THE TRIPLE POINT SPECTRUM OF CLOSED ORIENTABLE 3-MANIFOLDS

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ABSTRACT. The triple point numbers and the triple point spectrum of a closed 3-manifold are topological invariants that give a measure of the complexity of the 3-manifold using the number of triple points of minimal filling Dehn surfaces. Basic properties of these invariants are presented, and the triple point spectra of $\mathbb{S}^2 \times \mathbb{S}^1$ and $\mathbb{S}^3$ are computed.

1. INTRODUCTION

Through the whole paper all 3-manifolds and surfaces are assumed to be orientable and closed, that is, compact connected and without boundary. All objects are assumed to be in the smooth category: manifolds have a differentiable structure and all maps are assumed to be smooth. By simplicity a genus g surface, $g = 0, 1, 2, \ldots$, is called a *g-torus*.

Let M be a 3-manifold.

A subset $\Sigma \subset M$ is a *Dehn surface* in M [14] if there exists a surface S and a general position immersion $f : S \to M$ such that $\Sigma = f(S)$. In this situation we say that S is the *domain* of Σ and that f *parametrizes* Σ. If S is a 2-sphere, a torus, or a g-torus then Σ is a *Dehn sphere*, a *Dehn torus*, or a *Dehn g-torus* respectively. For a Dehn surface $\Sigma \subset M$, its singularities are divided into *double points*, where two sheets of Σ intersect transversely, and *triple points*, where three sheets of Σ intersect transversely, and they are arranged along *double curves* (see Section 2 below for definitions and pictures).
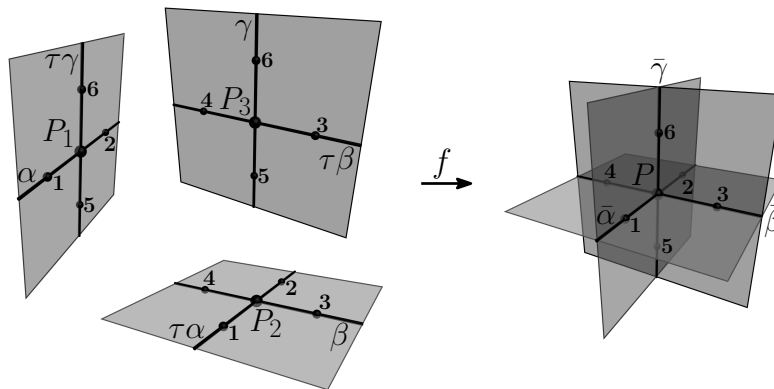
A Dehn surface $\Sigma \subset M$ *fills* M [13] if it defines a cell-decomposition of M in which the 0-skeleton is the set of triple points of Σ, the 1-skeleton is the set of double and triple points of Σ, and the 2-skeleton is Σ itself. Filling Dehn spheres are introduced (following ideas of W. Haken [8]) in [13], where it is proved that every closed, orientable 3-manifold has a filling Dehn sphere. In [13] filling Dehn spheres and their Johansson diagrams (cf. Section 2) are proposed as a suitable way for representing all closed orientable 3-manifolds. A weaker version of filling Dehn surfaces are the *quasi-filling Dehn surfaces* defined in [3]. A quasi-filling Dehn surface in M is a Dehn surface whose complementary set in M is a disjoint union of open 3-balls. In [6] it is proved that every 3-manifold has a quasi-filling Dehn sphere.

Fig. 1. A triple point of Σ and its triplet in S

As it is pointed out in [3], filling Dehn surfaces are closely related with *cubulations* of 3-manifolds: cell decompositions where the building blocks are cubes (see for instance [1], [4] or [7]). The dual decomposition of that defined by a filling Dehn surface is a cubulation and vice versa, the dual cell decomposition of a cubulation is a filling Dehn surface (perhaps with nonconnected or nonorientable domain).

The number of triple points of filling Dehn surfaces measures the complexity of 3-manifolds. A filling Dehn surface $\Sigma \subset M$ with domain S is *minimal* if there is no filling Dehn surface in M with domain S with less triple points than Σ. In [16] the *genus g triple point number* $t_g(M)$ of M is defined as the number of triple points of a minimal filling g-torus of M, and the *triple point spectrum* of M as the sequence of numbers

$$\mathscr{T}(M) = (t_0(M), t_1(M), t_2(M), \ldots).$$

The genus 0 triple point number of M is the *Montesinos complexity* of M [12]. These invariants are related to the *surface-complexity* of M, defined in [3] as the minimal number of triple points of a quasi-filling Dehn surface of M (perhaps with nonconnected or nonorientable domain).

Filling Dehn surfaces are an almost unexplored field with many natural and intriguing open questions. For example, it is interesting to find an effective characterization of minimal filling Dehn surfaces. We wonder which classical 3-manifold invariants can be efficiently computed from filling Dehn surfaces, or which new invariants can be constructed out of them, as in [2] for example.

We want to clarify how much information about M encloses $\mathscr{T}(M)$. Since the original work [16], nothing is known about the triple point numbers and the triple point spectrum apart from their definitions. In this paper we introduce some properties of $\mathscr{T}(M)$ (Section 3), and we present the two unique known triple point spectra

(1.1) $$\mathscr{T}(\mathbb{S}^2 \times \mathbb{S}^1) = (2, 1, 3, 5, \ldots, 2g - 1, \ldots),$$

(1.2) $$\mathscr{T}(\mathbb{S}^3) = (2, 4, 6, 8, \ldots, 2g + 2, \ldots).$$
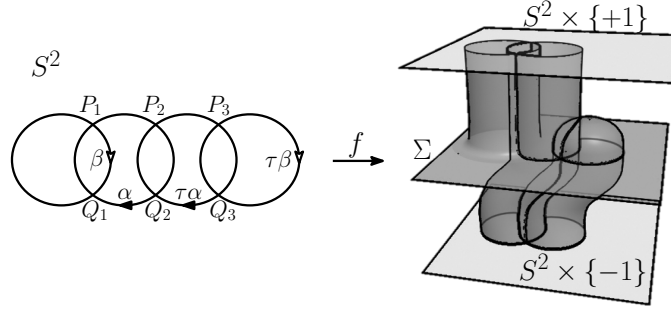
FIG. 2. The "Audi" diagram represents $\mathbb{S}^2 \times \mathbb{S}^1$.

The proof of (1.1) relies on two previously known examples of minimal filling Dehn surfaces of $\mathbb{S}^2 \times \mathbb{S}^1$, together with the inequality (Lemma 3.1)

$$(1.3) \qquad\qquad t_{g+1}(M) \leq t_g(M) + 2, \ g = 0, 1, 2, \ldots,$$

that arises from a simple surgery operation (the *handle piping*, see Figure 3). This will be detailed in Section 3.

