SUSPENSIONS OF BERNOULLI SHIFTS

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ABSTRACT. We show that for a given finitely generated group, its Bernoulli shift space can be equivariantly embedded as a subset of a space of pointed trees with Gromov-Hausdorff metric and natural partial action of a free group. Since the latter can be realised as a transverse space of a foliated space with leaves Riemannian manifolds, this embedding allows us to obtain a suspension of such Bernoulli shift. By a similar argument we show that the space of pointed trees is universal for compactly generated expansive pseudogroups of transformations.

1. INTRODUCTION

Let G be a semigroup with the identity element, that is, G is a set equipped with an associative binary operation. Let S be a finite set, and consider the set of all maps $\Sigma(G, S) = \{\sigma : G \to S\}$. Give the set $\Sigma(G, S)$ the product topology in the standard way, and define an action of G on $\Sigma(G, S)$ by

$$\sigma(g') \cdot g = \sigma(gg').$$

Thus one obtains a dynamical system $(\Sigma(G, S), G)$, called the *Bernoulli shift* [10]. The most well-known cases are $G = \mathbb{Z}$, $G = \mathbb{N}$ and $G = \mathbb{Z}^n$ (see [18] for a survey). In the cases $\mathbb{Z}$ and $\mathbb{N}$ one can study shifts using graphs generated by finite automata. This article proposes a way to represent the dynamics of a shift system using graphs in the case when G is any finitely generated group.

We will do that by constructing an equivariant embedding of $(\Sigma(G, S), G)$ into a metric space of pointed trees with dynamics given by a pseudogroup action. This space is obtained as follows.

For a symmetric set G^1 of generators of G, consider a graph $\mathcal{G}$ with unoriented labeled edges, which is essentially the Cayley graph of G with respect to G^1. We then denote by X the set of all non-compact subgraphs of $\mathcal{G}$ with only trivial loops, that is, every such subgraph $T \subset X$ is a tree. We also require that the identity element e is a vertex of T, and therefore (T, e) is a pointed metric space with the standard length metric. Distances between elements of X are measured with the help of the Gromov-Hausdorff metric d_{GH}, and it turns out that (X, d_{GH}) is a compact totally disconnected space [11, 4, 20, 22]. There is a natural partial action of the free group F_n on X, where n is the half cardinality of G^1, and this action gives rise to the action of a pseudogroup $\mathfrak{G}$ on X. One can suspend the action of $\mathfrak{G}$ on X to obtain a smooth foliated space $\mathfrak{M}_G$ with 2-dimensional leaves [11, 4, 20, 22]. By this construction, for $T \in X$ the corresponding leaf $L_T \subset \mathfrak{M}_G$ can be thought of as the two-dimensional boundary of the thickening of a graph of the orbit of T under the action of $\mathfrak{G}$. Following [22], we call the closure $\mathcal{M} = \overline{L}$ of a leaf $L \subset \mathfrak{M}_G$ a *graph matchbox manifold*. The term 'matchbox manifold' refers to the fact that (X, d_{GH}) is a totally disconnected space, and stems from the study of flows on 1-dimensional continua [1, 2]. In the latter case

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a foliation chart is homeomorphic to the product of an open interval and a zero-dimensional space, and so can be thought of as a box of matches, where each match corresponds to a path-connected component of the chart.

The construction of a foliated space transversely modeled on the space of pointed trees with pseudogroup action was introduced by R. Kenyon and used by É. Ghys [11] in the case $G = \mathbb{Z}^2$ to obtain an example of a space foliated by Riemannian manifolds such that each leaf is dense and there are leaves with different conformal types. Blanc [4] considered a similar construction for $G = F_n$, a free group on n generators, and the first author [20], also with F. Alcalde Cuesta and M. Macho Stadler [3], studied this construction for the case of an arbitrary finitely generated group. In the case $G = \mathbb{Z}^2$ the first author [21] found examples with interesting ergodic properties. Blanc [4] found an example of a graph matchbox manifold with specific asymptotic properties of leaves.

A systematic study of the dynamical and topological properties of $\mathfrak{M}_G$ was done by the second author in [22]. In particular, in [22, Theorem 1.3] the second author studied a partial order on $\mathfrak{M}_G$ given by inclusions, which is equivalent to the study of the orbit structure of the pseudogroup dynamical system $(X, \mathfrak{G})$. Using the notion of a level of a leaf, initially introduced for codimension 1 foliations by J. Cantwell and L. Conlon [8] (see also [9, 14, 15, 23, 24]) she constructed hierarchies of graph matchbox manifolds at infinite levels. The results of [22] also show that the orbit structure of $(X, \mathfrak{G})$ is reminiscent of the orbit structure of Bernoulli shifts, for example there is a meager subset of points with finite orbits, and a residual subset of points with dense orbits. Therefore, it is natural to seek the relation between the two. The second author has been asked this question by M. Barge, S. Hurder and A. Clark, when presenting her work at conferences. Our main theorem gives the answer to this question.

Write the symmetric generating set G^1 as $G^{1+} \sqcup G^{1-}$ where G^{1+} is the *positive* set of generators and $G^{1-} = (G^{1+})^{-1}$ is the *negative* set.

Theorem 1.1. *Let G be a finitely generated group, and $(\Sigma(G, S), G)$ be the Bernoulli shift. Given a set of generators G^1 of G, and an injective map $\alpha : G^{1+} \times S \to F_n^{1+}$, there exists an embedding*

$$\Phi_\alpha : \Sigma(G, S) \to X_n,$$

which is an orbit equivalence. Moreover, for any $\sigma \in \Sigma(G, S)$ and $g \in G^{1+}$

$$\Phi_\alpha(\sigma \cdot g) = \Phi_\alpha(\sigma) \cdot \alpha(g, \sigma(e)).$$

and if $g^{-1} \in G^{1+}$ then

$$\Phi_\alpha(\sigma \cdot g) = \Phi_\alpha(\sigma) \cdot \left[\alpha(g^{-1}, \sigma(g))\right]^{-1}.$$

Since X_n can be realised as a transverse space of a foliated space with leaves Riemannian manifolds, this embedding allows us to obtain a suspension of a Bernoulli shift for arbitrary G and S.

Recall [10] that a dynamical system (Ω, G), where (Ω, d_Ω) is a metric space, is ϵ-expansive if for any $x, y \in \Omega$ there exists $g \in G$ such that $d_\Omega(g(x), g(y)) \geq \epsilon$. Bernoulli shifts provide examples of ϵ-expansive (for some ϵ) dynamical systems. It is known [10, Proposition 2.6] that every expansive action of a group G on a compact metrizable space Ω is a quotient of the G-action on a closed G-invariant subspace of $\Sigma(G, S)$ for some finite set S, i.e. of a subshift. If Ω is totally disconnected, then the quotient map $\pi : \Sigma(G, S) \to \Omega$ can be made a conjugacy [10, Proposition 2.8]. This conjugacy is not unique, and there is no canonical way to choose the set S.

