

Insertion-tolerance and repetitiveness of random graphs

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Abstract

Bond percolation on Cayley graphs provides examples of random graphs. Other examples arise from the dynamical study of proper repetitive subgraphs of Cayley graphs. In this paper we demonstrate that these two families have mutually singular laws as a corollary of a general lemma about countable Borel equivalence relations on first countable Hausdorff spaces.

Keywords: Borel equivalence relations, probability measures, random graphs.

AMS MSC 2010: 05C80, 37A20, 60B05, 60C05.

1 Introduction

The notion of *random rooted graph* (as described by D. Aldous and R. Lyons in [2]) arises in substance from the paper [8] of R. Lyons, R. Pemantle, and Y. Peres on Galton-Watson trees. Bernoulli bond percolation on a Cayley graph \mathbb{G} provides the basic example of random rooted graph, which is obtained by keeping each edge with constant probability p independently to other edges, see Figure 1(a). This kind of random graphs (and other obtained by bond percolation on Cayley graphs or unimodular transitive graphs) enjoy the important property of *insertion-tolerance* introduced by R. Lyons and O. Schramm in Definition 3.2 of [9]: the measure of a nonnull Borel set after adding an edge is nonnull.

However, this property does not hold for other examples of random graphs arising as orbit closures of repetitive subgraphs of \mathbb{G} in the Gromov-Hausdorff space (described in [7]; see also [1] and [4]), see Figure 1(b). The *repetitiveness* of a subgraph of \mathbb{G} is equivalent to the minimality of its orbit closure.

In this paper, we show that the two properties above cannot occur simultaneously. More precisely, any random subgraph of a Cayley graph \mathbb{G} with insertion-tolerant, ergodic in restriction to the set of infinite states and quasi-invariant law and the orbit closure of any proper repetitive subgraph of \mathbb{G} are mutually singular.

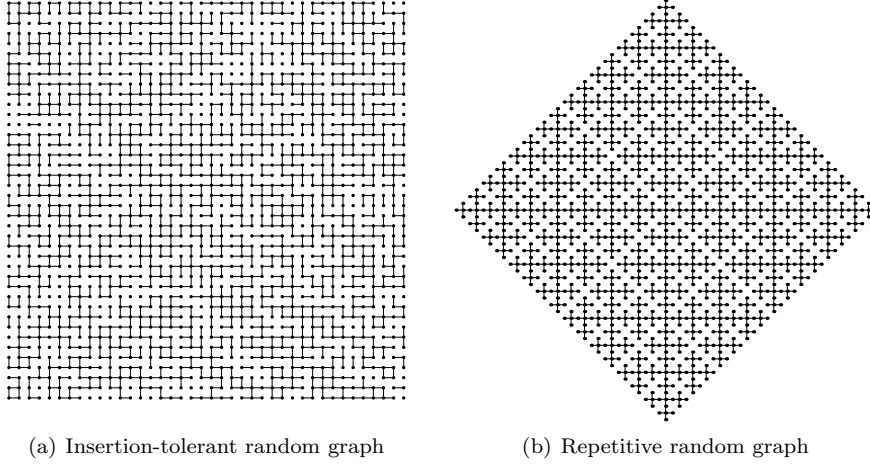


Figure 1: Random subgraphs of the Cayley graph of \mathbb{Z}^2 .

2 A lemma for Borel equivalence relations

Let X be a Borel space and let $\mathcal{R} \subset X \times X$ be a countable Borel equivalence relation. For each $x \in X$, we define $\mathcal{R}[x] = \{y \in X \mid (x, y) \in \mathcal{R}\}$, and similarly for each Borel set $B \subset X$,

$$\mathcal{R}[B] = \bigcup_{x \in B} \mathcal{R}[x].$$

A Borel probability measure μ on X is \mathcal{R} -invariant if $\varphi_*(\mu|_A) = \mu|_B$ for any partial transformation $\varphi : A \rightarrow B$ of \mathcal{R} (i.e. φ is a measurable bijection whose graph is contained in \mathcal{R}), where $\mu|_A$ and $\mu|_B$ are the measures restricted to A and B respectively. If only μ -null sets are preserved, μ is \mathcal{R} -quasi-invariant and \mathcal{R} is μ -nonsingular. The measure μ is \mathcal{R} -ergodic if either $\mu(\mathcal{R}[B]) = 0$ or $\mu(\mathcal{R}[B]) = 1$ for every Borel set $B \subset X$.

