On the subadditivity of Montesinos complexity of closed orientable 3-manifolds

Álvaro Lozano · Rubén Vigara

Abstract A filling Dehn sphere Σ in a closed 3-manifold M is a sphere transversely immersed in M that defines a cell decomposition of M. Every closed 3-manifold has a filling Dehn sphere (Montesinos-Amilibia, 2000). The Montesinos complexity of a 3-manifold M is defined as the minimal number of triple points among all the filling Dehn spheres of M. A sharp upper bound for the Montesinos complexity of the connected sum of two 3-manifolds is given.

1 Introduction

Throughout the paper all 3-manifolds are assumed to be closed, that is, compact, connected and without boundary, and orientable.

Let M be a 3-manifold.

A *Dehn sphere* in M is 2-sphere transversely immersed in M, and thus having only double point and triple point singularities. A Dehn sphere in M is *filling* if it naturally defines a cell decomposition of M (see Section 2 for details). Following (Haken, 1973), in (Montesinos-Amilibia, 2000) it is proved that every closed, orientable 3-manifold has a filling Dehn sphere, and filling Dehn spheres and their Johansson diagrams are proposed as a suitable way for representing all closed, orientable 3-manifolds. A weaker version of filling Dehn spheres are the so called *quasi-filling Dehn spheres* in the notation introduced in (Amendola, 2010). A quasi-filling Dehn sphere in M is a Dehn sphere whose complementary set in M is a disjoint union of open 3-balls. In (Fenn and Rourke, 1977) it is proved that every 3-manifold has a quasi-filling Dehn sphere.

Centro Universitario de la Defensa - Zaragoza
Ctra. de Huesca s/n, 50090 Zaragoza (Spain)
Tel. 976 739 500, Fax: 976 739 824 · I.U.M.A. - Universidad de Zaragoza
E-mail: alozano@unizar.es · rvigara@unizar.es

A simple check using Euler characteristics shows that the number of triple points of a Dehn sphere in M is always even. The filling Dehn sphere Σ in M is *minimal* if there is no filling Dehn sphere in M with less triple points than Σ. We define the *Montesinos complexity* of M, $mc(M)$, as the number of triple points of a minimal filling Dehn sphere of M.

Montesinos complexity has been introduced in (Vigara, 2006) with a different name (see Section 6), and it is closely related with G. Amendola's *surface-complexity* $sc(M)$ introduced in (Amendola, 2010).

Surface-complexity is subadditive under connected sums, that is,

$$sc(M_1 \# M_2) \leq sc(M_1) + sc(M_2),$$

were $M_1 \# M_2$ denotes the connected sum of the 3-manifolds M_1 and M_2.

Unlike in the previous case, Montesinos complexity is not subadditive.

Theorem 1 *For any* 3*-manifolds* M_1 *and* M_2 *we have*

$$mc(M_1 \# M_2) \leq mc(M_1) + mc(M_2) + 2\,.$$

The aim of this paper is to prove Theorem 1, and that the upper bound given there is sharp.

The proof of Theorem 1 relies on a surgery operation, similar to the one developed in (Vigara, 2012), which will be described in Section 3.

A filling Dehn sphere, and its Johansson diagram, provides a presentation of the fundamental group of the filled 3-manifold. We describe this presentation in Section 4. In Section 5 we briefly analyze the case of filling Dehn spheres with at most 4 triple points, proving the following theorem:

Theorem 2 *If* Σ *is a filling Dehn sphere of* M *with at most* 4 *triple points, then the first homology group* $H_1(M, \mathbb{Z})$ *cannot be isomorphic to* $\mathbb{Z}_3 \oplus \mathbb{Z}_3$.

It is known that the lens space $L(3,1)$ has Montesinos complexity 2 (cf. (Vigara, 2005)). Thus, by Theorem 1, $mc(L(3,1) \# L(3,1)) \leq 6$. The first homology group of $L(3,1) \# L(3,1)$ is known to be isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, but by Theorem 2, it cannot be $mc(L(3,1) \# L(3,1)) = 4$. Therefore,

Corollary 1 *The Montesinos complexity of* $L(3,1) \# L(3,1)$ *is* 6.

Corollary 2 *The upper bound of Theorem 1 is sharp.*

We plan to classify all the fundamental groups of the manifolds with Montesinos complexity up to 4 in a subsequent paper.

2 Dehn spheres and their Johansson diagrams

We will introduce some basic facts about Dehn spheres and their Johansson diagrams. We refer to (Vigara, 2006, 2005) for details.

In the following a *curve* in the 2-sphere S^2 or M is the image of an immersion from S^1 into S^2 or M, respectively. A *Dehn sphere* in M is a subset $\Sigma \subset M$ such that there exists a transverse immersion $f : S^2 \to M$ such that $\Sigma = f(S^2)$ (cf.
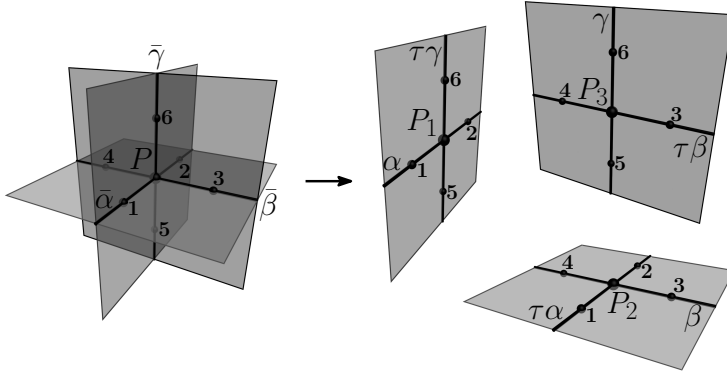
Fig. 1 A triple point of Σ and its triplet in S^2

(Papakyriakopoulos, 1957)). In this situation we say that f is a *parametrization* of Σ.