As a consequence of Theorem 1.1 we extend this result to pseudogroup dynamical systems (Ω, Γ), where Ω is a 0-dimensional compact metrizable space and Γ admits a *compact generating system* [12, 13]. The latter property means that Ω contains a relatively compact open set Y meeting all orbits of Γ and the reduced pseudogroup $\Gamma|_Y$ is generated by a finite set Λ_Y of elements of Γ such that each $\gamma \in \Lambda_Y$ is the restriction of an element $\bar{\gamma} \in \Gamma$ and such that the closure of $\operatorname{dom} \gamma$ is contained in $\operatorname{dom} \bar{\gamma}$. Recall [16] that a pseudogroup Γ on Ω is ϵ-*expansive* if for all $w \neq w' \in \Omega$ with $d(w, w') < \epsilon$ there exists $\gamma \in \Gamma$ with $w, w' \in \operatorname{dom} h$ such that $d(\gamma(w), \gamma(w')) \geq \epsilon$.

Theorem 1.2. *Let Ω be a 0-dimensional compact metrizable space, and Γ be an ϵ-expansive pseudogroup of transformations on Ω which admits a compact generating system. Then there is an embedding $\Omega \to X_n$ equivariant with respect to the actions of Γ and $\mathfrak{G}_n$, for a large enough n.*

In other words, (Ω, Γ) is conjugate to the $\mathfrak{G}_n$-action on a closed $\mathfrak{G}_n$-invariant subset of X_n.

The rest of the paper is organized as follows. In Section 2 we recall definitions and basic properties of Bernoulli shifts, and sketch the construction of a foliated space of graph matchbox manifolds. In Section 3 we present a proof of Theorem 1.1. Section 4 gives a proof of Theorem 1.2, and in Section 5 we show that our method is applicable to the semigroup $G = \mathbb{N}_0$, but not to general semigroups.

2. Preliminaries

2.1. **Bernoulli shifts.** Let Γ be a semigroup with the identity element, that is, Γ is a set equipped with an associative binary operation, and there is $e \in \Gamma$ such that for every $\gamma \in \Gamma$, $\gamma e = e\gamma = \gamma$. Let S be a finite set, and consider the set $\Sigma = \Sigma(\Gamma, S)$ of maps $\sigma : \Gamma \to S$, with the right action of Γ given by

(1) $$\sigma(\gamma') \cdot \gamma = \sigma(\gamma \gamma').$$

We give S discrete topology, and Σ the product topology, that is, given a finite set $F \subset \Gamma$, and an element $\sigma' \in \Sigma$, a basic open set is given by

$$C_{F,\sigma'} = \left\{ \sigma \in \Sigma \mid \sigma(\gamma) = \sigma'(\gamma) \text{ for all } \gamma \in F \right\}.$$

Then Σ with this topology is metrizable, compact and totally disconnected and, if $\operatorname{card} S$ is strictly greater than 1, perfect (see, for instance, [10]).

Definition 2.1 ([10]). *The space Σ equipped with the action of Γ given by* (1) *is called a* Bernoulli shift. *A closed Γ-invariant subspace of Σ is called a* subshift.

For basic properties of Bernoulli shifts we refer the reader to [10], stating here only that Bernoulli shifts are expansive, i.e. the following definition is satisfied.

Definition 2.2 ([10]). *A dynamical system (Ω, Γ), where (Ω, d_Ω) is a metric space, is ϵ-expansive if for any $x, y \in \Omega$ there exists $\gamma \in \Gamma$ such that $d_\Omega(\gamma(x), \gamma(y)) \geq \epsilon$.*

Example 2.3. Let $S = \{0, \ldots, N-1\}$, and let $\Gamma = \mathbb{Z}$ (resp. $\Gamma = \mathbb{N}_0$). In this case Σ can be thought of as a set of 2-ended (resp. 1-ended) sequences, and the action of Γ is given by

$$\sigma \cdot n = (\ldots \sigma(-1).\sigma(0)\sigma(1) \ldots) \cdot n = (\ldots \sigma(n-1).\sigma(n)\sigma(n+1) \ldots)$$

if $\Gamma = \mathbb{Z}$, and if $\Gamma = \mathbb{N}$, it is given by

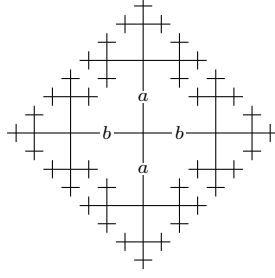$$\sigma \cdot n = (\sigma(0)\sigma(1) \ldots) \cdot n = (\sigma(n)\sigma(n+1) \ldots).$$

FIGURE 1. The Cayley graph $\mathcal{F}_2$ of the free group $F_2 = \langle a, b \rangle$.

A metric d_Σ on Σ compatible with the topology is given, for example, by

$$d_\Sigma(\sigma, \sigma') = \sum_{i=-\infty}^{\infty} \frac{1}{2^{|i|}} |\sigma(i) - \sigma'(i)|,$$

if $\Gamma = \mathbb{Z}$, and similarly in the case $\Gamma = \mathbb{N}_0$, with summation starting from $i = 0$. Then for $\epsilon < 1/2$ and two distinct elements σ, σ' there exists $|n| > 0$ such that $\sigma(n) \neq \sigma'(n)$, and $\tilde{\sigma}, \tilde{\sigma}'$ such that $\tilde{\sigma} = \sigma - n$, and $\tilde{\sigma}' = \sigma' - n$, that is, $\tilde{\sigma}(0) = \sigma(n) \neq \sigma'(n) = \tilde{\sigma}'(n)$, which shows that $d_\Sigma(\tilde{\sigma}, \tilde{\sigma}') > \epsilon$, and (Σ, Γ) is ϵ-expansive.

2.2. **Graph matchbox manifolds.** We give a brief outline of the construction of Kenyon and Ghys. The content of this section can also be found in [22, 11, 20, 3].

2.2.1. *Space of pointed trees with Gromov-Hausdorff metric.* Let $V(T)$ be the set of vertices, and $E(T)$ be the set of edges of a graph T. An edge $w \in E(T)$ may be given an orientation by specifying its starting and its ending vertex, denoted by $s(w)$ and $t(w)$ respectively. If we are not interested in the orientation of a directed edge, or if an edge is undirected, then we denote the set of its vertices by $V(w)$.

A subgraph of T is a graph T' with the set of vertices $V(T') \subseteq V(T)$ and the set of edges $E(T') \subset E(T)$. Edges of a graph T are labeled by symbols contained in a set $\mathcal{A}$, and sometimes we will use the notation w_a for an edge $w \in E(T)$ labeled by $a \in \mathcal{A}$.

Let G be a group with a symmetric set of generators G^1, that is, $(G^1)^{-1} = G^1$, and G^1 does not contain the identity element. We consider the action of G on itself on the right. Let $\mathcal{G}'$ be the Cayley graph of G with respect to the set of generators G^1; more precisely, set $V(\mathcal{G}') = G$, and to each pair $g_1, g_2 \in V(\mathcal{G}')$ such that $g_1 h = g_2$ for some $h \in G^1$, associate an edge $w_h \in E(\mathcal{G}')$ with $s(w_h) = g_1$ and $t(w_h) = g_2$. Since G^1 is a symmetric generating set, the Cayley graph $\mathcal{G}'$ is *undirected*, that is, $g_1, g_2 \in V(\mathcal{G}')$ are joined by an edge w_h if and only if they are joined by an edge $w_{h^{-1}}$. We divide G^1 into a *positive* part G^{1+} and a *negative* part G^{1-}, that is, if $h \in G^{1+}$, then $h^{-1} \in G^{1-}$. This division is, of course, arbitrary.

