

# REPRESENTING KNOTS BY FILLING DEHN SPHERES

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ABSTRACT. We prove that any knot or link in any 3-manifold can be nicely decomposed (*splitted*) by a filling Dehn sphere. This has interesting consequences in the study of branched coverings over knots and links. We give an algorithm for computing Johansson diagrams of filling Dehn surfaces out from coverings of 3-manifolds branched over knots or links.

## 1. INTRODUCTION

Through the whole paper all 3-manifolds are assumed to be orientable and closed, that is, compact connected and without boundary. On the contrary, surfaces are assumed to be orientable, compact and without boundary, but they could be non-connected.  $M$  generically denotes a 3-manifold and  $S$  a surface. Although all the constructions can be adapted to work in the topological or PL categories, we work in the smooth category: manifolds have a differentiable structure and all maps are assumed to be smooth.

Let  $M$  be a 3-manifold.

A *Dehn surface* in  $M$  [14] is a surface (the *domain* of  $\Sigma$ ) immersed in  $M$  in general position: with only double curve and triple point singularities. The Dehn surface  $\Sigma \subset M$  *fills*  $M$  [12] if it defines a cell-decomposition of  $M$  in which the 0-skeleton is the set  $T(\Sigma)$  of triple points of  $\Sigma$ , the 1-skeleton is the set  $S(\Sigma)$  of double and triple points of  $\Sigma$ , and the 2-skeleton is  $\Sigma$  itself (the notation  $T(\Sigma)$ ,  $S(\Sigma)$  is similar to that introduced in [15]). If  $\Sigma$  is a Dehn surface in  $M$ , a connected component of  $S(\Sigma) - T(\Sigma)$  is an *edge* of  $\Sigma$ , a connected component of  $\Sigma - S(\Sigma)$  is a *face* of  $\Sigma$ , and a connected component of  $M - \Sigma$  is a *region* of  $\Sigma$ . The Dehn surface  $\Sigma$  fills  $M$  if and only if all its edges, faces and regions are open 1, 2 or 3-dimensional disks respectively.

Following ideas of [6], in [12] it is proved that every 3-manifold has a filling Dehn sphere (see also [16], and specially [1] where an extremely short and elegant proof of this result can be found), and filling Dehn spheres and their Johansson diagrams are proposed as a suitable way for representing all 3-manifolds. A weaker version of filling Dehn surfaces are the *quasi-filling Dehn surfaces* defined in [2], which are Dehn surfaces whose complementary set in  $M$  is a disjoint union of open 3-balls. In [5] it is proved that every 3-manifold has a quasi-filling Dehn sphere. Since [12], some other papers have been appeared on this subject (cf. [1, 2, 9, 10, 16, 17, 19]), applying filling Dehn surfaces to different aspects of 3-manifold topology. This is the first of a series of papers where we will give some tools for applying filling Dehn surfaces to knot theory.

Let  $K$  be a tame knot or link in  $M$ .

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2000 *Mathematics Subject Classification*. Primary 57N10, 57N35.

*Key words and phrases*. 3-manifold, knots, links, immersed surface, filling Dehn surface, branched coverings, Johansson diagrams.

Both authors have been supported by the European Social Fund and Diputación General de Aragón (Grant E15 Geometría), and by Spanish Government's *Programa Estatal de Fomento de la Investigación Científica y Técnica de Excelencia* (research projects MTM2013-46337-C2 and MTM2013-45710-C2 for the first and second author respectively) .

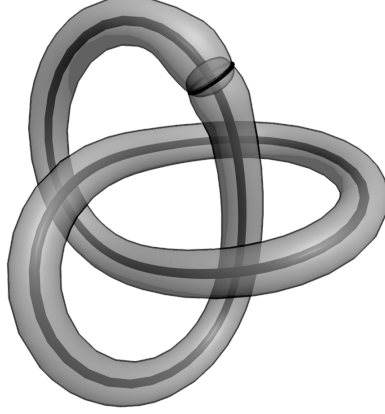


FIG. 1. A Dehn sphere containing the trefoil knot

**Definition 1.1.** The filling Dehn surface  $\Sigma$  of  $M$  *splits*  $K$  if:

- (1)  $K$  intersects  $\Sigma$  transversely in a finite set of non-singular points of  $\Sigma$ ;
- (2)  $K - \Sigma$  is a disjoint union of open arcs;
- (3) for each region  $R$  of  $\Sigma$ , if the intersection  $R \cap K$  is non-empty it is exactly one arc, unknotted in  $R$ ;
- (4) for each face  $F$  of  $\Sigma$ , the intersection  $F \cap K$  contains at most one point.