Let Σ be a Dehn sphere in M, and consider a parametrization f of Σ.

The *singularities* of Σ are the points $x \in \Sigma$ such that $\# f^{-1}(x) > 1$. The *singularity set* $S(\Sigma)$ of Σ is the set of singularities of Σ. As f is transverse, the singularities of Σ are arranged along *double curves*, and can be divided into *double points* ($\# f^{-1}(x) = 2$), where two sheets of Σ intersect transversely, and *triple points* ($\# f^{-1}(x) = 3$), where three sheets of Σ intersect transversely.

Because S^2 is compact and without boundary, the double curves of Σ are closed and there is a finite number of them. The triple points of Σ are isolated and there is a finite number of them. Following (Shima, 1997), we denote by $T(\Sigma)$ the set of triple points of Σ.

The preimage under f in S^2 of the singularity set of Σ, together with the information about how its points become identified by f in Σ is the *Johansson diagram* $\mathscr{D}$ of Σ (see (Montesinos-Amilibia, 2000)).

Because S^2 and M are orientable, the preimage under f of a double curve of Σ is the union of two different closed curves in S^2, and we will say that these two curves are *sister curves* of $\mathscr{D}$. Thus, the Johansson diagram of Σ is composed by an even number of different closed curves in S^2. Indeed, we will identify $\mathscr{D}$ with the set of different curves that compose it. For any curve $\alpha \in \mathscr{D}$ we denote by $\tau\alpha$ the sister curve of α in $\mathscr{D}$. This defines an involution $\tau : \mathscr{D} \to \mathscr{D}$ that sends each curve of $\mathscr{D}$ into its sister curve of $\mathscr{D}$.

The curves of $\mathscr{D}$ intersect with others or with themselves transversely at the *crossings* of $\mathscr{D}$. The crossings of $\mathscr{D}$ are the preimage under f of the triple points of Σ. If P is a triple point of Σ, the three crossings of $\mathscr{D}$ in $f^{-1}(D)$ compose *the triplet of P* (see Figure 1). Two points on S are *related* if they project onto the same point of Σ

The Dehn sphere Σ *fills* M if it defines a cell decomposition of M whose 0-skeleton is the set of triple points of Σ, the 1-skeleton is the set of singularities of Σ, and the 2-skeleton is Σ itself. Equivalently, Σ fills M iff

(F1) $S(\Sigma) - T(\Sigma)$ is a disjoint union of open arcs;
(F2) $\Sigma - S(\Sigma)$ is a disjoint union of open 2-disks;
(F3) $M - \Sigma$ is a disjoint union of open 3-balls.

In particular, if Σ is filling each double curve must cross at least one triple point and the Johansson diagram of Σ must be connected. A weaker version of filling Dehn spheres are the quasi-filling Dehn spheres, for which only condition **(F3)** is required.

If we are given an *abstract diagram*, i.e., an even collection of curves in S^2 coherently identified in pairs, it is possible to know if this abstract diagram is *realizable*: if it is actually the Johansson diagram of a Dehn sphere in a 3-manifold (see (Johansson, 1935, 1938; Vigara, 2005)). It is also possible to know if the abstract diagram is *filling*: if it is the Johansson diagram of a filling Dehn sphere of a 3-manifold (see (Vigara, 2005)). If Σ fills M, it is possible to build M out of the Johansson diagram of Σ. As every 3-manifold has a filling Dehn sphere, filling Johansson diagrams represent all closed, orientable 3-manifolds.

In Figure 2 we have depicted the simplest Johannson diagrams of filling Dehn spheres. In any case the curves must be identified in such a way that crossings are identified with crossings and the arcs labelled with the same arrow become identified in the obvious way. The graphs and the arrows give enough information about how all the points of the diagram must become identified in Σ. Nevertheless, for clarifying the pictures we have labelled with the same name crossings that belong to the same triplet. The diagram of Figure 2(a) is the classical diagram of I. Johansson (Johansson, 1935). The 3-sphere S^3 has only 3 (up to isotopy) filling Dehn spheres with 2 triple points. They are part of A. Shima's spheres given in (Shima, 1997). The corresponding Johansson diagrams are those of Figures 2(a), 2(b) and 2(c).

The Johansson diagrams of Figures 2(d) and 2(e) appeared in (Vigara, 2005). They are, respectively, the Johansson diagram of a filling Dehn sphere of $S^2 \times S^1$, and of $L(3,1)$. The Johansson diagrams of Figure 2 correspond to the only filling Dehn spheres of orientable 3-manifolds with 2 triple points. Therefore, $S^3, S^2 \times S^1$, and $L(3,1)$ are the only orientable 3-manifolds with Montesinos complexity 2.

It is well known that two closed curves in S^2 having nonempty transverse intersection must have an even number of intersection points. Along the text we will refer to this property as the *even intersection property*.

Lemma 1 *If the Dehn sphere Σ has p triple points and its Johannson diagram $\mathscr{D}$ is connected, it can have at most $(2 + 3p)/4$ double curves.*

Proof We define an *intersecting pair* of $\mathscr{D}$ as a pair of different curves of $\mathscr{D}$ having nonempty intersection. Although $\mathscr{D}$ has $3p$ crossings, by the even intersection property it can have at most $3p/2$ distinct intersecting pairs. As $\mathscr{D}$ is connected, it can have at most $1 + 3p/2$ different curves, and so Σ can have at most $(1 + 3p/2)/2 = (2 + 3p)/4$ double curves.