If X is also equipped with a topology, a closed subset $Z \subset X$ is \mathcal{R} -minimal if every class $\mathcal{R}[x]$ is dense in Z . Under some additional assumptions, we have the following general lemma:

Lemma 2.1. *Let X be a first countable Hausdorff space and let $\mathcal{L} \subset X \times X$ be a countable Borel equivalence relation. Let \mathcal{R} be a nonsingular equivalence subrelation of \mathcal{L} equipped with an ergodic quasi-invariant probability measure μ . Suppose that there exists a point x in the support of μ such that $\mathcal{L}[x] = \{x\}$. If Z is a closed \mathcal{L} -minimal subset of X , then either $Z = \{x\}$ or $\mu(Z) = 0$.*

Proof. Since X is a first countable Hausdorff space and x belongs to the support of μ , there is a decreasing sequence of open neighborhoods U_n of x such that

$$\bigcap_{n \in \mathbb{N}} U_n = \{x\} \quad \text{and} \quad \mu(U_n) > 0 \text{ for all } n \in \mathbb{N}.$$

As μ is \mathcal{R} -quasi-invariant and \mathcal{R} -ergodic, $\mu(\mathcal{R}[U_n]) = 1$ for all $n \in \mathbb{N}$ and hence

$$Y = \bigcap_{n \in \mathbb{N}} \mathcal{R}[U_n]$$

is also a conull Borel set. Let Z be a closed \mathcal{L} -minimal subset of X and assume that there is a point $z \in Z \cap Y$. Therefore, since $\mathcal{R} \subset \mathcal{L}$, the point

$$z \in Z \cap \mathcal{R}[U_n] \subset Z \cap \mathcal{L}[U_n]$$

for all $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$, we can find a point $z_n \in Z \cap U_n$. The sequence of points $z_n \in Z$ converges to x , but as Z is closed, it follows that $x \in Z$. By the \mathcal{L} -minimality of Z , we conclude that $Z = \{x\}$. Finally, if $Z \cap Y = \emptyset$, then $\mu(Z) = 0$ since Y is conull. \square

Actually, assuming X is compact, we can replace $\{x\}$ by any \mathcal{L} -minimal set \mathcal{M} contained in the support of μ . If there is a point $z \in Z \cap Y$, we still obtain a sequence of points $z_n \in X \cap U_n$. By compactness, passing to a subsequence if necessary, we can assume that z_n converges to point $x \in \mathcal{M}$. But as before, since Z is closed, the limit point $x \in Z$ and hence $Z = \mathcal{M}$ by minimality.

3 Random subgraphs of Cayley graphs

Let G be a countable group G with a finite and symmetric generating set $S \subset G$. The *Cayley graph* $\mathbb{G} = (V, E)$ associated to (G, S) is defined by $V = G$ and $E = \{(g, h) \in G \times G \mid hg^{-1} \in S\}$. It has a natural right G -action by graph automorphisms that combines the natural right action of G on itself and the right diagonal action of G on $E \subset G \times G$. Let 2^E the power set of all subsets of E , equipped with the product topology. This is a compact metrizable space, so in particular first countable and Hausdorff. It also have a natural left G -action by homeomorphisms, which is induced by the natural right G -action on E , namely

$$(g.\omega)(e) = \omega(eg)$$

for all $g \in G$, $\omega \in 2^E$ and $e \in E$. Note that E is a fixed point for this action. Given any finite and symmetric subset $F \subset E$, we consider the open set

$$U_F = \{\omega \in 2^E \mid F \subset \omega\}.$$

and thus we have that

$$\{E\} = \bigcap_{n \in \mathbb{N}} U_{F_n}$$

for any increasing exhaustion $\{F_n\}$ of E by finite and symmetric subsets.

