Wirtinger curves, Artin groups, and hypocycloids

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Con cariño para nuestra maestra y compañera Maite Lozano

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Abstract The computation of the fundamental group of the complement of an algebraic plane curve has been theoretically solved since Zariski-van Kampen, but actual computations are usually cumbersome. In this work, we describe the notion of Wirtinger presentation of such a group relying on the real picture of the curve and with the same combinatorial flavor as the classical Wirtinger presentation; we determine a significant family of curves for which Wirtinger presentation provides the required fundamental group. The above methods allow us to compute that fundamental group for an infinite subfamily of hypocycloids, relating them with Artin groups.

Introduction

In [15,16], W. Wirtinger introduced his well-known method to compute the fundamental group of the complement of a knot. His primary aim was to apply this method to algebraic knots and links [6], i.e., links obtained as the transversal intersection of an algebraic curve (in $\mathbb{C}^2$) with a *small* sphere centered at a singular point. His method also works for any link and it is most useful for such computations. One of its interesting features is that it provides a simple combinatorial method to compute this group from the diagram of a knot or link, while keeping track of its geometrical definition. The other practical method to compute this group comes from Artin braid groups [4,5]; this is the idea behind Zariski-van Kampen's method [17,8] in order to give a presentation of the fundamental group of the global complement of a plane algebraic curve (later formalized as braid monodromy by Moishezon [11]).

In this paper, we are going to adapt Wirtinger's method to compute the fundamental group of the complement of some plane algebraic curves. These curves must have a real equation and a *rich* algebraic picture. Our goal is to provide (for a relatively small though significant family of curves) a combinatorial method for the computation of this group. When implemented, Zariski-van Kampen's

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method tends to rely generically on heavy numerical computations using floating-point arithmetics (see [10] for a reasonably efficient implementation in `Sagemath` that assures an exact output), this is why some theoretical methods applying to infinite families of curves are needed.

The origin of our interest in this method started in [3], where the fundamental group of the complement of some small-degree complexified hypocycloids was studied. Our techniques were applied to curves where the real picture gave a lot of information. This is not the case for the whole family of hypocycloids, but the use of their symmetries allowed us to use similar techniques for their resulting quotients whenever a rich real picture was obtained.

Following the notation introduced in [3], a hypocycloid is determined by two integers $0 < \ell < k$, $\gcd(k, \ell) = 1$ as the real curve traced by a fixed point on a circumference of radius ℓ while rolling inside a circumference of radius $N := k + \ell$. Such real curves admit a real algebraic equation. The fundamental group of the complement of their complexified version $\mathcal{C}_{k,\ell}$ is the focus of our interest here.

Some partial results treated in [3] and all the hypocycloids for $N \leq 11$ (using the `Sirocco` package by Marco and Rodríguez [10]) lead to the following conjecture describing them as Artin groups. We must emphasize that there is no hope that the techniques presented in [3] as well as the computational methods used in small degrees may be generalized to the whole family.

Conjecture 1. *For any pair of coprime integers $0 < \ell < k$, $N := k + \ell$, the fundamental group for $\mathcal{C}_{k,\ell}$ is the Artin group of the N-gon.*

In a forthcoming paper the conjecture will be proved for hypocycloids of type $(k, k-2)$ (k odd), using quotient singularities.

The paper is organized as follows: in section 1 the original Zariski-van Kampen method is recalled as way to provide a presentation for the fundamental group of the complement of an affine complex curve $\mathcal{C}$. This presentation has an extra property proved by A.Libgober in [9] which assures that the homotopy type of this complement coincides with that of the CW-complex associated with the given presentation. Section 2 is devoted to introducing the concept of curves of Wirtinger type as a complexified real curve satisfying certain properties with respect to a projection. Associated with the real picture of such curves one can define a finite presentation *à la Wirtinger*. The resulting group is not necessarily isomorphic to the expected group $\pi_1(\mathbb{C}^2 \setminus \mathcal{C})$ as illustrated by a series of examples. However, under certain additional conditions they are indeed isomorphic. This is stated in the main Theorem 2.6. The proof of this result is given in section 3, where the Zariski-van Kampen presentation is transformed into the Wirtinger presentation preserving the homotopy type of the associated CW-complex in Corollary 4.5. The purpose of section 4 is to state and prove Conjecture 1 for $\ell = k - 1$ –see Corollary 4.5. A brief discussion on Artin groups and first properties of hypocycloids completes the section. Finally, a series of examples in section 5 exhibit how the conditions of Theorem 2.6 can be relaxed at the cost of understanding the so-called obstruction points. This idea results in a more sophisticated version of the Wirtinger presentation, whose description goes beyond the scope of this paper and will be presented somewhere else.

1 The Zariski-van Kampen method

Let $\mathcal{C} \subset \mathbb{C}^2$ be a plane algebraic curve. We assume that for a given coordinate system the equation of $\mathcal{C}$ is given by a polynomial $f(x, y) \in \mathbb{C}[x, y]$ such that $\deg_y f = d$ and the coefficient of f in y^d as polynomial in $\mathbb{C}[x][y]$ is 1. As we are only interested in the zero locus, we can assume $\mathcal{C}$ to be reduced, i.e., f is a square-free polynomial. In particular $D(x) := \operatorname{Disc}_y(f) \in \mathbb{C}[x]$ is a non-zero polynomial.

The geometrical characterization for f being monic in y is that $\mathcal{C}$ contains neither a vertical line nor a vertical asymptote. By a *vertical asymptote* we mean a vertical line that is tangent to the curve at infinity. Let us consider $p : \mathbb{C}^2 \to \mathbb{C}$ be the vertical projection $(x, y) \mapsto x$; the restriction $p_{|\mathcal{C}}$ fails to be a covering only at the points of $\Delta := \{t \in \mathbb{C} \mid D(t) = 0\}$. As a consequence, if the vertical line $x = t$ is denoted by L_t,

$$p_| : \mathbb{C}^2 \setminus \left(\mathcal{C} \cup \bigcup_{t\in\Delta} L_t\right) \to \mathbb{C} \setminus \Delta$$

is a locally trivial fibration. Let us denote by r the cardinality of the discriminant Δ. Providing a suitable section of this fibration (over a big enough closed disk) (e.g. using some horizontal line), the following theorem holds.