In particular, a filling Dehn sphere with 2 triple points can have at most 2 double curves, and a filling Dehn sphere with 4 triple points can have at most 3 double curves.

3 Surgery on minimal Dehn spheres. Proof of Theorem 1

Let M_1 and M_2 be two 3-manifolds, and let Σ_1 and Σ_2 be a filling Dehn sphere of M_1 and a filling Dehn sphere of M_2, respectively. Assume that Σ_1 and Σ_2 are
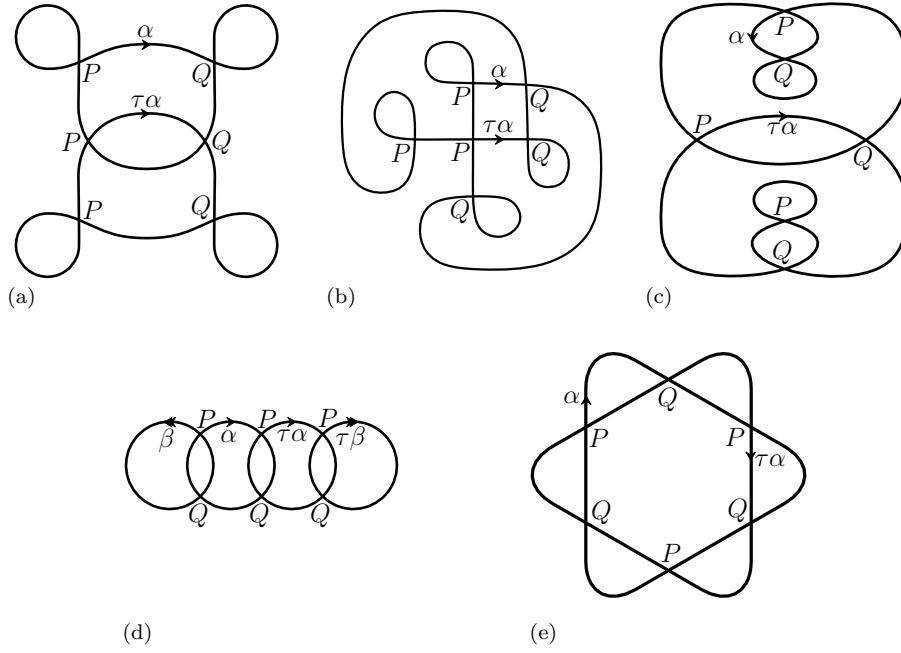
Fig. 2 Filling Johansson diagrams with two triple points

minimal in M_1 and M_2, respectively. The connected sum $M_1 \# M_2$ is performed by removing the interior of two closed 3-balls B_1 and B_2 lying in M_1 and M_2 respectively. After that, in the disjoint union of $M_1 \setminus int(B_1)$ and $M_2 \setminus int(B_2)$ the boundaries of B_1 and B_2 become identified by a homeomorphism.

A connected component of $\Sigma_1 - S(\Sigma_1)$ is *separating* if it is incident with different connected components of $M_1 - \Sigma_1$. A *separating disk* in Σ_1 is a closed disk in a separating component of $\Sigma_1 - S(\Sigma_1)$. Because $M_1 - \Sigma_1$ has more than one connected component (Vigara, 2006, Lemma 43), Σ_1 has a separating disk. The same argument applies to Σ_2.

If we choose the 3-balls B_1 and B_2 not intersecting Σ_1 and Σ_2, respectively, the Dehn spheres Σ_1 and Σ_2 are transformed after the connected sum into a pair of disjoint Dehn spheres of $M_1 \# M_2$, and the connected component of $M_1 \# M_2 \setminus (\Sigma_1 \cup \Sigma_2)$ lying between them is homeomorphic to $S^2 \times I$, where I is any open interval. We can remove a small separating disk from Σ_1 and from Σ_2 (Figure 3(a)) in order to connect them along a *piping* as in Figure 3(b). After that, we obtain a Dehn sphere $\Sigma_1 \# \Sigma_2$ which is not filling, but it is quasi-filling: the complementary set of $\Sigma_1 \# \Sigma_2$ in $M_1 \# M_2$ is a disjoint union of open 3-balls. The Dehn sphere $\Sigma_1 \# \Sigma_2$ is not filling because after the piping we have created a connected component of $\Sigma_1 \# \Sigma_2 \setminus S(\Sigma_1 \# \Sigma_2)$ which is topologically an open annulus. This obstruction can be removed by throwing two *fingers* (Figure 3(c)) along the piping between Σ_1 and Σ_2 until they intersect as in Figure 3(d), creating two new triple points. The resulting Dehn sphere $\widetilde{\Sigma_1 \# \Sigma_2}$ is now a filling one, and it has $p_1 + p_2 + 2$ triple points, where p_1 and p_2 are the number of triple points of Σ_1 and Σ_2 respectively.
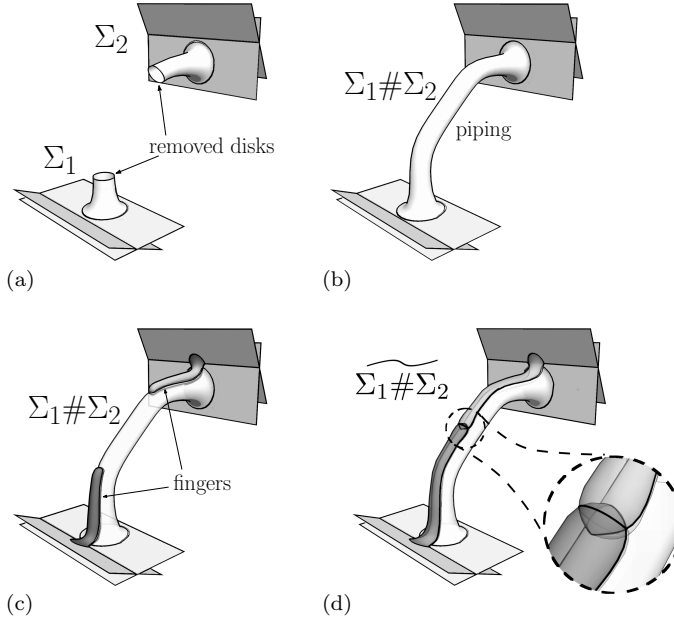
Fig. 3 Surgery between disjoint Dehn surfaces