The characterization of the triple point spectrum of $\mathbb{S}^3$ needs more work. The identity (1.2) is proposed in [12] as an open question. Using that $t_0(\mathbb{S}^3) = 2$ [12], the handle piping provides filling Dehn g-tori in $\mathbb{S}^3$ with $2 + 2g$ triple points for all $g = 1, 2, \ldots$. The harder part is to prove that all these filling Dehn surfaces in $\mathbb{S}^3$ are minimal. To do so, in Section 4 we introduce a presentation of the fundamental group of a Dehn g-torus. This is a generalization of a presentation of the fundamental group of Dehn spheres due to W. Haken and detailed in [16] (cf. [17, Chapter 4]). Using it, in Section 5 we prove:

Theorem 5.6. *If M is a $\mathbb{Z}/2$-homology 3-sphere*

$$t_g(M) \geq 2 + 2g.$$

This theorem completes the proof of (1.2).

2. Dehn surfaces and Johansson's diagrams

Let Σ be a Dehn surface in M and consider a parametrization $f : S \to M$ of Σ.

The *singularities* of Σ are the points $x \in \Sigma$ such that $\#f^{-1}(x) > 1$. The *singularity set* $S(\Sigma)$ of Σ is the set of singularities of Σ. As f is in general position, the singularities of Σ can be divided into double points ($\#f^{-1}(x) = 2$), and triple points ($\#f^{-1}(x) = 3$). Following [15], we denote by $T(\Sigma)$ the set of triple points of Σ. The preimage under f in S of the singularity set of Σ, together with the information about how its points become identified by f in Σ is the *Johansson diagram* $\mathcal{D}$ of Σ (see [10, 13]). We say that two points of S are *related* if they project onto the same point of Σ.

In the following a *curve* in S, Σ or M is the image of an immersion (a *parametrization* of the curve) from $\mathbb{S}^1$ or $\mathbb{R}$ into S, Σ or M, respectively. A *path* in S, Σ or M is a map η from the interval $[0, 1]$ into S, Σ or M, respectively. We say that η *joins* $\eta(0)$ *with* $\eta(1)$, or that η *starts* at $\eta(0)$ and *ends* at $\eta(1)$, and if $\eta(0) = \eta(1)$ we say that η is a *loop*. A double curve of Σ is a curve in M contained in $S(\Sigma)$.

Because S is closed, double curves are closed and there is a finite number of them. The number of triple points is also finite. Because S and M are orientable, the preimage under f of a double curve of Σ is the union of two different closed curves in S, and we will say that these two curves are *sister curves* of $\mathcal{D}$. Thus, the Johansson diagram of Σ is composed by an even number of different closed curves in S, and we will identify $\mathcal{D}$ with the set of different curves that compose it. For any curve $\alpha \in \mathcal{D}$ we denote by $\tau\alpha$ the sister curve of α in $\mathcal{D}$. This defines a free involution $\tau : \mathcal{D} \to \mathcal{D}$, the *sistering* of $\mathcal{D}$, that sends each curve of $\mathcal{D}$ into its sister curve of $\mathcal{D}$.

The curves of $\mathcal{D}$ intersect with others or with themselves transversely at the *crossings* of $\mathcal{D}$. The crossings of $\mathcal{D}$ are the preimage under f of the triple points of Σ. If P is a triple point of Σ, the three crossings of $\mathcal{D}$ in $f^{-1}(D)$ compose *the triplet of P* (see Figure 1). In Section 4 we will consider paths contained in curves of $\mathcal{D}$ or in double curves. In this special case, we will consider that paths are also immersed.

If Σ is a Dehn surface in M, a connected component of $M - \Sigma$ is a *region* of Σ, a connected component of $\Sigma - S(\Sigma)$ is a *face* of Σ, and a connected component of $S(\Sigma) - T(\Sigma)$ is an *edge* of Σ. The Dehn surface Σ fills M if and only if all its edges, faces and regions are open 1, 2 or 3-dimensional disks respectively.

In Figure 2 we have depicted (left) one of the simplest Johansson diagrams of filling Dehn spheres. This is the diagram of a Dehn sphere Σ (right) that fills $\mathbb{S}^2 \times \mathbb{S}^1$. In this figure we consider $\mathbb{S}^2 \times \mathbb{S}^1$ as $\mathbb{S}^2 \times [-1, 1]$ where the top and bottom covers $\mathbb{S}^2 \times \{+1\}$ and $\mathbb{S}^2 \times \{-1\}$ are identified by the vertical projection. This example appears completely detailed in [17, Section 7.2.1].

If we are given an *abstract diagram*, i.e., an even collection of curves in S coherently identified in pairs, it is possible to know if this abstract diagram is *realizable*: if it is actually the Johansson diagram of a Dehn surface in a 3-manifold (see [10, 11, 17]).

It is also possible to know whether the abstract diagram is filling, i.e., whether it is the Johansson diagram of a filling Dehn surface of a 3-manifold (see [17]). If Σ fills M, it is possible to build M out of the Johansson diagram of Σ. Thus, filling Johansson diagrams represent all closed, orientable 3-manifolds. It must be noted that when a diagram $(\mathcal{D}, \tau)$ in S is not realizable, the quotient space of S under the equivalence relation defined by the diagram is something very close to a Dehn surface: it is a 2-dimensional complex with simple, double and triple points, but it cannot be embedded in any 3-manifold. We reserve the name *pseudo Dehn surface* for these objects. Many constructions about Dehn surfaces, as the presentation of their fundamental group given in Section 4, for example, are also valid for pseudo Dehn surfaces.

3. THE TRIPLE POINT SPECTRUM

In order to understand the structure of the sequence $\mathcal{T}(M)$, we need to relate the different triple point numbers of M. For a shorter notation we will omit the word "filling" in the expression "minimal filling Dehn surface". If we are given a filling Dehn g-torus Σ_g of M with q triple points, we can always obtain a filling Dehn $(g+1)$-torus Σ_{g+1} of M with $q+2$ triple points by applying to Σ_g the *handle piping* modification depicted in Figure 3 in a small neighbourhood of a triple point. If the original Σ_g is minimal, we have:
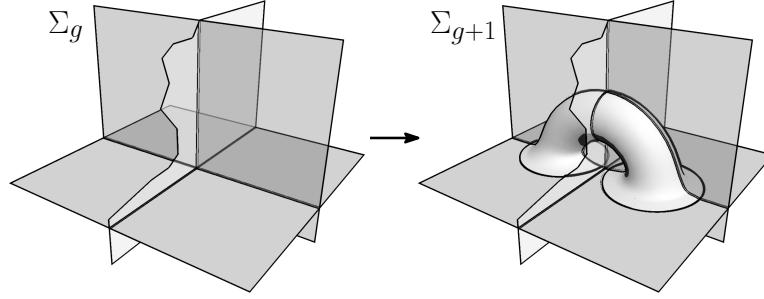
FIG. 3. Handle piping

Lemma 3.1. *For any $g = 0, 1, 2, \ldots$, the following holds*

$$(3.1) \qquad t_{g+1}(M) \leq t_g(M) + 2. \qquad \square$$

This lemma suggests the following definitions.