Theorem 1.1. *Under the above hypotheses,*

$$\pi_1\left(\mathbb{C}^2 \setminus \left(\mathcal{C} \cup \bigcup_{t\in\Delta} L_t\right); (x_0, y_0)\right) = \left\langle \mu_1, \ldots, \mu_d, \alpha_1, \ldots, \alpha_r \,\middle|\, \underset{1\leq i\leq d,\quad 1\leq j\leq r}{\alpha_j^{-1} \cdot \mu_i \cdot \alpha_j = \mu_i^{\tau_j}} \right\rangle.$$

The loops μ_j correspond to a *geometric* basis of the free group $\mathbb{F}_d := \pi_1(L_{x_0} \setminus \mathcal{C}; (x_0, y_0))$ (i.e., each element is a meridian and the reversed product is homotopic to the boundary of a big disk, see [11,1]); the loops α_i correspond to the lift to the horizontal line $y = y_0$ of a *geometric* basis of the free group $\pi_1(\mathbb{C} \setminus \Delta; x_0)$. By the continuity of roots, these loops, provide braids $\tau_j \in \mathbb{B}_d$ and the right action in the statement corresponds to the standard right action of $\mathbb{B}_d$ on $\mathbb{F}_d$. We identify the braid group with the fundamental group of $(\mathbb{C}^d \setminus \mathcal{D})/\Sigma_d$ with base point $p_{|\mathcal{C}}^{-1}(x_0) \subset \mathbb{C}$, where $\mathcal{D} = \{(x_1, ..., x_d) \in \mathbb{C}^d \mid x_i = x_j \text{ for some } i < j\}$ and the quotient is given by the group action of the permutation group Σ_d acting on the coordinates $\sigma \cdot (x_1, ..., x_d) = (x_{\sigma(1)}, ..., x_{\sigma(d)})$ [4,5]. The braid group is generated by the standard *half-twists* $\sigma_1, \ldots, \sigma_{d-1}$ and the action on the free group is defined by

$$\mu_i^{\sigma_j} := \begin{cases} \mu_i & \text{if } j \neq i, i-1 \\ \mu_{i+1} & \text{if } j = i \\ \mu_i \cdot \mu_{i-1} \cdot \mu_i^{-1} & \text{if } j = i-1, \end{cases} \qquad \mu_i^{\sigma_j^{-1}} = \begin{cases} \mu_i & \text{if } j \neq i, i-1 \\ \mu_i^{-1} \cdot \mu_{i+1} \cdot \mu_i & \text{if } j = i \\ \mu_{i-1} & \text{if } j = i-1. \end{cases} \tag{1.1}$$

For the sake of completeness, the action of the inverse of the standard half-twists have been added.

Assume for simplicity that for each $t \in \Delta$, $p_{|\mathcal{C}}$ fails only at one point $(t, y(t))$ to be a covering over t. The main ideas behind the Zariski-van Kampen method are the following ones. On one side, α_j will be null-homotopic in $\mathbb{C}^2 \setminus \mathcal{C}$; on the other side, if the braid τ_j correspond to $t_j \in \Delta$ (denoting $y_j := y(t_j)$), then τ_j can be written as $\eta_j^{-1} \cdot \delta_j \cdot \eta_j$, where δ_j is a positive braid involving only the m_j strings close to (t_j, y_j) (and whose conjugacy class is determined by the topological type of $\mathcal{C}$ at (t_j, y_j)). Note that, without loss of generality, one might assume that exactly the first m_j strings are involved in δ_j.

Corollary 1.2.

$$\pi_1\left(\mathbb{C}^2 \setminus \mathcal{C}; (x_0, y_0)\right) = \langle \mu_1, \ldots, \mu_d \mid \mu_i^{\delta_j \cdot \eta_j} = \mu_i^{\eta_j}, \quad 1 \leq j \leq r, \quad 1 \leq i < m_j \rangle.$$

Moreover, in case $\deg f = \deg_y f = d$, *a presentation for the fundamental group of the complement of the Zariski closure of* $\mathcal{C}$ *in* $\mathbb{P}^2$ *is obtained by adding the relation*

$$\mu_d \cdot \ldots \cdot \mu_1 = 1.$$

Remark 1.3. The fact that the only needed relations come from $1 \leq j < m_j$ is due to the fact that $(\mu_{m_j} \cdot \ldots \cdot \mu_1)^{\delta_j} = \mu_{m_j} \cdot \ldots \cdot \mu_1$ and $\mu_i^{\delta_j} = \mu_i$ for $m_j < i \leq d$. Note that for double points, the corresponding m_j equals 2, and hence only one relation is required. For meridians x_1, x_2 close to the singular point of $p_{|\mathcal{C}}$, the following relation is satisfied:

$$x_1 = x_2 \ (m = 0), \quad [x_1, x_2] = 1 \ (m = 1), \quad x_1 \cdot x_2 \cdot x_1 = x_2 \cdot x_1 \cdot x_2 \ (m = 2).$$

This comes from the action of the local braid σ_1^{m+1} on x_1, x_2 as described in (1.1).

2 Wirtinger curves

In this section the new concept of Wirtinger curves is defined. For this class of curves, a presentation of the fundamental group of their complement can be combinatorially obtained from their real picture. We start first by defining curves of *Wirtinger type*, which will be those which are candidates to be Wirtinger curves.

An affine curve $\mathcal{C} \subset \mathbb{C}^2$ is called of *Wirtinger type* if:

(W1) the ramification points of the vertical projection $p(x, y) = x$ are all real, that is,

$$R = \{P = (t, y_t) \in \mathcal{C} \mid p^{-1}(t) \text{ and } \mathcal{C} \text{ do not intersect transversally at } P\} \subset \mathbb{R}^2,$$

in particular, $\Delta = p(R) \subset \mathbb{R}$;

(W2) the local branches of $\mathcal{C}$ at $P \in R$ are all real,

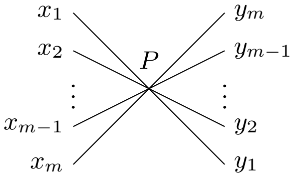
(W3) the vertical fiber $L = p^{-1}(x_0)$ of p at some $x_0 \in \mathbb{R}$ intersects the real part of $\mathcal{C}$ with maximal cardinality, that is, $\#(p|_{\mathcal{C}}^{-1}(x_0) \cap \mathbb{R}^2) = \deg_y f$ –note that this is not necessarily the degree of the curve,

(W4) it contains no vertical asymptotes and no vertical lines, and simple tangencies at smooth points are the only vertical lines in the tangent cone of $\mathcal{C}$ at any point.

(W5) the only singularities of $\mathcal{C}$ are either double –type $\mathbb{A}_m$– or ordinary (i.e., smooth branches with pairwise distinct tangent cones).

Given a curve $\mathcal{C}$ of Wirtinger type, we consider its *diagram* $\mathcal{C}_{\mathbb{R}} = \mathcal{C} \cap \mathbb{R}^2$. Denote by $\mathcal{V}_{\mathcal{C}}$ the set of singular points of $\mathcal{C}$, that is, $\mathcal{V}_{\mathcal{C}} = \operatorname{Sing} \mathcal{C} = \mathbb{R}^2 \cap \operatorname{Sing} \mathcal{C}$; they will be referred to as the *vertices* of $\mathcal{C}$. The *edges* of $\mathcal{C}$ are the closures of the connected components of $\mathcal{C}_{\mathbb{R}} \setminus \mathcal{V}_{\mathcal{C}}$. The set of edges will be denoted as $E_{\mathcal{C}}$. Our purpose is to describe a presentation of a group based on the vertices and edges of $\mathcal{C}_{\mathbb{R}}$. A *Wirtinger presentation* $G_{\mathcal{C}}$ associated with a curve of Wirtinger type $\mathcal{C}$ is given by a generating system parametrized by $E_{\mathcal{C}}$, that is, $\{x_\ell \mid \ell \in E_{\mathcal{C}}\}$. In addition, to each $P \in \mathcal{V}_{\mathcal{C}}$ the following relations are associated:

