_{c_3}\right) - \left(\frac{N_{c_1}}{N_o} - \frac{N_{c_2}}{N_\omega}\frac{N_{c_3}}{N_o}\right)(i + N_o - 1)\Bigg], \qquad (28)
\end{aligned}$$

and using the relation $k' = k + N_{c_3} \mod (N_\omega)$, we can operate in the former expression, obtaining:

$$\begin{aligned}
&\text{mod}\left(\mathcal{G}_M(j^*) - 1 + S_{M\omega}(k - 1 + N_{c_3}) + S_{M\Omega}(i + N_o - 1), Sym(\mathcal{G}_M)\right) - \\
&-\text{mod}\left(\mathcal{G}_M(j^*) - 1 + S_{M\omega}(k-1) + S_{M\Omega}(i-1), Sym(\mathcal{G}_M)\right) - N_{c1} = 0; \qquad (29)
\end{aligned}$$

and expanding the modular operators, an equivalent relation is obtained:

$$A Sym(\mathcal{G}_M) = S_{M\Omega}N_o - \left(N_{c_1} - S_{M\omega}N_{c_3}\right), \qquad (30)$$

where A is an unknown integer. As before, Equation (30) can be represented as:

$$Sym(\mathcal{G}_M) \mid S_{M\Omega}N_o - \left(N_{c_1} - S_{M\omega}N_{c_3}\right), \qquad (31)$$

which is again another Diophantine equation that relates the shifting parameter $S_{M\Omega}$ with the parameters of the Hermite Normal Form and the symmetry of the necklace ($Sym(\mathcal{G}_M)$). As before, the solutions of the shifting parameter ($S_{M\Omega}$) are integer values in the range $[0, Sym(\mathcal{G}_M) - 1]$. However, there is a difference in this case with respect Equation (23). As it can be seen, the equation is also related to the value of the shifting parameter $S_{M\omega}$ and thus, it generates a hierarchy of parameters, having to solve in advance Equation (23) and applying its conditions to Equation (31).

Symmetric configurations In this subsection the final results of the theory are summarized in order to present all the methodology in a more compact and clear way. All possible distributions of a particular necklace $\mathcal{G}_M$ can be described by the set of expressions:

$$\begin{aligned}
\Delta\Omega_{ij^*k} &= \frac{2\pi}{N_o}(i-1), \\
\Delta\omega_{ij^*k} &= \frac{2\pi}{N_\omega}(k-1) - \frac{2\pi}{N_\omega}\frac{N_{c_3}}{N_o}(i-1), \\
\Delta M_{ij^*k} &= \frac{2\pi}{N_{fso}}\Bigg[\text{mod}\left(\mathcal{G}_M(j^*) - 1 + S_{M\omega}(k-1) + S_{M\Omega}(i-1), Sym(\mathcal{G}_M)\right) - \\
&- \frac{N_{c_2}}{N_\omega}(k-1) - \left(\frac{N_{c_1}}{N_o} - \frac{N_{c_2}}{N_\omega}\frac{N_{c_3}}{N_o}\right)(i-1)\Bigg], \qquad (32)
\end{aligned}$$

where in order to obtain symmetric configurations, the values of the shifting parameters $S_{M\omega}$ and $S_{M\omega}$ have to fulfill the following relations:

$$\begin{aligned}
&Sym(\mathcal{G}_M) \mid S_{M\omega}N_\omega - N_{c_2}, \\
&Sym(\mathcal{G}_M) \mid S_{M\Omega}N_o - \left(N_{c_1} - S_{M\omega}N_{c_3}\right). \qquad (33)
\end{aligned}$$

Example of application

A constellation consisting of 30 satellites is created as an example of application. First, we have to define how those satellites are distributed in space. Let the constellation be distributed in five orbital planes ($N_o = 5$), with three different orbits in each orbital plane ($N_w = 3$), and two satellites in each of those orbits $N_{rso} = 2$. Based on this configuration, we have to decide how much we want to expand the search space for configurations, that is, how 'big' the parameter N_{fso} is. To maintain things simple, only one fictitious constellation is built with parameters $N_o = 5$, $N_w = 3$ and $N_{fso} = 4$. That means that from four possible locations to position satellites, we choose only two of them, that is, we are generating two times the positions that the constellation needs to fulfill.

With this definition, a necklace of two elements has to be built. The available positions in this case are $N_{fso} = 4$, which result in two different necklaces: $\mathcal{G}_M = \{1, 3\}$ and $\tilde{\mathcal{G}}_M = \{1, 2\}$. From these two possibilities, only $\mathcal{G}_M = \{1, 3\}$ is considered, just to keep the explanation as simple as possible.

Using Equation (4) the computation of the symmetry of the necklace $\mathcal{G}_M = \{1, 3\}$ can be done, obtaining $Sym\,(\mathcal{G}_M) = 2$. Now, a search of all the possible combinations of N_{c_1}, N_{c_2}, N_{c_3}, $S_{M\omega}$ and $S_{M\omega}$ that makes the configuration symmetric is performed. Doing that search, 75 different possibilities are obtained.