Definition 3.2. A minimal Dehn g-torus of M is *exceptional* if it has less than $t_{g-1}(M) + 2$ triple points.

Definition 3.3. The *height* $\mathscr{H}(M)$ of M is the highest genus among all exceptional Dehn g-tori of M. If M has no exceptional Dehn g-torus, $\mathscr{H}(M) = 0$.

It is clear that if $\mathscr{H}(M)$ is finite, the equality in (3.1) holds for all $g \geq \mathscr{H}(M)$.

An important result for filling Dehn surfaces which is of great interest here is the following.

Proposition 3.4. *Given a filling Dehn g-torus in M with q triple points and r regions, then*

$$(3.2) \qquad r = q + (2 - 2g).$$

This proposition follows from [16, Lemma 43] (see also [17, Theorem 3.25]). Its proof relies on Euler's characteristic techniques and it has strong consequences in this context:

Remark 3.5. A filling Dehn g-torus has at least $2g - 1$ triple points.

Remark 3.6. After a handle piping on a filling Dehn g-torus the number of regions remains unchanged. More generally, if Σ_g is a filling Dehn g-torus of M with q triple points and Σ_{g+1} is a filling Dehn $(g+1)$-torus of M with $q + 2$ triple points, the number of regions of Σ_g and Σ_{g+1} is the same.

Remark 3.7. Exceptional Dehn g-tori reduce the number of regions, that is, an exceptional Dehn g-torus in M has less regions than any minimal Dehn g'-torus in M with $g' < g$.

Since the number of regions is bounded from below, by Remark 3.7 there cannot be exceptional Dehn g-tori in M with arbitrarily high genus. Therefore,

Theorem 3.8. *The height of M is finite.* $\qquad \square$

This solves a question posed to the second author by H. Morton.

In [17, Section 7.2.2] it is shown a filling Dehn torus of $\mathbb{S}^2 \times \mathbb{S}^1$ with just 1 triple point. It is clearly a minimal Dehn torus, and by Proposition 3.4 it has only one

(a) (b)
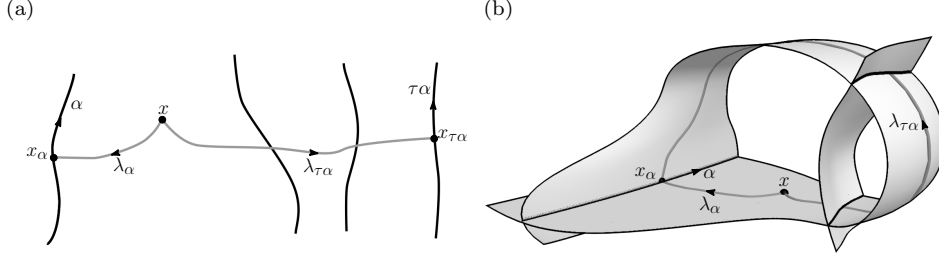
FIG. 4. Dual curves to the curves of $\mathcal{D}$

region. By Remark 3.7, there cannot be an exceptional Dehn g-torus in $\mathbb{S}^2 \times \mathbb{S}^1$ of genus $g > 1$. On the other hand, a Dehn sphere in any 3-manifold has an even number of triple points [8], and therefore the filling Dehn sphere of Figure 2 is minimal. It turns out that $\mathscr{H}(\mathbb{S}^2 \times \mathbb{S}^1) = 1$ and

Theorem 3.9. $\mathscr{T}(\mathbb{S}^2 \times \mathbb{S}^1) = (2, 1, 3, 5, \ldots, 2g - 1, \ldots).$ □

4. The fundamental group of a Dehn g-torus

In this section the manifold M containing Σ is no longer needed. Although we still talk about "Dehn surfaces", the construction only makes use of the Johansson diagram $\mathcal{D}$, so it is valid for pseudo Dehn surfaces.

Let $f : S \to M$ be a Dehn surface in M and denote by $\mathcal{D}$ its Johansson diagram.

Fix a simple point $x \notin \mathcal{D}$ as the base point of the fundamental group $\pi_1(\Sigma)$ of Σ. We identify x with its preimage under f.

A path in Σ is *surfacewise* if it is the image of a path on S through f. We denote by $\pi_S = f_*\pi_1(S)$ the subgroup of $\pi_1(\Sigma)$ generated by the sufacewise loops based at x. In general we will use the same notation for a surfacewise path in Σ and its preimage path in S.

Let $\alpha, \tau\alpha$ be two sister curves of $\mathcal{D}$, and take two paths $\lambda_\alpha, \lambda_{\tau\alpha}$ in S, starting from x and arriving to related points on $\alpha, \tau\alpha$, respectively (see Figure 4).

Definition 4.1. We say that the loop $\lambda_\alpha \lambda_{\tau\alpha}^{-1}$ in Σ based at x is *dual* to α.

Proposition 4.2. *Surfacewise loops based at x and loops dual to the curves of $\mathcal{D}$ generate $\pi_1(\Sigma)$.*

Proof. Consider a loop σ in Σ based at x. Up to homotopy, we can assume that all the intersections of σ with the singular set of Σ are of "crossing type" or of "corner type" as in Figure 5(a) and 5(b) respectively. Therefore, σ can be written as a product $\sigma = b_1 b_2 \cdots b_k$ of surfacewise paths (Figure 6). If $k = 1$ or 2, either σ is surfacewise or it is dual to a curve of $\mathcal{D}$, and so there is nothing to prove.

Otherwise, for each $i = 2, \ldots, k - 1$ we choose a midpoint x_i of b_i and a path c_i joining x with x_i (Figure 6). We write $b_i = b_i^- b_i^+$, with b_i^- ending and b_i^+ starting at x_i. The loop σ is homotopic to

$$b_1 b_2^- c_2^{-1} c_2 b_2^+ b_3^- \cdots c_{k-2} b_{k-2}^+ b_{k-1}^- c_{k-1}^{-1} c_{k-1} b_{k-1}^+ b_k,$$

and this expression is a product of loops dual to curves of $\mathcal{D}$. □

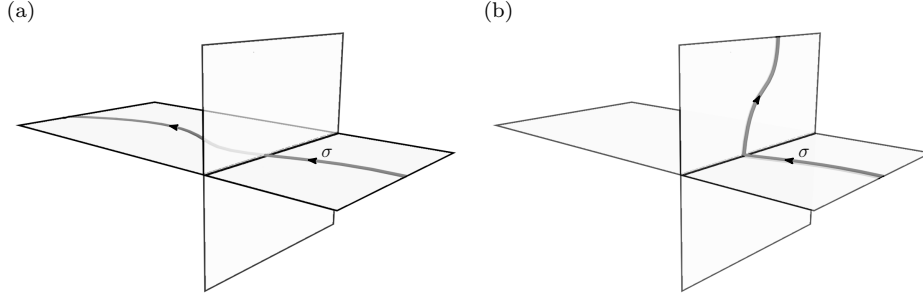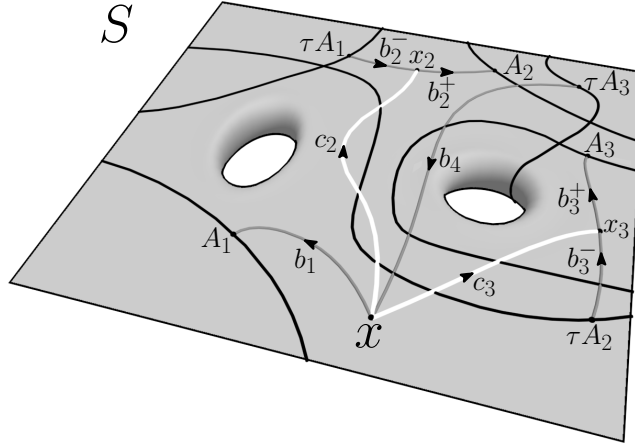(a)   (b)