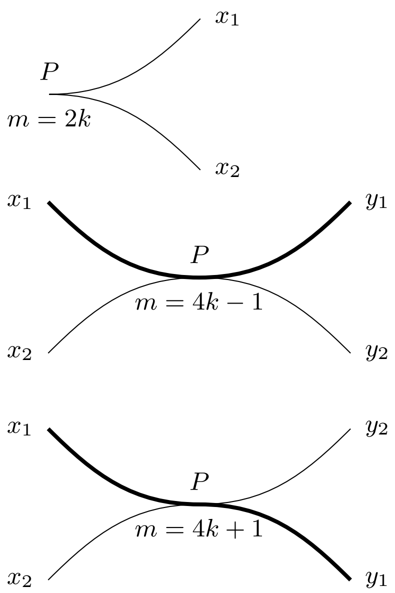
(R1) If P is an ordinary real singular point of multiplicity m as in (2.1), then the edges associated with P can be sorted out in two groups $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_m\}$ as shown below. Define $\bar{x}_k = x_k x_{k-1} \cdots x_1$ and $\bar{y}_k = y_1 y_2 \cdots y_k$. Then



$$\begin{cases} [\bar{x}_m, x_j] = 1 \\ y_j = \bar{x}_j^{-1} \cdot x_j \cdot \bar{x}_j = \bar{x}_{j-1}^{-1} \cdot x_j \cdot \bar{x}_{j-1} \end{cases} \tag{2.1}$$

Note that $\bar{x}_m = \bar{y}_m$ is a consequence and the relations could be written also from left to right.

(R2) If P is of type $\mathbb{A}_m$, then the following relations are added:



$$x_1(x_2x_1)^k = (x_2x_1)^k x_2,$$

$$\begin{cases} (x_1x_2)^{\frac{m+1}{2}} = (x_2x_1)^{\frac{m+1}{2}}, \\ y_i = (x_2x_1)^{-k} x_i (x_2x_1)^k, \quad i = 1, 2 \end{cases} \tag{2.2}$$

Remark 2.1. Note that relations in (2.1) and (2.2) involving only x_i's correspond with the local braid-monodromy relations described in Remark 1.3.

The remaining relations describe meridians on one side of the singularity in terms of meridians on the other side as elements of the local fundamental group of the singular point, whenever there are real branches on both sides of the singular point. For a $\mathbb{A}_{4k-1}$-singularity, the local braid monodromy is given by σ_1^{4k}, hence the relation $y_i = (x_2x_1)^{-k} x_i (x_2x_1)^k$, $i = 1, 2$ in (2.2) is nothing but $y_i = x_i^{\sigma_1^{-2k}}$ as described in (1.1). For an ordinary singular point of multiplicity m, the local braid monodromy is given by Δ_m^2 (where Δ_m represents a half-full twist in m strands); the second line of (2.1) is nothing but $y_i = x_i^{\Delta_m^{-1}}$.

Remark 2.2. Note that the case $m = 1$ is in both the family of ordinary points and double points. One can check that relations (2.1) become

$$[x_2x_1, x_1] = 1, y_1 = x_1, y_2 = x_1^{-1} x_2 x_1 = x_2.$$

whereas relations (2.2) become

$$x_1x_2 = x_2x_1, y_1 = x_1, y_2 = x_2.$$

Therefore both sets of relations are trivially equivalent to

$$[x_1, x_2] = 1, y_1 = x_1, y_2 = x_2.$$

In certain cases, $G_{\mathcal{C}}$ is a presentation of $\pi_1(\mathbb{C}^2 \setminus \mathcal{C})$, but not necessarily.

Example 2.3. For any real smooth curve $\mathcal{C}$ of Wirtinger type, note that $G_{\mathcal{C}}$ is a presentation of the free group of rank r, where r is the number of connected components of $\mathcal{C}_{\mathbb{R}}$, however $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}) = \mathbb{Z}$. It applies to $\mathcal{C} : y^2 - x(x^2 - 1) = 0$ ($r = 2$).

Example 2.4. Let $\mathcal{C}$ be a strongly real line arrangement (with no vertical lines), that is, a finite union of lines where each line has a real equation. In particular $\mathcal{C} = \{\ell_1 \cdot \ldots \cdot \ell_r = 0\}$ where $\ell_i \in \mathbb{R}[x,y]$, $i = 1,\ldots,r$ are pairwise non-proportional linear forms. Note that in this case $G_{\mathcal{C}}$ gives the Salvetti presentation [14] of $\pi_1(\mathbb{C}^2 \setminus \mathcal{C})$ and it can be reduced to the Zariski-van Kampen presentation associated with the vertical projection, with Tietze transformations of type I and IIa.

Example 2.5. Consider the affine tricuspidal quartic whose line at infinity is bitangent (the *deltoid*). This curve has a real equation

$$\mathcal{C} := \{(x,y) \in \mathbb{C}^2 \mid 3(x^2+y^2)^2 + 24x(x^2+y^2) + 6(x^2+y^2) - 32x^3 - 1 = 0\}.$$

Its diagram is a triangle whose vertices are the three cusps. Therefore $G_{\mathcal{C}} = \langle x_1, x_2, x_3 : x_1x_2x_1 = x_2x_1x_2, x_2x_3x_2 = x_3x_2x_3, x_3x_1x_3 = x_1x_3x_1\rangle$ is the standard presentation of the Artin group of the triangle, which coincides with $\pi_1(\mathbb{C}^2 \setminus \mathcal{C})$ –see [13,3].

The following result offers a wide collection of examples of curves of Wirtinger type whose Wirtinger presentation is a presentation of the fundamental group $\pi_1(\mathbb{C}^2 \setminus \mathcal{C})$. In order to state the conditions one needs to introduce the simple concept of real branches *facing* a vertical line. Consider a vertical real line $L_{\mathbb{R}}$ and a singular point of the vertical projection $P \notin L_{\mathbb{R}}$. The vertical line through P separates the real plane in two half-planes, one of them say H^+ containing $L_{\mathbb{R}}$. If H^+ contains real branches at P, then these branches at P are said to *face* $L_{\mathbb{R}}$.

Theorem 2.6. *Let $\mathcal{C}$ be a curve of Wirtinger type and such that the real part of each irreducible component is connected. Let $L = p^{-1}(x_0)$ be a line satisfying* (W3) *and let $B \subset \mathbb{R}^2$ be a closed topological disk (with piecewise smooth boundary) such that:*

(1) *$B \cup \mathcal{C}_{\mathbb{R}} \cup L_{\mathbb{R}}$ is simply connected.*
(2) *There is a parallel* real *plane $H_\varepsilon = \mathbb{R} \times (\mathbb{R} + \varepsilon\sqrt{-1})$ to $\mathbb{R}^2$ with $\varepsilon \neq 0$ such that*

$$B_\varepsilon \cap \mathcal{C} = \emptyset, \text{ where } B_\varepsilon = \{(x, y+\varepsilon\sqrt{-1}) \in H_\varepsilon \mid (x,y) \in B\} \subset H_\varepsilon. \tag{2.3}$$

(3) *All singularities of the vertical projection face $L_{\mathbb{R}}$.*

Then $G_{\mathcal{C}}$ is a presentation of $\pi_1(\mathbb{C}^2 \setminus \mathcal{C})$.