As Σ_1 and Σ_2 are minimal we have that $p_1 = mc(M_1)$ and $p_2 = mc(M_2)$. This proves Theorem 1.

In Figure 4 we have illustrated the modifications to be made on the Johansson diagrams of Σ_1 and Σ_2 in order to obtain the Johansson diagram of $\widetilde{\Sigma_1 \# \Sigma_2}$ when M_1 and M_2 are two copies of $L(3,1)$ an Σ_1 and Σ_2 are two identical copies of the filling Dehn sphere of $L(3,1)$ whose Johansson diagram is that of Figure 2(e). The Johansson diagrams of Σ_1 and Σ_2 are two copies of the diagram of Figure 2(e) depicted in two different 2-spheres S_1 and S_2. We have assumed that M_1 and M_2, Σ_1 and Σ_2, and B_1 and B_2 respectively are exact copies of each other and that the homeomorphism that identifies ∂B_1 with ∂B_2 is the identity map. With this assumptions, Σ_1 and Σ_2 become two specular copies of each other in $M_1 \# M_2$. If the disks removed from Σ_1 and Σ_2 during the piping were also identical, we must paste two specular copies of the same Johansson diagram as in Figure 4 in order to obtain the Johansson diagram of $\widetilde{\Sigma_1 \# \Sigma_2}$ (Figure 4(d)).

4 The diagram group

Let Σ be a filling Dehn sphere on M, let f be a parametrization of Σ, and let $\mathscr{D}$ be the Johansson diagram of Σ.

The Johansson diagram of Σ provides a presentation of the fundamental group of M (see (Vigara, 2006, 2005)). Let τ be the involution on the set of curves of $\mathscr{D}$ that relates each curve with its sister curve. If $\alpha_1, \alpha_2, \ldots, \alpha_k$ are the curves of $\mathscr{D}$,
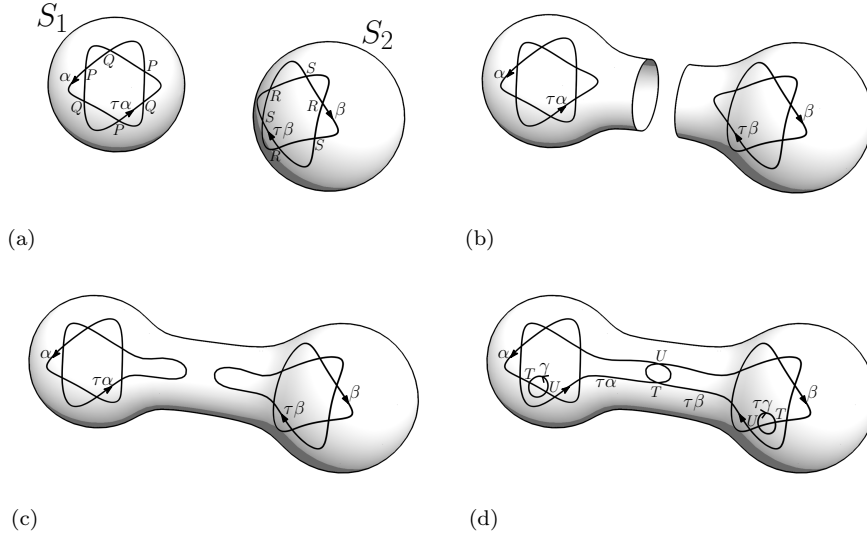
Fig. 4 Surgery between Johansson diagrams

then $\pi_1(M)$ is isomorphic to the *diagram group*:

$$\pi(\mathscr{D}) = \langle \alpha_1, \ldots, \alpha_k | \alpha_1 \cdot \tau\alpha_1 = \cdots = \alpha_k \cdot \tau\alpha_k = r_1 = \ldots = r_p = 1 \rangle, \qquad (1)$$

where the relators $r_i, i = 1, \ldots, p$ are the *triple point relators* of $\mathscr{D}$ because they are given by the triple points of Σ: if P is a triple point of Σ and we label its three preimages in S^2 and the curves of $\mathscr{D}$ intersecting at them as in Figure 1, the corresponding relation is $r = \alpha\beta\gamma = 1$. This presentation of $\pi_1(M)$ is due to W. Haken (see Problem 3.98 of (Kirby, 1995)). As α_i and $\tau\alpha_i$, with $i = 1, \ldots, k$, are inverse to each other in $\pi(\mathscr{D})$, we will use the notation α_i^{-1} instead of $\tau\alpha_i$ when we were talking about elements of $\pi(\mathscr{D})$. If α is a curve of $\mathscr{D}$, the generator α of $\pi(\mathscr{D})$ corresponds in $\pi_1(M)$ to the homotopy class of a *dual path* of α in Σ in the sense of (Vigara, 2005, Sec. 4.1.4).