However, instead of showing all those configurations, we choose a particular combination of N_{c_1}, N_{c_2} and N_{c_3}, and a constellation will be built based on those parameters. Let $N_{c_1} = 3$, $N_{c_2} = 1$ and $N_{c_3} = 2$ be the configuration numbers of the constellation, then, using Equation (23) and applying the particular values of this example:

$$Sym(\mathcal{G}_M) \mid S_{M\omega}N_w - N_{c_2} \Rightarrow 2 \mid 3S_{M\omega} - 1 \tag{34}$$

which has a solution of $S_{M\omega} = 1$. On the other hand, the value of $S_{M\omega}$ is obtained from Equation (31):

$$Sym(\mathcal{G}_M) \mid S_{M\Omega}N_o - (N_{c_1} - S_{M\omega}N_{c_3}) \Rightarrow 2 \mid 5S_{M\Omega} - 1, \tag{35}$$

where the solution in this case is $S_{M\Omega} = 1$.

With these results, Equation (32) can now be used to generate the distribution of the constellation. The graphical representation of the lattice of this configuration can be seen in Figure 3 where the distribution is represented in three torus. As it can be observed, the Ω–ω torus has all the available positions fulfilled because there is no necklace in those variables, however the Ω–M and ω–M torus present available positions that are empty and despite of that, the configuration is consistent with a closed lattice.

Moreover, it is interesting to see the representation in the (Ω, ω, M)-space shown on Figure 4. In this image, it can be observed how the satellites (colored circles) are distributed over the available positions (circles) maintaining the symmetry of the configuration in all directions.

The only thing left to define in the constellation are the values of the semi-major axis (a), the inclination (i) and the eccentricity (e). As it can be seen, the necklace generation and the study of the lattice of the constellation is completely independent of the values of semi-major axis, inclination and eccentricity. Let the constellation be defined with an eccentricity of $e = 0.1$, an inclination equal to the critical inclination $i = 63.43^o$ and a semi-major axis $a = 14420\,km$. With these orbital parameters, an inertial configuration as shown in Figure 5 is obtained.

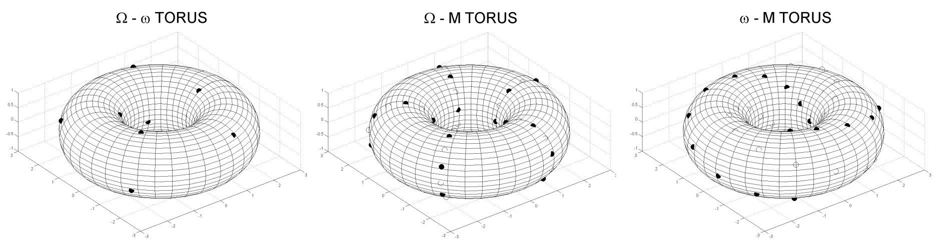

Figure 3. Torus representation of the constellation distribution.

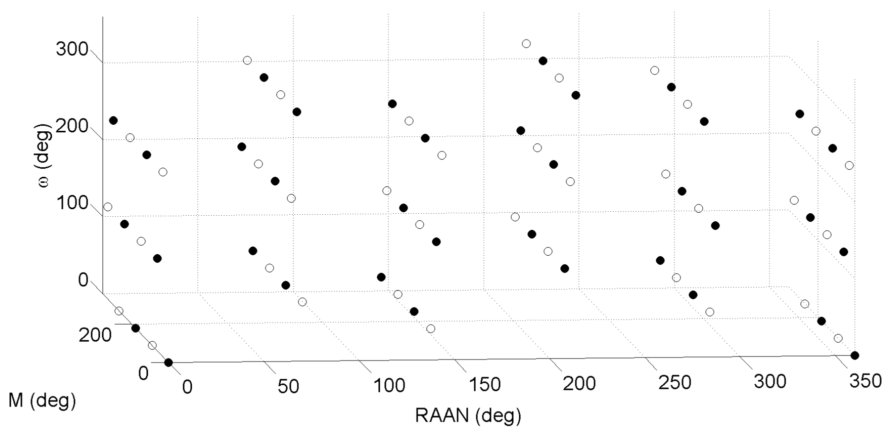

Figure 4. (Ω, ω, M)-space representation of the constellation.

As it can be seen in Figure 5 all the constellation is distributed in five different orbital planes, each containing three different orbits, each of one containing two satellites. These were the initial parameters of design that were selected at the beginning and as it can be observed, they have been fulfilled in the designed constellation.

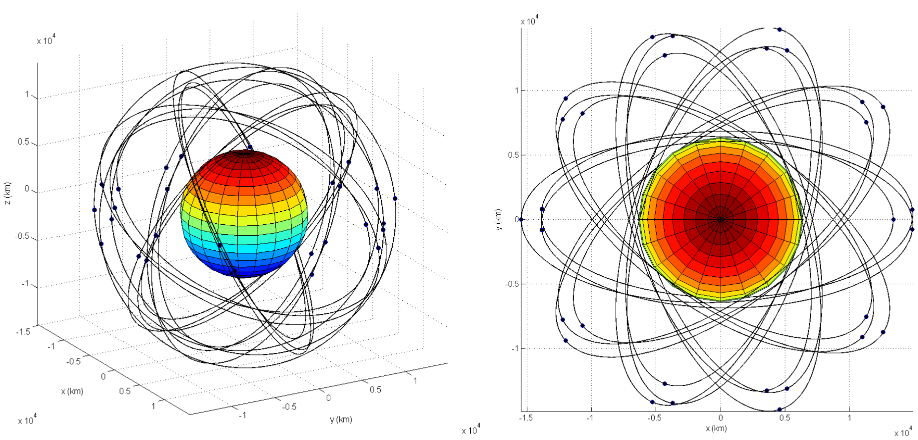

Figure 5. Inertial orbits of the constellation.