**Theorem 1.2.** *Every link  $K$  in a 3-manifold  $M$  can be splitted by a filling Dehn sphere of  $M$ .*

*Moreover, this filling Dehn sphere can be chosen such that it intersects exactly twice each connected component of  $K$ .*

*Proof.* The proof is a direct consequence of the *inflating of triangulations*, construction introduced in [11, 17, 18].

Take a smooth triangulation  $T$  of  $M$  [20] such that  $K$  is simplicial in  $T$ . This can be done because  $K$  is tame. In [11, 17, 18] it is shown how we can “inflate” the 0,1 and 2-dimensional simplices of  $T$  in order to obtain a filling collection of spheres in  $M$ , and how we can connect them until we obtain a unique filling Dehn sphere  $\Sigma_T$  of  $M$ . By construction it is easy to check that  $\Sigma_T$  splits  $K$ . In fact  $\Sigma_T$  splits every link which is a subcomplex of  $T$ .

For the second statement, we modify  $\Sigma_T$  slightly around each connected component  $K_i$  of  $K$ . Instead of introducing one sphere for each vertex and for each edge of the triangulation  $T$  belonging to  $K_i$ , we take a unique Dehn sphere containing  $K_i$  as in Figure 1, while the rest of  $\Sigma_T$  remains the same. With this modification,  $\Sigma_T$  intersects  $K_i$  twice. By repeating the same operation around each connected component of  $K$  we get the desired result.  $\square$

If the filling Dehn surface  $\Sigma$  splits  $K$ , when we draw the points  $P_1, P_2, \dots, P_k$  of  $\Sigma \cap K$  on the domain  $S$  of  $\Sigma$  together with the Johansson diagram  $\mathcal{D}$  of  $\Sigma$ , we obtain the *Johansson diagram*  $(\mathcal{D}, \{P_1, P_2, \dots, P_k\})$  of the knot or link  $K$ . Because the manifold  $M$  can be built up from  $\mathcal{D}$  [11, 18] and the arcs of  $\Sigma - K$  are unknotted, we have:

**Proposition 1.3.** *It is possible to recover the pair  $(M, K)$  from the Johansson diagram  $(\mathcal{D}, \{P_1, P_2, \dots, P_k\})$  of the knot  $K$ .*

If  $K$  is a knot, we will say that a filling Dehn sphere  $\Sigma$  that splits  $K$  is *diametral* for  $K$  or that *diametrically splits*  $K$  if it intersects  $K$  exactly twice. In this case,

if  $\mathcal{D}$  is the Johansson diagram of  $\Sigma$  and  $\Sigma \cap K = \{P_1, P_2\}$ , the pair  $(\mathcal{D}, \{P_1, P_2\})$  is a *diametral Johansson diagram* of the knot  $K$ . As it will be shown in Section 4, diametral filling Dehn spheres are specially useful for studying the branched covers of  $M$  with branching set  $K$ . For a given knot  $K$ , it should be interesting to have an algorithm that provides its simplest diametrically splitting filling Dehn sphere. Of course, before that we would have to define “simplest”, perhaps in terms of the number of triple points and/or double curves of the filling Dehn sphere.

Algorithms for computing the Johansson diagram of a filling Dehn sphere of  $M$  from a Heegard diagram of  $M$  are provided in [12, 16]. In Section 3 we outline an algorithm for computing such Johansson diagrams from branched covers of  $M$  with branching set a tame knot or link  $K$  in  $M$ . As an application of this algorithm, we apply it in Section 4 to the coverings of  $S^3$  branched over the unknot.

## 2. DEHN SURFACES AND THEIR JOHANSSON’S DIAGRAMS

We refer to [17, 9, 10] for more detailed definitions about Dehn surfaces and Johansson diagrams, and to [4, 13] as a basic reference about branched coverings.

A subset  $\Sigma \subset M$  is a Dehn surface in  $M$  if there exists a surface  $S$  and a general position immersion  $f : S \rightarrow M$  such that  $\Sigma = f(S)$ . In this situation we say that  $S$  is the *domain* of  $\Sigma$  and that  $f$  *parametrizes*  $\Sigma$ . If  $S$  is a 2-sphere, a torus, ... then  $\Sigma$  is a *Dehn sphere*, a *Dehn torus*, ... respectively.

