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An example of a non uniquely ergodic lamination

ÁLVARO LOZANO-ROJO‡†

‡ *Universidad del País Vasco. Facultad de CC. Económicas y Empresariales. Dpto. de Economía Aplicada III. Av. Lehendakari Aguirre, 83. 48015 Bilbao, Spain.*
(e-mail: alvaro.lozano@ehu.es)

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Abstract. This paper presents an example of Riemann surface lamination with at least two ergodic invariant measures. The generic leaves for those measures are of different growth and have different number of ends.

1. *Introduction*

A *foliation* is a decomposition of a compact manifold into submanifolds of a given dimension such that looks locally like a product. The study of foliations leads naturally to the theory of minimal saturated sets which are examples of compact laminations. A *lamination* is a decomposition of a locally compact Hausdorff space into manifolds of a given dimension with the same triviality condition as foliations.

One of the points of view in the study laminations is to pay attention to the behavior of "most leaves" of the lamination, that is, to properties shared by a *generic* set of leaves. There are two kinds of genericity in this context: 1) *topological*, a subset of a compact set is generic if it is a countable intersection of open dense sets, that is, a residual set, and; 2) *measurable*, a subset of a probability space is generic if it is a Borel set of full measure.

In [**8**], É. Ghys showed that generic leaves for ergodic harmonic measure have the same number of ends: 0, 1, 2 or a Cantor set of ends. Later, J. Cantwell and L. Conlon proved the analogous result in the topological setting [**4**].

Hence, a natural question arises: does there exists a minimal lamination which exhibit different generic properties depending on the considered invariant measure? In his PhD Thesis [**3**] E. Blanc constructs such a *mixed* minimal lamination. It is a subspace of the Gromov-Hausdorff graph foliated space of the subtrees of the Cayley graph of the free group with three generators, a huge space with lots of leaves of exponential growth. In fact, Blanc's example contains a generic set of leaves with this growth.

The exponential growth is very permissive with the kind of phenomena it allows. So, is it possible to construct a similar example dropping exponential growth allowing only the

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polynomial one? The aim of this paper is to construct such an example of mixed lamination in this more restrictive setting. In fact, the allowed growth will be subquadratic:

THEOREM 1.1. *There exists a minimal Riemann surface lamination with two transverse invariant measures ν and ν' such that: 1) the generic leaves for ν are the two-ended an linear ones and; 2) the generic leaves for ν' are the one-ended ones with the growth of $f(x) = x^{\ln 5/\ln 3}$. Moreover, the set of one-ended leaves is residual.*

If such spaces can be realized as a minimal set in a foliation is also discussed. This will lead directly to proof of the following result:

COROLLARY 1.1. *There is a foliation containing a minimal set as the one of Theorem 1.1.*

The paper is organized in the following way: Section 2 presents the construction of the *Gromov-Hausdorff foliated space*. Section 3 is devoted to the construction of two trees on Gromov-Hausdorff foliated space. The minimal lamination generated by these trees proves Theorem 3.1. In Section 4 the transverse dynamics of the Gromov-Hausdorff foliated space is reduced to the one given by a group action on the Cantor set. This fact allows to prove Theorem 4.1.

The results on this paper are part of the author's PhD thesis [**12**] defended at Universidad del País Vasco/Euskal Herriko Unibertsitatea on June 6th, 2008.

2. *The Gromov-Hausdorff foliated space*

2.1. *The Gromov-Hausdorff foliated space.* Let $\mathcal{Z}^2$ be the Cayley graph of $\mathbb{Z}^2$ respect to the usual generating system $S = \{(\pm 1, 0), (0, \pm 1)\}$. Denote by $\mathcal{T}$ the set of infinite subtrees of $\mathcal{Z}^2$ containing the origin as a vertex. Write $B_T(x, r)$ for the open ball in the graph T centered at x with radius r with respect the usual distance on a graph: the distance between two vertices is the least number of edges one has to go through to get from one vertex to another, and the edges (without their extremes) are isometric to the open unit interval $(0, 1)$. The *Gromov-Hausdorff distance* between two trees T and $T' \in \mathcal{T}$ is given by $d(T, T') = e^{-R(T,T')}$, where $R(T, T') = \sup\{N \geq 1 \mid B_T(0, N) = B_{T'}(0, N)\}$ or $R(T, T') = 0$ if the supremum is not defined. With this distance, $\mathcal{T}$ becomes a Cantor set (see [**7, 12**] for details).