We now consider a graph $\mathcal{G}$ such that $V(\mathcal{G}') = V(\mathcal{G})$, and two vertices g_1 and g_2 are joined by an edge if and only if they are joined by an edge in $\mathcal{G}'$. Thus for a pair of edges w_h and $w_{h^{-1}}$ in $\mathcal{G}'$ joining the same vertices we obtain a single unoriented edge in $\mathcal{G}$ labeled by $h \in G^{1+}$.

We assume that $\mathcal{G}$ is given the length structure ℓ in the usual way, so that every edge in $\mathcal{G}$ has length 1. Then the complete length metric D on $\mathcal{G}$ is defined by

$$D(x, y) = \inf_\delta \left\{ \ell(\delta) \mid \delta : [0, 1] \to \mathcal{G}, \, \delta(0) = x, \, \delta(1) = y \right\},$$

and $(\mathcal{G}, D)$ becomes a *length* metric space.

Consider the set X of all subgraphs of $\mathcal{G}$ which are non-compact, connected and simply connected, and which also contain the identity $e \in G$. A tree $T \in X$ is a subset of $\mathcal{G}$, and therefore has an induced length structure ℓ. The length structure ℓ gives rise to the length metric d on T, and the pair $(T, e) \in X$ with metric d becomes a pointed metric space.

The distance between pointed metric spaces $(T, e), (T', e) \in X$ is measured with the help of the Gromov-Hausdorff metric d_{GH} [5] (here for the purpose of this metric we have to restrict to maps preserving labeling of edges), and the set X has metric topology induced from this metric. It is also customary to consider the *box metric* d_X on X, which gives this space the same topology as d_{GH}, and which is easier to work with. To introduce this metric we first need the following definition.

Definition 2.4. *Let (A, g_1) and (B, g_2) be two pointed (not necessarily non-compact) subgraphs of $\mathcal{G}$ with induced metric d. We say that (A, g_1) and (B, g_2) are* isomorphic *if there exists an isometry $(A, g_1) \to (B, g_2)$ which maps g_1 into g_2 and preserves the labeling of edges.*

Then for a pair of pointed metric spaces $(T, e), (T', e) \in X$ we define

$$(2) \qquad d_X(T, T') = e^{-r(T, T')},$$

where

$$r(T, T') = \max\{r \in \mathbb{N} \cup \{0\} \mid \exists \, B_T(e, r) \text{ and } B_{T'}(e, r) \text{ are isomorphic}\}.$$

Using the box metric, one can prove [11, 20, 22] that (X, d_X) is totally disconnected, compact, and to derive a criterion for (X, d_X) to be a perfect space. In particular, if $G = F_n$, a free group on n generators with $n > 1$, then the space of pointed trees (X_n, d_X) is perfect and, therefore, is a Cantor set.

2.2.2. *Pseudogroup action on the space of pointed trees.* Recall [12] that a *pseudogroup of transformations on a space* Ω is a family of homeomorphisms from open sets to open sets in Ω closed under restriction to open sets, composition, inversion and extension (gluing finitely many homeomorphisms to form another homeomorphism). We now define a pseudogroup action on X.

Let $\mathcal{P}_e(T)$ be the set of paths $\delta : [0, 1] \to T$ such that $\delta(0) = e$, $\delta(1) = g \in V(T)$ and δ is the shortest path between e and g in T. It follows that δ does not have self-intersections, and the image of δ in T is the union of edges

$$w_{h_{i_1}} \cup w_{h_{i_2}} \cup \cdots \cup w_{h_{i_m}}, \text{ where } h_{i_k} \in G^{1+} \text{ for } 1 \leq k \leq m.$$

Thus δ defines a word $\tilde{h}_{i_1} \tilde{h}_{i_2} \cdots \tilde{h}_{i_m} \in F_n$, which is composed as follows. Denote by g_{k_1} and g_{k_2} vertices of $w_{h_{i_k}}$ in such a way that $d_T(e, g_{k_1}) < d_T(e, g_{k_2})$, that is, g_{k_1} is 'closer' to the identity than g_{k_2}. Then set

$$\begin{cases} \tilde{h}_{i_k} = h_{i_k}, & \text{if } g_{k_2} = g_{k_1} h_{i_k}, \\ \tilde{h}_{i_k} = h_{i_k}^{-1}, & \text{if } g_{k_2} = g_{k_1} h_{i_k}^{-1}. \end{cases}$$

Then $\tilde{h}_1 \tilde{h}_2 \cdots \tilde{h}_n$ gives a representation of an element in G in terms of the set of generators G^1, and each distinct path defines a distinct element. Therefore, the described procedure defines a well-defined map

$$p : \mathcal{P}_e(T) \to F_n : \delta \mapsto \tilde{h}_{i_1} \tilde{h}_{i_2} \cdots \tilde{h}_{i_m}.$$

Definition 2.5 ([22])**.** *Let $n < \infty$ be the cardinality of a set G^{1+} of generators of G, and (X, d_X) be the corresponding set of pointed trees. Let $g \in F_n$. Then*

    (1) *$(T, e) \cdot g$ is defined if and only if there exists a path $\delta \in \mathcal{P}_e(T)$ such that $p(\delta) = g$.*
    (2) *$(T', e) = (T, e) \cdot g$ if and only if (T, g) and (T', e) are isomorphic.*

Now for $r > 0$ denote by $D_X(T, r)$ an open and closed subset of diameter e^{-r} about (T, e), that is,

$$D_X(T, r) = \left\{ (T', e) \mid d_X(T, T') \leq e^{-r} \right\}.$$

For each $g \in F_n$ let $\ell_g = d_{\mathcal{F}_n}(e, g)$, and form the union of clopen subsets

$$(3) \qquad D = \bigcup \left\{ D_X(T, \ell_g) \mid T \in X, \ \delta \in \mathcal{P}_e(T) \text{ such that } p(\delta) = g \right\}.$$

The set D is clopen since G^1 is a finite set and so (3) is a finite union. The action of g is defined on D, so we can define a map

$$\tau_g : D \to X_n : (T, e) \mapsto (T, e) \cdot g$$

which is a homeomorphism onto its image. Then

$$\mathfrak{G} = \langle \tau_g \mid g \in F_n \rangle$$

is a pseudogroup of local homeomorphisms with a generating set $\mathfrak{G}^1 = \{ \tau_g \mid g \in G^1 \}$.

2.2.3. Suspension of the pseudogroup action on the space of pointed trees. The pseudogroup dynamical system $(X, \mathfrak{G})$ can be realised as the holonomy system of a smooth foliated space $\mathfrak{M}_G$ as in the following theorem.

Theorem 2.6 ([11, 20, 22])**.** *Let G be a finitely generated group, and (X, d_X) be the corresponding space of pointed trees with the action of a pseudogroup $\mathfrak{G}$. Then there exists a compact metric space $\mathfrak{M}_G$, and a finite smooth foliated atlas $\mathcal{V} = \{\phi_i : V_i \to U_i \times \mathfrak{X}_i\}_{1 \leq i \leq \nu}$, where $U_i \subset \mathbb{R}^2$ is open, with associated holonomy pseudogroup Γ, such that the following holds.*

    (1) *The leaves of $\mathfrak{M}_G$ are Riemann surfaces.*
    (2) *There is a homeomorphism onto its image*

$$t : X \to \bigcup_{1 \leq i \leq \nu} \mathfrak{X}_i,$$

    such that $t(X)$ is a complete transversal for the leaves of the foliation, and $\Gamma|_{\tau(X)} = t_* \mathfrak{G}$, where $t_* \mathfrak{G}$ is the pseudogroup induced on $t(X)$ by $\mathfrak{G}$.