Let  $\Sigma$  be a Dehn surface in  $M$  and consider a parametrization  $f : S \rightarrow M$  of  $\Sigma$ . The preimage under  $f$  in  $S$  of the singularity set  $S(\Sigma)$  of  $\Sigma$ , together with the information about how its points become identified by  $f$  in  $\Sigma$  is the *Johansson diagram*  $\mathcal{D}$  of  $\Sigma$  (see [7, 12]).

Because  $S$  is compact and without boundary, double curves are closed and there is a finite number of them. The number of triple points is also finite. Because  $S$  and  $M$  are orientable, the preimage under  $f$  of a double curve of  $\Sigma$  is the union of two different closed curves in  $S$ , and we will say that these two curves are *sister curves* of  $\mathcal{D}$ . Thus, the Johansson diagram of  $\Sigma$  is composed by an even number of different closed curves in  $S$ , and we will identify  $\mathcal{D}$  with the set of different curves that compose it. For any curve  $\alpha \in \mathcal{D}$  we denote by  $\tau\alpha$  the sister curve of  $\alpha$  in  $\mathcal{D}$ . This defines a free involution  $\tau : \mathcal{D} \rightarrow \mathcal{D}$ , the *sistering* of  $\mathcal{D}$ , that sends each curve of  $\mathcal{D}$  into its sister curve of  $\mathcal{D}$ .

If we are given an *abstract diagram*, i.e., an even collection of curves in  $S$  coherently identified in pairs, it is possible to know if this abstract diagram is *realizable*, that is, if it is actually the Johansson diagram of a Dehn surface in a 3-manifold (see [7, 8, 18]). It is also possible to know if the abstract diagram is *filling*: if it is the Johansson diagram of a filling Dehn surface of a 3-manifold (see [18]). If  $\Sigma$  fills  $M$ , it is possible to build  $M$  out of the Johansson diagram of  $\Sigma$ . Thus, filling Johansson diagrams represent all closed, orientable 3-manifolds. When a diagram  $\mathcal{D}$  in  $S$  is not realizable, the quotient space of  $S$  under the equivalence relation defined by the diagram is something very close to a Dehn surface: it is a 2-dimensional complex with simple, double and triple points, but it cannot be embedded in any 3-manifold. We reserve the name *pseudo Dehn surface* for these objects. Many constructions about Dehn surfaces, as the presentation of their fundamental group given in [10], for example, are also valid for pseudo Dehn surfaces.

A special case of filling Dehn surfaces is when the domain  $S$  of the filling Dehn surface  $\Sigma$  is a disjoint union of 2-spheres. In this case, we say that  $\Sigma$  is a *filling collection of spheres*. Starting from a filling collection of surfaces  $\Sigma$  in  $M$  we can always reduce the number of connected components of the domain  $S$  of  $\Sigma$  without losing the filling property by applying surgery modifications (see [3, 18, 11]).

Eventually, starting from a filling collection of surfaces in  $M$  we can always obtain a filling Dehn sphere of  $M$ .

### 3. SPLITTED KNOTS AND BRANCHED COVERINGS

Let  $K$  be a tame knot or link in  $M$ .

Assume that the filling Dehn surface  $\Sigma$  of  $M$  splits  $K$ . Let  $R_1, R_2, \dots, R_m$  be all the different regions of  $\Sigma$  whose intersection with  $K$  is empty, and take one point  $Q_i$  in  $R_i$  for  $i = 1, 2, \dots, m$ . Then,  $\Sigma - K$  is a strong deformation retract of

$$M - (K \cup Q_1, Q_2, \dots, Q_m),$$

and therefore,

**Proposition 3.1.** *The fundamental group of  $M - K$  is isomorphic to the fundamental group of  $\Sigma - K$ .*  $\square$

Our main interest on knots splitted by filling Dehn surfaces is due to the following theorem.

**Theorem 3.2.** *If  $p : \hat{M} \rightarrow M$  is a branched covering with downstairs branching set  $K$ , then  $\hat{\Sigma} = p^{-1}(\Sigma)$  is a filling Dehn surface of  $\hat{M}$ .*

*Proof.* As  $\Sigma \cap K$  contains no singular point of  $\Sigma$ ,  $\hat{\Sigma}$  is a Dehn surface in  $\hat{M}$  and all the edges of  $\hat{\Sigma}$  are open 1-arcs.