FIG. 5. Crossing-type and corner-type intersections with $S(\Sigma)$

FIG. 6. Loops as product of dual loops

We know that π_S is finitely generated because so is $\pi_1(S)$. The following lemma allows us to find a finite set of generators for $\pi_1(\Sigma)$.

Lemma 4.3. *If a, a' are loops on Σ dual to $\alpha \in \mathcal{D}$, they are* surfacewise conjugate, *that is, there are two surfacewise loops s, t based at x such that $a = sa't$.*

Proof. Put $a = \lambda_\alpha \lambda_{\tau\alpha}^{-1}$, $a' = \mu_\alpha \mu_{\tau\alpha}^{-1}$, where $\lambda_\alpha, \lambda_{\tau\alpha}, \mu_\alpha, \mu_{\tau\alpha}$ are surfacewise paths as in Definition 4.1. Let $x_\alpha, x_{\tau\alpha}, x'_\alpha, x'_{\tau\alpha}$ be the endpoints of $\lambda_\alpha, \lambda_{\tau\alpha}, \mu_\alpha, \mu_{\tau\alpha}$ in $\mathcal{D}$, respectively (Figure 7). Let d be one path in S contained in α joining x_α with x'_α, and let τd be the sister path of d inside $\tau\alpha$ joining $x_{\tau\alpha}$ with $x'_{\tau\alpha}$. In Σ the paths d and τd coincide, and so we have:

$$a = \lambda_\alpha \lambda_{\tau\alpha}^{-1} = \lambda_\alpha d(\tau d)^{-1} \lambda_{\tau\alpha}^{-1} = \lambda_\alpha d\mu_\alpha^{-1} \mu_\alpha \mu_{\tau\alpha}^{-1} \mu_{\tau\alpha}(\tau d)^{-1} \lambda_{\tau\alpha}^{-1} = sa't,$$

where $s = \lambda_\alpha d\mu_\alpha^{-1}$ and $t = \mu_{\tau\alpha}(\tau d)^{-1}\lambda_{\tau\alpha}^{-1}$ are surfacewise loops in Σ based at x. $\square$

From now on, we will fix a set of generators $s_1, s_2, \ldots, s_{2g}$ of π_S, and we will also fix a set of preferred dual loops to the curves of $\mathcal{D}$ as follows. For each diagram curve $\alpha \in \mathcal{D}$ we choose a *basepoint* x_α and a joining arc λ_α from x to x_α, such that
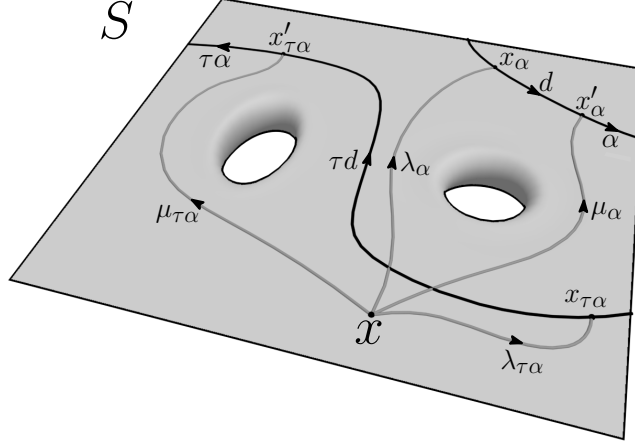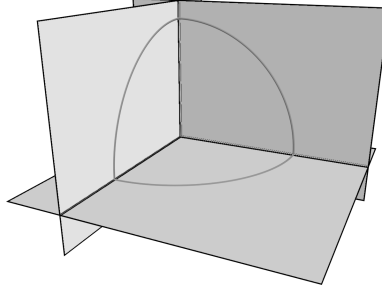
FIG. 7. Duals to α are surfacewise conjugate

FIG. 8. Contractible loop near a triple point

the basepoints of sister curves are related (see Figure 4). The preferred dual loop of $\alpha \in \mathcal{D}$ is

$$a = \lambda_\alpha \lambda_{\tau\alpha}^{-1}.$$

If a curve of $\mathcal{D}$ is denoted by lowercase greek letters

$$\alpha, \beta, \gamma, \ldots, \alpha_1, \alpha_2, \ldots, \alpha_i, \ldots, \tau\alpha,$$

its preferred dual loop will be denoted by the corresponding lowercase roman letters

$$a, b, c, \ldots, a_1, a_2, \ldots, a_i, \ldots, \tau a.$$

By Proposition 4.2 and Lemma 4.3, we have:

Theorem 4.4. *If $\alpha_1, \alpha_2, \ldots, \alpha_{2k}$ are the curves of $\mathcal{D}$, then $\pi_1(\Sigma)$ is generated by*

$$s_1, s_2, \ldots, s_{2g}, a_1, a_2, \ldots, a_{2k}. \qquad \square$$

In order to obtain a presentation of $\pi_1(\Sigma)$, we need to establish a set of relators associated to the generators of Theorem 4.4. These relators will be a natural extension of Haken's relators for Dehn spheres (see also [12, 17]).
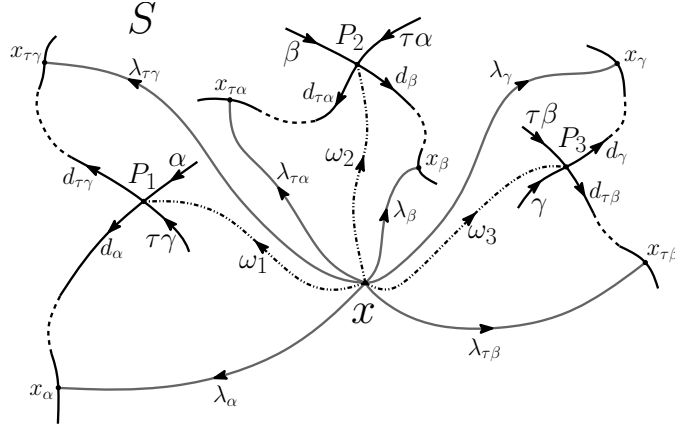
FIG. 9. Triple point relation (diagram view)

(R1) *Dual loop relations.* By construction, if $\alpha, \tau\alpha$ are two sister curves of $\mathcal{D}$, their dual loops $a, \tau a$ verify $\tau a = a^{-1}$.