At the triple point P of Σ one, two or three different double curves of Σ could intersect, and in each case we say that P is a triple point of *type I*, *type II* or *type III*, respectively. We will analyze these cases with more detail. Let P_1, P_2, P_3 be the triplet of P.

Type I triple points If the three arcs of double curve that intersect at P belong to the same double curve of Σ, two things could happen:

– **Type I.1**: one of the crossings P_1, P_2, P_3 is a self-intersection point of a curve of $\mathscr{D}$. If P_1, for example, is a self-intersecting point of a curve α of $\mathscr{D}$, then we are necessarily in the situation of Figure 5(a): the other two crossings of the triplet must be an intersection point of α with $\tau\alpha$ and a self-intersection point of $\tau\alpha$. In this case the corresponding relation is $\alpha\alpha^{-1} = 1$, which implies that $\alpha = 1$. We say also that the relation of $\pi(\mathscr{D})$ obtained from P is of *type I.1*.

– **Type I.2**: none of the crossings P_1, P_2, P_3 is a self-intersection point of a curve of $\mathscr{D}$. In this case, the three crossings P_1, P_2, P_3 are intersection points of a curve α of $\mathscr{D}$ with its sister curve $\tau\alpha$ (Figure 5(b)). The corresponding *type I.2 relation* is $\alpha^3 = 1$.

Type II triple points If two, but not three, of the three arcs of double curve that intersect at P belong to the same double curve $\bar{\alpha}$ of Σ, and $\alpha, \tau\alpha$ are the curves of $\mathscr{D}$ that project onto $\bar{\alpha}$ under f, two possibilities arise:

– **Type II.1**[1]: one of the three crossings P_1, P_2, P_3 is a self-intersection point of α or $\tau\alpha$. If, for example, P_1 is a self-intersection point α, the other two points P_2, P_3 must be intersection points of $\tau\alpha$ with β and $\tau\beta$, where $\beta, \tau\beta$ are curves of $\mathscr{D}$ different from α and $\tau\alpha$. Up to interchanging β and $\tau\beta$ we get a situation similar to that of Figure 5(c), where the corresponding *type II.1 relation* is $\alpha\beta\alpha^{-1} = 1$, which is equivalent to $\beta = 1$.
– **Type II.2**: if none of the three crossings P_1, P_2, P_3 is a self-intersection point of α or $\tau\alpha$, one of them, say P_1 must be a intersection point of α with $\tau\alpha$. Then, we have the configuration of Figure 5(d), whose corresponding *type II.2 relation* is $\alpha\beta\alpha = 1$, which is equivalent to $\beta = \alpha^{-2}$.

Type III triple points If the three arcs of double curve that intersect at P belong to different double curves, then we can label the curves of $\mathscr{D}$ to obtain a configuration as that of Figures 1 and 5(e), whose *type III relation* is $\alpha\beta\gamma = 1$.

We will denote by 1 the trivial group and by $\mathbb{Z}_q$, with $q = 2, 3, \ldots$, the cyclic group with q elements. For any two groups H, G, we write $H \cong G$ when the groups are isomorphic, and $H \lesssim G$ when H is isomorphic to a subgroup of G.

Theorem 3 *If Σ is a filling Dehn sphere of M with at most two double curves, then if $\pi_1(M)$ is isomorphic to 1, $\mathbb{Z}$ or $\mathbb{Z}_p$ with $p \leq 6$.*

Proof If Σ has only one double curve, there is one generator of the $\pi(\mathscr{D})$, and all triple points are of type I. It is $\pi(\mathscr{D}) = 1$ or $\pi(\mathscr{D}) \cong \mathbb{Z}_3$.

Assume now that Σ has two double curves and let $\alpha, \tau\alpha, \beta, \tau\beta$ be the curves of the Johansson diagram $\mathscr{D}$ of Σ. Because Σ is filling, the singularity set of Σ and the diagram $\mathscr{D}$ are connected, and so there must be at least one type II triple point P in Σ.

If there is one triple point of type II.1, we can assume that the relation $\beta = 1$ holds for the diagram group $\pi(\mathscr{D})$. Consequently, $\pi(\mathscr{D})$ is the cyclic group generated by α. Suppose that there is a type II relation not equivalent to $\beta = 1$:

– the relator $\alpha\beta\alpha$ or $\alpha\beta^{-1}\alpha$ implies $\alpha^2 = 1$, and so it is $\pi(\mathscr{D}) \lesssim \mathbb{Z}_2$;
– $\beta\alpha\beta$, $\beta\alpha^{-1}\beta$, $\beta\alpha\beta^{-1}$ or $\beta\alpha^{-1}\beta^{-1}$ leads to $\alpha = 1$, so $\pi(\mathscr{D})$ is trivial.

Assume that all the type II relations in $\pi(\mathscr{D})$ are equivalent to $\beta = 1$. If there's no type I relation involving the generators α or $\tau\alpha = \alpha^{-1}$, the generator α would be free and so it is $\pi(\mathscr{D}) \cong \mathbb{Z}$. If there are type I relators involving α or α^{-1}, we would have $\pi(\mathscr{D}) \cong 1$ or $\pi(\mathscr{D}) \cong \mathbb{Z}_3$.