The action of $\mathbb{Z}^2$ over $\mathcal{Z}^2$ induces a pseudogroup of transformations over $\mathcal{T}$: each $v \in \mathbb{Z}^2$ defines a partial transformation $\tau_v : T \mapsto T - v$ defined on the open set of all trees having v as vertex. All these maps generate a pseudogroup of transformations Γ over $\mathcal{T}$. As $\mathbb{Z}^2$ is generated by S, each element $\gamma \in \Gamma$ can be locally written as $\gamma = \tau_{s_1} \circ \cdots \circ \tau_{s_k}$, with $s_1, \ldots, s_k \in S$. Suppose that τ_s is defined over the clopen set $D_s = \{T \in \mathcal{T} \mid \text{the edge } [0, s] \subset T\}$, then the local writing holds, so Γ is finitely generated by $\{\tau_s\}$. The election of a generating set defines a graph structure on $\mathcal{T}$ in the following way: there is a edge (labeled by $s \in S$) from $T \in D_s$ to $\tau_s(T)$.

As the maps τ_s generate Γ, the connected components of the graph structure on $\mathcal{T}$ are the orbits $\Gamma[T] = \{T - v \mid v \in T\}$. Each $T' \in \Gamma[T]$ can be thought as "T rooted at v instead of 0". With this idea of *change of root* in mind, it is obvious that the geometric realization

$\overline{\Gamma}[T]$ of $\Gamma[T]$ can be identified with the quotient graph $T/\mathrm{Iso}(T)$, where $\mathrm{Iso}(T) \subset \mathbb{Z}^2$ denotes the *isotropy group of* T.

Remember that the *degree of a vertex* v in a graph is the number of edges incident with it. In this case, each tree T is a vertex in the graph $\Gamma[T]$, so makes sense to consider its degree, for which one has $\deg(T) = \deg_{\Gamma[T]}(T) = \deg_T(0)$. Hence the degree is constant on the ball $B_{\mathcal{T}}(T, e^{-r}) = \{T' \in \mathcal{T} \mid B_T(0,r) = B_{T'}(0,r)\}$, so $\deg : \mathcal{T} \to \mathbb{N}$ is a continuous map. In [**7**] É. Ghys used implicitly the continuity of this map to construct a Riemann surface lamination which has $\mathcal{T}$ as complete transversal and Γ as holonomy pseudogroup. This construction can be divided in two steps. Firstly, construct the *geometric realization* of all the graphs $\Gamma[T]$ at once, obtaining a graph foliated space:

THEOREM 2.1 (STEP 1: GEOMETRIC REALIZATION THEOREM [**12, 13**]) *There exists a compact transversely* 0*-dimensional graph foliated space* $(\mathcal{G}_{\mathcal{T}}, \mathcal{F})$ *such that a)* $\mathcal{T}$ *is the complete closed transversal of the vertices, b) the holonomy pseudogroup of* $\mathcal{F}$ *on* $\mathcal{T}$ *is* Γ *and c) the leaf* L_T *through* T *is the graph* $\overline{\Gamma}[T]$. *The graph foliated space* $(\mathcal{G}_{\mathcal{T}}, \mathcal{F})$ *is called* Gromov-Hausdorff *foliated space.* □

Secondly, replace local graph structure by a thickened version of the local graph structure:

THEOREM 2.2 (STEP 2: THICKENING THEOREM [**7, 12, 13**]) *There exists a compact Riemann surface lamination* $(M_{\mathcal{T}}, \mathcal{L})$ *of bounded geometry (see [**10**]) such that* $\mathcal{T}$ *is a closed complete transversal of* $\mathcal{L}$ *and the holonomy pseudogroup restricted to* $\mathcal{T}$ *is* Γ. *Additionally, the leaves of* $\mathcal{L}$ *are quasi-isometric (in a uniform way) to the corresponding leaves of* $\mathcal{F}$. □

See [**12, 13**] for a complete proof of these results in a more general setting.