(R2) *Triple point relations.* The idea behind this relation is that any small circuit around a triple point as that of Figure 8 is homotopically trivial. Assume that all curves of $\mathcal{D}$ are oriented, with sister curves coherently oriented. Let P be a triple point of Σ and let P_1, P_2, P_3 be the three crossings of $\mathcal{D}$ in the triplet of P. Label the curves of $\mathcal{D}$ intersecting at these crossings as in Figure 9. Consider three paths $\omega_1, \omega_2, \omega_3$ joining x with P_1, P_2, P_3 respectively. At P_1, consider the path d_α contained in α that travels from P_1 along the branch labelled α in the positive sense until it reaches the basepoint x_α of α. Consider also the similarly defined paths d_β, d_γ contained in β, γ and joining P_2, P_3 with x_β, x_γ respectively, and the sister paths $d_{\tau\alpha}, d_{\tau\beta}, d_{\tau\gamma}$ of $d_\alpha, d_\beta, d_\gamma$ respectively, that join P_2, P_3, P_1 with $x_{\tau\alpha}, x_{\tau\beta}, x_{\tau\gamma}$ along $\tau\alpha, \tau\beta, \tau\gamma$, respectively. If we introduce the loops

$$t_\alpha = \omega_1 d_\alpha \lambda_\alpha^{-1} \qquad\qquad t_\beta = \omega_2 d_\beta \lambda_\beta^{-1} \qquad\qquad t_\gamma = \omega_3 d_\gamma \lambda_\gamma^{-1}$$

$$t_{\tau\alpha} = \omega_2 d_{\tau\alpha} \lambda_{\tau\alpha}^{-1} \qquad\qquad t_{\tau\beta} = \omega_3 d_{\tau\beta} \lambda_{\tau\beta}^{-1} \qquad\qquad t_{\tau\gamma} = \omega_1 d_{\tau\gamma} \lambda_{\tau\gamma}^{-1}$$

we get the triple point relation around P:

$$(4.1) \qquad\qquad t_\alpha a t_{\tau\alpha}^{-1} t_\beta b t_{\tau\beta}^{-1} t_\gamma c t_{\tau\gamma}^{-1} = 1.$$

Note that the loops $t_\alpha, t_{\tau\alpha}, t_\beta, t_{\tau\beta}, t_\gamma$ and $t_{\tau\gamma}$ are surfacewise elements of Σ and so they can be expressed as words in $s_1, s_2, \ldots, s_{2g}$. See Figure 10.

(R3) *Double curve relations.* In [12, 16] this relators did not appear because in $\mathbb{S}^2$ all loops are homotopically trivial, while in a g-torus this is not true. Let $\alpha, \tau\alpha$ be a pair of sister curves of $\mathcal{D}$. We orient both curves coherently starting from their basepoints $x_\alpha, x_{\tau\alpha}$. The loops $\lambda_\alpha \alpha \lambda_\alpha^{-1}$ and $\lambda_{\tau\alpha} \tau\alpha \lambda_{\tau\alpha}^{-1}$ in S are related in Σ because α and $\tau\alpha$ coincide in Σ:

$$\lambda_\alpha \alpha \lambda_\alpha^{-1} = \lambda_\alpha \lambda_{\tau\alpha}^{-1} \lambda_{\tau\alpha} \tau\alpha \lambda_{\tau\alpha}^{-1} \lambda_{\tau\alpha} \lambda_\alpha^{-1}.$$

If we also denote by α and $\tau\alpha$ the surfacewise elements of $\pi_1(\Sigma)$ represented by $\lambda_\alpha \alpha \lambda_\alpha^{-1}$ and $\lambda_{\tau\alpha} \tau\alpha \lambda_{\tau\alpha}^{-1}$, respectively, this relation can be written as

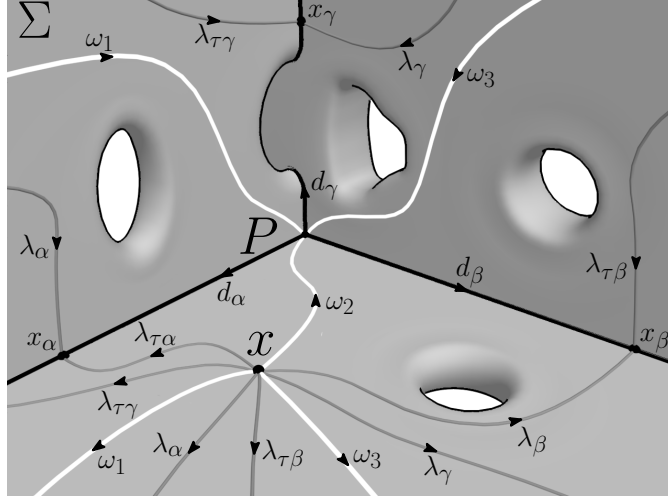$$(4.2) \qquad\qquad \alpha = a \, \tau\alpha \, a^{-1}.$$

FIG. 10. Triple point relation (Dehn surface view)

(R4) Surface relations. Those are the relations that the surfacewise generators $s_1, s_2, \ldots, s_{2g}$ verify when considered as elements of $\pi_1(S)$.

Any relation among those specified above define an associated *dual loop relator*, *triple point relator*, *double curve relator* or *surface relator* in the natural way.

Theorem 4.5. *If $\mathcal{D} = \{\alpha_1, \alpha_2, \ldots, \alpha_{2k}\}$, the fundamental group $\pi_1(\Sigma)$ is isomorphic to*

$$\pi(\mathcal{D}) = \langle\, s_1, s_2, \ldots, s_{2g}, a_1, a_2, \ldots, a_{2k} \mid R1,\ R2,\ R3 \text{ and } R4 \text{ relators}\,\rangle.$$

We need to introduce some notation and a previous result before being able to give the proof of this theorem. If we consider the generators of $\pi(\mathcal{D})$ as elements of $\pi_1(\Sigma)$, we are defining implicitly a homomorphism $\varepsilon : \pi(\mathcal{D}) \to \pi_1(\Sigma)$ which is surjective by Theorem 4.4. To prove that ε is also injective we will prove first that $\pi(\mathcal{D})$ behaves exactly as $\pi_1(\Sigma)$ when dealing with covering spaces: the covering spaces of Σ are completely characterized by representations of $\pi(\mathcal{D})$ on a permutation group. For $m \in \mathbb{N}$ we denote by Ω_m the group of permutations of the finite set $\{1, 2, \ldots, m\}$, and we denote by Ω_∞ the group of permutations of the countable set $\mathbb{N}$.