If a face  $\hat{F}$  of  $\hat{\Sigma}$  does not intersect the lift  $\hat{K} := p^{-1}(K)$  of the link  $K$ , then  $\hat{F}$  is a regular covering of a face of  $\Sigma$ . In the other case,  $\hat{F}$  is a branched covering of a face  $F$  with branching set the unique point of  $F \cap K$ . In both cases  $\hat{F}$  must be an open 2-disk. In the same way, a region  $\hat{R}$  of  $\hat{\Sigma}$  is a regular covering space of a region of  $\Sigma$ , if  $\hat{R}$  does not intersect  $\hat{K}$ , or it is a branched covering space of a region  $R$  of  $\Sigma$  whose branching set in  $R$  is the unknotted arc  $R \cap K$ . In both cases,  $\hat{R}$  must be an open 3-ball. This implies that  $\hat{\Sigma}$  fills  $\hat{M}$ .  $\square$

In the previous theorem the filling Dehn surfaces  $\Sigma$  and  $\hat{\Sigma}$  could have different domains. Let  $S$  be the domain of  $\Sigma$ , let  $f : S \rightarrow M$  be a parametrization of  $\Sigma$  and let  $f^{-1}(K)$  be the set of points  $\{P_1, P_2, \dots, P_k\}$ . We will denote by  $M_K$  and  $S_K$  the sets

$$M - K \quad \text{and} \quad S - \{P_1, P_2, \dots, P_k\},$$

respectively. Take a non-singular point  $x$  of  $\Sigma$  as the base point of the fundamental group  $\pi_K := \pi_1(M_K, x)$  of  $M_K$ . We denote also by  $x$  the preimage of  $x$  under  $f$ , and we choose it as base point of the fundamental group  $\pi_{S_K} := \pi_1(S_K, x)$  of  $S_K$ . The map  $f_K$  defined as

$$f_K = f|_{S_K} : S_K \rightarrow M_K$$

induces an homomorphism

$$(f_K)_* : \pi_{S_K} \rightarrow \pi_K.$$

On the other hand, the branched covering  $p : \hat{M} \rightarrow M$  has an associated monodromy homomorphism  $\rho : \pi_K \rightarrow \Omega_n$  into the group  $\Omega_n$  of permutations of  $n$  elements, with  $n = 1, 2, \dots$

In [10], in order to give a presentation of the fundamental group of a Dehn surface, it is made a detailed study of unbranched coverings of Dehn surfaces. In particular, for a covering  $p : \hat{\Sigma} \rightarrow \Sigma$  it is shown how to construct the domain  $\hat{S}$  of  $\hat{\Sigma}$  and the Johansson diagram of  $\hat{\Sigma}$  on  $\hat{S}$ . Although the results given in [10] are stated for orientable Dehn surfaces of genus  $g$  (*Dehn  $g$ -tori*), the construction is also valid for more general surfaces, as the punctured surface  $\Sigma_K$ .

**Theorem 3.3.** *The domain  $\hat{S}$  of  $\hat{\Sigma}$  is a branched covering space  $p_S : \hat{S} \rightarrow S$  with downstairs branching set  $\{P_1, P_2, \dots, P_k\}$  and monodromy homomorphism  $\rho \circ (f_K)_*$ .*

*Proof.* The proof follows from [10, Sec. 4].  $\square$

The branched covering  $p_S : \hat{S} \rightarrow S$  is called the *domain branched covering* of  $p$  for  $\Sigma$  and the monodromy homomorphism  $\rho_S := \rho \circ (f_K)_*$  is called the *domain monodromy* of  $p$  for  $\Sigma$ .

The previous results provide an algorithm for obtaining the Johansson diagram of a filling Dehn sphere of the coverings of  $M$  branched over  $K$ . This algorithm can be summarized as follows:

**Algorithm 3.4.** (1) Find a filling Dehn surface  $\Sigma_K$  in  $M$  that splits  $K$ ;  
 (2) draw the Johansson diagram  $\mathcal{D}_K$  of  $\Sigma_K$  and mark the points  $P_1, P_2, \dots, P_r$  of  $\Sigma_K \cap K$  on it;  
 (3) find a presentation of the knot group  $\pi_K$  of  $K$  in terms of  $\Sigma_K$  (cf. [10]);  
 (4) for each transitive representation  $\rho_K$  of  $\pi_K$  into a permutation group  $\Omega_n$ , find the domain monodromy  $\rho_S$  of  $\pi_{\Sigma_K}$  into  $\Omega_n$ ;  
 (5) build the branched covering of  $p_S : \hat{S} \rightarrow S$  with branching set  $\{P_1, \dots, P_r\}$  and monodromy  $\rho_S$ , and lift the diagram  $\mathcal{D}_K$  to a diagram  $\hat{\mathcal{D}}$  on  $\hat{S}$ ;  
 (6) by identifying the curves of  $\hat{\mathcal{D}}$  using the images under the representation  $\rho$  of the loops dual to the to the curves of  $\mathcal{D}$  we obtain the Johansson diagram of a filling Dehn of the branched covering  $p^3 : \hat{M} \rightarrow M$  with branching set  $K$  and monodromy  $\rho_K$ .