2.2. *Minimal sets in the Gromov-Hausdorff foliated space.* Roughly speaking, a graph is *repetitive* when one cannot know on what point one is at, taking only into account a finite neighborhood of that point. This property is based on the analogous properties of repetitiveness [**2**] or local isomorphism [**15**] for tilings. The notion for graphs appears implicitly on E. Blanc's PhD thesis [**3**].

Given two trees $T, T' \in \mathcal{T}$, the ball $B_T(x,r)$ is said that *embeds faithfully* into T', written as $B_T(x,r) \hookrightarrow T'$, if $B_T(x,r) + v = B_{T'}(x+v,r)$ for some $v \in \mathbb{Z}^2$. If additionally $B_{T'}(x+v,r) \subset B_{T'}(x',r')$, is said that $B_T(x,r)$ *embeds faithfully* into $B_{T'}(x',r')$, writing $B_T(x,r) \hookrightarrow B_{T'}(x',r')$ in this case.

Definition 2.1. A tree $T \in \mathcal{T}$ is *repetitive* if for all $r > 0$, there exists $R(r) > 0$ such that $B_T(0,r) \hookrightarrow B_T(x,R)$ for any $x \in T$.

Remark 2.1. There is a *uniform* notion of repetitiveness. In that case, one asks for the existence of $R > 0$, for a given $r > 0$, such that any ball of radius r can be embedded faithfully into any ball of radius R. This notion is equivalent to the repetitiveness (see [**1, 12**]).

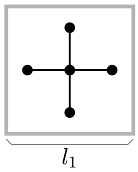

FIGURE 1. The tree G_1.

The following Theorem gives a very convenient criterion for minimality in the Gromov-Hausdorff space in terms of repetitive trees.

PROPOSITION 2.1 ([**1, 3, 12**]) *A subset of $\mathcal{T}$ is minimal if and only if it is the closure of the orbit of a repetitive tree.* □

COROLLARY 2.1. *A subspace of $(\mathcal{G}_{\mathcal{T}}, \mathcal{F})$ or $(M_{\mathcal{T}}, \mathcal{L})$ is minimal if and only if it is the closure of the leaf through a repetitive tree.*

3. *A non uniquely ergodic example*

This section is devoted to the construction of two aperiodic subtrees of $\mathcal{Z}^2$. These trees will be created using the same finite patterns, so they will be locally indistinguishable, but the global properties will be different.

As they share the same building blocks, they belong to the same minimal set. On the other hand, the different properties of the trees will translate into two distinct measures with different generic leaves. In fact, those measures will be constructed by means of the averaging sequences used on the construction.

3.1. *Construction of the trees.*

The construction is done inductively. Let G_1 be the cross-shaped tree of width 2 contained in a square of side $l_1 = 3$, like the one in Figure 1. Let T_1 be the infinite horizontal straight line through the origin. Let P_1 be the ball in T_1 centered at the origin of radius $(l_1 - 1)/2 = 1$.

The tree G_2 is constructed mixing G_1 and P_2: put four copies of G_1 around G_1 itself and glue them all with edges. In this way a bigger cross-shaped tree is obtained. On the both sides of each vertical arm add a copy of P_1, joint with an edge. The resulting tree is G_2, which is contained in a square of side $l_2 = 3l_1 = 3^2$. The process is shown in Figure 2.

Lets define T_2. Choose an even integer r_1 such that

$$\frac{\#G_1}{r_1} \leq \frac{1}{l_1}.$$

Now, insert copies of G_1 periodically on intervals of length r_1 along T_1, in the way shown in Figure 3. The resulting tree is T_2. As before, P_2 denotes the ball $\overline{B}_{T_2}(\mathbf{0}, (l_2 - 1)/2)$.

Continue in this way: construct G_3 from P_2 and G_2 putting five copies of G_2 making a cross and then gluing four copies of P_2 on the vertical arms, just like in the case of G_2 (see Figure 4). Obviously G_3 is contained in a square of side $l_3 = 3l_2 = 3^3$. To construct T_3

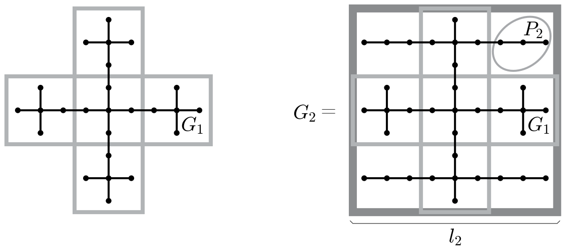

FIGURE 2. Constructing G_2 with G_1 and P_1.