One important thing to note is that this is just a particular case of study in which only certain values of the parameters have been taken into account. The search space can be expanded as much as required, increasing the amount of possible configurations for a particular constellation as much as desired.

THE 3D LATTICE FLOWER CONSTELLATIONS USING NECKLACES

In the previous section, the methodology to include necklaces in the mean anomaly is introduced showing an example of application. The objective now is to generalize the generation of necklaces to not just the mean anomaly, but also the argument of perigee. That means that an expansion of the search space can be done by introducing necklaces in the argument of perigee in the same way it was done for the mean anomaly. However, the addition of this new degree of freedom generates more dependences between parameters.

In general, two different necklaces can be defined in a 3D Lattice Flower Constellation, one in the mean anomaly, and the other in the argument of perigee. It is possible to generate necklaces in the right ascension of the ascending node with the 3D Lattice Flower Constellation configuration, but this is equivalent to generate the distribution and keeping just the orbital planes that we are interested in. For this reason, we do not consider this case, as it is not required the use of necklaces for these cases.

Let $\mathcal{G}_\omega$ be a necklace defined in the argument of perigee with a number of elements equal to $N_{r\omega} = |\mathcal{G}_\omega|$, the number of real orbits per plane and a number of available positions equal to $N_{f\omega}$ which correspond to the size of the space in the fictitious constellation. This necklace is defined as a vector in the same way as $\mathcal{G}_M$:

$$\mathcal{G}_\omega = \{\mathcal{G}_\omega(1), \dots, \mathcal{G}_\omega(k^*), \dots, \mathcal{G}_\omega(N_{r\omega})\}, \tag{36}$$

with

$$1 \leq \mathcal{G}_\omega(1) \leq \dots \leq \mathcal{G}_\omega(k^*) \leq \dots \leq \mathcal{G}_M(N_{r\omega}) \leq N_{f\omega}, \tag{37}$$

where the index k^* is an integer modulo $N_{r\omega}$. This allows to define an application (T3) that points to the positions occupied by the necklace from the available positions:

$$\begin{array}{rccl} \text{T3}: & \mathbb{Z}_{N_{r\omega}} & \longrightarrow & \mathbb{Z}_{N_{f\omega}} \\ & k^* & \longmapsto & \mathcal{G}_\omega(k^*), \end{array} \tag{38}$$

which is used to refer to $\mathcal{G}_\omega(k^*)$, where the integer parameter $k^* \in \{1, \dots, N_{r\omega}\}$ represents the movement inside the necklace defined. Moreover, the necklace represents a ring of integers, thus there exist a modular arithmetic inside the necklace:

$$\mathcal{G}_\omega(k^*) = \mathcal{G}_\omega(\text{mod}(k^* + N_{r\omega}, N_{r\omega})), \tag{39}$$

which is equivalent to a complete loop in the available positions in the argument of perigee:

$$k = k + L_\omega \mod (L_\omega), \tag{40}$$

as both are two formulations for the same movement, one using the parametrization of the necklace and the other using the parametrization of the fictitious constellation.

Now, an application (T4) has to be defined which relates the distribution indexes (i, j^*, k^*) from the necklace, with the indexes of the available positions (i, j, k):

$$\begin{aligned} \text{T4}: \quad \mathbb{Z}_{N_o} \times \mathbb{Z}_{N_{rso}} \times \mathbb{Z}_{N_{f\omega}} &\longrightarrow \mathbb{Z}_{N_o} \times \mathbb{Z}_{N_{fso}} \times \mathbb{Z}_{N_{f\omega}} \\ (i, j^*, k^*) &\longmapsto (i, j, k), \end{aligned} \tag{41}$$

where the effects of the possible movement respect to the right ascension of the ascending node and the argument of perigee are introduced in the formulation by the use of the three shifting parameters, $S_{\omega\Omega}$ the shifting parameter that relates the argument of perigee with the right ascension of the ascending node, $S_{M\Omega}$ the shifting parameter that relates the mean anomaly and the right ascension of the ascending node, and $S_{M\omega}$ the shifting parameter that relates the mean anomaly and the argument of perigee. That way, the possible movements of the integer k are described as:

$$k = \mathcal{G}_\omega(k^*) + S_{\omega\Omega}(i-1), \tag{42}$$

while the integer j is defined as:

$$j = \mathcal{G}_M(j^*) + S_{M\omega}(k-1) + S_{M\Omega}(i-1). \tag{43}$$

We now subtract one unit of each expression to relate to the original formulation provided by Equation (5), obtaining:

$$\begin{aligned} k-1 &= \mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), \\ j-1 &= \mathcal{G}_M(j^*) - 1 + S_{M\omega}(k-1) + S_{M\Omega}(i-1). \end{aligned} \tag{44}$$

Both expressions present modular arithmetic with respect the symmetries of their necklaces, thus:

$$\begin{aligned} k-1 &= \mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1) \mod Sym(\mathcal{G}_\omega), \\ j-1 &= \mathcal{G}_M(j^*) - 1 + S_{M\omega}(k-1) + S_{M\Omega}(i-1) \mod Sym(\mathcal{G}_M). \end{aligned} \tag{45}$$

However, j depends on k, and we require a dependency over k^*, thus, a substitution of k is performed in the second expression, leading to:

$$\begin{aligned} j-1 &= \mathcal{G}_M(j^*) - 1 + S_{M\omega}\text{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) + \\ &+ S_{M\Omega}(i-1) \mod Sym(\mathcal{G}_M), \end{aligned} \tag{46}$$

where it can be seen that the movement in j depends also on the necklace in the argument of perigee.