Lemma 4.6. *Every representation $\rho : \pi(\mathcal{D}) \to \Omega_m$, with $m \in \mathbb{N} \cup \{\infty\}$, factors through ε, i.e., there exists a representation $\hat{\rho} : \pi_1(\Sigma) \to \Omega_m$ such that the following diagram commutes.*

$$
\begin{array}{ccc}
\pi(\mathcal{D}) & \xrightarrow{\ \rho\ } & \Omega_m \\[2pt]
{\scriptstyle \varepsilon}\big\downarrow & \nearrow {\scriptstyle \hat{\rho}} & \\[2pt]
\pi_1(\Sigma) & &
\end{array}
$$

Proof. Fix an $m \in \mathbb{N} \cup \{\infty\}$, and consider a representation $\rho : \pi(\mathcal{D}) \to \Omega_m$. As $\pi(\mathcal{D})$ verifies the surface relations we have a natural homomorphism

$$\zeta : \pi_1(S) \to \pi(\mathcal{D}).$$

The map $\rho_S = \rho \circ \zeta$ is a representation of $\pi_1(S)$, it is the monodromy homomorphism of a m-sheeted covering map $h : \hat{S} \to S$.

Let $x^1, x^2, \ldots$ be the different points of $\hat{S}$ on the fiber of h over x, labelled in such way that for any $s \in \pi_1(S)$ the lift of s starting in x^i has its end in x^j, where $j = \rho_S(s)(i)$.

If we lift the diagram $\mathcal{D}$ to $\hat{S}$, we obtain another diagram $\hat{\mathcal{D}}$ in $\hat{S}$. We will define a "sistering" $\hat{\tau} : \hat{\mathcal{D}} \to \hat{\mathcal{D}}$ between the curves of $\hat{\mathcal{D}}$, with the expected compatibility with τ. For any curve $\alpha \in \mathcal{D}$, we call λ_α^i the lift of λ_α starting at x^i, and x_α^i the endpoint of λ_α^i. The sistering $\hat{\tau}$ is defined as follows: $\hat{\tau}$ sends the curve of $\hat{\mathcal{D}}$ passing through x_α^i to the curve of $\hat{\mathcal{D}}$ passing through $x_{\tau\alpha}^j$, where $j = \rho(a)(i)$. The points of both curves are related by $\hat{\tau}$ starting from $x_\alpha^i, x_{\tau\alpha}^j$ lifting the relation among the points of α and $\tau\alpha$ near x_α and $x_{\tau\alpha}$ in the natural way. If we think about the circle $\mathbb{S}^1$ as the real numbers modulo 2π, we can take parametrizations $\alpha, \tau\alpha : \mathbb{R} \to S$ of the curves $\alpha, \tau\alpha$ respectively such that $\alpha(0) = x_\alpha$, $\tau\alpha(0) = x_{\tau\alpha}$ and such that $\alpha(t)$ is related by τ with $\tau\alpha(t)$ for all $t \in \mathbb{R}$. Taking lifts $\alpha^i, \tau\alpha^j : \mathbb{R} \to \hat{S}$ of these parametrizations with $\alpha^i(0) = x_\alpha^i$, $\tau\alpha^j(0) = x_{\tau\alpha}^j$ we state that $\alpha^i(t)$ is related by $\hat{\tau}$ with $\tau\alpha^j(t)$ for all $t \in \mathbb{R}$. We want to prove that $(\hat{\mathcal{D}}, \hat{\tau})$ is an abstract Johansson diagram on $\hat{S}$.

Claim 1. *The sistering $\hat{\tau}$ is well defined: if $x_\alpha^i, x_\alpha^{i'}$ are different lifts of x_α lying on the same curve $\hat{\alpha}$ of $\hat{\mathcal{D}}$, then $x_{\tau\alpha}^j$ and $x_{\tau\alpha}^{j'}$ lie on the same curve of $\hat{\mathcal{D}}$, where $j = \rho(a)(i)$ and $j' = \rho(a)(i')$.*

Proof of Claim 1. Assume first that x_α^i and $x_\alpha^{i'}$ are consecutive in $\hat{\alpha}$: we can travel along $\hat{\alpha}$ from x_α^i to $x_\alpha^{i'}$ without crossing any other point of $h^{-1}(x_\alpha)$. Choosing appropiately the orientation of α or the indices i, i', we can assume that the lift of α starting at x_α^i ends at $x_\alpha^{i'}$, and in terms of $\pi(\mathcal{D})$ this implies that $\rho(\alpha)(i) = i'$. Therefore,

$$\rho(\tau\alpha)(j) = \rho(\tau\alpha)(\rho(a)(i)) = \rho(a\tau\alpha)(i)$$
$$= \rho(\alpha a)(i) = \rho(a)(\rho(\alpha)(i)) = \rho(a)(i') = j',$$

because by the double curve relation (4.2) it is $\alpha a = a\tau\alpha$. The points $x_{\tau\alpha}^j$ and $x_{\tau\alpha}^{j'}$ are consecutive in a lift of $\tau\alpha$ in $\hat{\mathcal{D}}$.

By repeating this argument, the statement is also true if x_α^i and $x_\alpha^{i'}$ are not consecutive in $\hat{\alpha}$ $\qquad\square$

Claim 2. *The sistering $\hat{\tau}$ is an involution on $\hat{\mathcal{D}}$.*

Proof of Claim 2. We have $\tau a = a^{-1}$ in $\pi(\mathcal{D})$ by dual loop relations. Then $\rho(\tau a) = \rho(a)^{-1}$ because ρ is a homomorphism. Therefore if we have $\rho(a)(i) = j$, it is also $\rho(\tau a)(j) = i$. In other words, the sister of the curve of $\hat{\mathcal{D}}$ passing through x_α^i is the curve of $\hat{\mathcal{D}}$ passing through $x_{\tau\alpha}^j$, and the sister of the curve of $\hat{\mathcal{D}}$ passing through $x_{\tau\alpha}^j$ is the curve of $\hat{\mathcal{D}}$ passing through x_α^i. $\qquad\square$

The sistering $\hat{\tau}$ defines an equivalence relation into the points of $\hat{\mathcal{D}}$. By Claims 1 and 2, each point of $\hat{\mathcal{D}}$ which is not a crossing has exactly another point related with it.

Claim 3. *The crossings of $\hat{\mathcal{D}}$ are related in triplets by $\hat{\tau}$.*

Proof of Claim 3. Let P be a triple point of Σ, and let P_1, P_2, P_3 be the three crossings of $\mathcal{D}$ in the triplet of P. We label the curves of $\mathcal{D}$ intersecting at P_1, P_2, P_3 as in Figure 9, and we consider the paths

$$\omega_1, \omega_2, \omega_3, \quad d_\alpha, d_{\tau\alpha}, d_\beta, d_{\tau\beta}, d_\gamma, d_{\tau\gamma}, \quad t_\alpha, t_{\tau\alpha}, t_\beta, t_{\tau\beta}, t_\gamma, t_{\tau\gamma}$$

as in the construction of the triple point relations (Figure 9).