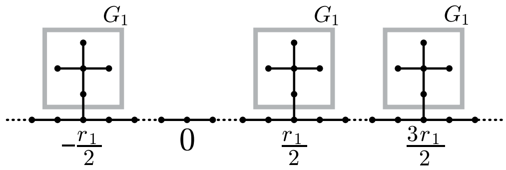

FIGURE 3. Constructing the infinite tree T_2, adding copies of G_1.

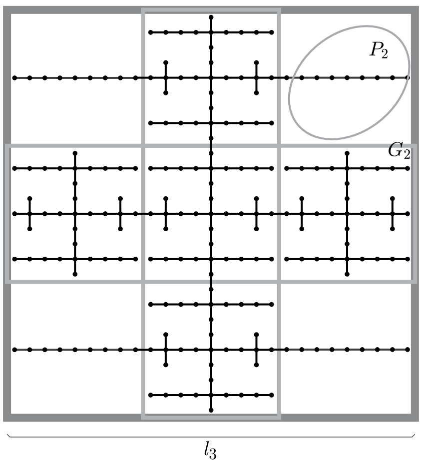

FIGURE 4. Third step. The tree G_3.

choose r_2, a multiple of r_1, such that

$$\frac{\#G_2}{r_2} \leq \frac{1}{l_2}.$$

Add copies of G_2 to T_2, but now with periodicity r_2. Some copies of G_2 will replace copies of G_1, like in Figure 5. Notice that in this step (and in the next steps) there is no need to add more edges to connect the copies of G_2 to T_2.

Recursively one gets two increasing sequences of finite trees $\{P_n\}$ and $\{G_n\}$, which converges to $P_\infty = \bigcup_{n\geq 1} P_n = \bigcup_{n\geq 1} T_n$ and $G_\infty = \bigcup_{n\geq 1} G_n$ respectively. As these trees share all of theirs finite patches, it is straightforward to proof that:

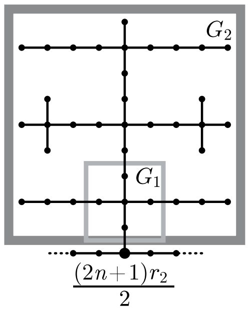

FIGURE 5. Detail in the construction of T_3. A copy of G_2 replaces one of G_1.

LEMMA 3.1. *Both P_∞ and G_∞ are repetitive and $\mathcal{X} = \overline{\Gamma[P_\infty]} = \overline{\Gamma[G_\infty]}$.*

3.2. *Inequalities derived from construction.* The construction implies some inequalities for the cardinals of P_n and G_n. In a horizontal straight segment of length l_{n-1} exists, at most, $\frac{l_{n-1}-r_1}{r_1} \le \frac{l_{n-1}}{r_1}$ copies of G_1 within P_∞. For G_2 there are, at most, $\frac{l_{n-1}-r_2}{r_2} \le \frac{l_{n-1}}{r_2}$ copies and so on. So

$$\begin{aligned} l_{n-1} + 1 \le \#P_n \le l_{n-1} + 1 + \sum_{i=1}^{\infty} \frac{l_{n-1}}{r_i} \#G_i \\ \le l_{n-1} + 1 + l_{n-1} \sum_{i=1}^{\infty} \frac{1}{3^i} \le \frac{3}{2} l_{n-1} + 1. \end{aligned} \tag{1}$$

On the other hand, it is clear that

$$\#G_n = 5\#G_{n-1} + 4\#P_{n-1} = \cdots = 5^{n-1}\#G_1 + 4\sum_{i=0}^{n-2} 5^i \#P_{n-i}. \tag{2}$$

Using (1) on (2), it follows that

$$5^n \le \#G_n \le 5^n\left(1 + 6\sum_{i=0}^{n-2} \frac{3^{n-i-1}+1}{5^{n-i}}\right) \le 16 \cdot 5^{n-1}. \tag{3}$$

The inequality (1) shows that P_∞ has linear growth. From (3) it is also obvious that the growth of G_∞ is like the one of $f(x) = x^{\ln 5/\ln 3}$.