Once the distribution over each index is performed, we introduce Equations (45) and (46) into Equation (5), resulting in:

$$\begin{aligned} \Delta\Omega_{ij^*k^*} &= \frac{2\pi}{N_o}(i-1), \\ \Delta\omega_{ij^*k^*} &= \frac{2\pi}{N_{f\omega}}\left[\text{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) - \frac{N_{c_3}}{N_o}(i-1)\right], \\ \Delta M_{ij^*k^*} &= \frac{2\pi}{N_{fso}}\left[\text{mod}\left(\mathcal{G}_M(j^*) - 1 + S_{M\omega}\text{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) + \right.\right. \\ &+ \left. S_{M\Omega}(i-1), Sym(\mathcal{G}_M)\right) - \frac{N_{c_2}}{N_{f\omega}}\text{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) - \\ &- \left.\left(\frac{N_{c_1}}{N_o} - \frac{N_{c_2}}{N_{f\omega}}\frac{N_{c_3}}{N_o}\right)(i-1)\right], \end{aligned} \tag{47}$$

which describes the possible movements of the two necklace defined ($\mathcal{G}_M$ and $\mathcal{G}_\omega$) inside the distribution created in the fictitious constellation.

Equation (47) allows, not only to make the distribution of the satellites in the lattice, but also to find all symmetric configurations using the necklace theory. An important thing to notice is that, in the expression for ΔM_{ijk}, the necklace in the argument of perigee appears, which means that properties in this necklace are now affecting the distribution of the constellation in the mean anomaly. This effect is also seen in the conditions for the shifting parameters of the configuration as it will be seen later.

Symmetry in the 3D Lattice Flower Constellations

As done with the necklace in the mean anomaly, the condition of symmetry has to be imposed in the constellation distribution in order to obtain the values of the shifting parameters that make the configuration symmetric.

Symmetry with respect to the mean anomaly The conditions for symmetry in the three variables when a complete rotation in the mean anomaly is performed are:

$$\begin{aligned} \Delta\Omega_{ij^*k^*} &= \Delta\Omega_{i(j^*+N_{rso})k^*}, \\ \Delta\omega_{ij^*k^*} &= \Delta\omega_{i(j^*+N_{rso})k^*}, \\ \Delta M_{ij^*k^*} &= \Delta M_{i(j^*+N_{rso})k^*}, \end{aligned} \tag{48}$$

where all expressions are automatically fulfilled as $\Delta\Omega_{ij^*k^*}$ and $\Delta\omega_{ij^*k^*}$ do not depend on the movement of the mean anomaly whilst $\Delta M_{ij^*k^*}$ is also achieved due to the modular arithmetic nature of the problem seen in Equation (18).

Symmetry with respect to the argument of perigee In order to have symmetry in the argument of perigee, the configuration of the constellation has to fulfill the following conditions:

$$\begin{aligned} \Delta\Omega_{ij^*k^*} &= \Delta\Omega_{ij(k^*+N_{r\omega})}, \\ \Delta\omega_{ij^*k^*} &= \Delta\omega_{ij^*(k^*+N_{r\omega})}, \\ \Delta M_{ij^*k^*} &= \Delta M_{ij^*(k^*+N_{r\omega})}. \end{aligned} \tag{49}$$

where the first equation is always true as it does not depend on the movement in the argument of perigee. On the other hand, the other two equations depend on k^* and, as such, they have to be studied.

Taking the condition in $\Delta\omega_{ij^*k^*}$, and from the equivalences in the definition between Equations (39) and (40), we can conclude that the operation $k^* + N_{r\omega}$ is equivalent to a full rotation in the argument of perigee, that is:

$$\Delta\omega_{ij^*k^*} + 2\pi = \Delta\omega_{ij^*(k^*+N_{r\omega})}, \tag{50}$$

which applied to the expression of the argument of perigee, leads to:

$$\begin{aligned} &\frac{2\pi}{N_{f\omega}}\left[\mathrm{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) - \frac{N_{c_3}}{N_o}(i-1)\right] + 2\pi = \\ &= \frac{2\pi}{N_{f\omega}}\left[\mathrm{mod}\left(\mathcal{G}_\omega(k^* + N_{r\omega}) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) - \frac{N_{c_3}}{N_o}(i-1)\right], \end{aligned} \tag{51}$$

from where a relation between the two modular operators can be established:

$$\begin{aligned}&\text{mod}\left(\mathcal{G}_\omega(k^*+N_{r\omega})-1+S_{\omega\Omega}(i-1),Sym(\mathcal{G}_\omega)\right)-\\&-\text{mod}\left(\mathcal{G}_\omega(k^*)-1+S_{\omega\Omega}(i-1),Sym(\mathcal{G}_\omega)\right)=N_{f\omega},\end{aligned}\tag{52}$$

where this equation will be used later in order to impose the condition of symmetry in the mean anomaly with respect to the argument of perigee.

