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FUNDAMENTAL GROUP OF RATIONAL HOMOLOGY DISK SMOOTHINGS OF SURFACE SINGULARITIES

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ABSTRACT. It is known that there are exactly three triply-infinite and seven singly-infinite families of weighted homogeneous normal surface singularities admitting a rational homology disk smoothing, i.e., having a Milnor fibre with Milnor number zero. Some examples are found by an explicit "quotient construction", while others require the "Pinkham method". The fundamental group of the Milnor fibre has been known for all except three exceptional families. In this paper, we settle these cases. We present a new explicit construction for one of the exceptional families, showing the fundamental group is non-abelian (as occurred previously only for three families). We show that the fundamental groups for the remaining two exceptional families are abelian, hence easily computed; using the Pinkham method here requires precise calculations for the fundamental group of the complement of a plane curve.

INTRODUCTION

Let $(X,0)$ be the germ of a complex normal surface singularity, with neighborhood boundary (or link) Σ. A *smoothing of* $(X,0)$ is a morphism $f:(\mathcal{X},0)\to(\mathbb{C},0)$, with $(\mathcal{X},0)$ a three-dimensional isolated Cohen-Macaulay singularity, equipped with an isomorphism

$$(f^{-1}(0),0)\simeq(X,0).$$

The Milnor fibre M is the general fibre $f^{-1}(\delta)$, a 4-manifold with boundary Σ. The second Betti number of M is called μ, the Milnor number of the smoothing; the first Betti number always vanishes [5]. We say f is a $\mathbb{Q}$HD (or *rational homology disk*) smoothing if $\mu=0$, i.e., the Euler characteristic $\chi(M)=1$. In such a case, the 3-manifold Σ has a particularly interesting filling (e.g., it is Stein).

Example 1. Such smoothings occur for cyclic quotient singularities of type $\frac{n^2}{nq-1}\equiv\frac{1}{n^2}(1,nq-1)$, where $0<q<n,(n,q)=1$ ([15, (2.7)]). One proceeds as follows; if $f(x,y,z)=xz-y^n$, then $f:\mathbb{C}^3\to\mathbb{C}$ is a smoothing of the A_{n-1} singularity, whose Milnor fibre M is simply connected, with Euler characteristic n. Let $G\subset GL(3,\mathbb{C})$ by the diagonal cyclic group generated by $[\zeta,\zeta^q,\zeta^{-1}]$, where $\zeta=\exp\frac{2\pi i}{n}$. The group G acts freely on $\mathbb{C}^3\setminus\{0\}$ and f is G-invariant; so the induced map $f:\mathbb{C}^3/G\to\mathbb{C}$ is a smoothing of the cyclic quotient singularity A_{n-1}/G, which has type $\frac{n^2}{nq-1}$. The new Milnor fibre is the free quotient M/G, of Euler characteristic 1, hence is a $\mathbb{Q}$HD. We call this class $\mathcal{G}_{n,q}$.

In the early 1980's, the second named author produced 9 other families (e.g., [16, (5.9.2)], but mainly unpublished). As with the family $\mathcal{G}_{n,q}$, some examples can be produced by an explicit "quotient construction".

Start with a smoothing $f:(\mathcal{Y},0)\to(\mathbb{C},0)$ of some 2-dimensional germ $(Y,0)$, with simply connected Milnor fibre M. Assume G is a group of automorphisms of $(\mathcal{Y},0)$ acting fixed point

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freely off 0, with f G-invariant. Then $f : (\mathcal{Y}/G, 0) \to (\mathbb{C}, 0)$ is a smoothing of Y/G with Milnor fibre M/G. If $\chi(M) = |G|$, then $\chi(M/G) = 1$, so M/G is a $\mathbb{Q}$HD.

This method can produce equations for the families eventually named $\mathcal{W}(p, q, r)$, $\mathcal{N}(p, q, r)$, $\mathcal{A}^4(p)$, $\mathcal{B}^4(p)$, and $\mathcal{C}^4(p)$ (here $p, q, r \geq 0$); the $\mathcal{Y}$ involved could be $\mathbb{C}^3$, a hypersurface singularity in $\mathbb{C}^4$, or the cone over a del Pezzo surface in $\mathbb{P}^6$. All these singularities are weighted homogeneous, and a superscript denotes the valence of the central curve in the graph of the minimal good resolution.

It turns out that other (and in fact all) weighted homogeneous $\mathbb{Q}$HD examples can be constructed by using the Pinkham method of "smoothing of negative weight" [12], [18]:

- Consider the projective $\mathbb{C}^*$-compactification of the singularity;
- resolve at infinity to obtain a curve configuration E';
- if possible, smooth the projective surface keeping E' fixed.
- The projective general fibre Z is a rational surface containing a configuration D' isomorphic to E', obtained by blowing-up $\mathbb{P}^2$ along an appropriate plane curve D so that D' is the total transform of D minus several curves.
- Then $Z \setminus D'$ is the Milnor fibre of the smoothing.

The Milnor fibre is a $\mathbb{Q}$HD when the components of D' rationally span $\operatorname{Pic} Z$. Given some cohomological vanishing conditions, Pinkham's construction allows one to go backwards from a given pair (Z, D') to a $\mathbb{Q}$HD smoothing of a weighted homogeneous surface singularity. The group $\pi_1(Z \setminus D')$ is difficult to compute in general, unless one already knows that $\pi_1(\mathbb{P}^2 \setminus D)$ is abelian; this occurs for types $\mathcal{W}, \mathcal{N}$, and $\mathcal{M}$, since here D can be taken to be four lines in general position.

The possible resolution graphs of any $(X, 0)$ admitting a $\mathbb{Q}$HD smoothing were greatly restricted by the results of [13], which also gave names to the known examples. For $(X, 0)$ weighted homogeneous, these turned out to be the only ones, via:

Theorem 2 (Bhupal-Stipsicz Theorem [1])**.** *The weighted homogeneous surface singularities admitting a* $\mathbb{Q}$HD *smoothing are the following families:* $\mathcal{G}_{n,q}$, $\mathcal{W}(p, q, r)$, $\mathcal{N}(p, q, r)$, $\mathcal{M}(p, q, r)$, $\mathcal{B}^3_2(p)$, $\mathcal{C}^3_2(p)$, $\mathcal{C}^3_3(p)$, $\mathcal{A}^4(p)$, $\mathcal{B}^4(p)$, *and* $\mathcal{C}^4(p)$.