Remark 2.7. Given a curve of Wirtinger type, the set of points $H_\varepsilon \cap \mathcal{C} \subset H_\varepsilon$ projected onto $\mathbb{R}^2$ via the real-part map will be referred to as *obstruction points*. Before we prove this result, we will describe strategies to determine the position of the obstruction points based on the real picture $\mathcal{C}_{\mathbb{R}}$ in order to check property (2.3) without actual computations. First note that a smooth branch transversal to the vertical line can be locally parametrized by $y = 0$ after a real change of coordinates. Therefore a parallel plane H_ε near $P = (0,0)$ will be locally disjoint to $\mathcal{C}$, since $y = v + \varepsilon\sqrt{-1} = 0$ has no solution for $v \in \mathbb{R}$. Since the ordinary singularities as well as $\mathbb{A}_{2k+1}$ are a product of smooth branches, this forces the same local property $H_\varepsilon \cap \mathcal{C} = \emptyset$ near P. The remaining two cases are either simple vertical tangencies or $\mathbb{A}_{2k}$. In the simple tangency case $y^2 = x$, note that locally in a ball B_P around P,

$$\begin{aligned}(H_\varepsilon \cap \mathcal{C})_P &= \{(u, v+\varepsilon\sqrt{-1}) \in B_P \mid u,v \in \mathbb{R}, (v+\varepsilon\sqrt{-1})^2 = v^2 - \varepsilon^2 + 2v\varepsilon\sqrt{-1} = u\} \\ &= \{(-\varepsilon^2, \varepsilon\sqrt{-1})\}.\end{aligned}$$

Analogously, if $y^2 = -x$, then $(H_\varepsilon \cap \mathcal{C})_P = \{(\varepsilon^2, \varepsilon\sqrt{-1})\}$. The position of this obstruction point relative to the curve is depicted in Figure 1.

Finally, at an irreducible double singularity of type $\mathbb{A}_{2k}$ of local equation $y^2 = x^{2k+1}$ one can check that $(H_\varepsilon \cap \mathcal{C})_P = \{(-\varepsilon^{\frac{2}{2k+1}}, \varepsilon\sqrt{-1})\}$, where $\varepsilon^{\frac{2}{2k+1}}$ represents the only real $2k+1$ root of ε^2. The position of this obstruction point relative to the curve is also depicted in Figure 1.

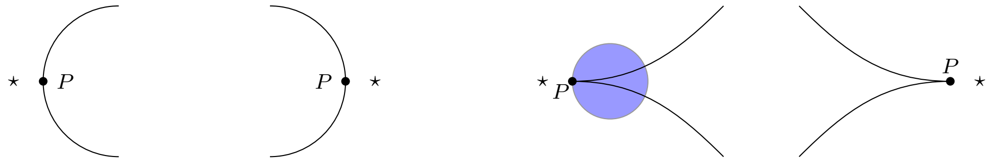


Fig. 1: Obstruction points at vertical tangencies and $\mathbb{A}_{2k}$-singular points

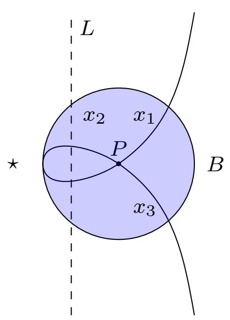
Remark 2.8. The compactness of B makes condition (2) in Theorem 2.6 of a combinatorial nature because of the discussion in Remark 2.7.

Example 2.9. It is straightforward to check that the curves $\mathcal{C} : y^2 - x^{m+1} = 0$ are of Wirtinger type. We apply Theorem 2.6 by choosing B as in Figure 1.

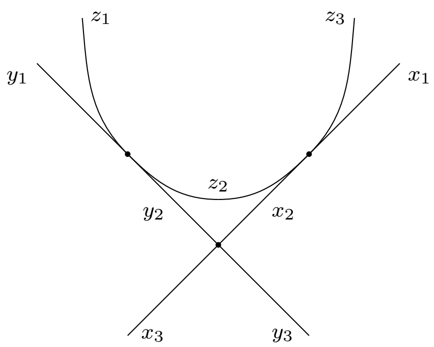
Example 2.10. As a simple application of Theorem 2.6 and Remark 2.7, note that the affine nodal cubic $\mathcal{C} = \{y^2 = x^2(x+1)\}$ is a curve of Wirtinger type. According to the discussion above, there is only an obstruction point (see Figure 2a) and the given B and L satisfy the conditions of Theorem 2.6. Since the diagram $\mathcal{C}_{\mathbb{R}}$ contains three edges and only one vertex (associated with the nodal point P), $G_{\mathcal{C}}$ has three generators x_1, x_2, x_3 and only one set of relations as given in (2.1):

$$x_2^2 = x_3x_1, [x_2^2, x_1] = [x_2^2, x_2] = 1, x_3 = x_2, x_1 = x_2,$$

and hence $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}) = \mathbb{Z}$.



(a) Nodal cubic



(b) Parabola and tangent lines

Fig. 2

Example 2.11. Consider the parabola $y = x^2$ together with two parallel lines as in Figure 2b. The union of these irreducible components is an affine curve $\mathcal{C}$ of Wirtinger type. Choosing as B a big enough rectangle centered at the origin containing all singularities and a vertical line L placed at the left-most edge of B, one can trivially check they satisfy the hypotheses of Theorem 2.6.

The following is a complete set of relations obtained from the diagram $\mathcal{C}_{\mathbb{R}}$, namely,

$$\begin{cases} (z_1y_1)^2 = (y_1z_1)^2 \\ z_2 = (y_1z_1)^{-1}z_1(y_1z_1) \\ y_2 = z_1^{-1}y_1z_1 \end{cases} \qquad \begin{cases} x_3y_2 = y_3x_2 \\ y_2 = y_3 \\ [y_2, x_3] = 1 \\ x_2 = y_2^{-1}x_3y_2 = x_3 \end{cases} \qquad \begin{cases} (z_2x_2)^2 = (x_2z_2)^2 \\ z_3 = (x_2z_2)^{-1}z_2(x_2z_2) \\ x_1 = z_2^{-1}x_2z_2 \end{cases}$$

Using the relations $x = x_2 = x_3$, $y = y_2 = y_3$, $x_1 = z_2^{-1}xz_2$, $z_1 = yz_2y^{-1}$ $y_1 = z_1yz_1^{-1} = yz_2yz_2^{-1}y^{-1}$, $z_3 = xz_2x^{-1}$ the presentation $G_{\mathcal{C}}$ can be reduced to

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}) = \langle x, y, z_2 : xy = yx, (yz_2)^2 = (z_2y)^2, (xz_2)^2 = (z_2x)^2 \rangle$$

which is the presentation of the Euclidean Artin group $(4, 4, 2)$.