On the other hand, regarding the condition in ΔM_{ijk}, from the system of Equations (49), we can derive the following expression:

$$\begin{aligned}&\frac{2\pi}{N_{fso}}\Bigg[\text{mod}\left(\mathcal{G}_M(j^*)-1+S_{M\omega}\text{mod}\left(\mathcal{G}_\omega(k^*)-1+S_{\omega\Omega}(i-1),Sym(\mathcal{G}_\omega)\right)+\right.\\
+&\quad \left.S_{M\Omega}(i-1),Sym(\mathcal{G}_M)\right)-\frac{N_{c_2}}{N_{f\omega}}\text{mod}\left(\mathcal{G}_\omega(k^*)-1+S_{\omega\Omega}(i-1),Sym(\mathcal{G}_\omega)\right)-\\
-&\quad \left(\frac{N_{c_1}}{N_o}-\frac{N_{c_2}}{N_{f\omega}}\frac{N_{c_3}}{N_o}\right)(i-1)\Bigg]=\\
=&\quad \frac{2\pi}{N_{fso}}\Bigg[\text{mod}\left(\mathcal{G}_M(j^*)-1+S_{M\omega}\text{mod}\left(\mathcal{G}_\omega(k^*+N_{r\omega})-1+S_{\omega\Omega}(i-1),Sym(\mathcal{G}_\omega)\right)+\right.\\
+&\quad \left.S_{M\Omega}(i-1),Sym(\mathcal{G}_M)\right)-\frac{N_{c_2}}{N_{f\omega}}\text{mod}\left(\mathcal{G}_\omega(k^*+N_{r\omega})-1+S_{\omega\Omega}(i-1),Sym(\mathcal{G}_\omega)\right)-\\
-&\quad \left(\frac{N_{c_1}}{N_o}-\frac{N_{c_2}}{N_{f\omega}}\frac{N_{c_3}}{N_o}\right)(i-1)\Bigg];\end{aligned}\tag{53}$$

which can be simplified to:

$$\begin{aligned}&\text{mod}\left(\mathcal{G}_M(j^*)-1+S_{M\omega}\text{mod}\left(\mathcal{G}_\omega(k^*)-1+S_{\omega\Omega}(i-1),Sym(\mathcal{G}_\omega)\right)+\right.\\
&\left.+S_{M\Omega}(i-1),Sym(\mathcal{G}_M)\right)-\frac{N_{c_2}}{N_{f\omega}}\text{mod}\left(\mathcal{G}_\omega(k^*)-1+S_{\omega\Omega}(i-1),Sym(\mathcal{G}_\omega)\right)=\\
&=\text{mod}\left(\mathcal{G}_M(j^*)-1+S_{M\omega}\text{mod}\left(\mathcal{G}_\omega(k^*+N_{r\omega})-1+S_{\omega\Omega}(i-1),Sym(\mathcal{G}_\omega)\right)+\right.\\
&\left.+S_{M\Omega}(i-1),Sym(\mathcal{G}_M)\right)-\frac{N_{c_2}}{N_{f\omega}}\text{mod}\left(\mathcal{G}_\omega(k^*+N_{r\omega})-1+S_{\omega\Omega}(i-1),Sym(\mathcal{G}_\omega)\right).\end{aligned}\tag{54}$$

Moreover, expanding the modular operator in $Sym(\mathcal{G}_M)$ and using Equation (52) leads to:

$$ASym(\mathcal{G}_M)=S_{M\omega}N_{f\omega}-N_{c_2};\tag{55}$$

where A is an unknown integer number. As previously, this equation can be also represented as:

$$Sym(\mathcal{G}_M)\mid S_{M\omega}N_{f\omega}-N_{c_2}.\tag{56}$$

Equation (56) is the first condition for the shifting parameters of the configuration. As it can be seen, it depends on the symmetry of the necklace, and some elements from the Hermite Normal Form. Note that the shifting parameter of the mean anomaly respect to the argument of perigee ($S_{M\omega}$) depends on the number of fictitious orbits per orbital plane and not the real number. The reason for that is the actual expansion of the configuration that has been made, a property that increases the number of possibilities in the configuration.

Symmetry with respect to the right ascension of the ascending node The conditions of symmetry that we have to impose respect to the right ascension of the ascending node are the following:

$$\begin{aligned} \Delta\Omega_{ij^*k^*} &= \Delta\Omega_{(i+N_o)j^*k^*}, \\ \Delta\omega_{ij^*k^*} &= \Delta\omega_{(i+N_o)j^*k^*}, \\ \Delta M_{ij^*k^*} &= \Delta M_{(i+N_o)j^*k^*}. \end{aligned} \tag{57}$$

where each one of these conditions is treated separately.

The condition in the right ascension of the ascending node is automatically fulfilled as:

$$\Delta\Omega_{ij^*k^*} = \frac{2\pi}{N_o}(i-1) = \frac{2\pi}{N_o}(i-1) + 2\pi \mod (2\pi), \tag{58}$$

which is independent of any of the shifting parameters of the problem.

From the condition in the argument of perigee:

$$\frac{N_{f\omega}}{2\pi}\Delta\omega_{ij^*k^*} = \frac{N_{f\omega}}{2\pi}\Delta\omega_{(i+N_o)j^*k^*}, \tag{59}$$

that can be used to obtain the following expression:

$$\begin{aligned} &\mathrm{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) - \frac{N_{c3}}{N_o}(i-1) = \\ &= \mathrm{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1) + S_{\omega\Omega}N_o, Sym(\mathcal{G}_\omega)\right) - \frac{N_{c3}}{N_o}(i-1) - N_{c3}, \end{aligned} \tag{60}$$

which can be simplified, leading to:

$$\begin{aligned} &\mathrm{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1) + S_{\omega\Omega}N_o, Sym(\mathcal{G}_\omega)\right) - \\ &-\mathrm{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) = N_{c3}, \end{aligned} \tag{61}$$

where Equation (61) is a relation that is used later and allows to solve the symmetries in the mean anomaly.