Some further results are as follows:

1. By earlier results of Laufer ([8]), for the first seven families, the analytic type is uniquely determined by the graph of the singularity. For the valence 4 examples, there is in each case a unique cross-ratio for which a $\mathbb{Q}$HD smoothing exists ([2]).
2. In the base space of the semi-universal deformation of these singularities, a $\mathbb{Q}$HD smoothing component has dimension one, and there are one or two such components ([2], [19, (7.2)]).
3. The first homology of the Milnor fibre is isomorphic to a self-isotropic subgroup of $H_1(\Sigma)$, the discriminant group of the singularity ([9]).
4. Every $\mathbb{Q}$HD smoothing arises from a quotient construction $\mathcal{Y} \to \mathcal{Y}/G$, where $\mathcal{Y}$ is canonical Gorenstein and $G = \pi_1(M)$ [18]. In general, the embedding dimension of $\mathcal{Y}$ can be arbitrarily large, so one cannot expect explicit equations, and only the Pinkham method seems available.
5. The fundamental group of the Milnor fibre is abelian for the families $\mathcal{G}$, $\mathcal{W}$, $\mathcal{N}$, $\mathcal{M}$, and non-abelian metacyclic for $\mathcal{A}^4, \mathcal{B}^4, \mathcal{C}^4$ ([2], [17], [19]).

The main results of this paper consider the three exceptional cases for which $\pi_1(M)$ was unknown, namely $\mathcal{B}^3_2(p)$, $\mathcal{C}^3_2(p)$, and $\mathcal{C}^3_3(p)$. We state the results and list the resolution dual graphs (with the convention that no weight means -2).

Theorem 3. *The fundamental group of the* $\mathbb{Q}$HD *smoothing of* $\mathcal{C}^3_2(p)$ *is cyclic, of order* $3(p+3)$.

Theorem 4. *The fundamental group of the* $\mathbb{Q}$HD *smoothing of* $\mathcal{C}^3_3(p)$ *is cyclic, of order* $2(p+4)$.

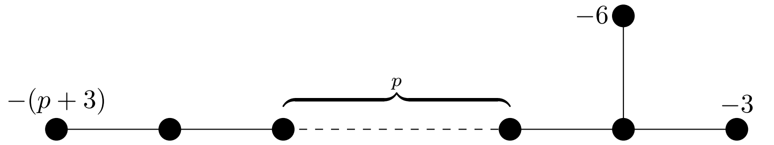

FIGURE 1. Resolution dual graph of $\mathcal{C}_2^3(p)$

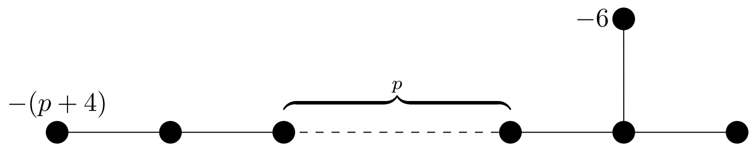

FIGURE 2. Resolution dual graph of $\mathcal{C}_3^3(p)$

Theorem 5. *The fundamental group of the* $\mathbb{Q}$HD *smoothing of* $\mathcal{B}_2^3(p)$ *is non-abelian of order* $4(p+2)(p+3)$*, with an index* 2 *cyclic subgroup and abelianization of order* $4(p+3)$*. There is an explicit quotient construction, with* $\mathcal{Y}$ *a hypersurface singularity in* $\mathbb{C}^4$*.*

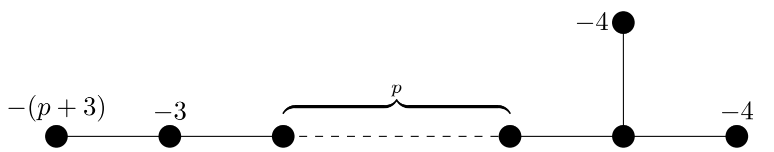

FIGURE 3. Resolution dual graph of $\mathcal{B}_2^3(p)$

In all three cases, one can use the plane curves D and their blow-ups to produce the pair (Z, D') for which one must compute the fundamental group of the complement. The proofs can be found at §2.2, §2.3 and §2.4.

For $\mathcal{B}_2^3(p)$, the precise description of the fundamental group has allowed us to find a direct quotient construction in §1.2.The first author initially did the computation in case $p = 0$, discovering that the group was non-abelian, of order 24. The second author used this unexpected result to first construct a non-abelian cover of degree 24 of the original singularity, a complete intersection in $\mathbb{C}^4$, and then to find a fixed-point free 4-dimensional representation of the group leaving this cover and its smoothing invariant. The first author later extended his computation of the fundamental group for all p, while independently the second author extended the explicit construction for all p, as in Theorem 1.1 below. Once a quotient construction for all p is obtained, the results of Fowler [2] imply there is only one $\mathbb{Q}$HD smoothing, so the fundamental group computations using D become unnecessary. Nonetheless, a detailed presentation of these results is included in §2.2, as the computational methods are important and illustrate a basic method.

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1. The family $\mathcal{B}_2^3(p)$

From Figure 3 we see that the continued fraction expansion of the long arm, starting from the outside, arises from $2(p+2)^2/(2p+3)$, and the discriminant group has order $16(p+3)^2$; it follows that the first homology group of the Milnor fibre has order $4(p+3)$.

Theorem 1.1. *For each $p \geq 0$, there is a hypersurface singularity $(\mathcal{Y}, 0) \subset (\mathbb{C}^4, 0)$, a group $G \subset SL(4, \mathbb{C})$ acting freely on $\mathcal{Y} \setminus \{0\}$, and a G-invariant function f, so that $f : \mathcal{Y}/G \to \mathbb{C}$ provides the* $\mathbb{Q}$HD *smoothing of a singularity of type $\mathcal{B}_2^3(p)$.*

Writing down explicitly the equations and the representation of G, it is straightforward to construct a $\mathbb{Q}$HD smoothing whose Milnor fibre has G as fundamental group. What takes extensive computation is the verification that the smoothed singularity is of type $\mathcal{B}_2^3(p)$.

1.1. The group G.

Let $m \geq 2$ be an integer, $N = 2m(m+1)$, ω a primitive N^{th} root of 1. Consider the diagonal linear transformation of $\mathbb{C}^4$ given by

$$S(a, b, c, d) = (\omega a,\ \omega^{-(2m+1)} b,\ \omega^{2m+1} c,\ \omega^{-1} d).$$

The action S can be also be written

$$S = \frac{1}{N}[1,\ -(2m+1),\ 2m+1,\ -1].$$

This allows one to quickly write down S^p when $N = pN'$; replace $\frac{1}{N}$ by $\frac{1}{N'}$, and then reduce the entries in $[\bullet, \bullet, \bullet, \bullet]$ mod N'.

Let $\zeta = \omega^m$ a primitive $(2m+2)^{th}$ root of 1, and define

$$T(a, b, c, d) = (\zeta b, a, d, \zeta^{-1} c).$$

One easily finds

$$S^N = I,\ TST^{-1} = S^{-(2m+1)},\ T^2 = S^m = \frac{1}{2m+2}[1, 1, -1, -1].$$

Let $G = G_m \subset SL(4, \mathbb{C})$ be the group generated by S and T.