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[1] In this situation, in the notation of (Haken, 1973) it is said that the double curve $f(\beta)$ of Σ is *compensated*. A Dehn sphere such that each double curve is compensated is simply connected.

If there is no triple point of type II.1 we can assume, by renaming the curves of $\mathscr{D}$ if necessary, that the relation $\alpha\beta\alpha = 1$ holds in $\pi(\mathscr{D})$. Again, $\pi(\mathscr{D})$ is the cyclic group generated by α. If there is another relation in $\pi(\mathscr{D})$ not equivalent to $\beta = \alpha^{-2}$:

- $\alpha\beta^{-1}\alpha = 1$ leads to $\alpha^4 = 1$, and so $\pi(\mathscr{D}) \lesssim \mathbb{Z}_4$;
- $\beta\alpha\beta = 1$ implies that $\alpha^3 = 1$ and so $\pi(\mathscr{D}) \lesssim \mathbb{Z}_3$;
- $\beta\alpha^{-1}\beta = 1$ gives $\alpha^5 = 1$ and so $\pi(\mathscr{D}) \lesssim \mathbb{Z}_5$;
- $\alpha = 1$ makes $\pi(\mathscr{D})$ trivial;
- $\alpha^3 = 1$ implies that $\pi(\mathscr{D}) \lesssim \mathbb{Z}_3$;
- $\beta = 1$ makes $\pi(\mathscr{D}) \lesssim \mathbb{Z}_2$;
- $\beta^3 = 1$ leads to $\alpha^6 = 1$, and so $\pi(\mathscr{D}) \lesssim \mathbb{Z}_6$.

5 3-manifolds with Montesinos complexity 4

Let Σ be a filling Dehn sphere on M, f a parametrization of Σ, and $\mathscr{D}$ the Johansson diagram of Σ.

Lemma 2 *If Σ has three double curves and four triple points, it has no type I triple point.*

Proof Let P be a type I triple point of Σ, and let Q, R, S be the other three triple points of Σ. Let $\bar{\alpha}$ be the double curve of Σ through P, and let $\alpha, \tau\alpha$ be the curves of $\mathscr{D}$ that are projected onto $\bar{\alpha}$ under f. The triple point P can be of type I.1 or of type I.2, but in both cases there is an odd number of intersection points between α and $\tau\alpha$ in the triplet of P (see Figures 5(a) and 5(b)). By the even intersection property, there must be another intersection point Q_1 of α with $\tau\alpha$ out of the triplet of P. We can assume that Q_1 belongs to the triplet of Q.

The unique types of triple points where a curve of $\mathscr{D}$ intersects its sister curve are types I and II.2 (see Figure 5). If Q is a type II.2 triple point, we can assume also that the curves $\alpha, \tau\alpha, \beta, \tau\beta$ intersect at the triplet of Q as in Figure 5(d), where $\beta, \tau\beta$ are two curves of $\mathscr{D}$ which project onto a double curve $\bar{\beta}$ of Σ different from $\bar{\alpha}$. Let $\gamma, \tau\gamma$ be the other two curves of $\mathscr{D}$ different form $\alpha, \tau\alpha, \beta$ and $\tau\beta$. As $\mathscr{D}$ is connected, by the even intersection property, among the remaining six crossings of $\mathscr{D}$ lying in the triplets of R and S there must be at least:

1. another intersection point of $\tau\alpha$ with β;
2. another intersection point of α and $\tau\beta$;
3. two intersection points of γ with one of the curves $\alpha, \tau\alpha, \beta, \tau\beta$; and
4. two intersection points of $\tau\gamma$ with the sister curve of the one intersecting with γ.

It is not difficult to check that with this restrictions, each of the remaining two triplets of $\mathscr{D}$ must involve the six curves of $\mathscr{D}$. Call R the triple point whose triplet contains a point of $\tau\alpha \cap \beta$, after renaming $\gamma, \tau\gamma$ if necessary we can assume that the intersection of the curves of $\mathscr{D}$ around $f^{-1}(R)$ is as that of Figure 5(e). Then, $f^{-1}(S)$ must contain: an intersection point of α and $\tau\beta$, an intersection point of γ and $\tau\beta$ and an intersection point of α and $\tau\gamma$, which is impossible because there fail to appear $\tau\alpha$ and β. This means that Q cannot be a type II.2 triple point.