Expanding now the modular operator from Equation (61) and simplifying:

$$A\,Sym(\mathcal{G}_\omega) = S_{\omega\Omega}N_o - N_{c3}; \tag{62}$$

where A is an unknown integer. This expression is equivalent to:

$$Sym(\mathcal{G}_\omega) \mid S_{\omega\Omega}N_o - N_{c3}. \tag{63}$$

Equation (63) is the second condition for the shifting parameters. As it can be observed, it relates the shifting of the argument of perigee with respect the right ascension of the ascending node $S_{\omega\Omega}$, with the symmetries of the necklace in the argument of perigee $Sym(\mathcal{G}_\omega)$ and particular elements of the Hermite Normal Form.

Once the problem of symmetry in the argument of perigee is solved, we impose the condition of symmetry in the mean anomaly by the use of its condition from Equation (57):

$$\frac{N_{fso}}{2\pi}\Delta M_{ij^*k^*} = \frac{N_{fso}}{2\pi}\Delta M_{(i+N_o)j^*k^*}, \tag{64}$$

from where we can derive:

$$\begin{aligned}
&\text{mod}\left(\mathcal{G}_M(j^*) - 1 + S_{M\omega}\text{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) + \right.\\
&\left. + S_{M\Omega}(i-1), Sym(\mathcal{G}_M)\right) - \frac{N_{c_2}}{N_{f\omega}}\text{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) - \\
&- \left(\frac{N_{c_1}}{N_o} - \frac{N_{c_2}}{N_{f\omega}}\frac{N_{c_3}}{N_o}\right)(i-1) = \\
&= \text{mod}\left(\mathcal{G}_M(j^*) - 1 + S_{M\omega}\text{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1) + \right.\right.\\
&\left. + S_{\omega\Omega}N_o, Sym(\mathcal{G}_\omega)\right) + S_{M\Omega}(i-1) + S_{M\Omega}N_o, Sym(\mathcal{G}_M)\big) - \\
&- \frac{N_{c_2}}{N_{f\omega}}\text{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1) + S_{\omega\Omega}N_o, Sym(\mathcal{G}_\omega)\right) - \\
&- \left(\frac{N_{c_1}}{N_o} - \frac{N_{c_2}}{N_{f\omega}}\frac{N_{c_3}}{N_o}\right)(i-1) - \left(N_{c_1} - \frac{N_{c_2}N_{c_3}}{N_{f\omega}}\right),
\end{aligned} \tag{65}$$

which, using Equation (61) can be simplified to:

$$\begin{aligned}
&\text{mod}\left(\mathcal{G}_M(j^*) - 1 + S_{M\omega}\text{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1) + \right.\right.\\
&\left. + S_{\omega\Omega}N_o, Sym(\mathcal{G}_\omega)\right) + S_{M\Omega}(i-1) + S_{M\Omega}N_o, Sym(\mathcal{G}_M)\big) - \\
&- \text{mod}\left(\mathcal{G}_M(j^*) - 1 + S_{M\omega}\text{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) + \right.\\
&\left. + S_{M\Omega}(i-1), Sym(\mathcal{G}_M)\right) = N_{c_1}
\end{aligned} \tag{66}$$

Now we expand the modular operator and apply again the relation from Equation (61) in order to obtain:

$$A Sym(\mathcal{G}_M) = S_{M\Omega}N_o - (N_{c_1} - S_{M\omega}N_{c_3}) \tag{67}$$

which is equivalent to:

$$Sym(\mathcal{G}_M) \mid S_{M\Omega}N_o - (N_{c_1} - S_{M\omega}N_{c_3}) \tag{68}$$

Equation (68) is the third condition for the shifting parameters. As we can see, this relation has a particularity, $S_{M\Omega}$ depends also on other shifting parameter, $S_{M\omega}$ which generates a logical order in the generation of the shifting parameters.

Symmetric configurations In this subsection the final results of the theory are summarized in order to present all the methodology in a more compact and clear way. All possible distributions of a particular necklace $\mathcal{G}$ can be described by the set of expressions:

$$\begin{aligned}
\Delta\Omega_{ij^*k^*} &= \frac{2\pi}{N_o}(i-1), \\
\Delta\omega_{ij^*k^*} &= \frac{2\pi}{N_{f\omega}}\left[\text{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) - \frac{N_{c_3}}{N_o}(i-1)\right], \\
\Delta M_{ij^*k^*} &= \frac{2\pi}{N_{fso}}\left[\text{mod}\left(\mathcal{G}_M(j^*) - 1 + S_{M\omega}\text{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) + \right.\right. \\
&+ \left. S_{M\Omega}(i-1), Sym(\mathcal{G}_M)\right) - \frac{N_{c_2}}{N_{f\omega}}\text{mod}\left(\mathcal{G}_\omega(k^*) - 1 + S_{\omega\Omega}(i-1), Sym(\mathcal{G}_\omega)\right) - \\
&- \left.\left(\frac{N_{c_1}}{N_o} - \frac{N_{c_2}}{N_{f\omega}}\frac{N_{c_3}}{N_o}\right)(i-1)\right],
\end{aligned} \tag{69}$$

where the values of the parameters of shifting $S_{\omega\Omega}$, $S_{M\omega}$ and $S_{M\omega}$ have to fulfill the following relations in order to obtain symmetric configurations:

$$\begin{aligned} Sym(\mathcal{G}_\omega) &\mid S_{\omega\Omega}N_o - N_{c_3}, \\ Sym(\mathcal{G}_M) &\mid S_{M\omega}N_{f\omega} - N_{c_2}, \\ Sym(\mathcal{G}_M) &\mid S_{M\Omega}N_o - (N_{c_1} - S_{M\omega}N_{c_3}) \,. \end{aligned} \tag{70}$$