3 Proof of Theorem 2.6

Proof of Theorem 2.6. Let us consider a curve of Wirtinger type $\mathcal{C}$, a topological disk B and a vertical line $L = p^{-1}(t_0)$ satisfying the hypotheses. For simplicity, we assume $\varepsilon > 0$. The strategy of the proof is to inductively transform a Zariski-van Kampen presentation of $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}; P_0)$ into the Wirtinger presentation $G_{\mathcal{C}}$.

Before we start, a general method to construct loops is described as follows. Let $\ell \in E_{\mathcal{C}}$, $p_\ell \in E_{\mathcal{C}}$ a smooth point, and Δ_ℓ a disk of radius ε centered at p_ℓ and transversal to $\mathcal{C}$. Let q_ℓ be the unique point in $\Delta_\ell \cap B_\varepsilon$. The meridian μ_ℓ is defined taking a path ρ_ℓ in B_ε from P_0 to q_ℓ, running $\partial\Delta_\ell$ counterclockwise and coming back to P_0 via ρ_ℓ^{-1}. A key remark is that this construction defines a unique meridian μ_ℓ independently of the choice of ρ_ℓ and p_ℓ by condition (2.3).

We will start with an appropriate Zariski-van Kampen presentation for a suitable base point P_0 on L. Let us write $L \cap \mathcal{C} = L_{\mathbb{R}} \cap \mathcal{C}_{\mathbb{R}} = \{p_{\ell_1} = (t_0, y_1), \ldots, p_{\ell_d} = (t_0, y_d)\}$ where p_{ℓ_i} is a smooth point in $\ell_i \in E_{\mathcal{C}}$ with $y_1 > \cdots > y_d$ and choose $y_0 \in \mathbb{R}$, $y_0 \geq y_1$ such that $(t_0, y_0) \in B \cap L_{\mathbb{R}}$. This is possible since $L_{\mathbb{R}} \cap B$ is an interval, $L_{\mathbb{R}} \cap \mathcal{C}_{\mathbb{R}}$ is a finite set of points, $B \cup L_{\mathbb{R}} \cup \mathcal{C}_{\mathbb{R}}$ is simply connected, by condition (1), and hence $L_{\mathbb{R}} \cap \mathcal{C}_{\mathbb{R}} \subset B$. The point $P_0 = (t_0, y_0 + \varepsilon\sqrt{-1}) \in B_\varepsilon \cap L$ will be taken as a base point.

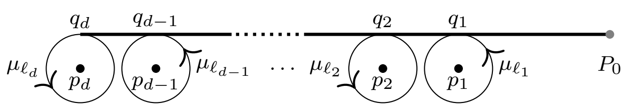


Fig. 3: Generators in the fiber

As was mentioned above, the idea of the proof is to transform the Zariski-van Kampen presentation of $\pi_1(\mathbb{C}^2 \setminus \mathcal{C})$ into the Wirtinger presentation $G_{\mathcal{C}}$. To simplify this procedure one can transform slightly the Wirtinger presentation by considering an extended diagram where the vertices in $\tilde{\mathcal{V}}_{\mathcal{C}}$ contain $\mathcal{V}_{\mathcal{C}}$ and the vertical tangencies of $\mathcal{C}$ and considering such points as $\mathbb{A}_0$-singular points. The resulting relation is provided in (2.2) for $m = 0$, i.e. the two generators coincide. The set of resulting edges will be denoted by $\tilde{E}_{\mathcal{C}}$.

Let us start from the Zariski-van Kampen presentation G_0 of Corollary 1.2 generated by the meridians $\mu_{\ell_1}, \ldots, \mu_{\ell_d}$ as in Figure 3, where ℓ_j is the edge containing p_j. Recall that the relators in G_0 correspond to the singular points of the projection $p|_{\mathcal{C}}$, i.e. with the vertices of the modified Wirtinger presentation $\tilde{G}_{\mathcal{C}}$. Let us order the set $\tilde{\mathcal{V}}_{\mathcal{C}} = \{P_1, \ldots, P_r\}$ of singular points of the projection $p|_{\mathcal{C}}$ by its distance to $L_{\mathbb{R}}$. Denote by L_j, $j = 1, \ldots, r$ (resp. L_0) the vertical line containing P_j (resp. $L_{\mathbb{R}}$). An inductive procedure will be presented to transform G_0 into $G_r = \tilde{G}_{\mathcal{C}}$ using only Tietze transformations of type I and IIa (without IIb [7], i.e. the homotopy type is preserved). At each step $j \in \{0, \ldots, r\}$, a presentation will be given whose generators are associated with the edges in $\tilde{E}_{\mathcal{C}} \cap (L_0 \cup \cdots \cup L_j)$ and whose relations associated with $P_1, \ldots, P_j$ coincide with those of $\tilde{G}_{\mathcal{C}}$ while the ones associated with $P_{j+1}, \ldots, P_r$ are still those of Zariski-van Kampen presentation.

For $j = 0$, the result is trivial using G_0. Assume G_j is constructed and consider the point P_{j+1} and its associated relations. The only new edges in $\tilde{E}_{\mathcal{C}} \cap L_{j+1}$ might come from adjacent edges to P_{j+1}. If P_{j+1} is of type $\mathbb{A}_{2k}$ and since P_{j+1} faces $L_{\mathbb{R}}$ (by condition (3)), no new edges arise. Let $x_{\ell'}$ be any generator associated with an edge ℓ' adjacent to P_{j+1}, then $x_{\ell'} = x_\ell^{\eta_j}$ for some generator x_ℓ in G_0, see Corollary 1.2. The local braid δ_j described before Corollary 1.2 is σ_1^{2k+1}. Hence relation (R2) produces $x_{\ell'} = x_{\ell'}^{\delta_j}$, which becomes $x_\ell^{\eta_j} = x_\ell^{\delta_j \eta_j}$, that is, the Zariski-van Kampen relation associated with P_{j+1}, which is replaced by the Wirtinger relation (R2) in G_{j+1}.

For the remaining cases ($\mathbb{A}_{2k+1}$ and ordinary), there are new edges in $\tilde{E}_{\mathcal{C}}$ adjacent to L_{j+1} and the local braids δ_j are squares, say $\delta_j = \tilde{\delta}_j^2$. As above, the relations involving the Zariski-van Kampen relation can be analogously replaced in G_j by those in (R1) and (R2) involving only the old edges (denoted by x's). As above let x be any generator associated with an edge adjacent to P_{j+1} and facing $L_{\mathbb{R}}$ and let y be the corresponding new edge on the same irreducible component. We will further transform G_j by adding a new generator y and a relation $y = x^{\tilde{\delta}_j}$ (see Remark 2.1). This process is continued until $G_r = \tilde{G}_{\mathcal{C}}$ is obtained. □

Remark 3.1. As a consequence of the beginning of the proof, a homomorphism $h : G_{\mathcal{C}} \to \pi_1(\mathbb{C}^2 \setminus \mathcal{C})$ can be defined as follows. Given x_ℓ the generator of $G_{\mathcal{C}}$ corresponding to $\ell \in E_{\mathcal{C}}$ as in § 2, then $h(x_\ell) := \mu_\ell$. The rest of the proof shows that h is in fact an isomorphism.