Proposition 1.2. *The following properties hold for G:*

(G1) $|G| = 2N = 4m(m+1)$.
(G2) *The abelianization of G has order $4(m+1)$, is cyclic when m is odd, and is*

$$\mathbb{Z}/(2(m+1)) \times \mathbb{Z}/(2) \text{ if } m \text{ is even.}$$

(G3) *The center of G is the cyclic group generated by S^m, of order $2(m+1)$.*
(G4) *$S^i T$ has even order > 2.*
(G5) *G acts freely on $\mathbb{C}^4$ off the origin.*

Proof. The first two statements are straightforward. One sees that no $S^i T$ commutes with S, so the center is generated by a power of S, easily seen to be S^m. The powers of ω that occur in S are all primitive roots of 1, so the subgroup generated by S acts freely on $\mathbb{C}^4 \setminus \{0\}$. Note

$$(S^i T)^2 = S^{-m(2i-1)},$$

which can not equal the identity, so itself has no fixed points; thus, $S^i T$ has no fixed points. □

Remark 1.3. One could also consider the simpler linear transformation T' defined by

$$T'(a, b, c, d) = (b, -a, d, -c),$$

and the group $G' \subset SL(4, \mathbb{C})$ that S and T generate. One now has

$$S^N = I,\ T'^2 = S^{m(m+1)} = -I,\ T'ST'^{-1} = S^{-(2m+1)}.$$

When m is even, G' is isomorphic to G, as seen by setting $T' = S^{\frac{m+2}{2}}T$; one may use this representation to consider later simpler polynomials $xw - yz$ and $zw - x^{2m} + y^{2m}$.

However, when m is odd, G' is not isomorphic to G, and does not act freely on $\mathbb{C}^4 \setminus \{0\}$ because $S^{\frac{m+1}{2}}T'$ has order 2 and fixed points. The relevant representation of G is now more complicated than in the even case.

We make some additional remarks about the group G, which however are not used later on.

Define the *generalized quaternion 2-group* Q_r by generators and relations as

$$Q_r : A^{2^{r-1}} = 1,\ A^{2^{r-2}} = B^2,\ BAB^{-1} = A^{-1}.$$

Then $|Q_r| = 2^r$, $Q_{r-1} \subset Q_r$ (use generators A^2 and B), and Q_3 is the usual quaternion group of order 8.

Proposition 1.4. *For G as above, write $N = 2m(m+1) = 2^{r+1}p$, where p is odd.*

(Q1) *$H = \langle S^{2^{r+1}} \rangle$ is a cyclic normal subgroup of order p, consisting of all elements of odd order.*

(Q2) *If m is odd, write $m+1 = 2^r(2u-1)$, and $J = \langle S^u T \rangle$. Then J is a cyclic Sylow 2-subgroup of order 2^{r+2}, and G is the semi-direct product of H and J.*

(Q3) *If $m = 2^r q$ is even (with q odd), the Sylow 2-subgroup*

$$J = \langle S^{q(m+1)}, S^{\frac{m+2}{2}}T \rangle$$

is isomorphic to Q_{r+2}, and G is the semi-direct product of H and J.

(Q4) *The 2-Sylow subgroup of G is normal if and only if m is a power of 2, in which case G is the direct product of $H = \langle S^m \rangle$ and $J = \langle S^{m+1}, S^{\frac{m+2}{2}}T \rangle$.*

Proof. (Q1) is straightforward. For (Q2), note S has order $N = 2^{r+1}m(2u-1)$ and $(S^u T)^2 = S^{-m(2u-1)}$, so J is cyclic of order 2^{r+2}, hence is a 2-Sylow subgroup. Since $G = HJ$, $H \cap J = \{I\}$, and H is normal, one has a semi-direct product.

In (Q3), one checks that the given generators of J match the generators and relations of Q_{r+2}; it follows as before that there is a semi-direct product decomposition.

For (Q4), normality of J in the case of m odd would imply G is abelian, which is never true. For m even, normality of J implies that the conjugate $S \cdot S^{\frac{m+2}{2}}T \cdot S^{-1}$ is of the form $S^{iq(m+1)} \cdot S^{\frac{m+2}{2}}T$, for some i. A calculation shows this is equivalent to $S^{(m+1)(qi-2)} = I$, so $2^{r+1}q$ divides $qi-2$. Since q, which is odd, divides 2, we have $q=1$, and one can set $i=2$. One easily checks that the T-conjugate of $S^{\frac{m+2}{2}}T$ is also in J. $\square$

Remark 1.5. The groups G, with a fixed-point free representation, have the familiar property (seen for instance in [20]) that odd order Sylow subgroups are cyclic, and the 2-Sylow is either cyclic or contains an index two cyclic subgroup. An avid reader might wish to locate the groups above in the complete chart in [20, Section 7.2].

1.2. The equations.

The representation above of G acting on $\mathbb{C}^4$ has its contragredient representation acting on the coordinate functions x, y, z, w, via

$$S(x,y,z,w) = (\omega^{-1}x,\ \omega^{2m+1}y,\ \omega^{-(2m+1)}z,\ \omega w)$$
$$T(x,y,z,w) = (\zeta^{-1}y,\ x,\ w,\ \zeta z).$$

Proposition 1.6. *The group G acts freely on $\mathbb{C}^4$ off the origin, leaves invariant the hypersurface singularity*

$$\mathcal{Y} = \{zw + x^{2m} + \zeta y^{2m} = 0\} \subset \mathbb{C}^4,$$

and fixes the polynomial

$$f(x,y,z,w) = xw + yz.$$

Thus $f : \mathcal{Y}/G \to \mathbb{C}$ is a smoothing of the G-quotient of the isolated complete intersection singularity $Y = \mathcal{Y} \cap \{f = 0\} \subset \mathbb{C}^4$.

Proof. The only new item needed is the simple calculation that Y has an isolated singularity at the origin. □

The map $f : \mathcal{Y} \to \mathbb{C}$ gives a smoothing of Y. By Hamm-Lê (e.g., [6]), the Milnor fibre $M = f^{-1}(\delta)$ is simply connected. The Euler characteristic can be computed from the Greuel-Hamm formula [4] for weighted homogeneous complete intersections, yielding

$$\chi(M) = 1 + \mu = 4m(m + 1).$$

The group G acts freely on $\mathcal{Y} \setminus \{0\}$ and M. As $\chi(M) = |G|$, M/G has Euler characteristic 1, hence is a rational homology disk whose fundamental group is isomorphic to G.

Proposition 1.7. *The map $f : \mathcal{Y}/G \to \mathbb{C}$ gives a rational homology disk ($\mathbb{Q}$HD) smoothing of the singularity Y/G, whose Milnor fibre has non-abelian fundamental group G.*

The following section is devoted to the proof of the following proposition.

Proposition 1.8. *The singularity Y/G is of type $\mathcal{B}^3_2(m - 2)$.*

A priori, one knows the quotient is a rational singularity with discriminant the square $[4(m + 1)]^2$. The standard approach (e.g., [12] or [17]) is to lift the action of G from Y to its *Seifert partial resolution* $\mathcal{S} \to Y$, the result of weighted blow-up, which has a smooth central curve C along which are cyclic quotient singularities. The quotient $\mathcal{S}/G$ will be the Seifert resolution of Y/G. The resolution space $\mathcal{S}$ has a covering $\{\mathcal{S}_i\}$ by 4 open affines, corresponding to weighted inversion of the coordinates. One identifies singular points and fixed points of the action of G along C on each affine, as well as on certain partial quotients. At the end, one finds three singular points on a rational curve, whose self-intersection on its minimal resolution is computable from knowledge of the discriminant.