We refer for a proof to [11, 20, 22], and only outline the idea of the construction here. Namely, X can be covered by a finite number of clopen sets $D_X(\mathbf{a}, 1)$ of radius 1, where $\mathbf{a}$ is a subgraph of a compact ball of radius 1 in $\mathcal{G}$. To each of these sets we associate a compact surface with boundary $\Sigma_{\mathbf{a}}$ which can be thought of as the two-dimensional boundary of the thickening of $\mathbf{a}$, and is homeomorphic to a 2-sphere with at most 4 open disks taken out. We then parametrise the surfaces near their boundaries, and obtain a foliated space

$$(4) \qquad \mathfrak{M}_G = \bigsqcup_{\mathbf{a} \in \mathbf{A}} D_X(\mathbf{a}, 1) \times \Sigma_{\mathbf{a}} / \sim$$

by imposing an appropriate equivalence relation on the disjoint union $\bigsqcup_{\mathbf{a} \in \mathbf{A}} D_X(\mathbf{a}, 1) \times \Sigma_{\mathbf{a}}$. Namely, this equivalence relation 'glues' the surfaces $\Sigma_{\mathbf{a}}$ along the parametrised regions near their boundaries. The resulting foliated space $\mathfrak{M}_G$ has Riemannian leaves and is modeled transversely on the totally disconnected space X.

Definition 2.7 ([22])**.** *Let G be a finitely generated group, and $\mathfrak{M}_G$ be a suspension of $(X, \mathfrak{G})$ as in Theorem 2.6. Then a* graph matchbox manifold *is the closure $\mathcal{M} = \overline{L}$ of a leaf L in $\mathfrak{M}_G$.*

2.2.4. *Properties of graph foliated spaces.* Some general properties of graph matchbox manifolds were shown in [22]. The first property we mention is the *universal property*, which allows to reduce the study of all variety of foliated spaces $\mathfrak{M}_G$ to the cases $G = F_n$, a free group on n generators. If $G = F_n$, we denote the space of pointed trees by X_n, the pseudogroup by $\mathfrak{G}_n$ and the corresponding suspension by $\mathfrak{M}_n$.

Proposition 2.8 ([22])**.** *Given a group G with a set of generators $G^1 = G^{1+} \cup G^{1-}$, $\mathrm{card}\, G^{1+} \leq n$, there exists a foliated embedding*

$$\Phi : \mathfrak{M}_G \to \mathfrak{M}_n,$$

where $\mathfrak{M}_G$ and $\mathfrak{M}_n$ are suspensions of $(X, \mathfrak{G})$ and $(X_n, \mathfrak{G}_n)$ as in Theorem 2.6.

2.2.5. *Bernoulli shifts versus graph matchbox manifolds.* As it has already been mentioned in Section 1, the question about the relation between transverse dynamics of the space of graph matchbox manifolds and Bernoulli shifts is a natural one, as the following properties of graph matchbox manifolds show. Recall that the definition of an ϵ-expansive dynamical system can be generalised for pseudogroup dynamical systems as follows.

Definition 2.9 ([16])**.** *The pseudogroup Γ on Ω is ϵ-expansive if for all $w \neq w' \in \Omega$ with $d(w, w') < \epsilon$ there exists $\gamma \in \Gamma$ with $w, w' \in \mathrm{dom}\,\gamma$ such that $d(\gamma(w), \gamma(w')) \geq \epsilon$.*

Bernoulli shifts $\Sigma(G, S)$ are expansive [10], and it was shown in [22] that $(X_n, \mathfrak{G}_n)$ is expansive. Besides, [22] proves that that the set of periodic orbits in $(X_n, \mathfrak{G}_n)$ (equivalently, the set of compact leaves in $\mathfrak{M}_n$) is dense, and there are points with dense orbits in $(X_n, \mathfrak{G}_n)$ (equivalently, dense leaves in $\mathfrak{M}_n$). Therefore, it is natural to study the relation between Bernoulli shifts (see [17] on dynamics of shifts) and pseudogroup dynamical systems $(X_n, \mathfrak{G}_n)$.

3. Embedding shift spaces into the space of pointed trees

We now give a proof of Theorem 1.1, that is, for a finitely generated group G, a finite set of symbols S and a given injective map $\alpha : G^{1+} \times S \to F_n^{1+}$, where G^{1+} denotes a 'positive' part of a symmetric set of generators G^1 of G, there exists an embedding with appropriate equivariance properties

$$\Phi_\alpha : \Sigma(G, S) \to X_n,$$

for n large enough.

Let $G = \langle\, G^1 \mid R \,\rangle$ be a presentation of G, where G^1 is a finite symmetric set of generators, and R is a set of relations, possibly infinite. Denote by M the cardinality of G^{1+} and let $f : F_M \to F_M/R \cong G$ be a homomorphism. Then there is an induced map $\hat{f} : \Sigma(G, S) \to \Sigma(F_M, S)$ between the corresponding shift spaces given by

$$\hat{f}(\sigma)(w) = \sigma(f(w)), \ w \in F_M.$$

The map $\hat{f}$ is continuous, indeed, for a finite subset $F \subset F_M$ and a $\sigma' \in \Sigma(F_M, S)$ we have

$$\hat{f}^{-1}(C_{F, \sigma'}) = \big\{\, \sigma \in \Sigma(G, S) \,\big|\, \sigma(f(w)) = \sigma'(w) \text{ for any } w \in F \,\big\}.$$

The map $\hat{f}$ is also injective: given two different elements $\sigma, \sigma' \in \Sigma(G, S)$ there is $g \in G$ such that $\sigma(g) \neq \sigma'(g)$, and therefore $\hat{f}(\sigma) \neq \hat{f}(\sigma')$ as they differ on the set $f^{-1}(g)$. Since

Bernoulli shift spaces are compact and Hausdorff, $\hat{f}$ is an embedding. Finally, since f is a homomorphism, $\hat{f}$ is equivariant, that is,

$$\left[\hat{f}(\sigma) \cdot \gamma\right](w) = \hat{f}(\sigma)(\gamma w) = \sigma(f(\gamma w)) = \left[\sigma \cdot f(\gamma)\right](f(w)) = \hat{f}[\sigma \cdot f(\gamma)](w)$$

for any $w \in F_M$. This argument shows that to prove Theorem 1.1, it is enough to show that the embedding exists for free groups. This embedding will depend on the choice of the map α, as in the proposition below.