(a) $\alpha\alpha\alpha^{-1} = 1 \iff \alpha = 1$

(b) $\alpha^3 = 1$

(c) $\alpha\beta\alpha^{-1} = 1 \iff \beta = 1$

(d) $\alpha\beta\alpha = 1 \iff \beta = \alpha^{-2}$
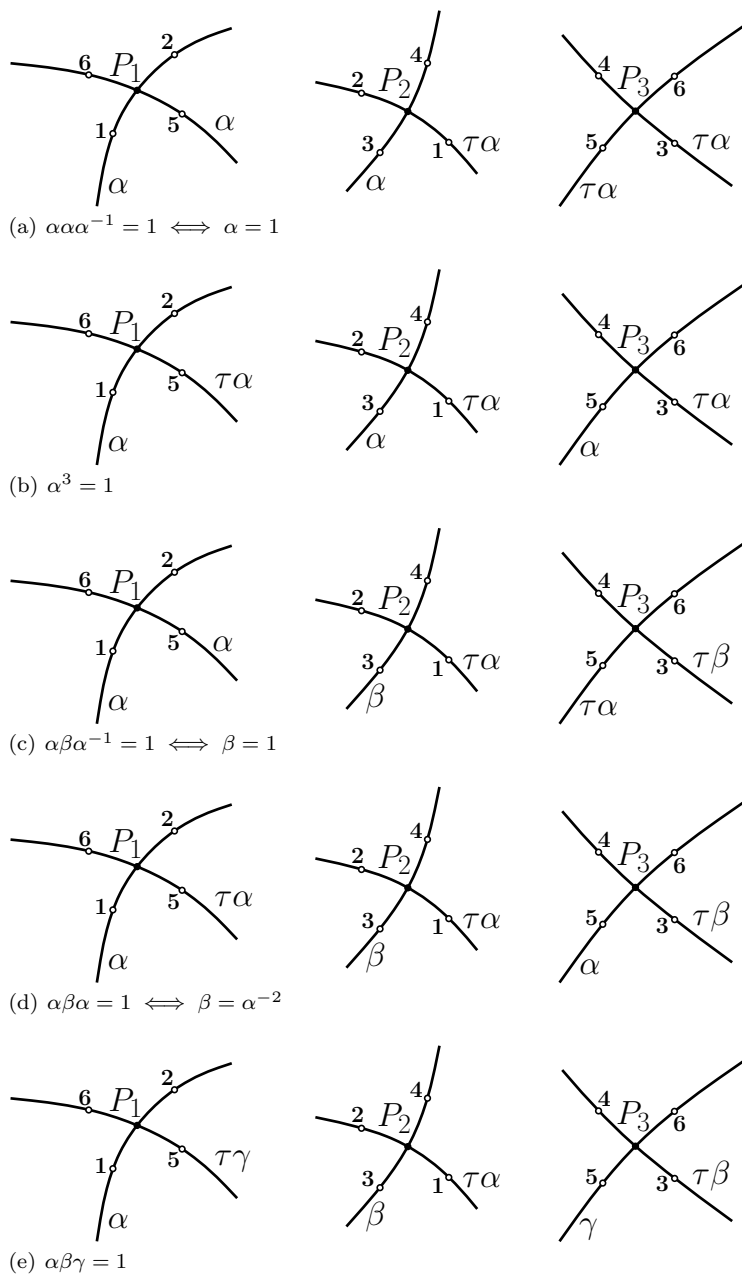

(e) $\alpha\beta\gamma = 1$

Fig. 5 The curves of $\mathscr{D}$ around a triplet of crossings

Therefore, Q is a type I triple point. By the even intersection property, for creating a connected diagram with $\alpha \cup \tau\alpha$ and $\beta, \tau\beta, \gamma, \tau\gamma$ we need to introduce at least eight crossings, but there are only six remaining crossings in $\mathscr{D}$. This leads to a contradiction, and so there cannot be a triple point of type I.

Proof (Proof of Theorem 2) As $\pi(\mathscr{D})$ is isomorphic to $\pi_1(M)$, the abelianized $\mathscr{A}(\mathscr{D})$ of $\pi(\mathscr{D})$ is isomorphic to $H_1(M, \mathbb{Z})$. We will work with $\mathscr{A}(\mathscr{D})$, and we will show that it cannot be isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, the abelianized group of $\mathbb{Z}_3 * \mathbb{Z}_3$. We will use the same names for the generators of $\mathscr{A}(\mathscr{D})$ and $\pi(\mathscr{D})$, and we will give the same names (type I.1, I.2, II.1, II.2 and III) to the abelianized relations in $\mathscr{A}(\mathscr{D})$ as their original relations in $\pi(\mathscr{D})$. By Theorem 3, we can restrict our analysis to the case when Σ has three double curves, and by Lemma 2, in this case there is no type I triple point in Σ. Therefore all the relations in $\pi(\mathscr{D})$ are of type II or III.

If $\pi(\mathscr{D})$ has a type III relation, we can assume that in $\mathscr{A}(\mathscr{D})$ holds the relation

$$\alpha + \beta + \gamma = 0. \tag{2}$$

If all the relations of $\mathscr{A}(\mathscr{D})$ are equivalent to (2), then $\mathscr{A}(\mathscr{D})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

If $\mathscr{A}(\mathscr{D})$ has another type III relation not equivalent to (2), after renaming the curves of $\mathscr{D}$ we can assume that this relation is

$$\alpha - \beta - \gamma = 0.$$

This relation, together with (2) gives $2\alpha = 0$. If α is trivial in $\mathscr{A}(\mathscr{D})$, then by (2) it is $\beta = -\gamma$ and $\mathscr{A}(\mathscr{D})$ is cyclic. If α is not trivial, $\mathscr{A}(\mathscr{D})$ has an element of order two, and so $\mathscr{A}(\mathscr{D})$ cannot be isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_3$.

If $\mathscr{A}(\mathscr{D})$ has a type II relation, we can assume that it is $\beta = 0$ or $\beta = -2\alpha$. In any case, this relation, together with (2), implies that $\mathscr{A}(\mathscr{D})$ is cyclic.

Thus, if $\mathscr{A}(\mathscr{D})$ has a type III relation, it cannot be isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_3$.

Assume now that all relators are of type II. If there is a type II.1 relation, we can assume that the relation $\beta = 0$ holds in $\mathscr{A}(\mathscr{D})$ (see Figure 5(c)). We have that:

- If the remaining three relations are equivalent to $\beta = 0$, $\mathscr{A}(\mathscr{D})$ is a free abelian group of rank two.
- If there is a relation of the type $\alpha = 0$, $\alpha = \pm 2\gamma$, $\alpha = \pm 2\beta$, perhaps interchanging α with γ, the group $\mathscr{A}(\mathscr{D})$ is cyclic.
- If there is a relation of the type $\beta = \pm 2\alpha$, then $\mathscr{A}(\mathscr{D})$ is cyclic (if $\alpha = 0$) or it has elements of order two (if $\alpha \neq 0$). The same holds if we have $\beta = \pm 2\gamma$.