1.3. **Resolution of Y/G.**

If $\mathbb{C}^t$ has coordinates z_i with positive integer weights n_i (without common divisor), the weighted blow-up is a map $\mathcal{U} \to \mathbb{C}^t$, with fibre over the origin the weighted projective space $\mathbb{P}_{\mathbf{n}} = \mathbb{P}_{(n_1,\cdots,n_t)}$. The space $\mathcal{U}$ has an open affine covering U_i, each of which is a quotient of an affine space V_i by a cyclic group of order n_i. For instance, V_1 has coordinates $A_1, \dots, A_t$, related to the z_i via

$$z_1 = A_1^{n_1},\ z_2 = A_1^{n_2} A_2, \dots,\ z_t = A_1^{n_t} A_t;$$

the quotient U_1 equals V_1 modulo the action on the A_i's of the cyclic group generated by

$$\frac{1}{n_1}[-1,\ n_2, \dots,\ n_t].$$

Weighted blow-up of $\mathbb{C}^4$, with coordinates x, y, z, w and weights $1, 1, m, m$, induces a weighted blow-up $\mathcal{S} \to Y$, covered by 4 affines $\mathcal{S}_i$. The exceptional fibre is a smooth projective curve $C \subset \mathbb{P}_{\mathbf{m}} = \mathbb{P}_{(1,1,m,m)}$.

Inverting first x, $\mathcal{S}_1$ has coordinates

$$x = A_1,\ y = A_1 A_2,\ z = A_1^m A_3,\ w = A_1^m A_4,$$

with equations

$$A_4 = -\, A_2 A_3$$
$$A_3 A_4 = -(1 + \zeta A_2^{2m}).$$

Thus in coordinates A_1, A_2, A_3, $\mathcal{S}_1$ is defined by

$$A_2 A_3^2 - 1 - \zeta A_2^{2m} = 0,$$

while C is given by $A_1 = 0$. Both the surface and the curve are smooth. Rewriting C as

$$(A_2A_3)^2 = A_2(1 + \zeta A_2^{2m}),$$

its function field is a double cover of the affine line branched at $2m+1$ points; so, C is a hyperelliptic curve of genus m.

The group G lifts to a group of automorphisms of $\mathcal{S}$. On $\mathcal{S}_1$,

$$S(A_3) = S(zx^{-m}) = (\omega^{-(2m+1)}z)(\omega^{-1}x)^{-m} = \omega^{-(m+1)}zx^{-2m} = \omega^{-(m+1)}A_3,$$

while

$$T(A_3) = T(z)T(x)^{-m} = w(\zeta^{-1}y)^{-m} = \zeta^m A_4 A_2^{-m} = \zeta^{-1}A_3A_2^{1-m}.$$

We summarize as

$$\begin{aligned} S(A_1, A_2, A_3) =& (\omega^{-1}A_1,\ \omega^{2m+2}A_2,\ \omega^{-(m+1)}A_3) \\ T(A_1, A_2, A_3) =& (\zeta^{-1}A_1A_2,\ \zeta A_2^{-1},\ \zeta^{-1}A_3A_2^{1-m}). \end{aligned}$$

The group G acts on $\mathcal{S}_1$ (A_2 is never 0 there). Note S^{2m} is a pseudo-reflection, sending A_1 to $\omega^{-2m}A_1$, leaving A_2 and A_3 fixed. Then $\bar{\mathcal{S}}_1 = \mathcal{S}_1/\langle S^{2m}\rangle$ has as coordinates the invariants $A_1^{m+1} \equiv \bar{A}_1$, A_2, and A_3, with equation

$$A_2A_3^2 = 1 + \zeta A_2^{2m}$$

and central curve C defined by $\bar{A}_1 = 0$. Then $\bar{G} = G/\langle S^{2m}\rangle$ acts on $\bar{\mathcal{S}}_1$ as follows: Let $\eta = \omega^{-(m+1)}$, a primitive $(2m)^{\text{th}}$ root of 1. Then

$$\begin{aligned} S(\bar{A}_1, A_2, A_3) =& (\eta\bar{A}_1,\ \eta^{-2}A_2,\ \eta A_3) \\ T(\bar{A}_1, A_2, A_3) =& (-\bar{A}_1A_2^{m+1},\ \zeta A_2^{-1},\ \zeta^{-1}A_3A_2^{1-m}). \end{aligned}$$

We describe all fixed points of elements of $\bar{G}$ and their orbits.

First, S^m fixes all points of the form $(0, a, 0)$, where a is a $(2m)^{th}$ root of $-\zeta^{-1} = \zeta^m$. Defining a square root of ζ by $\tau^2 = \zeta$, a is of the form $\tau\eta^k$, $k = 0, \cdots, 2m-1$. These $2m$ fixed points are permuted by powers of S, which sends for instance $a = \tau\eta^k$ to $\tau\eta^{k+2}$. In particular, there are 2 G-orbits of these fixed points, corresponding to $a = \tau$ and $a = \tau\eta$.

A calculation shows that

$$S^kT(\bar{A}_1, A_2, A_3) = (\eta^{m-k}\bar{A}_1A_2^{m+1},\ \zeta\eta^{2k}A_2^{-1},\ \zeta^{-1}\eta^{-k}A_3A_2^{1-m})$$

has two fixed points as above, where $a = \pm\tau\eta^k$; note that in $\bar{G}$, $(S^kT)^2 = S^m$. Thus the isotropy subgroup of $\bar{G}$ at $a = \tau$ is the cyclic group of order 4 generated by T, and similarly the isotropy group at $a = \tau\eta$ is generated by ST. At the point $(0, \tau, 0)$, generators for the local ring are $\bar{A}_1$ and A_3, with C given by $\bar{A}_1 = 0$. As

$$T(\bar{A}_1, A_2, A_3) = (-\bar{A}_1A_2^{m+1},\ \zeta A_2^{-1},\ \zeta^{-1}A_3A_2^{1-m}),$$

the action on the tangent space when $A_2 = \tau$ is checked to be scalar multiplication by $\tau^{-(m+1)}$, which is a primitive 4^{th} root of 1. The quotient is the singularity $\frac{1}{4}[1,1]$ whose minimal resolution is a single smooth rational curve of self-intersection -4. The same calculation holds at the fixed point when $A_2 = \tau\eta$. These two orbits are the only fixed points of $\bar{G}$ on $\bar{\mathcal{S}}_1$, so the quotient has two (-4)-singularities along the central curve. The rationality of the central curve is known because one has a $\mathbb{Q}$HD smoothing, or can be seen directly as follows: Invariants of S^m on $\bar{\mathcal{S}}_1$ are $M = \bar{A}_1{}^2, N = \bar{A}_1A_3, P = A_3^2$, and A_2, so the quotient is defined by equations

$$MP = N^2,\ A_2P = 1 + \zeta A_2^{2m}.$$

The image of C is given by $M = N = 0$ and the plane curve $A_2P = 1 + \zeta A_2^{2m}$, which is clearly rational (P is a function of A_2).