As it can be seen, the set of Equations (69) and (70) leads to the ones presented in Section if the necklace in the argument of perigee $\mathcal{G}_\omega$ is the complete configuration, that is, there is no necklace defined in this variable. Moreover, if there is no necklace in the configuration, the 3D Lattice Flower Constellations distributions are obtained instead, which contain in their formulation the 2D Lattice Flower Constellations.[7] Regarding the 2D Lattice Flower Constellations using necklaces,[8] the shifting parameter in the mean anomaly was defined as:

$$Sym(\mathcal{G}) \mid S_{M\Omega}N_o - N_c, \tag{71}$$

where $\mathcal{G}$ is a necklace in the mean anomaly and N_c is the configuration number for the 2D Lattice Flower Constellations which corresponds to the N_{c_1} parameter in the 3D Lattice Flower Constellations. This relation is equivalent to the last condition in Equation (70) when the argument of perigee is not a variable of the configuration, thus, the 3D Necklace Flower Constellations also includes the 2D Lattice Flower Constellations using necklaces.

Therefore, Equations (69) and (70) constitute the generalization of the necklace theory for the 3D Lattice Flower Constellations, which include all former Lattice Flower Constellations: 2D Lattice Flower Constellations, 2D Lattice Flower Constellations using necklaces, 3D Lattice Flower Constellations and now 3D Lattice Flower Constellations using necklaces.

In the next section a detailed example is presented in order to show, in a clear manner, the methodology to generate 3D Necklace Flower Constellations.

Example of application

For this example, we assume that a constellation made of 42 satellites is chosen. Let suppose that the constellation is required to be built in 7 orbital planes, thus, $N_o = 7$, and each plane has to contain two orbits. That means that the parameters of number of real orbits per plane is $N_{rw} = 2$ and the number of real satellites per orbit is $N_{rso} = 3$.

Now, an expansion of the search space is done, choosing a fictitious constellation with parameters $N_{fw} = 6$ and $N_{fso} = 9$. That means that we are generating two different necklaces, one in the argument of perigee and the other one in the mean anomaly. Moreover, as it can be seen, the available positions both in mean anomaly and the argument of perigee have been trebled, being just the ninth part of all available positions real positions of satellites in the constellation.

Applying the 3D lattice flower constellations using necklaces to these parameters, we obtain 8820 different symmetrical configurations (compared to the 294 configurations obtained using just the 3D lattice flower constellations theory). Note that the number of configurations using necklaces can be increased even further by expanding the fictitious constellation or generating other fictitious constellations.

As there are too many configurations to analyze, we choose $N_{c_1} = 4$, $N_{c_2} = 3$ and $N_{c_3} = 6$ as combination numbers of the constellation, and $G_M = \{1, 4, 7\}$ and $G_\omega = \{1, 4\}$ as the necklaces in the mean anomaly and the argument of perigee respectively. Applying the definition of symmetry of a necklace from Equation (4), these results are obtained: $Sym(\mathcal{G}_M) = 3$ and $Sym(\mathcal{G}_\omega) = 3$.

With these parameters, we can use Equation (63) to obtain the shifting of the argument of perigee with respect the right ascension of the ascending node:

$$Sym(\mathcal{G}_\omega) \mid S_{\omega\Omega} N_o - N_{c_3} \Rightarrow 3 \mid 7S_{\omega\Omega} - 6, \tag{72}$$

which leads to $S_{\omega\Omega} = \{0\}$. On the other hand, the shifting parameter of the mean anomaly with respect of the argument of perigee can be computed using Equation (56):

$$Sym(\mathcal{G}_M) \mid S_{M\omega} N_{fw} - N_{c_2} \Rightarrow 3 \mid 6S_{M\omega} - 3, \tag{73}$$

which has three solutions, $S_{M\omega} = \{0, 1, 2\}$. Now, with this result, we apply Equation (68) to obtain the shifting parameter of the mean anomaly with respect the right ascension of the ascending node:

$$Sym(\mathcal{G}_M) \mid S_{M\Omega} N_o - (N_{c_1} - S_{M\omega} N_{c_3}) \Rightarrow 3 \mid 7S_{M\Omega} - (4 - 6S_{M\omega}), \tag{74}$$

which is $S_{M\Omega} = \{1\}$ no matter the value of $S_{M\omega} = \{0, 1, 2\}$ used.

As it can be seen, three configurations have been generated during the process due to the multiple solutions of $S_{M\omega}$. We choose $S_{\omega\Omega} = 0$, $S_{M\omega} = 2$ and $S_{M\Omega} = 1$ as the selected configuration. The lattice obtained from this configuration can be seen in Figure 6 where the (Ω, ω, M)-space of the distribution selected is shown. The circles represent available positions whilst the colored are the real satellites of the configuration.