Corollary 3.2. *The Wirtinger presentation of a curve of Wirtinger type satisfying the conditions of Theorem 2.6 has the homotopy type of its associated CW-complex.*

Proof. Since Tietze transformations of type III are used in the proof of Theorem 2.6, this corollary is a consequence of the proof of the main result in [9], where the transversality with the line at infinity is not needed in his own proof. Incidentally, despite all the strong genericity conditions stated in his result, only the non-existence of vertical asymptotes is actually required. □

The conditions of Theorem 2.6 are sufficient, but not necessary. The following examples illustrate that the conditions in the statement of this theorem are not only technical.

Example 3.3. Let $\mathcal{C}$ be the curve defined by $f(x, y) = y^3 - y^2 + 10x^2y + x^3$. A sketch of its real picture is in Figure 4a. Note that condition (3) in Theorem 2.6 is not fulfilled. It is straightforward to see that $G_{\mathcal{C}} = \langle x_1, x_2 \mid x_1 \cdot x_2 \cdot x_1 = x_2 \cdot x_1 \cdot x_2 \rangle$ while $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}) \cong \mathbb{Z}$ and hence $\pi_1(\mathbb{C}^2 \setminus \mathcal{C})$ and $G_{\mathcal{C}}$ are not isomorphic.

Example 3.4. Let $\mathcal{C}$ be the cardioid curve $\mathcal{C}$ defined by $f(x, y) = (y^2 + x^2 - 2x)^2 - 4(x^2 + y^2)$, see Figure 4b. It is not possible to find a simply connected region B satisfying the hypotheses of Theorem 2.6 (because of the obstruction point close to the cusp). It is straightforward to see that

$G_{\mathcal{C}} \cong \mathbb{Z}$. The projective curve is the tricuspidal quartic curve (with two cusps at infinity), which has a non-abelian fundamental group (as proved in [17]).

(a) Cuspidal cubic generic at infinity (b) Cardioid

Fig. 4

4 Artin groups and Hypocycloids

In this section Artin groups and basic properties of hypocycloids will be recalled in order to state and prove the main Theorem on the fundamental group of hypocycloid $\mathcal{C}_{k,k-1}$ curves. There are many conventions to define Artin groups, but for our purpose the Dynkin-diagram convention will be more suitable.

Definition 4.1. Let Γ be a graph, the Artin group associated to $\Gamma = (V, E)$ is defined as group G_Γ generated by the vertices $v \in V$ of Γ with the following relations:

$$v \cdot w \cdot v = w \cdot v \cdot w \text{ if } \{v, w\} \in E, \quad \text{and } [v, w] = 1 \text{ otherwise.} \tag{4.1}$$

Example 4.2. Let $G_N := G_{\tilde{A}_N}$ be the Artin group of an N-gon (an affine Dynkin diagram $\tilde{A}_N$). According to (4.1) a presentation of G_N can be written as

$$\langle x_j, j \in \mathbb{Z}_N \mid x_i \cdot x_{i+1} \cdot x_i = x_{i+1} \cdot x_i \cdot x_{i+1}, \quad x_i \cdot x_j = x_j \cdot x_i \text{ for } |i-j| \neq 1 \rangle,$$

where the subindices are considered in $\mathbb{Z}_N$ and $|i-j| \neq 1$ means $i - j \not\equiv \pm 1 \mod N$.

Consider $\mathbb{Z}/2 = \langle t \mid t^2 = 1 \rangle$ acting on G_N as $x_j^t := x_{-j}$. In the special case $N = 2k-1$, it is straightforward to check that the semidirect product $G_N \rtimes \mathbb{Z}/2$ admits a presentation generated by t, $x_0, \ldots, x_{k-1}$ and whose relations are:

(SD1) $t^2 = 1$;
(SD2) $x_j \cdot x_{j+1} \cdot x_j = x_{j+1} \cdot x_j \cdot x_{j+1}$ for $0 \leq j < k-1$;
(SD3) $(x_{k-1} \cdot t)^3 = (t \cdot x_{k-1})^3$;
(SD4) $[x_0, x_j] = 1$ for $1 < j \leq k-1$;
(SD5) $[x_i, x_j] = 1$ for $0 < i, j \leq k-1$ and $j - i > 1$;
(SD6) $[x_i, t \cdot x_j \cdot t] = 1$ for $0 < i \leq j \leq k-1$ and $(i, j) \neq (k-1, k-1)$.

Let us recall some of the main properties of the hypocycloids. We follow the notation of [3] – see [12] for details too. A hypocycloid is a real curve associated with each pair $k,\ell \in \mathbb{Z}$ such that $0 < \ell < k$ and $\gcd(k,\ell) = 1$. Consider $N = k + \ell$ and any pair of positive integers r, R such that $\frac{r}{R} = \frac{\ell}{N}$ (or $\frac{k}{N}$), the hypocycloid is the real curve obtained as the trace of a fixed point on a circumference of radius r when rolling inside a circumference of radius R. This real curve admits an algebraic equation and thus one can consider its complexification $\mathcal{C}_{k,\ell}$ as the complex curve in $\mathbb{C}^2$ defined by this algebraic equation.

The complex curve $\mathcal{C}_{k,\ell}$ is rational and has degree $2k$ and its projective closure contains two points at infinity −the so-called concyclic points. As a summary of its algebraic properties:

(C1) $\mathcal{C}_{k,\ell}$ has N ordinary cuspidal singular points.
(C2) $\mathcal{C}_{k,\ell}$ has $N(k-2)$ ordinary double points. In its classical presentation, $N(\ell-1)$ of them are real points with real tangent lines while the other $N(k-\ell-1)$ are complex.
(C3) The two points at infinity of $\bar{\mathcal{C}}_{k,\ell}$ have local equations topologically equivalent to $u^{k-\ell} + v^k = 0$ (tangent to the line at infinity with contact order k).

These properties are classical and imply that the curve is rational, see e.g. [3] for a modern description, precise formulæ, and parametrizations.