3.3. *From the trees to a non uniquely ergodic lamination.* As Lemma 3.1 says the trees P_∞ and G_∞ are repetitive and belong to the same minimal set $\mathcal{X} \subset \mathcal{T}$. Obviously one gets the same result for the geometric realization and thickening of $\mathcal{X}$. Hence, a minimal graph foliated space $(\mathcal{G}_{\mathcal{X}}, \mathcal{F})$ and a minimal Riemann surface lamination $(M_{\mathcal{X}}, \mathcal{L})$ are obtained, which are the closure of the leaf through P_∞ or G_∞ in $(\mathcal{G}_{\mathcal{T}}, \mathcal{F})$ and $(M_{\mathcal{T}}, \mathcal{L})$ respectively. By definition the spaces of transverse invariant measures of those spaces agree with the space $\mathfrak{M}(\Gamma)$ of Γ-invariant measures over X.

PROPOSITION 3.1. *The space $\mathfrak{M}(\Gamma)$ contains at least two invariant measures.*

Proof. It is obvious that $\mathfrak{G} = \{G_n\}_{n\geq 1}$ and $\mathfrak{P} = \{P_n\}_{n\geq 1}$ are two Følner sequences. Therefore, they define two Γ-invariant measures $\mu_{\mathfrak{G}}$ and $\mu_{\mathfrak{P}}$ (see [**9**]).

Lets see that there exists a Borelian set where these two measures differ. The sets $\mathcal{X}_{G_n} = \{T \in \mathcal{X} \mid G_n - p \subset T$ for some $p \in G_n\}$ and $\mathcal{X}_{G_n,0} = \{T \in \mathcal{X} \mid G_n \subseteq T\} \subseteq \mathcal{X}_{G_n}$ are clopen sets. Now, define $A_n = \bigcap_{i\leq n} \mathcal{X}_{G_i}$. These sets form a decreasing family of clopen sets of non empty intersection: $G_\infty \in \bigcap_n A_n$. As

$$A_n = \{T \in \mathcal{X} \mid G_n - p \subseteq T \text{ with } p \text{ in the copies of } G_1 \text{ within } G_n\},$$

it is clear that $\mu(A_n) = 5^n \mu(\mathcal{X}_{G_n,0})$ for any $\mu \in \mathfrak{M}(\Gamma)$.

Claim: $\mu_{\mathfrak{P}}(\bigcap_n A_n) = 0$. Within P_p there exists, at most, $\frac{l_{p-1}}{r_n}$ copies of G_n. There are also at most $\frac{l_{p-1}}{r_{n+1}}$ copies of G_{n+1} which has 5 copies of G_n, an so on. Therefore

$$\mu_{\mathfrak{P}}(\mathcal{X}_{G_n,0}) = \lim_{p\to\infty} \frac{\#(\mathcal{X}_{G_n,0} \cap P_p)}{\# P_p} \leq \lim_{p\to\infty} \frac{l_{p-1} \sum_{i\geq n} \frac{5^{i-n}}{r_i}}{l_{p-1}+1} \leq \sum_{i\geq n} \frac{5^{i-n}}{r_i}.$$

Using this, the inequality (1), the property about the measure of A_n and the relations between r_i, $\# G_n$ and l_i one has that $\mu_{\mathfrak{P}}(A_n) \leq \sum_{i\geq n} \frac{1}{l_i}$. Hence $\mu_{\mathfrak{P}}(\bigcap_n A_n) = 0$.