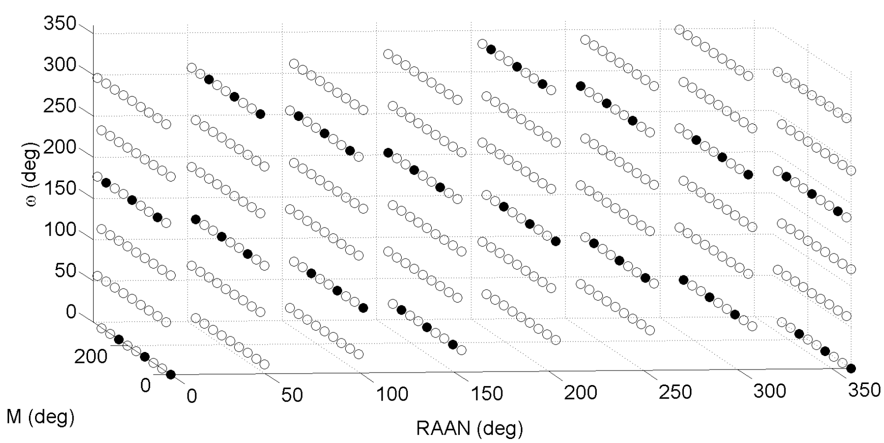

Figure 6. (Ω, ω, M)**-space representation of the constellation.**

Moreover, it is interesting to study the representation of this lattice using torus. This can be observed in Figure 7 where the three torus that define the distribution are shown. As it can be seen from Figure 6 and Figure 7, the distribution is symmetrical in all three orbital parameters: the right ascension of the ascending node, the argument of perigee and the mean anomaly.

Now, we this configuration is applied to the configuration of a satellite constellation. We choose an eccentricity of $e = 0.3$, an inclination equal to the critical inclination $i = 63.43^o$ and a semi-major axis equal to $a = 12770\,km$. With these orbital parameters, an inertial configuration as shown in Figure 8 is obtained.

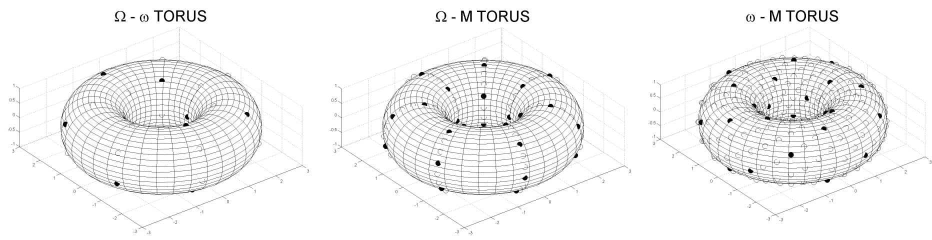

Figure 7. Torus representation of the constellation distribution.

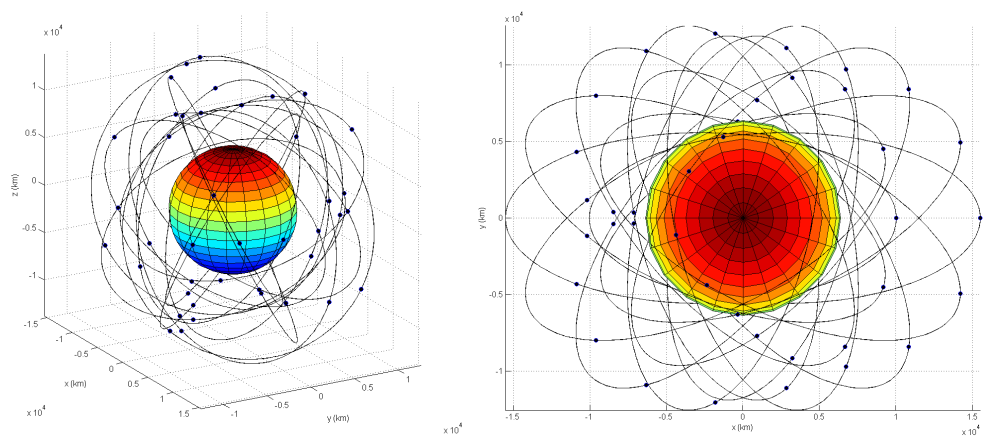

Figure 8. Inertial orbits of the constellation.

This constellation is just an example of the possibilities that the application of necklaces into the 3D lattice flower constellations theory can bring. As it has been said, the number of possibilities can be increase indefinitely, being the only constraint the computational power available.

CONCLUSIONS

3D Lattice Flower Constellations is a powerful tool that allows the generation of constellations with symmetric configurations and minimum parametrization. The distribution obtained with this methodology is fixed to certain positions which is a constraint in the number of possible configurations that the theory can generate.

This paper introduces the concept of necklaces in the formulation of 3D Lattice Flower Constellations, increasing the number of possible symmetric configurations, being the only limitation the computational power available. This is achieved by an expansion of the space of search of the constellation and applying the necklace to fit the configuration again to the one sought. Moreover, all the configurations obtained by this methodology maintain the properties of the former Flower Constellations, presenting symmetry in the lattice of the right ascension of the ascending node, the argument of perigee and the mean anomaly of all the satellites in the constellation.

Furthermore, the 3D Lattice Flower Constellations using necklaces includes all the former Lattice Flower Constellations design, being as such, a generalization of the Lattice Flower Constellations theory. That means that 3D Lattice Flower Constellations using necklaces is able to generate all former configurations (2D, 3D Lattice Flower Constellations and 2D Lattice Flower Constellations using necklaces), and create new distributions using the necklace theory.

Finally, it is important to note that the expansion of the search space can be increased as much as desired, providing more possibilities of design as the size of the fictitious constellation gets larger. Moreover, this expansion can also be done in an n dimension Lattice instead of just a 2D or a 3D Lattice and apply the same concepts. This further generalization will be treated in future works.

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