For each $n = 1, 2, 3$ and $i = 1, 2, \ldots$, let ω_n^i be the lift of ω_n starting at x^i, and let P_n^i be the endpoint of ω_n^i. If we choose one of the lifts P_1^i of P_1, it is related by $\hat{\tau}$ with one lift P_2^j of P_2 by means of the sister curve of the lift of α passing through P_1^i. Let us label by α^i the lift of α passing through P_1^i. In order to find P_2^j we need to find the sister curve of α^i, and to do this we follow α^i until we reach a lift of x_α. This is made by taking the lift of d_α starting at P_1^i, which ends at a lift $x_\alpha^{i'}$ of x_α. The lift of λ_α ending at $x_\alpha^{i'}$ starts at $x^{i'}$, where $i' = \rho(t_\alpha)(i)$. The sister curve of α^i is the lift of $\tau\alpha$ passing through $x_{\tau\alpha}^{j'}$, where $j' = \rho(a)(i')$. Finally, P_2^j is located at the starting point of the lift of $d_{\tau\alpha}$ ending at $x_{\tau\alpha}^{j'}$, and it verifies that $j = \rho(t_{\tau\alpha}^{-1})(j')$. In other words, it is

$$j = \rho(t_\alpha a t_{\tau\alpha}^{-1})(i).$$

By the same argument, we have that P_2^j is related with P_3^ℓ, with $\ell = \rho(t_\beta b t_{\tau\beta}^{-1})(j)$ and that P_3^ℓ is related with $P_1^{i^*}$, where $i^* = \rho(t_\gamma c t_{\tau\gamma}^{-1})(\ell)$. Triple point relation (4.1) at P implies that $i^* = i$, and thus there is no more points related with P_1^i different from P_2^j and P_3^ℓ. This construction is valid in general and so the crossings of $\mathcal{D}$ are related in triples by $\hat{\tau}$.   □

We have proved that $(\hat{\mathcal{D}}, \hat{\tau})$ is an abstract Johansson diagram in $\hat{S}$: points in $\hat{\mathcal{D}}$ are identified in pairs with the exception of crossings, which are identified in triplets. Moreover, this equivalence relation is compatible with f, that is, if $\hat{Y}, \hat{Z} \in \hat{S}$ are related by $\hat{\tau}$, their projections $h(\hat{Y}), h(\hat{Z}) \in S$ are related by τ. The quotient space $\hat{\Sigma}$ of $\hat{S}$ under the equivalence relation defined by $\hat{\tau}$ is a (pseudo) Dehn surface and we can define a map $\hat{h} : \hat{\Sigma} \to \Sigma$ making the following diagram commutative:

$$
\begin{array}{ccc}
\hat{S} & \xrightarrow{\hat{f}} & \hat{\Sigma} \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \hat{h}} \\
S & \xrightarrow{f} & \Sigma
\end{array}
$$

where $\hat{f} : \hat{S} \to \hat{\Sigma}$ is the quotient map. The map $\hat{h}$ is in fact an m-sheeted covering map whose monodromy homomorphism $\hat{\rho}$ verifies by construction $\rho = \hat{\rho} \circ \varepsilon$.   □

Proof of Theorem 4.5. Due to Cayley's Theorem [9, p. 9], every group admits an injective representation into the group of permutations of its own elements. Because $\pi(\mathcal{D})$ is finite or countable, the group of permutations of its elements is isomorphic to Ω_m, for an $m \in \mathbb{N} \cup \{\infty\}$. Thus, we can construct a faithful representation $\rho : \pi(\mathcal{D}) \to \Omega_m$, and by Lemma 4.6, this implies that ε must be injective.   □

Hurewicz's Theorem implies that the first $\mathbb{Z}$-homology group of Σ is isomorphic to the abelianization of its fundamental group. Let us see how to adapt the presentation of Theorem 4.5 to the abelian case. Following the notation above:

(AR1) *Dual loop relations.* These relations remains the same, if α and $\tau\alpha$ are sister curves, their preferred dual loops satisfy $a = -\tau a$.

(AR2) *Triple point relations.* The triple points relations can be simplified using commutativity to
$$a + b + c = w,$$
where w is a word in surfacewise generators and a, b and c are preferred dual loops of double curves.

(AR3) *Double curve relations.* Conjugacies on commuting elements are equalities, so the double curve relation can be written as $\alpha = \tau\alpha$.

(AR4) *Surface relations.* The generators of the fundamental group of the g-torus S are related by a product of commutators. In the abelian context, this relation is trivial.

Relations AR1 allow us to select k generators $a_1, a_2, \ldots, a_k$ from the set of preferred dual loops, dropping k of them. Finally,

Theorem 4.7. *The abelian group $H_1(\Sigma)$ is isomorphic to*
$$H_1(\mathcal{D}) = \langle\, s_1, s_2, \ldots, s_{2g}, a_1, a_2, \ldots, a_k \mid AR2 \text{ and } AR3 \text{ relations}\,\rangle.$$

Remark 4.8. If Σ is filling, it is the 2-skeleton of M as a CW-complex, so $\pi_1(\Sigma) = \pi_1(M)$. Hence $H_1(\Sigma) = H_1(M)$, and therefore Theorem 4.7 characterizes completely the homology of the closed orientable manifold M.

5. Checkerboards and homology spheres

Consider a closed orientable 3-manifold M and a Dehn surface $S \to \Sigma \subset M$. We say M *is checkered* by Σ if we can assign a color from $\{0, 1\}$ to each region of Σ such that if two of them share a face they have different colors. In the same way, S is checkered by the Johansson diagram $\mathcal{D}$ of Σ if we can assign a color from $\{0, 1\}$ to each face of $S - \mathcal{D}$ such that if two of them share an edge they have different colors. Checkerboards are a usual tool in knot theory (see [5], for example).

Lemma 5.1. *The manifold M is checkered by Σ if and only if any loop on M meeting Σ transversely on the faces intersects Σ in an even number of points.*

Proof. Given the decomposition of M in regions of Σ, we can build a graph G_Σ describing it combinatorially. Each vertex corresponds to a region and there is an edge connecting two vertices if the corresponding regions share a face. It is clear that M is checkered by Σ if and only if G_Σ is 2-colourable.

Each path γ in M meeting Σ transversely on the faces has an associated path c_γ in G_Σ that encodes the sequence of "crossings" of γ between different regions of Σ across the faces of Σ. The number of points of $\gamma \cap \Sigma$ is just the number of edges of c_γ, and if γ is closed, c_γ is closed too.