So, the quotient $\mathcal{S}_1/G$ consists of a surface with exactly two (-4)-singularities along a rational curve.

Since $\mathcal{S}_1 \cap C$ consists of all points of C with first quasi-homogeneous coordinate non-0, there remains to consider only the behavior of $\mathcal{S}$ and the group action near the other points of C, namely $[0:0:1:0]_{\mathbf{m}}$ and $[0:0:0:1]_{\mathbf{m}}$. As T permutes these points, we need only look at the action of $\langle S\rangle$ on $\mathcal{S}_3$ near the first of these.

So, consider the weighted blow-up from inverting z. One has an affine space with coordinates

$$z = B_1^m,\ x = B_1B_2,\ y = B_1B_3,\ w = B_1^m B_4,$$

divided by the diagonal action of

$$\frac{1}{m}[-1, 1, 1, m].$$

The proper transforms of the two equations defining $\mathcal{X}$ yield

$$B_2B_4 + B_3 = 0,\ B_4 + B_2^{2m} + \zeta B_3^{2m} = 0,$$

which define a non-singular surface $\mathcal{S}_3'$, given in coordinates B_1, B_2, B_4 by

$$B_4 + B_2^{2m}(1 + \zeta B_4^{2m}) = 0,$$

with central curve given by $B_1 = 0$. To reach the surface $\mathcal{S}_3$, one divides by the diagonal action, giving invariants $B_1^m = z$, $B_1B_2 = x$, $B_2^m \equiv M$, and B_4, now satisfying

$$zM = x^m,\ B_4 + M^2(1 + \zeta B_4^{2m}) = 0,$$

with central curve $z = x = 0$. Thus, $\mathcal{S}_3$ has a singularity of type A_{m-1} at the origin, corresponding to the point $[0:0:1:0]_{\mathbf{m}}$ of C. We conclude that the Seifert resolution $\mathcal{S}$ of the singularity consists of a central hyperelliptic curve of genus m along which are two A_{m-1} singularities.

Note S lifts to an action on $\mathcal{S}_3$, calculated to be

$$S(z, x, M, B_4) = (\omega^{-(2m+1)}z,\ \omega^{-1}x,\ \omega^{m+1}M,\ \omega^{2m+2}B_4).$$

To complete the description of $\mathcal{S}/G$, it suffices to consider the quotient of $\mathcal{S}_3$ by $\langle S\rangle$ at the singular point.

For this purpose, it is easier to extend S to the m-fold cover $\mathcal{S}_3'$, which is smooth. In the relevant coordinates B_1, B_2, and B_4 define

$$\bar{S} = \frac{1}{mN}[-(2m+1),\ m+1,\ 2m(m+1)].$$

This extension has the property that $\bar{S}^N = \frac{1}{m}[-1, 1, 0]$, so dividing gives $\mathcal{S}_3$; and

$$\bar{S}^m = \frac{1}{N}[-(2m+1), m+1, 0]$$

gives the same action above of S on the coordinates $z = B_1^m, x = B_1B_2, M = B_2^m$, and B_4 of $\mathcal{S}_3$.

So, it suffices to decipher the action of $\bar{S}$ on $\mathcal{S}_3'$. Now,

$$\bar{S}^{2m^2} = \frac{1}{m+1}[1, 0, 0]$$

is a pseudoreflection, multiplying B_1 by ω^{2m} and fixing B_2 and B_4. Dividing by the group it generates and letting $\bar{B}_1 = B_1^{m+1}$, one has coordinates $\bar{B}_1, B_2, B_4$, the same equation as before (with $\bar{B}_1 = 0$ the central curve), and group action

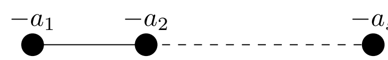
$$\bar{S} = \frac{1}{2m^2}[-(2m+1),\ 1,\ 2m]$$

on a smooth surface. This action is free except that $\bar{S}^m$ has fixed points when the first two coordinates are 0 (and hence so is the last). Local coordinates at this point are $\bar{B}_1$ and B_2, and the group action is $\frac{1}{2m^2}[-(2m+1),\ 1]$. To describe the resolution of this cyclic quotient singularity on $\mathcal{S}/G$ in relation to the curve which is the image of $\bar{B}_1 = 0$, we apply the following well-known result.

Lemma 1.9 ([7, pp. 9,10])**.** *Consider the cyclic action $(x, y) \mapsto (\mu x, \mu^q y)$ on $\mathbb{C}^2$, where μ is a primitive n^{th} root of* 1*,* $0 < q < n$*,* $(q, n) = 1$*. Write the continued fraction expansion*

$$\frac{n}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_{s-1} - \cfrac{1}{a_s}}}}.$$

Then the minimal resolution graph of the quotient is



and the curve on right intersects transversally the proper transform of the image of $x = 0$*.*

Combining what has already been proved about $\mathcal{S}/G$, one knows the resolution graph of the minimal resolution of Y/G, except for the self-intersection $-d$ of the central curve. The standard calculation of the order of the discriminant group involves d and the outward continued fraction expansions from the center; in this case, it is

$$4 \cdot 4 \cdot 2m^2 \left[d - \frac{1}{4} - \frac{1}{4} - \frac{2m^2 - 2m - 1}{2m^2} \right].$$

Comparing with the known discriminant value of $16(m + 1)^2$, one concludes that $d = 2$. One therefore has the graph $\mathcal{B}_2^3(m - 2)$.

Remark 1.10. Assuming the Bhupal-Stipsicz Theorem, one can by process of elimination conclude that Y/G has type $\mathcal{B}_2^3$. For, excluding types $\mathcal{C}^3$ and $\mathcal{B}^3$, the only example with non-abelian group and quotient smoothing of a complete intersection singularity is type $\mathcal{B}^4(p)$. However, its fundamental group is metacyclic, and the commutator subgroup has index and order incompatible with our G. To rule out $\mathcal{C}^3$, either use the other theorems in this paper; or, note that Y/G has a graded involution of order 2 (using a "square root" of S), whereas neither $\mathcal{C}^3$ type could have such a symmetry.

2. Zariski-van Kampen computations

2.1. **Fundamental group of the complement of a line arrangement.**
Let $\mathcal{L}$ be the projective line arrangement of $\mathbb{P}^2$ in Figure 4 (the line at infinity is not part of $\mathcal{L}$), which is the curve D of [2] for $\mathcal{B}_2^3(p)$. We are going to use the Zariski-van Kampen method to compute $\pi_1(\mathbb{P}^2 \setminus \mathcal{L})$ together with precise descriptions of meridians close to the singular points, in order to be able to find meridians of the exceptional components of successive blowing-ups.

Let us denote with small letters the standard meridians in the vertical dotted line of Figure 4. Let $\mathcal{A} := \{\ell_1, \ell_2, \ell_3, \ell_4, a_1, a_2, a_3\}$. We denote the multiple points as $P_{ij} := L_i \cap L_j$ and $R_i := A_j \cap A_k$ (where $\{i, j, k\} = \{1, 2, 3\}$).