If the four relators are of type II.2, we can assume that one of them gives the relation $\beta = 2\alpha$. Then

- If the remaining three relations are equivalent to $\beta = 2\alpha$, then $\mathscr{A}(\mathscr{D})$ is free abelian of rank two.
- If there is a relation of the type $\alpha = \pm 2\gamma$, the group $\mathscr{A}(\mathscr{D})$ is cyclic.
- If there is a relation of the type $\gamma = \pm 2\alpha$, $\mathscr{A}(\mathscr{D})$ is cyclic.
- If $\beta = \pm 2\gamma$ holds, we have that $2\alpha = \pm 2\gamma$. If it is $2\alpha = 2\gamma$, by taking $\alpha, \alpha - \gamma$ as generators of $\mathscr{A}(\mathscr{D})$ we have that $\mathscr{A}(\mathscr{D})$ must be cyclic (if $\alpha - \gamma = 0$) or it must contain elements of order two (if $\alpha - \gamma \neq 0$). The same argument can be applied when the relation $2\alpha = -2\gamma$ holds in $\mathscr{A}(\mathscr{D})$.

- If there is a relation of the type $\gamma = \pm 2\beta$, $\mathscr{A}(\mathscr{D})$ is cyclic.
- If there's no type II.2 relation involving γ, the generator γ of $\mathscr{A}(\mathscr{D})$ is free and so $\mathscr{A}(\mathscr{D})$ has rank at least 1.

  The proof is complete.

6 Comments

With a bit more effort, we can extend the techniques of the proofs of Theorems 3 and 2 for obtaining a list of candidates for fundamental groups of manifolds with Montesinos complexity up to 4. This will be made in a subsequent paper, where the following theorem will be proved:

Theorem 4 *If a 3-manifold M has Montesinos complexity $mc(M) \leq 4$, the fundamental group of M is isomorphic to either 1, $\mathbb{Z}$, $\mathbb{Z}_p$ with $p \leq 6$, $\mathbb{Z} \oplus \mathbb{Z}$ or to the group:*

$$K = \langle a, b | a^2 = b^2 \rangle .$$

The proof of this theorem relies in a combinatorial study of the groups having at most three generators and four triple point relations. The combinatorial properties of the filling Johansson diagrams: (i) connectedness; (ii) even intersection property; and (iii) the *symmetry* between sister curves (when performing a complete travel along sister curves we must cross the same number of remaining); impose strong combinatorial restrictions on the diagram groups. For an arbitrary group $\mathscr{G}$, we can wonder if there exists a *Haken presentation* of $\mathscr{G}$: a presentation similar to those of the diagram groups (generators and triple point relations) of filling Johansson diagrams with the same combinatorial restrictions as those imposed by properties (i), (ii) and (iii) above. For a given group $\mathscr{G}$ having a Haken presentation we can define its *Haken complexity* $hc(\mathscr{G})$ as the minimal number of triple point relators among all its Haken presentations. Of course, the fundamental group of a 3-manifold M has a Haken presentation and it is always $hc(\pi_1(M)) \leq mc(M)$. The question: *is it always $hc(\pi_1(M)) = mc(M)$?*, naturally arises. A possitive answer to this question would be highly nontrivial to prove because, in particular, it would imply a solution of the Poincaré Conjecture. We don't know if all the fundamental groups of the list of Theorem 4 actually occur as fundamental groups of manifolds with Montesinos complexity four. We have examples of filling Johansson diagrams with 4 triple points whose diagram groups are $\mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}$ or $\mathbb{Z}_q$ with $q \leq 5$, but we have found no examples for $\mathbb{Z}_6$ or K. It must be noted that K is isomorphic to the fundamental group of the Klein bottle[2].

The definition of filling Dehn spheres is naturally extended to *filling Dehn surfaces*, which are arbitrary compact immersed surfaces verifying the conditions **(F1)**, **(F2)** and **(F3)** of Section 2. If we require the Dehn surface to be an immersed orientable surface of genus g, we can talk about *genus g filling Dehn surfaces*. In (Vigara, 2005) it is defined the *triple point spectrum* of a 3-manifold m as the sequence

$$\mathscr{T}(M) = (t_0(M), t_1(M), t_2(M) \ldots) ,$$

where for all $g = 0, 1, 2, \ldots$ the number $t_g(M)$ is the *genus g triple point number* of M, i.e. the minimal number of triple points among all genus g filling Dehn

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[2] This fact was pointed out to the authors by the referee and by M. T. Lozano.

surfaces of M. Note that $mc(M) = t_0(M)$. A simple surgery operation shows that the genus g triple point numbers verify the inequality $t_{g+1}(M) \leq t_g(M) + 2$ for all $g = 0, 1, 2, \ldots$, but the equality does not necessarily hold because, for example, there are filling Dehn tori with just one triple point (see (Vigara, 2006, 2005)). Apart from this inequality, nothing is known about the triple point spectrum of any 3-manifold. A first question to answer in this context is if the triple point spectrum of S^3 is $(2, 4, 6, \ldots)$.

All these numbers, as Amendola's surface-complexity, can be used to give a census of 3-manifolds with increasing complexity. It should be interesting to investigate if it can be designed an efficient computer program for giving a list of 3-manifolds with bounded Montesinos complexity, as it has been done for the Matveev complexity (Matveev, 1990), for example.

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