As the simpler filling Dehn surfaces are filling Dehn spheres, we want to use this algorithm to obtain the Johansson diagram of a filling Dehn sphere of  $\hat{M}$ .

**Definition 3.5.** A covering  $p : \hat{M} \rightarrow M$  branched over the knot  $K$  is *locally cyclic* if the monodromy map  $\rho$  sends knot meridians onto  $n$ -cycles, where  $n$  is the number of sheets of the covering. This is equivalent to say that  $p : p^{-1}(K) \rightarrow K$  is a homeomorphism.

**Theorem 3.6.** In the hypotheses of Theorem 3.2, if  $K$  is a knot and  $\Sigma$  is a filling Dehn sphere in  $M$  which is diametral for  $K$ , then  $\hat{\Sigma}$  is a filling collection of spheres, and it is a Dehn sphere if and only if  $p$  is locally cyclic.

*Proof.* As  $\Sigma$  is diametral for  $K$ , by Theorem 3.3 the domain  $\hat{S}$  of  $\hat{\Sigma}$  is a branched covering space of the domain  $S$  of  $\Sigma$ , which is a 2-sphere, with two points as branching set. This implies that  $\hat{S}$  is a disjoint union of 2-spheres, and by Theorem 3.2,  $\hat{\Sigma}$  is a filling collection of spheres. The Dehn surface  $\hat{\Sigma}$  is a Dehn sphere if and only if its domain  $\hat{S}$  is connected, and this occurs if and only if the domain monodromy  $\rho_S$  acts transitively. This is equivalent to the fact that the branched covering  $p$  is locally cyclic.  $\square$

Therefore, after combining Algorithm 3.4 and surgery operations in filling collections of spheres, we can eventually obtain the Johansson diagram of a filling Dehn sphere of  $\hat{M}$ .

#### 4. EXAMPLE: JOHANSSON'S SPHERE AND THE UNKNOT

The Dehn sphere  $\Sigma_J$  depicted in Figure 2(a) is called *Johansson's sphere* in [18] because its Johansson diagram appears in [7]. In the Johansson diagram the two sister curves must be identified as indicated by the arrows. The Dehn sphere  $\Sigma_J$  can be obtained using an algorithm introduced in [16], after connecting by surgery an immersed torus-like sphere (Figure 3(b)) with an embedded sphere (Figure 3(c)) intersecting as in Figure 3(a). According to [15] and [18], it is one of the three unique filling Dehn spheres of  $S^3$  with only two triple points. If we consider the unknot  $K$  intersecting  $\Sigma_J$  as in the same Figure 2(a), it is clear that  $\Sigma_J$  splits  $K$ ,