As a graph is 2-colorable if and only if it is bipartite, the statement of the lemma is equivalent to: G_Σ is bipartite if and only if all loops have an even number of edges, which is trivially true. $\square$

Proposition 5.2. *A $\mathbb{Z}/2$-homology 3-sphere is checkered by any Dehn surface.*

Proof. Let M be a $\mathbb{Z}/2$-homology 3-sphere and let Σ be a Dehn surface in M. Let γ be a loop in M intersecting Σ transversely at the faces. Both γ and Σ are $\mathbb{Z}/2$-nullhomologous. Therefore the set of intersection points of γ and Σ must be a

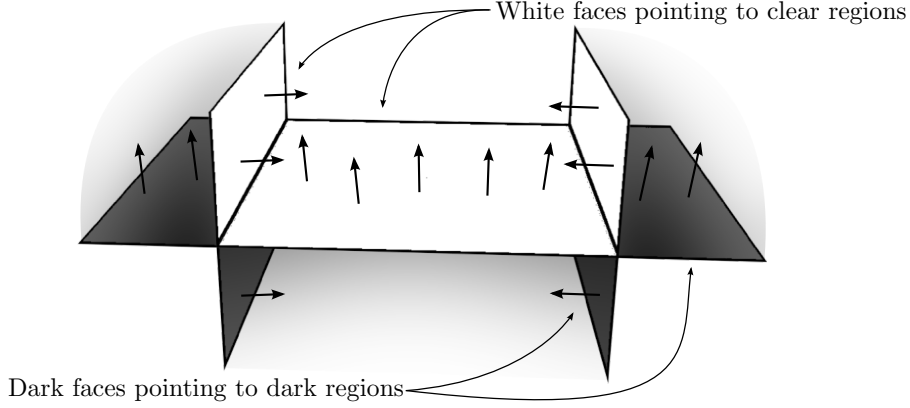White faces pointing to clear regions

Dark faces pointing to dark regions

FIG. 11. Coloring faces with region colors.

$\mathbb{Z}/2$-nulhomologous 0-cycle, and then, the number of intersection points should be even. Lemma 5.1 completes the proof. □

Since S is orientable, we can fix a well defined normal field on Σ in M.

If M is checkered by Σ, we assign to each face of the latter the color of the region pointed by the normal vector field on the face. The faces of S sharing an edge cannot share also the color, as the corresponding regions share a face (see Figure 11). Therefore:

Lemma 5.3. *If M is checkered by Σ, the surface S is also checkered by $\mathcal{D}$.*

The converse of this lemma does not hold. The 2-sphere $\mathbb{S}^2$ is checkered by the diagram of Figure 2, but $\mathbb{S}^2 \times \mathbb{S}^1$ is not checkered by the corresponding filling Dehn sphere. The existence of checkerboards has strong implications on the number of triple points.

Proposition 5.4. *If M is checkered by Σ, the number of triple points of Σ is even.*

Proof. For each curve of $\mathcal{D}$, consider a *neighbouring curve* running parallel to it and intersecting $\mathcal{D}$ transversely near the crossings of $\mathcal{D}$ (see [16]). As S is checkered by $\mathcal{D}$, by a similar argument to that of the proof of Lemma 5.1 each neighbouring curve must intersect $\mathcal{D}$ in an even number of points. Because the number of curves in $\mathcal{D}$ is even, the total amount of intersection points between neighbouring curves and diagram curves is a multiple of 4. Each crossing of the diagram corresponds to two of those intersections, so the number of crossings is even. Finally, the number of triple points is also even, as it is the number of crossings over three. □

From Remark 3.5 and Proposition 5.4 we get:

Theorem 5.5. *Let Σ be a filling Dehn g-torus of M. If M is checkered by Σ, the number of triple points of Σ is greater than or equal to $2g$.* □

All these results allow us to give a sharper lower bound for the triple point numbers of $\mathbb{Z}/2$-homology 3-spheres.

Theorem 5.6. *If M is a $\mathbb{Z}/2$-homology 3-sphere*

$$t_g(M) \geq 2 + 2g.$$

Proof. By Theorem 5.5 it is enough to show that the number of triple points cannot be $2g$. Let Σ be a Dehn g-torus in M with exactly $2g$ triple points. Let S be the domain of Σ.

Let us compute the first homology group of Σ with coefficients in $\mathbb{Z}/2$. By Theorem 4.7 this abelian group is generated by $s_1, s_2, \ldots, s_{2g}$, the curves generating $H_1(S)$, and the preferred dual loops $a_1, a_2, \ldots, a_k$ extracted from the k double curves of Σ. The relations come from the $2g$ AR2 relations and the k AR3 relations by just taking coefficients in $\mathbb{Z}/2$. With this presentation $H_1(\Sigma; \mathbb{Z}/2)$ has the same number of generators as of relators.

Consider only the k AR3 relators. If α and $\tau\alpha$ are sister curves, as they are surfacewise curves they can be written (up to homology) as

$$\alpha = \sum_{i=1}^{2g} n_\alpha^i s_i, \qquad\qquad \tau\alpha = \sum_{i=1}^{2g} m_\alpha^i s_i,$$

with $n_\alpha^i, m_\alpha^i \in \mathbb{Z}/2$. Therefore, the AR3 relation given by the pair $\alpha, \tau\alpha$ is

$$\sum_{i=1}^{2g} q_\alpha^i s_i = 0,$$

with $q_\alpha^i = n_\alpha^i + m_\alpha^i$. All this information can be written as a tableau

	s_1	s_2	$\cdots$	s_{2g}
α_1	$q_{\alpha_1}^1$	$q_{\alpha_1}^2$	$\cdots$	$q_{\alpha_1}^{2g}$
α_2	$q_{\alpha_2}^1$	$q_{\alpha_2}^2$	$\cdots$	$q_{\alpha_2}^{2g}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
α_k	$q_{\alpha_k}^1$	$q_{\alpha_k}^2$	$\cdots$	$q_{\alpha_k}^{2g}$

Rows in this tableau represent double curves relators. The number $q_{\alpha_j}^i$ is the intersection number of α_j and the dual loop of s_i (that is, a loop s_i' such that the intersection number between s_i' and s_ℓ is 1 if $\ell = i$ and 0 otherwise).

By Proposition 5.2 and Lemma 5.3 the surface S is checkered by $\mathcal{D}$, so the intersection number of each curve s_i' with the diagram should be even. This implies that the sum of each column is 0 (mod 2). In other words, one of the AR3 relators can be written as a linear combination of the others. Therefore, the group $H_1(\Sigma; \mathbb{Z}/2)$ has $2g + k$ generators and at most $2g + k - 1$ nontrivial relations, and it cannot be trivial. By Remark 4.8 this implies that Σ cannot fill the $\mathbb{Z}/2$-homology 3-sphere M. $\qquad\square$

It is well known that $t_0(\mathbb{S}^3) = 2$ (see [12,17]). From Lemma 3.1 and Theorem 5.6, $\mathbb{S}^3$ cannot have exceptional Dehn g-tori, therefore $\mathscr{H}(\mathbb{S}^3) = 0$ and finally

Corollary 5.7. $\mathscr{T}(\mathbb{S}^3) = (2, 4, 6, \ldots, 2g + 2, \ldots)$. $\qquad\square$

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