Let us consider equations such that a point in the real axis has a vertical tangency. Note that $p_{|\mathcal{C}_{k,\ell}} : \mathcal{C}_{k,\ell} \to \mathbb{C}$ is a proper $2k$-fold branched covering, extending to $\bar{p}_| : \bar{\mathcal{C}}_{k,\ell} \to \mathbb{P}^1 \equiv \mathbb{C} \cup \{\infty\}$, where $\bar{\mathcal{C}}_{k,\ell}$ is the normalization of its projective closure. Since the curve is rational, $\bar{p}_|$ has $2(2k-1)$ points of ramification, counted with multiplicity. Two of them lie at ∞, each one with multiplicity $k-1$. Since no tangent line to the cuspidal points is vertical, each one of the N cusps contributes with multiplicity 1. The remaining multiplicity accounts for the amount simple tangencies of the projection, namely,

$$2(2k-1) - 2(k-1) - (k+\ell) = k - \ell.$$

Whenever N (and hence $k-\ell$) is even only two of them are real, while in the odd case exactly one is real, leaving $k-\ell-1$ of them in pairs of complex conjugated tangencies. Thus, in the special case $\ell = k-1$ there must be only one vertical non-transversal line, which has to be real, N lines through the cusps, and the vertical lines passing through the nodes (all of them real). Since the horizontal axis is a symmetry axis, it intersects the curve −tangentially− at a cusp, at the $k-2$ nodes, and at the point of vertical tangency. Hence the nodes are in $k - 2 + \frac{N-1}{2}(k-2) = (k-2)k$ vertical lines. Despite the fact that the real picture of $\mathcal{C}_{k,k-1}$ contains all the topological information of its embedding in $\mathbb{C}^2$, Theorem 2.6 cannot be applied directly since it does not satisfy (3). However its quotient by the horizontal reflection will.

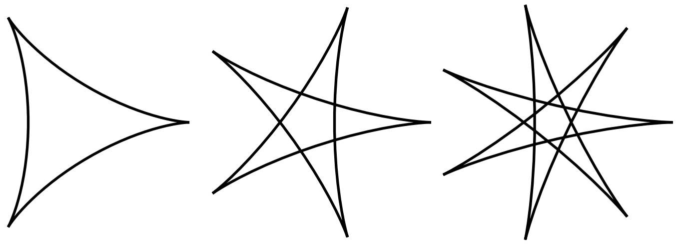


Fig. 5: Curves $\mathcal{C}_{k,k-1}$ for $k = 2, 3, 4$.

The equation $f(x,y)=0$ of $\mathcal{C}_{k,k-1}$ satisfies $f(x,y)=g(x,y^2)$ for some $g\in\mathbb{C}[x,y]$. The curve $\mathcal{D}_{k,k-1}$ defined by $h(x,y):=yg(x,y)$ will do the trick.

Lemma 4.3. *The curve $\mathcal{D}_{k,k-1}$ satisfies the hypotheses of Theorem 2.6.*

Proof. Note that the projection change the local type of branches intersecting the horizontal line, namely, it converts nodes into tangencies, branches with vertical tangencies into transversal branches with respect to the vertical line, and cusps into inflection points, see Figure 6. A topological disk B can be chosen as in Figure 6. The line L is close to the node in the horizontal axis. Since there is no vertical tangency and all the cusps are facing outwards, the result follows. □

Let us label the edges of the non-linear component $\mathcal{D}$ of the curve $\mathcal{D}_{k,k-1}$. Its real part $\mathcal{D}_\mathbb{R}$ is the image of a map $\mathbb{R}\to\mathbb{R}^2$, starting from the negative x-half plane (to the image of the transversal intersection to the horizontal line) to the part after the inflection point.

In order to label the edges we follow some conventions. First, we do not change the labels when passing through a node. We start with the label x_0 for the edge transversal to the horizontal line. We continue as follows:

1. If x_j ends in a tangency in the horizontal line then the next edge is x_{-j}.
2. If $x_{\varepsilon j}$, $j\geq 0$, $\varepsilon=\pm1$ (assume $\varepsilon=(-1)^k$ for $j=0$), then the next edge is $x_{\varepsilon(j+1)}$

With these conventions, the inflection edge is x_{k-1}. This procedure is illustrated in Figure 6 for $k=4$.

Actually, we are not interested in the group $\pi_1(\mathbb{C}^2\setminus\mathcal{D}_{k,k-1})=G_{\mathcal{D}_{k,k-1}}$ but in its quotient G_k obtained by *killing* the square of a meridian of the horizontal line. This is an orbifold fundamental group. If we consider the double cover we obtain that $\pi_1(\mathbb{C}^2\setminus\mathcal{C}_{k,k-1})$ is the kernel of the epimorphism $G_k\to\mathbb{Z}/2$ which sends the meridians of the line to 1 and the meridians of the other component to 0.

Proposition 4.4. *The group G_k is isomorphic to the semidirect product $G_N\rtimes\mathbb{Z}/2$, where G_N is the Artin group of the N-gon.*

Proof. Note that the group G_k is generated by $x_{2-k},\dots,x_{k-1}$ (the edges of the quotient of the hypocycloid) and t, which is the generator in the horizontal line corresponding to the edges adjacent to the normal crossing of the hypocycloid (x_0). Note that all the tacnodes to the left of t correspond to $(-1)^k$-labels, while the tacnodes located to the right of t correspond to $(-1)^{k+1}$-labels.

Let x_j, $j\neq 0,k-1$, be an edge of a tacnode in the side closer to t. Then, $x_{-j}=t'\cdot x_j\cdot t'$, and t' is obtained conjugating t by a product of x_i's, where all the indices i are of the same parity as j (and distinct form $k-1$). As a consequence, $x_{-j}=t\cdot x_j\cdot t$ and we can check that we obtain the presentation of $G_N\rtimes\mathbb{Z}/2$ generated by $t,x_0,\dots,x_{k-1}$ with relations (SD1)-(SD6). □

Corollary 4.5. *The fundamental group of the complement of hypocycloids $\mathcal{C}_{k,k-1}$ is the Artin group of the polygon in $N=2k-1$ vertices. In particular, $G_{\mathcal{C}_{k,k-1}}$ is a Wirtinger presentation of $\pi_1(\mathbb{C}^2\setminus\mathcal{C}_{k,k-1})$.*

5 Extending the method

There are several ways to improve this method to compute fundamental groups of complements of groups. The list of allowed singularities can be enlarged. Besides ordinary singular points (without vertical tangencies), any singular point where all the branches are real and smooth is allowed (as far as no branch has vertical tangency). As in (R1), we will add the local relations induced by

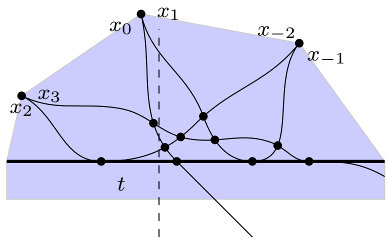


Fig. 6: Quotient curve of $\mathcal{C}_{4,3}$.

Corollary 1.2 and one relation for each branch in order to express the generators of one side in terms of the generators of the other side.

There are other allowed singular points besides ordinary and double points. Namely singular points whose local irreducible components are all double ($\mathbb{A}_{2m}$), real and must be on the same side with respect to the vertical line (and transversal to it). In fact, in this case, we can admit smooth (real) branches with vertical tangency, such that the intersection number with the vertical line is 2 (for each branch); note that we may allow larger intersection multiplicities between these smooth branches. If we want these curves to match with Theorem 2.6, they must satisfy the *facing* condition (3).

Finally, even vertical lines can be admitted as far as the following condition is fulfilled.