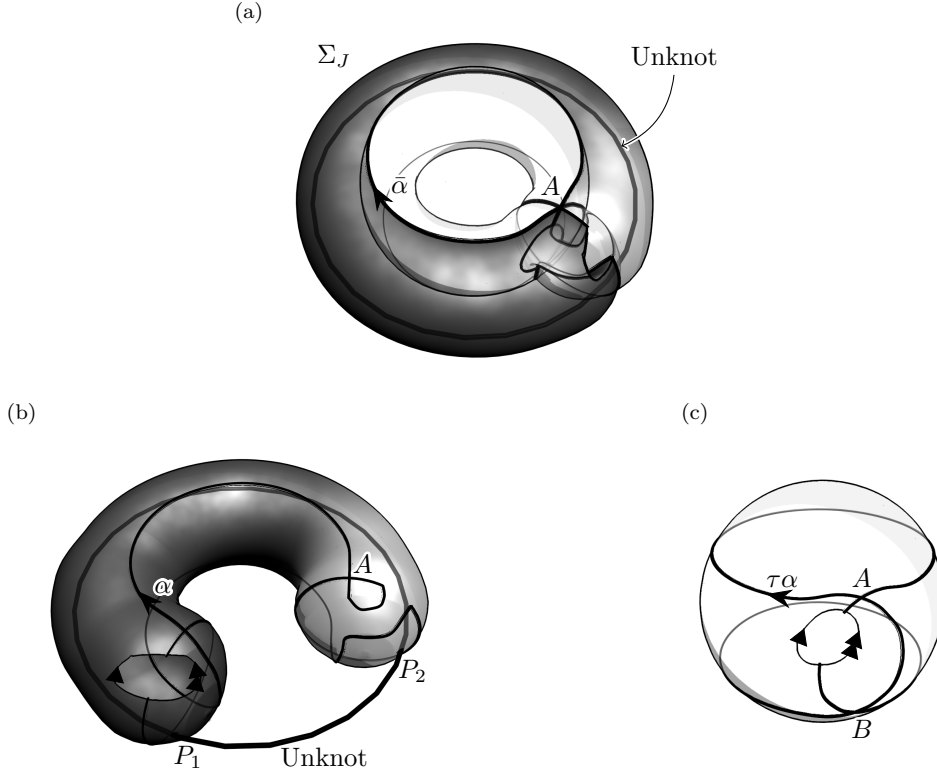


FIG. 2. Johansson's sphere splitting the unknot

and it can be checked using Example 7.1 of [16] that the two points  $P_1, P_2$  of  $\Sigma_J \cap K$  correspond to the points also labelled by  $P_1, P_2$  in the diagram of Figure 4(a).

By [10], the fundamental group of  $\Sigma_J - K$  based at  $x$  is generated by the loops  $m$  and  $a$ , where:

- $m$  is (the image in  $S^3$  of) the generator of the fundamental group of  $S^2 - \{P_1, P_2\}$  depicted in Figure 4(a);
- the loop  $a$  is *dual* to the curve  $\alpha$  of the diagram  $\mathcal{D}$ : it is the loop in  $\Sigma_J$  composed by the product of paths  $\lambda_\alpha * \lambda_{\tau\alpha}^{-1}$ , where  $\lambda_\alpha$  and  $\lambda_{\tau\alpha}$  are paths joining  $x$  with related points on  $\alpha$  and  $\tau\alpha$  (Figure 4(a)).

It must be noted that the loop  $m$  in  $\Sigma_J$  is homotopic to a meridian of  $K$ . Using [10], it can be checked also that the element  $a$  is trivial in  $\pi_K$  and that  $\pi_K$  is the infinite cyclic group generated by  $m$ .

As  $\pi_K$  is cyclic, the only coverings of  $S^3$  branched over  $K$  are the cyclic ones, and it is well known that the unique manifold that covers  $S^3$  branching over  $K$  is again  $S^3$ . Assume that  $p : S^3 \rightarrow S^3$  is the  $n$ -fold covering branched over  $K$ . By Theorems 3.2 and 3.6,  $\hat{\Sigma}_J = p^{-1}(\Sigma_J)$  is a filling Dehn sphere of  $S^3$ . By Theorem 3.3 and [10], the Johansson diagram  $\hat{\mathcal{D}}_J$  of  $\hat{\Sigma}_J$  is obtained lifting  $\mathcal{D}_J$  to the  $n$ -fold cover of  $S^2$  branched over  $\{P_1, P_2\}$ , and the sistering of the curves of  $\hat{\Sigma}_J$  is determined by the image of the dual loops of the curves of  $\mathcal{D}_J$  under the monodromy homomorphism induced by  $p$ . If we cut the 2-sphere of Figure 4(a) along an arc joining  $P_1$  and  $P_2$  and after that we send the point  $P_2$  through infinity, we obtain a *fan* as this depicted in Figure 4(b). By pasting ciclically  $n$  copies of this fan we obtain the Johansson diagram of  $\hat{\Sigma}_J$ . We have depicted the diagrams that arise in this situation for the