Proposition 3.1. *Let F_M be a free group. Given an injective map $\alpha : F_M^{1+} \times S \to F_n^{1+}$, there exists an embedding*

$$\Phi_\alpha : \Sigma(F_M, S) \to X_n,$$

equivariant with respect to the actions of F_M and F_n. More precisely, for any $\sigma \in \Sigma(F_M, S)$ and any $h \in F_M^{1+}$ we have

$$\Phi_\alpha(\sigma \cdot h) = \Phi_\alpha(\sigma) \cdot \alpha(h, \sigma(e)),$$

and if $h^{-1} \in F_M^{1+}$ then

$$\Phi_\alpha(\sigma \cdot h) = \Phi_\alpha(\sigma) \cdot \left[\alpha(h^{-1}, \sigma(e))\right]^{-1}.$$

Proof. Given $\sigma \in \Sigma(F_M, S)$, we construct $\Phi_\alpha(\sigma) = T_\sigma$ as a subgraph of $\mathcal{F}_n$ by induction.

For $j \geq 1$, let F_M^j be the set of words in F_M of length j. Also set $F_M^0 = \{e\}$. The set of words of length less or equal to j is denoted by $F_M^{\leq j} = \bigcup_{0 \leq i \leq j} F_M^i$. Define $K_0 = \{e\} \subset \mathcal{F}_n$ to be a graph consisting of just one vertex e and no edges. Finally we define $\kappa_0 : F_M^{\leq 0} = F_M^0 \to V(K_0) \subset V(\mathcal{F}_n)$ in the only possible way.

We proceed by induction. We assume that we are given the following data:

(1) A connected subgraph $K_{j-1} \subset \mathcal{F}_n$ containing e.
(2) A bijective map $\kappa_{j-1} : F_M^{\leq j-1} \to V(K_{j-1})$ such that if $w_k \in F_M^{\leq j-1}$, $k = 1, 2$, and $w_2 = w_1 h$ for some $h \in F_M^{1+}$, then

$$\kappa_{j-1}(w_2) = \kappa_{j-1}(w_1)\alpha(h, \sigma_1(e)),$$

where $\sigma_k = \sigma \cdot w_k$, $k = 1, 2$.

We now construct a set $F_M^{\leq j}$, a graph K_j and a map $\kappa_j : F_M^{\leq j} \to V(K_j)$. For any $w' \in F_M^{\leq j-1}$ set $\kappa_j(w') = \kappa_{j-1}(w')$. Since F_M is a free group, given $w_2 \in F_M^j$ there is a unique $w_1 \in F_M^{j-1}$ such that $w_2 = w_1 h$, and either $h \in F_M^{1+}$ or $h^{-1} \in F_M^{1+}$. In the first case set

(5) $\kappa_j(w_2) = \kappa_{j-1}(w_1)\alpha(h, \sigma_1(e)),$

and otherwise set

(6) $\kappa_j(w_2) = \kappa_{j-1}(w_1)\left[\alpha(h^{-1}, \sigma_2(e))\right]^{-1},$

where, as before, the juxtaposition denotes group multiplication in F_n. Let K_j be a subgraph of $\mathcal{F}_n$ with the set of vertices $\{\kappa_j(F_M^{\leq j})\}$. Define $T_\sigma = \bigcup_{j=0}^\infty K_j$. We notice that the graph K_j is in fact a closed ball $B_{T_\sigma}(e, j)$ of radius j in K_j.

We show that Φ_α is continuous. As the involved spaces are metrizable it is enough to show sequential continuity. Let $\{\sigma_k\} \to \sigma$, then (passing to a subsequence if necessary) there exists an increasing sequence of integers $\{r_k\}$ such that

(7) $\sigma_\ell(w) = \sigma(w)$ for all words $w \in F_M^{\leq k}$ and all $\ell \geq r_k$.

Now, by construction each tree is a union of finite graphs, that is, $T_{\sigma_k} = \bigcup_{j \geq 0} K_j^k$, and $T_\sigma = \bigcup_{j \geq 0} K_j$. Then (7) implies that $K_k^\ell = K_k$ for all $\ell \geq r_k$. Therefore, $\{T_{\sigma_k}\}$ converges to T_σ, and Φ_α is continuous.

A continuous map between a compactum and a Hausdorff space is always closed, and therefore, to show that Φ_α is an embedding, it is enough to show that it is injective. If $\sigma \neq \sigma' \in \Sigma(F_M, S)$, there is $w \in F_M^{\leq j}$ for some $j \geq 0$ such that $\sigma(w) \neq \sigma'(w)$. The definition of κ_j in (5) and (6) forces $K_j \neq K_j'$, where $T_\sigma = \bigcup_{j \geq 0} K_j$ and $T_{\sigma'} = \bigcup_{j \geq 0} K_j'$. Hence $T_\sigma \neq T_{\sigma'}$.

Finally, we show the equivariance of the embedding. We consider only the case $h \in F_M^{1+}$, as the argument in the case $h^{-1} \in F_M^{1+}$ is similar. Denote $\sigma' = \sigma \cdot h$. By the definition of the F_M-action on $\Sigma(F_M, S)$ and by the construction there is map

$$V(T_{\sigma'}) \to V(T_\sigma) : g \mapsto g\alpha(h, \sigma'(h^{-1}))^{-1},$$

which means that $(T_{\sigma'}, e)$ is isomorphic to $(T_\sigma, \alpha(h, \sigma(e)))$. Then

$$T_{\sigma'} = T_\sigma \cdot \alpha(h, \sigma(e)).$$

$\square$

Remark 3.2. Given a graph $T_\sigma \in \text{range}(\Phi_\alpha)$, it is useful to know how to recover $\sigma \in \Sigma(F_M, S)$. Therefore, we outline here the construction of the inverse Φ_α^{-1}. We will denote by pr_i the ith coordinate projection.

Let $T \in \text{range}\,\Phi_\alpha$, then every vertex in T has degree $2M$, and the labels of edges in T are necessarily contained in $\text{range}\,\alpha \subset F_n^{1+}$. Moreover, the following can be deduced from the properties of free groups and the construction.

Let $g \in V(T)$ be a vertex, w_{h_1} and w_{h_2} be edges such that $\{g\} = V(w_{h_1}) \cap V(w_{h_2})$ and let $g_i \in V(w_{h_i}) \smallsetminus \{g\}$, $i = 1, 2$, be another vertex. Suppose $g_i = gh_i$, $i = 1, 2$. Then

$$(8) \qquad pr_2 \circ \alpha^{-1}(h_1) = pr_2 \circ \alpha^{-1}(h_2) = s \in S,$$

and there is precisely M edges with this property.

By construction $T = \bigcup_{j=1}^\infty K_j = B_T(e, j)$. We construct σ by induction.

Set $F_M^0 = \{e\} \in F_M$, then $\lambda_0 : V(K_0) = \{e\} \to F_M^0$ is just the map of the identities. Let $E \subset E(T)$ be the set of $2M$ edges adjacent to the identity e. For $w_h \in E$, let $g \in V(w_h) \smallsetminus \{e\}$, then either $g = h$ or $g = h^{-1}$ for $h \in F_n^{1+}$. In any case, $h \in \text{range}\,\alpha$, and there is a unique pair $(\gamma, s) = \alpha^{-1}(h)$. If $g = h^{-1}$, define $\lambda_1 : V(K_1) \to F_M^{1-}$ and $\sigma|_{F_M^{1-}} : F_M^{1-} \to S$ by

$$\lambda_1(g) = \gamma^{-1}, \text{ and } \sigma_1(\lambda_1(g)) = s.$$

In the case $g = h$ set $\sigma(e) = s$, and $\lambda_1(g) = \gamma$. The value of $\sigma(e)$ is well-defined by the property (8). To define σ on F_M^{1+} we have to look at edges in $B_T(e, 2) \smallsetminus B_T(e, 1)$.