Let us clarify what we mean by *standard meridians*. In a punctured plane $\mathbb{C} \setminus \Delta$, where Δ is an ordered finite set $\{z_1, \dots, z_r\}$, a geometric basis of of $\pi_1(\mathbb{C} \setminus \Delta; *)$, where $z_0 \in \mathbb{R}$, $z_0 \gg 0$, is a family of meridians as in Figure 5 where the product $z_r \cdot \ldots \cdot z_1$ is homotopic to the counterclockwise boundary of a big disk.

We follow the classical method of Zariski-van Kampen. Let us fix a base point $p := (x_0, y_0)$, where x_0 is the coordinate of the dotted line in Figure 4, where the standard meridians lie. We know that $\mathcal{A}$ generates $G := \pi_1(\mathbb{P}^2 \setminus \mathcal{L}; p)$, and since the line at infinity is not part of $\mathcal{L}$, the following relation holds:

$$(\infty) \qquad 1 = \ell_3 \cdot a_2 \cdot a_3 \cdot \ell_4 \cdot \ell_2 \cdot a_1 \cdot \ell_1$$

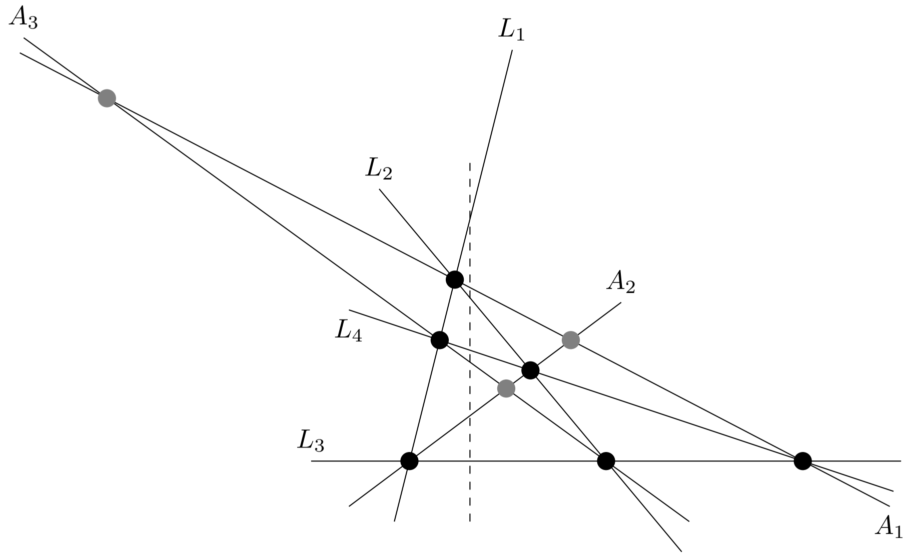


FIGURE 4. 7-line arrangement

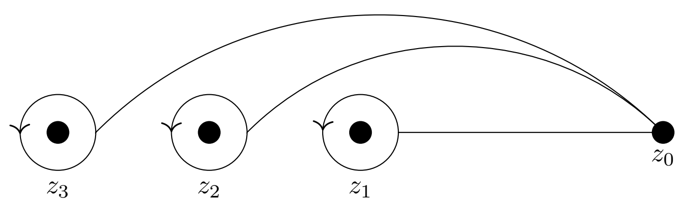


FIGURE 5. Geometric basis, $r = 3$

The rest of the relations will be found using the Zariski-van Kampen method. In order to find them we need to enlarge the concept of standard meridian. In a vertical line $x = x_1$ we can consider also a geometric basis with base point (x_1, y_0). These new meridians can be seen as elements of G if we conjugate them by a path joining p and (x_1, y_0) in the horizontal line $y = y_0$ which avoids the x-coordinates of the multiple points of $\mathcal{L}$. These new meridians can be written in terms of the original ones, and this is usually the difficult part of the Zariski-van Kampen method. The real picture in Figure 4 provides all the needed information.

Let us recall how this works for double and triple points using Figure 6. In both cases the standard generators to the left of the multiple point are expressed in terms of the generators to the right, taking into account the relations created by the singular point $[a, b] = 1$ for a double point and $c \cdot b \cdot a = b \cdot a \cdot c = a \cdot c \cdot b$ for a triple point; we will denote the second relation as $[c, b, a] = 1$. Note that the element e commutes with the involved meridians in each case. If we call E the exceptional component of the blowing-up of the multiple point, then e is a meridian of this component.

Let us consider the multiple points to the left of the dotted vertical line, see Figure 7. We obtain the following relations:

$$(\rho_{12}) \qquad [\ell_2, a_1, \ell_1] = 1$$

$$(\rho_{14}) \qquad [a_3, \ell_4, \ell_1] = 1$$

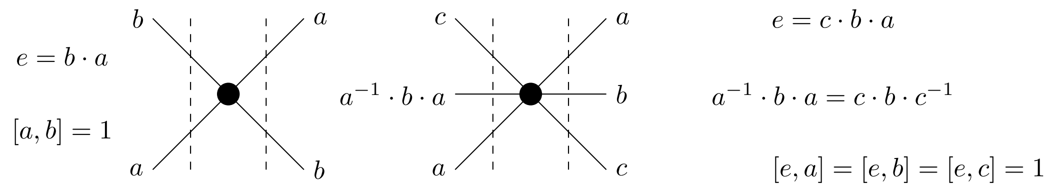


FIGURE 6. Local picture at double and triple points

$$[\ell_3, a_2, \ell_1] = 1 \tag{ρ_{13}}$$

$$[a_3, \ell_2 \cdot a_1 \cdot \ell_2^{-1}] = 1 \tag{ρ_2}$$

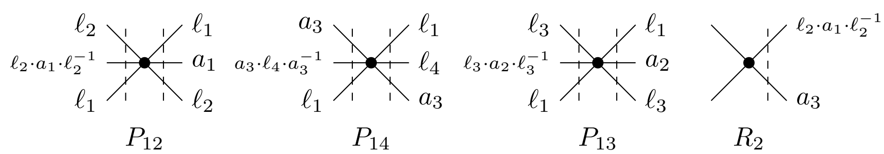


FIGURE 7. Multiple points to the left of the dotted vertical line

Let us continue with the multiple points to the right of the dotted line, see Figure 8. Since we always avoid the singular fiber running counterclockwise, the situation of Figure 6 applies, interchanging left and right.

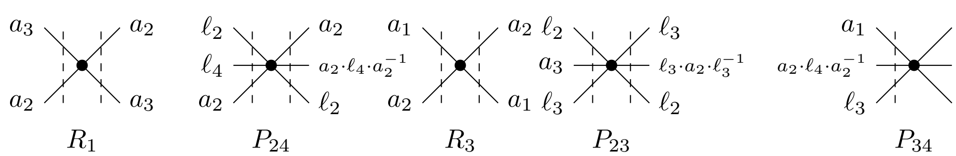


FIGURE 8. Multiple points to the right of the dotted vertical line

We obtain the following relations:

$$[a_2, a_3] = 1 \tag{ρ_1}$$

$$[a_2, \ell_4, \ell_2] = 1 \tag{ρ_{24}}$$

$$[a_1, a_2] = 1 \tag{ρ_3}$$

$$[\ell_3, a_3, \ell_2] = 1 \tag{ρ_{23}}$$

$$[\ell_3, a_2 \cdot \ell_4 \cdot a_2^{-1}, a_1] = 1 \tag{ρ_{34}}$$

Let us summarize these computations.