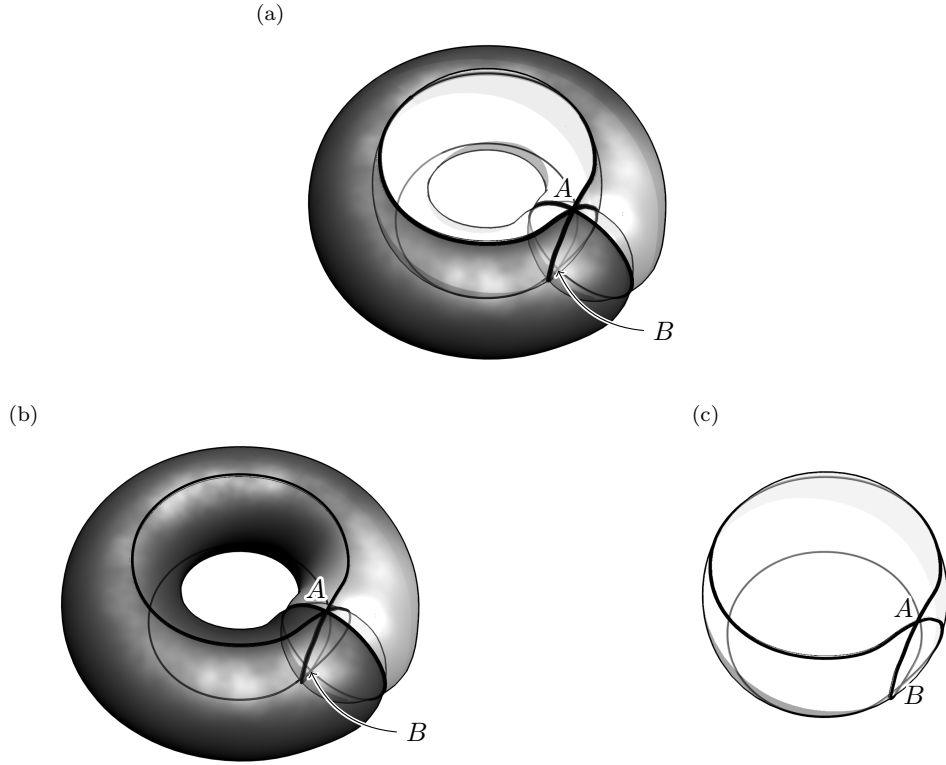


FIG. 3. Constructing Johansson's sphere

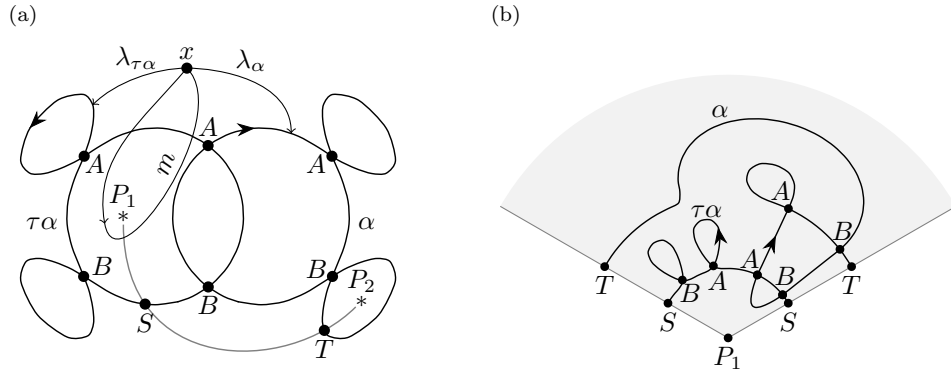


FIG. 4. Johansson's diagram of the unknot

cases  $n = 2, 3$  in Figure 4. Because  $a$  is trivial in  $\pi_K$ , the sistering of the curves of the diagram must be as indicated in this figure. In all cases we obtain a Johansson diagram with just two curves, both with self-intersections, and this implies that the Dehn spheres  $\hat{\Sigma}_J$  are all simply connected. If we had chosen  $K$  in a different relative position with respect to  $\Sigma_J$ , we would have obtained a different family of Johansson diagrams of  $S^3$  from these branched coverings.

Of course, the cyclic coverings of  $S^3$  branched over the unknot might not be very interesting, but *exactly the same construction can be done for the cyclic and locally*