For every $g \in V(K_1)$ such that $g = h \in F_n^{1+}$, let $E' \subset B_T(e, 2)$ be the set of edges adjacent to g. By the properties of T there exists $w_{h'}$ such that for $g' \in V(w_{h'}) \smallsetminus \{g\}$ we have $g' = gh'$. Let $(\gamma', s') = \alpha^{-1}(h')$, and set $\sigma(g) = s'$. Then $\sigma(g)$ is well-defined by the property (8).

To obtain the full map σ, proceed inductively. Given an injective map $\lambda_{j-1} : V(K_{j-1}) \to F_M^{\leq j-1}$, and $g \in V(K_{j-1}) \cap F_n^{j-1}$, consider the set of edges adjacent to g which do not lie in K_{j-1}. For such an edge w_h let g' be the vertex in $V(w_h) \smallsetminus \{g\}$, and if $g' = gh^{-1}$, define $\lambda_j(g') = \lambda_{j-1}(g)\left[pr_1 \circ \alpha^{-1}(h)\right]^{-1}$ and $\sigma(\lambda_j(g')) = pr_2 \circ \alpha^{-1}(h)$. If $g' = gh$, set $\lambda_j(g') = \lambda_{j-1}(g)\left[pr_1 \circ \alpha^{-1}(h)\right]$, and to determine $\sigma(\lambda_j(g'))$ consider the set of edges adjacent to
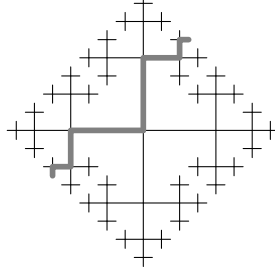
FIGURE 2. The tree T_σ

g'. Like at the first step of the inductive procedure, there is an edge $w_{h'}$ such that for $g'' = V(w_{h'}) \smallsetminus \{g'\}$ we have $g'' = g'h'$. Then let $\sigma(\lambda_j(g')) = pr_2 \circ \alpha^{-1}(h')$.

Remark 3.3. Given a set of generators G^{1+}, the tree T_σ is the universal covering space of the graph of the orbit of σ in $\Sigma(G, S)$.

Example 3.4. Let $G = \mathbb{Z}$ and $S = \{0, 1\}$, then $\Sigma(\mathbb{Z}, S)$ is the set of bi-infinite sequences of 0 and 1. The positive generating set of $\mathbb{Z}$ is $\{1\}$. Consider the free group with 2 generators F_2 with positive generating set $\{a, b\}$. Define

$$\alpha : \{1\} \times \{0, 1\} \to \{a, b\}$$

by $\alpha(1, 0) = a$ and $\alpha(1, 1) = b$.

Let $\sigma : \mathbb{Z} \to S$ be given by $\sigma(k) = 0$ if k is even and $\sigma(k) = 1$ if k is odd. Then the orbit of σ in $\Sigma(\mathbb{Z}, S)$ contain just two points, σ itself and $\sigma' = \sigma + 1$. One thinks of graphs $T_\sigma, T_{\sigma'} \subset \mathcal{F}_2$ as 'ladders' made up of a and b-edges. Each graph is a covering space of the graph of the orbit of T_σ in X_2, which consists of two vertices T_σ and $T_{\sigma'}$ joined by two edges, one marked by a and another one marked by b.

Remark 3.5. The embedding of Proposition 3.1 is highly non-unique. Given a symmetric set of generators F_M^1 and a set S, the embedding depends on the choice of $n \geq M$ card S, of a 'positive' set F_M^{1+} and of the map $\alpha : F_M^{1+} \times S \to F_n^{1+}$. Moreover, the embedding of $\Sigma(G, S)$ in Theorem 1.1 depends on the choice of the homomorphism $f : F_n \to G$.

Remark 3.6. Geometrically non-trivial relations in $G = \langle G^1 \mid R \rangle$ manifest themselves as symmetries of graphs T_σ. For example, if $G = \mathbb{Z}^2$, then for every $g \in F_2$ and every $\sigma \in \Sigma(F_M, S)$ we have $\sigma(g) = \sigma(aba^{-1}b^{-1}g)$, and for every $R > 0$ the ball $B_{T_\sigma}(e, R)$ is isometric to the ball $B_{T_\sigma}(aba^{-1}b^{-1}, R)$, and similarly for other elements in $\mathbb{Z}^2$ congruent to e.

4. UNIVERSAL SPACE FOR COMPACTLY GENERATED PSEUDOGROUPS

In order to prove Theorem 1.2, we adapt the proofs of [10, Propositions 2.6, 2.8] and Proposition 3.1. We first recall some background facts.

In [12, 13] A. Haefliger showed that the holonomy pseudogroup of a compact foliated manifold satisfies a finiteness condition, called *compact generation*.

Definition 4.1. *A pseudogroup of transformations Γ of a space Ω is compactly generated if Ω contains a relatively compact open set Y meeting all orbits of Γ, and the reduced pseudogroup $\Gamma|_Y$ is generated by a finite set Λ_Y of elements of Γ such that each element $\lambda \in \Lambda_Y$ is the restriction of an element $\overline{\lambda} \in \Gamma$ with the closure of $\operatorname{dom} \lambda$ contained in $\operatorname{dom} \overline{\lambda}$. Such generating system Λ_Y is called a* compact generation system (CGS) *of Γ on Y.*

In the totally disconnected case, it is possible to chose a CGS with compact domains, as the following lemma shows.

Lemma 4.2. *Let Γ be a pseudogroup of transformations of a 0-dimensional locally compact metrizable space Ω. If Γ is compactly generated there exists a CGS Λ_Y on a compact and open set $Y \subset \Omega$, where $\mathrm{dom}\,\lambda$ is compact and open for each $\lambda \in \Lambda_Y$.*

Proof. By definition there exists a CGS Λ_Z of Γ on an open relatively compact set Z which meets all Γ-orbits. For each $\lambda \in \Lambda_Z$, $\overline{\mathrm{dom}\,\lambda}$ is compact and contained in the open set $\mathrm{dom}\,\overline{\lambda}$ for some $\overline{\lambda} \in \Gamma$. By the hypothesis, the space Ω has a countable basis of compact and open sets, and so we can cover $\overline{\mathrm{dom}\,\lambda}$ with a finite number of compact open sets contained in $\mathrm{dom}\,\overline{\lambda}$. Then their union D_λ is again compact and open. We then define $\lambda' = \overline{\lambda}|_{D_\lambda}$ which satisfies $\lambda = \lambda'|_{\mathrm{dom}\,\lambda}$.

Since Λ_Z is finite, the set $Y = \bigcup_{\lambda \in \Lambda_Z} D_\lambda$ is compact and open. Since $Z \subset Y$, Y meets all Γ-orbits. But in general the finite family $\{\,\lambda' \mid \lambda \in \Lambda_Z\,\}$ is not a generating set for $\Gamma|_Y$, which we can amend for by adding a finite number of maps in the following way. Since Z meets every orbit, for each $y \in Y \smallsetminus Z$ we can find $\gamma_y \in \Gamma$ such that its domain is compact, open and contained in Y and its range is contained in Z. Since $Y \smallsetminus Z$ is compact, there is a finite family of maps $\{\gamma_i\}_{i \in I}$ such that the union of their domains covers $Y \smallsetminus Z$.