Proposition 2.1. *The group $G = \pi_1(\mathbb{P}^2 \setminus \mathcal{L})$ is generated by $\mathcal{A}$ with relations (∞), (ρ_{12}), (ρ_{13}), (ρ_{14}), (ρ_{23}), (ρ_{24}), (ρ_{34}), (ρ_1), (ρ_2), and (ρ_3).*

This is a non-abelian group, as it is the case for any line arrangement having *more-than-double* points. In fact, this presentation has redundant generators and relations but it will be useful to express the meridians of the exceptional components of the blow-ups. If we blow up a point P_{ij}, we denote by E_{ij} its exceptional component and by e_{ij} a suitable meridian. In the same way, if we blow up a point R_i, we denote by E_i its exceptional component and by e_i a suitable meridian.

Lemma 2.2. *The meridians e_{ij}, e_i have the following expressions in terms of the generators of G:*

$$\begin{aligned} e_{12} &:= \ell_2 \cdot a_1 \cdot \ell_1 & e_{13} &:= \ell_3 \cdot a_2 \cdot \ell_1 \\ e_{14} &:= \ell_1 \cdot a_3 \cdot \ell_4 & e_{23} &:= \ell_3 \cdot a_3 \cdot \ell_2 \\ e_{24} &:= \ell_2 \cdot a_2 \cdot \ell_4 & e_{34} &:= a_2 \cdot \ell_4 \cdot a_2^{-1} \cdot a_1 \cdot \ell_3 \\ e_1 &:= a_2 \cdot a_3 & e_2 &:= a_3 \cdot \ell_2 \cdot a_1 \cdot \ell_2^{-1} \\ e_3 &:= a_1 \cdot a_2. \end{aligned}$$

2.2. **Fundamental group of the complement of the Milnor fiber of a smoothing of $\mathcal{B}_2^3(p)$.**

In this subsection we perform the direct computation of the fundamental group of the Milnor fiber of a smoothing of $\mathcal{B}_2^3(p)$; the result coincides with the group described in §1.1.

Let $\pi : X_3^2 \to \mathbb{P}^2$ the composition of the following blowing-ups. First we blow-up all the points P_{ij}. In this intermediate surface, we blow up Q_1 (resp. Q_2), the intersection point of E_{23} (resp. E_{14}) and the strict transform of A_3. Finally we perform $p+1$ extra blowing-ups over R_3, all of them on the intersection point of the strict transform of A_1 and the previous exceptional component. It is obvious that $X_3^2 \setminus \pi^{-1}(\mathcal{L})$ is isomorphic to $\mathbb{P}^2 \setminus \mathcal{L}$. The curve $\pi^{-1}(\mathcal{L})$ has $16+p$ connected components (see Figure 9).

Let $\mathcal{B} \subset \pi^{-1}(\mathcal{L})$ the curve obtained as union of the strict transforms of L_j, A_i and E_{14}, E_{23}, E_{12} and the strict transforms of the first p exceptional components over R_3, i.e, $10+p$ connected components. It is a normal crossing divisor whose dual graph is shown in Figure 9.

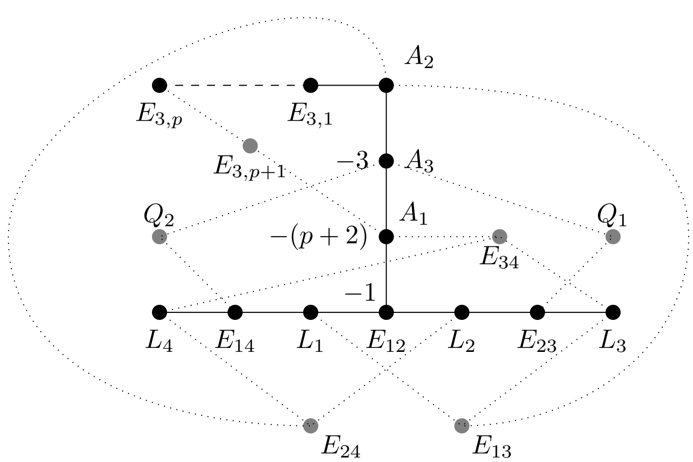


FIGURE 9. Missing self-intersections are -2 if black and -1 if gray.

The missing curves are E_{13}, E_{24}, E_{34}, the last exceptional component over R_3, and the exceptional divisors coming from Q_1, Q_2; the meridians of those ones are

$$\begin{aligned} e_{3,j} &:= a_1^j \cdot a_2 \\ q_1 &:= e_{23} \cdot a_3 = \ell_2 \cdot \ell_3 \cdot a_3^2 \\ q_2 &:= e_{14} \cdot a_3 = \ell_4 \cdot \ell_1 \cdot a_3^2. \end{aligned}$$

In order to compute $G_1 := \pi_1(X_3^2 \setminus \mathcal{B})$ we need the following classical result.

Proposition 2.3 ([3, Lemma 4.18])**.** *Let X be a smooth complex projective surface, $D\subset X$ a reduced divisor, and $D'=D\cup A_1\cup\dots\cup A_r$, where $A_1,\dots,A_r$ are the irreducible components of D' which are* not *in D.*

Let $i_*:\pi_1(X\setminus D')\to\pi_1(X\setminus D)$. Then i_* is surjective and $\ker i_*$ is generated by all the meridians of the components A_i in X.

This statement is sharper than Fujita's original one, where $r=1$ and A_1 must be transversal to D; the combination of induction and embedded resolution of D reduces this statement to the original one.

Therefore, G_1 is isomorphic to the quotient of G by the normal subgroup generated by e_{13}, e_{24}, e_{34}, $e_{3,p+1}$, q_1, and q_2. A presentation of G_1 is obtained from the presentation of G by adding the relations coming from killing the above meridians; we can forget the relations (ρ_{12}), (ρ_{24}), (ρ_{34}) and (ρ_3). Summarizing, G_1 is generated by $\mathcal{A}$, i.e., $\ell_1,\dots,\ell_4,a_1,a_2,a_3$, with the following relations:

$$(\rho_1)\qquad [a_2,a_3]=1$$
$$(\sigma_{24})\qquad a_2\cdot\ell_4\cdot\ell_2=1$$
$$(\sigma_{13})\qquad \ell_3\cdot a_2\cdot\ell_1=1$$
$$(\sigma_{34})\qquad \ell_3\cdot a_2\cdot\ell_4\cdot a_2^{-1}\cdot a_1=1$$
$$(\sigma_3)\qquad a_1^{p+1}\cdot a_2=1$$
$$(\infty)\qquad \ell_3\cdot a_2\cdot a_3\cdot\ell_4\cdot\ell_2\cdot a_1\cdot\ell_1=1\Longleftrightarrow a_3\cdot a_1=a_2$$
$$(\tau_1)\qquad \ell_2\cdot\ell_3\cdot a_3^2=1$$
$$(\tau_2)\qquad \ell_4\cdot\ell_1\cdot a_3^2=1$$
$$(\rho_{12})\qquad [\ell_2,a_1,\ell_1]=1$$
$$(\rho_{14})\qquad [a_3,\ell_4,\ell_1]=1\Leftrightarrow\ell_1\cdot a_3\cdot\ell_4\cdot a_3=1\Leftrightarrow[\ell_1,a_3]=[\ell_4,a_3]=1$$
$$(\rho_2)\qquad [a_3,\ell_2\cdot a_1\cdot\ell_2^{-1}]=1$$
$$(\rho_{23})\qquad [\ell_3,a_3,\ell_2]=1\Leftrightarrow[\ell_2,a_3]=[\ell_3,a_3]=1$$