For each $g \in G$ and each $\omega \in 2^E$, we denote by $C_g(\omega)$ the connected component or *cluster* of the graph (G, ω) that contains g . If we take g equal to the identity $\mathbb{1}$, the map associating to each element $\omega \in 2^E$ the cluster $C_{\mathbb{1}}(\omega) \subset \mathbb{G}$ is a continuous map. It takes values in the *Gromov-Hausdorff space* \mathcal{G} formed of all connected subgraphs \mathbb{H} of \mathbb{G} containing $\mathbb{1}$ (see [1] and [7]). In fact, this space is endowed with the ultrametric

$$d(\mathbb{H}, \mathbb{H}') = 1/\exp(\sup\{r \geq 0 \mid B_{\mathbb{H}}(\mathbb{1}, r) = B_{\mathbb{H}'}(\mathbb{1}, r)\}),$$

where $B_{\mathbb{H}}(\mathbb{1}, r)$ is the combinatorial closed ball of radius r centered at $\mathbb{1}$.

The *orbit equivalence relation* \mathcal{R}^G on 2^E is defined by

$$\mathcal{R}^G = \{(\omega, \omega') \in 2^E \times 2^E \mid \exists g \in G : \omega' = g.\omega\},$$

and the *cluster equivalence relations* \mathcal{R} on 2^E by

$$\mathcal{R} = \{ (\omega, \omega') \in 2^E \times 2^E \mid \exists g \in C_1(\omega) : \omega' = g.\omega \}.$$

On the quotient \mathcal{G} , we have another natural equivalence relation, abusively denoted by \mathcal{R} , which is defined by

$$\mathcal{R} = \{ (\mathbb{H}, \mathbb{H}') \in \mathcal{G} \times \mathcal{G} \mid \exists g \in G : \mathbb{H}' = g.\mathbb{H} = \mathbb{H}g^{-1} \}$$

and induced by the equivalence relation \mathcal{R} on 2^E . These two compatible actions are the orbital equivalence relations defined by the left actions of the pseudogroups generated by the local transformations $\omega \mapsto g.\omega$ and $\mathbb{H} \mapsto g.\mathbb{H}$ defined on the open subset of 2^E of all sets ω such that $g \in C_1(\omega)$ and the open subset of \mathcal{G} of all subgraphs \mathbb{H} containing the vertex g respectively.

Let μ be a probability measure on $X = 2^E$ that makes the cluster equivalence relation \mathcal{R} nonsingular. Assume that:

1. The \mathcal{R} -invariant set $X_\infty = \{ \omega \in 2^E \mid \text{card}(\mathcal{R}[\omega]) = \infty \}$ has positive measure, and the restriction of μ to this set is \mathcal{R} -ergodic.
2. The measure μ is *insertion-tolerant*, i.e. for each finite and symmetric set $F \subset E$, the map $i_F : 2^E \rightarrow 2^E$ given by

$$i_F(\omega) = \omega \cup F$$

preserves all Borel sets of positive μ -measure.

Since $\mu(X_\infty) > 0$, we can replace μ with itself conditioned to X_∞ , which is still denoted abusively by μ . By our second assumption, if $\{F_n\}$ is an increasing exhaustion of E , the open sets $i_{F_n}(X_\infty) = U_{F_n} \cap X_\infty$ have positive μ -measure for every n . It follows that all the open neighborhoods U_{F_n} of E have also positive μ -measure and hence E belong to the support of μ . Finally, since E is a fixed point of the G -action on X and $C_1(E) = \mathbb{G}$, we conclude that $\mathcal{R}[E] = \{E\}$. Lemma 2.1 applied to the first countable Hausdorff space $X = 2^E$ equipped with the equivalence relation $\mathcal{L} = \mathcal{R}$ and the G -fixed point $x = E$ yields:

Theorem 3.1. *Let μ be a probability measure on $X = 2^E$. If μ is quasi-invariant and ergodic with respect to \mathcal{R} and insertion tolerant, then any closed \mathcal{R} -minimal set $Z \subset X$ is either $Z = \{E\}$ or $\mu(Z) = 0$. \square*