Finally, we claim that $\Lambda_Y = \{\,\lambda', \gamma_i \mid \lambda \in \Lambda_Z, i \in I\,\}$ is CGS of Γ on Y. To see that, let $\gamma \in \Gamma|_Z$. We claim that γ locally belongs to $\langle \Lambda_Y \rangle$ and therefore it belongs $\langle \Lambda_Y \rangle$ globally by properties of pseudogroups. Indeed, if there is $z \in Z$ such that $\gamma(z) \in Z$ then this just follows from the hypothesis. If $z \in Z$ and $\gamma(z) = y \in Y \smallsetminus Z$, then there exists γ_i such that $\gamma_i(y) \in Z$ and locally $\gamma_i \circ \gamma \in \Gamma|_Z$ can be written as a Λ_Z-word. Then $\gamma = \gamma_i^{-1} \circ w$ which is a Λ_Y-word. The statement holds in the two other cases by a similar argument. $\square$

Recall from definition 2.9 (see also [16]) that a pseudogroup Γ on a Polish space Ω is ϵ-*expansive* if for all $w \neq w' \in \Omega$ with $d(w, w') < \epsilon$ there exists $\gamma \in \Gamma$ with $w, w' \in \mathrm{dom}\,h$ such that $d(\gamma(w), \gamma(w')) \geq \epsilon$, where d is a metric on Ω. We now prove Theorem 1.2, which shows that if Ω is compact, and Γ is compactly generated and ϵ-expansive, then the pseudogroup dynamical system (Ω, Γ) can be equivariantly embedded into the space of pointed trees X_n, for n large enough. Together with Theorem 1.1, this result can be seen as a generalization of the well-known fact that every expansive dynamical system on a totally disconnected compact space is conjugate to a subshift [10]. The proof of Theorem 1.2 below uses the ideas of [10, Propositions 2.6, 2.8] in conjunction with the method of Proposition 3.1.

Proof. (of Theorem 1.2). Let Γ be a compactly generated pseudogroup on a compact metrizable 0-dimensional space Ω with the CGS (Z, Λ_Z). By Lemma 4.2 we can construct a CGS (Y, Λ_Y) over a clopen set Y such that the domains of the elements of Λ_Y are compact, and we can use the same method to find a system of generators Λ_Ω with the same properties on Ω. We can also assume that Λ_Ω is a symmetric set of generators, and denote by Λ_Ω^+ its 'positive' part.

By assumption Γ is ϵ-expansive, and since Ω is compact, we can find a finite partition $\{B_s\}_{s \in S}$ of Ω by clopen pairwise disjoint sets of diameter smaller than ϵ. Let $m = \mathrm{card}\,S$. We now form the set $\mathbb{F}$ of all formal finite words of elements Λ_Ω, that is, $\mathbb{F}$ is a free group with $M = \mathrm{card}\,\Lambda_\Omega^+$ generators. Then, given a word $g \in \mathbb{F}$, there is a pair (θ_g, γ_g), where γ_g is a homeomorphism obtained as the composition of generators in g, and θ_g is the clopen domain of γ_g. Such compositions are not always defined, and we allow θ_g to be an empty set. This is the main difference of the considered situation from that in [10, Propositions

2.6, 2.8]. Let
$$\widetilde{S} = S \cup \{s_\emptyset\},$$
where $s_\emptyset$ would code the empty set, and let $\Sigma(\mathbb{F}, \widetilde{S}) = \{\sigma : \mathbb{F} \to \widetilde{S}\}$ be the shift space. Define
$$I(\sigma) = \bigcap_{g \in \mathbb{F}, \theta_g \neq \emptyset} \left\{ \gamma_g^{-1} \left(B_{\sigma(g)} \cap \gamma_g(\theta_g) \right) \right\},$$
and let $\Psi \subset \Sigma(\mathbb{F}, \widetilde{S})$ be the subset containing those σ for which $I(\sigma)$ is non-empty. Note that an arbitrarily large number of elements in $\mathbb{F}$ can be mapped by such a σ to $s_\emptyset$, which means that points in $I(\sigma)$ are not in the domain of homeomorphisms corresponding to these elements. If $\theta_g = \emptyset$, then for every gh, $h \in \mathbb{F}$, we have $\theta_{gh} = \emptyset$, and so $\sigma(gh) = s_\emptyset$.

By a similar argument as for dynamical systems, ϵ-expansivity of Γ and the choice of the covering $\{B_s\}_{s \in S}$ implies that each $I(\sigma)$ is a single point. More precisely, let $x, y \in I(\sigma)$. Then for any $g \in \mathbb{F}$ with $\theta_g \neq \emptyset$ we have $\gamma_g(x), \gamma_g(y) \in B_{\sigma(g)}$, which implies that $d_\Omega(\gamma_g(x), \gamma_g(y)) < \epsilon$ for all $g \in \mathbb{F}$, and so $x = y$. It follows that the map
$$\pi : \Psi \to \Omega : \sigma \mapsto I(\sigma)$$
is well-defined and injective. However, Ψ is not invariant under the action of $\mathbb{F}$. Indeed, it follows from definitions that if $\sigma(g) \neq s_\emptyset$, then $I(\sigma \cdot g) = \gamma_g(I(\sigma))$ and therefore $I(\sigma \cdot g) \in \Psi$. However, if $\sigma(g) = s_\emptyset$, then there exists $\sigma \cdot g \in \Sigma(\mathbb{F}, \widetilde{S})$, while $\gamma_g(I(\sigma))$ is not defined. We therefore define a *partial action* of $\mathbb{F}$ on Ψ by allowing $g \in \mathbb{F}$ to act on σ if and only if $\sigma(g) \neq s_\emptyset$. Then the map π is equivariant with respect to this partial action of $\mathbb{F}$ on Ψ and the Γ-action on Ω.

We have to show that π is continuous and surjective. Let $\{\sigma_n\} \in \Psi$ be a sequence converging to $\sigma \in \Psi$. Then for every $g \in \mathbb{F}$ we can find an integer n_g such that for all $n > n_g$ we have $\sigma_n(g) = \sigma(g)$, that is, $\sigma_n(g)$ and $\sigma(g)$ are in the same element of the partition. Since Ω is compact, by extracting a subsequence, we can assume that $x_n = \pi(\sigma_n)$ converges to an element $x \in \Omega$. If $\sigma(g) \neq s_\emptyset$, then for $n > n_g$ we have $x_n \in \gamma_g^{-1}(B_{\sigma(g)})$, and $x \in \gamma_g^{-1}(B_{\sigma(g)})$ since $\gamma_g^{-1}(B_{\sigma(g)})$ is closed. If $\sigma(g) = s_\emptyset$, then for $n > n_g$ we have $\sigma_n(g) = s_\emptyset$, that is, $x_n \notin \operatorname{dom} \gamma_g$. Since domains of homeomorphisms in Γ are open, their complements are closed and so $x \notin \operatorname{dom} \gamma_g$. It follows that $x \in I(\sigma)$, and π is continuous. Surjectivity follows from the fact that the domains of pseudogroup generators and their inverses form a clopen covering of Ω. Therefore π is a homeomorphism.