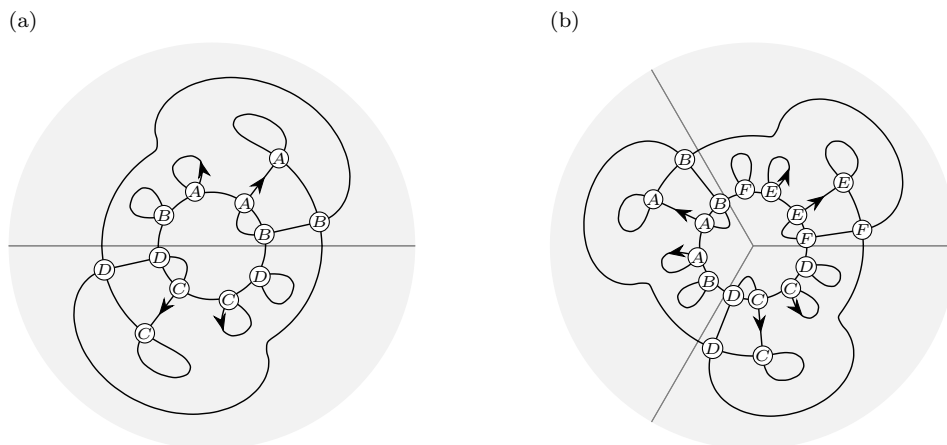


FIG. 5. Lifting  $\Sigma_J$  to cyclic branched coverings over the unknot

*cyclic branched coverings of any knot  $K$* , once a diametral Johansson diagram of  $K$  have been obtained.

## ACKNOWLEDGEMENTS

The authors want to thank Professors J.M. Montesinos and M.T. Lozano for their valuable advices about the content of this paper.

## REFERENCES

1. G. Amendola, *A local calculus for nullhomotopic filling Dehn spheres*, Algebr. Geom. Topol. **9** 2 (2009), 903–933.
2. G. Amendola, *A 3-manifold complexity via immersed surfaces*, J. Knot Theory Ramifications **19** 12 (2010), 1549–1569.
3. T. F. Banchoff, *Triple points and surgery of immersed surfaces*, Proc. Amer. Math. Soc. **46** 3 (1974), 407–413.
4. I. Bernstein, A. L. Edmonds, *On the construction of branched coverings of low-dimensional manifolds* Trans. Amer. Math. Soc. **247** (1979), 87–124.
5. R. Fenn and C. Rourke, *Nice spines on 3-manifolds*, Topology of low-dimensional manifolds, LNM 722 (1977), 31–36.
6. W. Haken, *Some special presentations of homotopy 3-spheres*, Topology Conference, LNM 375 (1973), 97–107.
7. I. Johansson, *Über singuläre Elementarflächen und das Dehnsche Lemma*, Math. Ann. **110** (1935), 312–320.
8. I. Johansson, *Über singuläre Elementarflächen und das Dehnsche Lemma II*, Math. Ann. **115** (1938), 658–669.
9. Á. Lozano, R. Vigara *On the subadditivity of Montesinos complexity of closed orientable 3-manifolds*, To appear in RACSAM Rev. R. Acad. A. DOI 10.1007/s13398-014-0179-1.
10. Á. Lozano, R. Vigara *The triple point spectrum of closed orientable 3-manifolds*, submitted, <http://arxiv.org/abs/1412.1637>
11. Á. Lozano, R. Vigara *Representing 3-manifolds by filling Dehn surfaces*, World Scientific Publishing Co. (to appear).
12. J. M. Montesinos-Amilibia, *Representing 3-manifolds by Dehn spheres*, Contribuciones Matemáticas: Homenaje a Joaquín Arregui Fernández, Editorial Complutense (2000), 239–247.
13. J. M. Montesinos, *Lectures on 3-fold simple coverings and 3-manifolds*, Combinatorial methods in topology and algebraic geometry (Rochester, N.Y., 1982), 157177, Contemp. Math., 44, Amer. Math. Soc., Providence, RI, 1985.
14. C. D. Papakyriakopoulos, *On Dehn's Lemma and the asphericity of knots*, Ann. Math. **2** 66 (1957), 1–26.



15. A. Shima, *Immersiones from the 2-sphere to the 3-sphere with only two triple points*. Topology of real singularities and related topics (Japanese) (Kyoto, 1997). Sūrikaiseikikenkyūsho Kōkyūroku **1006** (1997), 146–160.
16. R. Vigarà *A new proof of a theorem of J. M. Montesinos* J. Math. Sci. Univ. Tokyo **11** (2004), 325351.
17. R. Vigarà, *A set of moves for Johansson representation of 3-manifolds*, Fund. Math. **190** (2006), 245–288.
18. R. Vigarà. *Representación de 3-variedades por esferas de Dehn rellenantes*, PhD Thesis, UNED 2006.
19. R. Vigarà, *Lifting filling Dehn spheres*, Journal of Knot Theory and Its Ramifications **21** (2012) No. 8,1250082 7 pages.
20. J. H. C. Whitehead, *On  $C^1$ -complexes*, Ann. Math. **2** 41 (1940), 809-824.

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