In this result, we can replace the configuration space $X = 2^E$ with the Gromov-Hausdorff space \mathcal{G} so that a probability measure μ makes \mathcal{G} a *random subgraph* of the Cayley graph \mathbb{G} . If μ is quasi-invariant with respect to \mathcal{R} , we say the random graph (\mathcal{G}, μ) is *nonsingular*. Let \mathcal{G}_\bullet be the locally compact metrizable space of the isomorphism classes of locally finite connected rooted graphs (see [3]). Using the natural continuous map from \mathcal{G} to \mathcal{G}_\bullet sending each graph \mathbb{H} on its isomorphism class $[\mathbb{H}]$, we can interpret (\mathcal{G}, μ) as a *random graph* in the sense of [2]. On the other hand, as pointed out in the introduction, it is well known (see for example [1] and [4]) that the closed \mathcal{R} -minimal subsets of \mathcal{G} are in one-to-one correspondence with the orbit closures of repetitive subgraphs of \mathbb{G} . Now, we can state an equivalent version of the previous theorem:

Theorem 3.2. *Any nonsingular random subgraph of a Cayley graph \mathbb{G} whose law is insertion-tolerant and ergodic in restriction to the set of its infinite states and any orbit closure of a proper repetitive subgraph of \mathbb{G} are mutually singular. \square*

4 Examples

In this section, we assemble some examples to which Theorems 3.1 and 3.2 apply.

Example 4.1. *Bernoulli bond percolation* on a Cayley graph \mathbb{G} provides the basic example of random graph, which is obtained by keeping each edge with constant probability p independently to other edges. More precisely, the power set 2^E is equipped with the Bernoulli measure μ with constant survival parameter p (which means that μ is the product of the measure on $\{0, 1\}$ assigning probabilities $1 - p$ and p to 0 and 1 respectively). In the supercritical phase $p_c < p < 1$, if we denote by \mathbb{P} the law that governs the clusters $C_1(\omega)$, the set of infinite clusters has positive \mathbb{P} -measure. According to the cluster indistinguishability theorem proved by R. Lyons and O. Schramm in [9] (see also [6]), the measure \mathbb{P} conditioned to the set of infinite clusters is \mathcal{R} -ergodic and insertion-tolerant. Then the orbit closure of any proper repetitive subgraph \mathbb{H} of \mathbb{G} is \mathbb{P} -null.

Example 4.2. Since Theorem 3.3 of [9] is valid for any insertion-tolerant \mathcal{R} -invariant probability measure μ on 2^E , Theorem 3.2 also applies to random graphs obtained by this kind of G -invariant percolation. Recall that a G -invariant bond percolation process on \mathbb{G} is given by a measure preserving G -action on a standard Borel probability space (X, μ) together with a G -equivariant Borel map $\pi : X \rightarrow 2^E$ (see [5]). In the case under consideration, the map π is the identity id , but the law of the process is not longer the Bernoulli measure. However, the cluster indistinguishability theorem also holds for some G -invariant percolation processes with $\pi \neq id$, like the *percolation process with scenery* introduced in Remark 3.4 of [9] (see also Proposition 6 of [6]).

Example 4.3. Let $\mathbb{G} = (V, E)$ be a locally finite graph and let G be a unimodular closed group of automorphisms of \mathbb{G} acting transitively on V . Then the proof of Theorem 3.1 extends to this case. Thus, if μ is an insertion tolerant, \mathcal{R} -ergodic and \mathcal{R} -invariant probability measure on 2^E , any closed \mathcal{R} -minimal subset Z of 2^E is either $Z = \{E\}$ or $\mu(Z) = 0$. But Theorem 3.3 of [9] is also valid for any G -invariant insertion-tolerant bond percolation process on \mathbb{G} , and hence Theorem 3.2 applies to all unimodular random graphs obtained by this kind of percolation.

Acknowledgements

The Lemma 2.1 and its proof were provided by the referee based on the first version of the paper. Section 3 also benefited from their suggestions and comments. The authors were supported by Spanish Excellence Grant MTM2013-46337-C2-2-P, Galician Grant GPC2015/006 and the European Regional Development Fund. Second author was also supported by the European Social Fund and Diputación General de Aragón (Grant E15 Geometría).

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