We now apply a procedure similar to the one in Theorem 1.1 to embed Ψ into X_n for a large enough n. The difference with Theorem 1.1 is that if $\sigma(g) = s_\emptyset$ we do not allow g to act on σ, and therefore we do not add any edge to the graph. As before, we assume that a injective map $\alpha : \Lambda_\Omega^+ \times S \to F_n^{1+}$ has been chosen, for $n \geq Mm$.

We fix $\sigma \in \Psi$ and construct T_σ as a subgraph of $\mathcal{F}_n$ as follows. We proceed by induction. Let $\{e\} = K_0 \subset T_\sigma$ be a subgraph with a single vertex. As before, $\mathbb{F}^j$ and $\mathbb{F}^{\leq j}$ represent the set of words of length j and words of length less or equal to j in $\mathbb{F}$. We also define
$$\widetilde{\mathbb{F}}^j = \left\{ g \in \mathbb{F}^j \mid \sigma(g) \neq s_\emptyset \right\},$$
the set $\widetilde{\mathbb{F}}^{\leq j}$ is defined in a similar way. We assume that we are given the following data:

(1) A connected subgraph $K_{j-1} \subset \mathcal{F}_n$ containing e.
(2) A bijective map $\kappa_{j-1} : \widetilde{\mathbb{F}}^{\leq j-1} \to V(K_{j-1})$ on a subset $\widetilde{\mathbb{F}}^{\leq j-1} \subset \mathbb{F}^{\leq j-1}$, such that if $g_k \in \widetilde{\mathbb{F}}^{\leq j-1}$, $k = 1, 2$, and $g_2 = g_1 h$ for some $h \in \Lambda_\Omega$, then
$$\kappa_{j-1}(g_2) = \kappa_{j-1}(g_1)\alpha(h, \sigma_1(e)),$$

where $\sigma_k = \sigma \cdot g_k$.

Now we construct the graph K_j and the map $\kappa_j : \widetilde{\mathbb{F}}^{\leq j} \to V(K_j)$. For any $g' \in \widetilde{\mathbb{F}}^{j-1}$ set $\kappa_j(g') = \kappa_{j-1}(g')$. Since $\mathbb{F}$ is a free group, given $g_2 \in \widetilde{\mathbb{F}}^j$ there is a unique $g_1 \in \widetilde{\mathbb{F}}^{j-1}$ such that $g_2 = g_1 \cdot h$, and either $h \in \Lambda_\Omega^+$ or $h^{-1} \in \Lambda_\Omega^+$. In the first case set

$$\kappa_j(\sigma, g_2) = \kappa_{j-1}(\sigma, g_1)\alpha(h, \sigma_1(e)),$$

and otherwise set

$$\kappa_j(\sigma, g_2) = \kappa_{j-1}(\sigma, g_1)\big[\alpha(h^{-1}, \sigma_2(e))\big]^{-1}.$$

Let K_j be a subgraph of $\mathcal{F}_n$ with the set of vertices $\kappa_j(\widetilde{\mathbb{F}}^{\leq j})$. Define $T_\sigma = \bigcup_{j=0}^\infty K_j$. The rest of the proof proceeds similarly to that of Theorem 1.1. □

Remark 4.3. The main difference in the geometry of graphs in the images of embeddings of Bernoulli shifts, and embeddings of expansive compactly generated pseudogroups is that in the former case vertices of the graphs have constant degree, that is, the number of edges starting and ending at a vertex in T_σ is the same for every vertex. In the latter it may vary, with the highest degree always $\mathrm{card}(\Lambda_\Omega^+)$.

5. Bernoulli shifts on semigroups

In this section we explore the case of Bernoulli shifts on semigroups. Our constructions can be applied to the Bernoulli shift on $\mathbb{N}_0$, just because its dynamics is given by a pseudogroup of transformations generated by finitely many transformations of compact domain: if we fix $\Sigma = \Sigma(\mathbb{N}_0, S)$, we have the compact cylinders $C_s = \{\,\sigma \in \Sigma \mid \sigma(0) = s\,\}$ and the maps

$$1_s : C_s \to \Sigma : \sigma \mapsto \sigma \cdot 1,$$

that is the action of $1 \in \mathbb{N}_0$ restricted to C_s, with $s \in S$. It is straightforward to show $\{C_s\}_{s \in S}$ is a partition of Σ and the pseudogroup associated to the action of $\mathbb{N}_0$ on Σ is generated by the family $\{1_s\}_{s \in S}$. Therefore it is possible apply the Theorem 1.2 and equivariantly embed Σ in the space of pointed trees. But it is not always possible to extend those arguments to other semigroups.

The following argument gives a family of semigroups which cannot be equivariantly embedded in any foliated space, as their dynamics is not given by a pseudogroup of local homeomorphisms. This class of semigroups contains, for instance $\mathbb{N}_0 \times \mathbb{N}_0$.

Lemma 5.1. *Let G be a semigroup generated by a finite family of elements G^1, and fix a finite set of symbols S of cardinality greater than 1. Then if there exists a and $b \in G$ such that $\langle a \rangle \cong \mathbb{N}_0$ and $a^k \neq bw$ for any $w \in G$ and $k \in \mathbb{N}_0$, then the action of G on $\Sigma(G, S)$ cannot be conjugate to the pseudogroup of local homeomorphisms.*

Proof. Suppose there exist $a, b \in G$ as in the hypothesis. Fix an element $\sigma \in \Sigma(G, S)$ and a symbol $s_0 \in S$. Recall that the set $\Sigma(\langle a \rangle, S)$ contains an uncountable number of maps. Now for each $\varsigma \in \Sigma(\langle a \rangle, S)$ we will construct σ_ς such that $\sigma_\varsigma \cdot b = \sigma$. For that write

$$\sigma_\varsigma(w) = \begin{cases} \varsigma(w), & \text{if } w = a^k \text{ for some } k \in \mathbb{N}, \\ \sigma(w^\flat), & \text{if } w = bw^\flat \text{ for some } w^\flat \in G, \\ s_0, & \text{in other case.} \end{cases}$$

As $a^k \neq bw$ for any $w \in G$ and $k \in \mathbb{N}_0$, the map $\sigma_\varsigma : G \to S$ is well defined. Now, by definition

$$\sigma_\varsigma(w) \cdot b = \sigma_\varsigma(bw) = \sigma(w).$$

Clearly if $\varsigma \neq \varsigma'$, then $\sigma_\varsigma \neq \sigma_{\varsigma'}$, i.e. all σ_ς are distinct. Therefore, there are uncountably many elements of $\Sigma(G, S)$ which are mapped to σ by $b \in G$. Since every T_σ contains a countable number of vertices, the dynamics on the shift space $\Sigma(G, S)$ cannot be conjugate to the dynamics of $(X_n, \mathfrak{G}_n)$. The same holds for any open set in $\Sigma(G, S)$, as fixing the images of finitely many a^k does not break uncountability of $b^{-1}(\sigma)$. Thus the map associated to the action of b is not locally injective and cannot belong (even locally) to a pseudogroup of local homeomorphisms. $\square$

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