Claim: $\mu_{\mathfrak{G}}(\bigcap_n A_n) > 0$. If $p \geq n$, the number of copies of G_n in G_p is at least 5^{p-n}. From (3)

$$\mu_{\mathfrak{G}}(\mathcal{X}_{G_n,0}) = \lim_{p\to\infty} \frac{\#(\mathcal{X}_{G_n,0} \cap G_p)}{\# G_p} \geq \lim_{p\to\infty} \frac{5^{p-n}}{16 \cdot 5^{p-1}} = \frac{1}{16 \cdot 5^{1-n}}.$$

Therefore $\mu_{\mathfrak{G}}(A_n) \geq \frac{5}{16}$. So $\mu_{\mathfrak{G}}(\bigcap_n A_n) > 0$, and then $\mu_{\mathfrak{P}}$ and $\mu_{\mathfrak{G}}$ are different. □

3.4. *Generic leaves.* Consider the following tree of $\bigcap_n A_n$:

$$T = \bigcup_n \Big(G_n - \sum_{i=1}^{n-1} (l_i, 0) \Big).$$

This is a one-ended tree which covers the right half-plane and the horizontal right ray defines its end. The corresponding leaf has one end, therefore the set of all one-ended leaves is residual [**4**]. From the measurable point of view there are various sorts of generic leaves, depending on the considered measure:

THEOREM 3.1. *There exists two transverse invariant measures $\nu_{\mathfrak{G}}$ and $\nu_{\mathfrak{P}}$ on $(\mathcal{X}, \mathcal{L})$ such that: 1) the generic leaves for $\nu_{\mathfrak{G}}$ are the one-ended ones with the growth of $f(x) = x^{\ln 5/\ln 3}$, and; 2) the generic leaves for $\nu_{\mathfrak{P}}$ are the two-ended ones with linear growth.*

Proof. 1) Using the notation above, the saturation A of $\bigcap_n A_n$ has positive $\mu_{\mathfrak{G}}$-measure. Obviously, all the leaves $L \subseteq A$ have the same growth as G_∞, that is, the one of $f(x) = x^{\ln 5/\ln 3}$. So, there exits an invariant measure $\nu_{\mathfrak{G}}$ whose generic leaves are of the growth of $f(x) = x^{\ln 5/\ln 3}$. A result of G. Levitt says that the set of non-linear two-ended leaves is of null measure [**11**]. Thence, the residual leaves of $\nu_{\mathfrak{G}}$ have one end.

2) This is just an implication of a general result of D. Gaboriau [**6**]: *If there is a linear growth leaf, there exists an invariant measure with the two-ended leaves as generic*. In fact, the measure $\nu_{\mathfrak{P}}$ can be constructed from $\mu_{\mathfrak{P}}$ in a similar way that $\nu_{\mathfrak{G}}$ from $\mu_{\mathfrak{G}}$ [**3, 6**]. □

4. *Realization of the example as a minimal set in a foliation*

The aim of this section is to build a surface foliation of codimension two having a minimal set with the same transverse dynamics (i.e. with equivalent holonomy pseudogroup) and coarse geometry as $(\mathcal{G}_{\mathcal{T}}, \mathcal{F})$. Therefore any foliated subspace of $(\mathcal{G}_{\mathcal{T}}, \mathcal{F})$ can be *found* (in the previous sense) within a codimension 2 foliation. In fact, the construction can be applied as is to any transversally Cantor compact graph foliated space, because in such a space the vertex transversal is always homeomorphic to the Cantor set.

4.1. *The transverse dynamics of $\mathcal{G}_{\mathcal{T}}$ are given by a group action.* Consider the graph foliated space $(\mathcal{G}_{\mathcal{T}}, \mathcal{F})$. Define $\mathcal{Y} = \mathcal{T} \sqcup E$ where E is the set of edges of the graphs in $\mathcal{F}$. Identifying each edge in E with its middle point, $\mathcal{Y}$ becomes a closed complete transversal of $(\mathcal{G}_{\mathcal{T}}, \mathcal{F})$.

Proposition 4.1. *The holonomy pseudogroup of $\mathcal{F}$ over $\mathcal{Y}$ is the pseudogroup of a continuous group action of the free group with $2S$ generators $\mathbf{F}_{2S}$.*