1. The global intersection number of the vertical line and $\mathcal{C}_\mathbb{R}$ equals to $\deg_y \mathcal{C}$.
2. If one branch in L is smooth and transversal to the vertical line, then all branches are.
3. If one branch in L has intersection number 2 with L, the same arises for the other branches and all of them are in the same side. Moreover, they must satisfy condition (3) in order to apply Theorem 2.6.

For the cases where Theorem 2.6(3) cannot be avoided there are several ways to provide a combinatorial (and correct) presentation of $\pi_1(\mathbb{C}^2 \setminus \mathcal{C})$. One of them uses the the real part of the pairs of complex conjugate branches, but their computation may be involved, see [2].

If we use the ideas of the proof of Theorem 2.6, we can still recover a combinatorial description of the fundamental group. In order to do this, if $B \cap \mathcal{C} \neq \emptyset$, the construction of the loops associated to each edge of $\mathcal{C}_\mathbb{R}$ must be done taking into account this intersection points. Moreover, the relations must involved the right loops.

Remark 5.1. Let us describe the loops associated with one of these intersection points, namely one close to a a point of type $\mathbb{A}_{2m}$. Let us assume that in local coordinates it has local equation $y^2 + \alpha x^{2m+1} = 0$, $\alpha^2 = 1$. Following the computations in Remark 2.7, its intersection with H_ε is $(\alpha\eta, \sqrt{-1}\varepsilon)$ where $\eta = \varepsilon^{\frac{2}{2m+1}}$. Let us check if the intersection is transversal and which is the intersection number if the curve is naturally oriented as a complex curve and if H_ε has counterclockwise orientation.

An $\mathbb{R}$-basis of the tangent plane for H_ε is given by $\{(1,0),(0,\sqrt{-1})\}$. A $\mathbb{C}$-basis for the complex tangent line to the curve is given by $\{(2\sqrt{-1}\varepsilon), -(2m+1)\alpha\eta^{2m}\}$. The natural orientation of this plane is given by completing the basis with $\{(-2\varepsilon, -(2m+1)\alpha\eta^{2m}\sqrt{-1})\}$. If $\tilde{\eta} = (2m+1)\eta^{2m}$ the

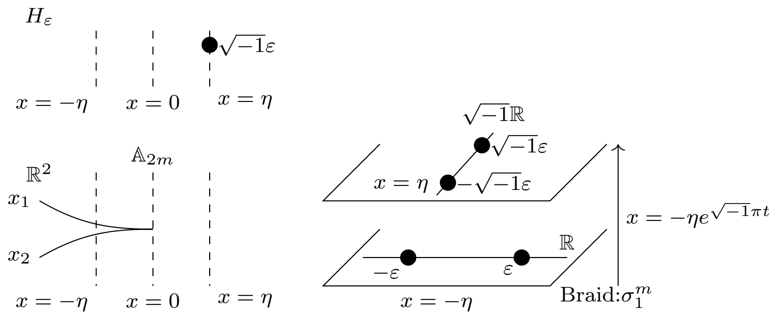


Fig. 7: Meridian at B_ε

orientation is given by the sign of

$$\det\begin{pmatrix}1 & 0 & 0 & -2\varepsilon\\ 0 & 0 & 2\varepsilon & 0\\ 0 & 1 & -\alpha\tilde{\eta} & 0\\ 0 & 0 & 0 & -\alpha\tilde{\eta}\end{pmatrix} = -\det\begin{pmatrix}2\varepsilon & 0\\ 0 & -\alpha\tilde{\eta}\end{pmatrix} = 2\varepsilon\alpha\tilde{\eta}\alpha,$$

which is the sign of α. Hence, if the cusp if left-sided, the sign is positive, while right sided is negative. What happens with the meridians of the intersections with B_ε? In the situation of Figure 7, the local loop in B_ε is $y_1 := x_1^{(\sigma_1^{-m})}$; this is related to the half-tour around the singularity from $x=-\eta$ to $x=\eta$ (counterclockwise from $-\eta$ to η and hence clockwise the other way around). If the $\mathbb{A}_{2m}$-point is on the other side the loop is y_1^{-1}.

Instead of describing a general procedure to compute this group, let us apply it to a couple of examples.

Example 5.2. Let us consider the curve of Example 3.4. We consider L to the right of the cusp in Figure 4b and we take the generators x_1, x_2 as in Figure 3. The relation $x_1 \cdot x_2 \cdot x_1 = x_2 \cdot x_1 \cdot x_2$ holds. The bitangent vertical line provides no actual new relation, but the right-hand side vertical tangency does. If we approach to it following the loop x_1, in order to apply the local relation, we have to consider the loop $z \cdot x_2 \cdot z^{-1}$, where z is a counterclockwise loop around the obstruction point. Following the discussion in Remark 5.1, $z = y_1^{-1}$ and $y_1 = x_1^{(\sigma_1^{-1})} = x_1^{-1} \cdot x_2 \cdot x_1$. Hence

$$x_1 = x_1^{-1} \cdot x_2^{-1} \cdot x_1 \cdot x_2 \cdot x_1^{-1} \cdot x_2 \cdot x_1 \Longrightarrow x_2 \cdot x_1 \cdot x_2^{-1} = x_1 \cdot x_2 \cdot x_1^{-1}$$

Applying the two relations we obtain:

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}) = \langle x_1, x_2 \mid x_1 \cdot x_2 \cdot x_1 = x_2 \cdot x_1 \cdot x_2, [x_1^2, x_2] = 1\rangle \cong \mathbb{Z} \rtimes \mathbb{Z}/3.$$

Example 5.3. Let us consider the union of two concentric circumferences. This curve is of Wirtinger type, see Figure 8 for the vertical tangencies and note that they are tangent at the concyclic points. It is clear that $G_{\mathcal{C}} = \mathbb{Z} * \mathbb{Z}$. The curve does not satisfy the hypotheses of Theorem 2.6. If we choose the generators as in the proof of Theorem 2.6, the tangencies of the inner curve, provide the equality $x_2 = x_3$.

We denote by z_1, z_2 the counterclockwise loops around the inner $\star$-points. From Figure 8, we deduce that $x_1 = z_1 \cdot x_4 \cdot z_1^{-1} = z_2^{-1} \cdot x_4 \cdot z_2$. Using Remark 5.1, we have $z_1 = x_2^{-1}$ and $z_2 = x_2$. Hence, only the relation $x_1 = x_2^{-1} \cdot x_4 \cdot z_2$ remains. As a consequence, $G_{\mathcal{C}}$ is isomorphic to $\pi_1(\mathbb{C}^2 \setminus \mathcal{C})$.

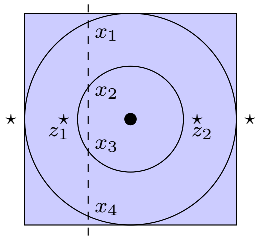


Fig. 8

Remark 5.4. Using the extended Wirtinger method one can prove that the CW-complex associated with the Artin group G_N of the N-gon has the homotopy type of $\mathbb{C}^2 \setminus \mathcal{C}_{k,k-1}$. By simple Euler-characteristic calculations, the case $\ell = k - 1$ is the only one where this can be expected.

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