Proof. The proof is similar to the Feldman and Moore's construction for Borel equivalence relations (see [**5**]). Each translation $\tau_s : D_s \subset \mathcal{T} \to R_s \subset \mathcal{T}$ can be written as $\tau_s = \tau_s^2 \circ \tau_s^1$ where $\tau_s^1 : D_s \subset \mathcal{T} \to E_s \subset E$ and $\tau_s^2 : E_s \subset E \to R_s \subset \mathcal{T}$ are two maps constructed as follows: the image of $T \in D_s$ through τ_s^1 is the edge $(T, s, T - s) \in E$, and the image of the edge $(T, s, T - s) \in E$ is $T - s$, and E_s is $\{(T, s, T - s) \mid T \in D_s\}$. Obviously, τ_s^1 and τ_s^2 are homeomorphisms between clopen disjoint subset of $\mathcal{Y}$. So, for each $s \in S$ and $i = 1, 2$ one can define the homeomorphism $f_s^i : \mathcal{Y} \to \mathcal{Y}$

$$f_s^i(y) = \begin{cases} \tau_s^i(y) & \text{if } y \in D_s, \\ (\tau_s^i)^{-1}(y) & \text{if } y \in E_s, \\ y & \text{in any other case.} \end{cases}$$

These homeomorphisms define an obvious action of the free group $\mathbf{F}_{2S}$ on Y which induces the pseudogroup of holonomy on Y. □

Remark 4.1. The free group $\mathbf{F}_{2S}$ can be replaced by $*_{i=1}^{2S} \mathbb{Z}/2\mathbb{Z}$, because $(f_s^i)^2 = 1_Y$.

The usual graph structure on $\mathbf{F}_{2S}$ induces another on the orbits in the obvious way. Applying the Geometric Realization Theorem one gets a graph foliated space $(\mathcal{G}_{\mathcal{Y}}, \mathcal{F})$ whose pseudogroup of holonomy restricted to the complete closed transversal $\mathcal{T}$ is Γ. This foliated space is obtained substituting each edge of $(\mathcal{G}_{\mathcal{T}}, \mathcal{F})$ by a flower-shaped graph and adding some cycles corresponding to the action of elements fixing those point (see Figure 6). It is straightforward to show that:

Lemma 4.1. *The inclusion $\iota : \mathcal{T} \to \mathcal{Y}$ is a Lipschitz map, when $\mathcal{T}$ and $\mathcal{Y}$ have the metric induced by the leaves of $\mathcal{G}_{\mathcal{T}}$ and $\mathcal{G}_{\mathcal{Y}}$ respectively.*

Applying the Thickening Theorem to $(\mathcal{G}_{\mathcal{Y}}, \mathcal{F})$ one obtains a Riemann surface lamination $(M_{\mathcal{Y}}, \mathcal{L})$ with these properties: 1) has $\mathcal{T}$ and $\mathcal{Y}$ as a complete closed transversals, 2) the holonomy pseudogroup reduced to $\mathcal{T}$ is Γ, 3) the leaf through a tree $T \in \mathcal{T}$ is quasi-isometric to $\overline{\mathcal{R}}[T]$, and 4) its dynamics reduced to $\mathcal{Y}$ are given by a group action.

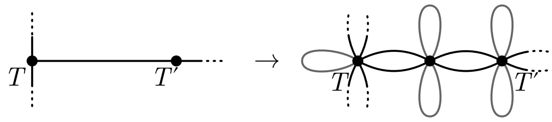

FIGURE 6. How the graph structure change from $\mathcal{G}_{\mathcal{T}}$ to $\mathcal{G}_{\mathcal{Y}}$.

4.2. *Realization in a foliation.*

THEOREM 4.1. *There exists a Riemann surface foliation $(M, \mathcal{F})$ of class $C^{\infty,0}$ and codimension 2 containing $(M_{\mathcal{Y}}, \mathcal{L})$ as a closed saturated set. Therefore, there exists a foliation containing a minimal set with the transverse dynamical and coarse geometrical properties of the example of Section 3.*

Proof. The set Y is a Cantor set, so the homeomorphisms $f_s^i : \mathcal{Y} \to \mathcal{Y}$ can be extended to homeomorphisms $\widetilde{f_s^i} : \mathbb{S}^2 \to \mathbb{S}^2$ (see Chapter 13 of [**14**]). Therefore an action of the free group $\mathbf{F}_{2S}$ on the sphere $\mathbb{S}^2$ appears. Again, the graph structure of $\mathbf{F}_{2S}$ defines a graph structure of constant degree on the orbits given by a generating set. Applying the Thickening Theorem a codimension two Riemann surface foliation of class $C^{\infty,0}$ is obtained, which obviously has $(M_{\mathcal{Y}}, \mathcal{L})$ as a minimal saturated set. □

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