This implies that a_3 is central, which replaces (ρ_{23}), (ρ_2), (ρ_{14}), and (ρ_1). Some generators can be eliminated:

$$a_2=a_1^{-(p+1)}\qquad a_3=a_1^{-(p+2)}\qquad \ell_2=\ell_1\cdot a_1^{-(p+3)}$$
$$\ell_3=\ell_1^{-1}\cdot a_1^{p+1}\qquad \ell_4=a_1^{2(p+2)}\cdot\ell_1^{-1}$$

Hence the group is generated by a_1,ℓ_1 with the following relations

$$[a_1^{p+2},\ell_1]=1\qquad [a_1,\ell_1^2]=1$$
$$\ell_1\cdot a_1\cdot\ell_1^{-1}\cdot a_1^{2p+5}=1\qquad \ell_1\cdot a_1\cdot\ell_1=a_1^{p+1}$$

As a consequence $a_1^{2(p+2)(p+3)}=1$, and then $\ell_1^2\cdot a_1^{p+6}=1$. Hence, calling $a:=a_1$, $\ell:=\ell_1$, and $q:=p+3$ we have:

$$G_1=\langle a,\ell\mid a^{2(q-1)q}=1,\ell^2=a^{3(q-1)},\ell\cdot a\cdot\ell^{-1}=a^{1-2q}\rangle. \tag{2.1}$$

Note that $\ell^2,a^{(q-1)}$ are central. The group fits in a short exact sequence

$$1\to C_{2q(q-1)}\to G_1\to C_2\to 1$$

where C_j is a cyclic group of order j. Any element of G_1 admits a unique representation of the form $\ell^\varepsilon\cdot a^m$, where $\varepsilon\in\{0,1\}$ and $m\in\{0,1,\dots,2(q-1)q-1\}$:

$$
\begin{array}{lll}
a_2 = a_1^{2-q} & a_3 = a_1^{1-q} & \ell_2 = \ell_1 \cdot a_1^{-q} \\
\ell_3 = \ell_1 \cdot a_1^{1-2q} & \ell_4 = \ell_1 \cdot a_1^{1-q} & e_{12} = a_1^{2(q-1)} \\
e_{14} = a^{q-1} & e_{23} = a^{q-1} &
\end{array}
$$

Let us compare this presentation with the one given in §1.1. In the notation $m = p+2 = q-1$. The element S in §1.1 corresponds with the element a in (2.1) while T corresponds with ℓa.

2.3. **Fundamental group of the complement of the Milnor fiber of a smoothing of $\mathcal{C}_3^2(p)$.**

We are going to study a projective curve $\mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{T}_\infty$, where $\mathcal{C}_3$ is a nodal cubic with node at $P \in \mathbb{P}^2$, $\mathcal{C}_2$ is a smooth conic, and $\mathcal{T}_\infty$ is a line satisfying:

- $\mathcal{C}_2 \cap \mathcal{C}_3 = \{Q\}$, where $P \neq Q$ (from Bézout's theorem $(\mathcal{C}_2 \cdot \mathcal{C}_3)_Q = 6$).
- $\mathcal{T}_\infty$ is one of the tangent lines to $\mathcal{C}_3$ at P.

It is not hard to see that there is only such a curve up to projective transformation. Equations can be given:

$$
\begin{aligned}
\mathcal{C}_3 &: y^2 z = x^2(x+z) \\
\mathcal{C}_2 &: y^2 = -(x+z)(2x+z) \\
\mathcal{T}_\infty &: y = x.
\end{aligned}
$$

The other tangent line to $\mathcal{C}_3$ at P is denoted by $\mathcal{T}_0$ and its equation is $y + x = 0$.

Remark 2.4. Fowler showed that $\mathcal{C}_3^2(p)$ had two distinct smoothing components (related by complex conjugation) which seems to be in contradiction with the projective rigidity of $\mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{T}_\infty$. In fact, there is no such contradiction; note that as shown in Figure 11 for the construction of the Milnor fiber we need to perform some blow-ups at one of the two points of $\mathcal{C}_2 \cap \mathcal{T}_\infty$ (which are complex conjugate with the above equations!), and this fact confirms the existence of two distinct smoothing components.

Unfortunately, the real picture in Figure 10 does not contain all the topological information of the curve, mainly due the fact that $\mathcal{C}_2$ and $\mathcal{T}_\infty$ do not intersect at real points.

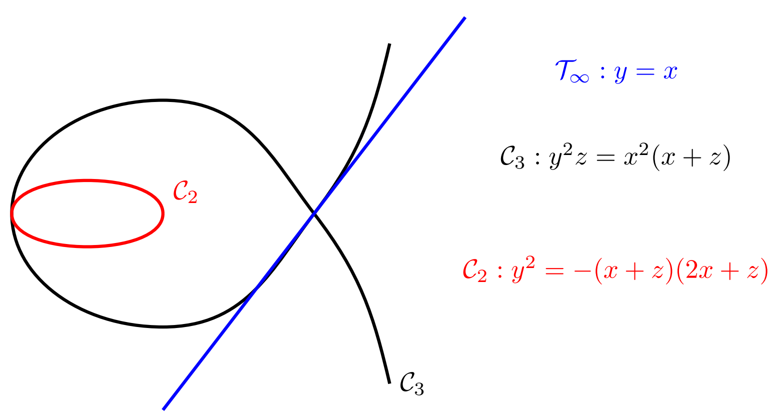


FIGURE 10. Real picture of $\mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{T}_\infty$.

Theorem 2.5. *The fundamental group of the complement of $\mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{T}_\infty$ in $\mathbb{P}^2$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.*

Before giving the proof of this theorem, let us show how the plane curve curve $\mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{T}_\infty$ is related to the Milnor fibre of the $\mathbb{Q}$HD-smoothing of $\mathcal{C}_3^2(p)$. This curve follows the ideas in [19] to find the curve at infinity for $\mathcal{C}_2^3(p)$, using conics. In [2], the author proceeds using a line arrangement with 9 lines: the McLane arrangement and one line joining two triple points.

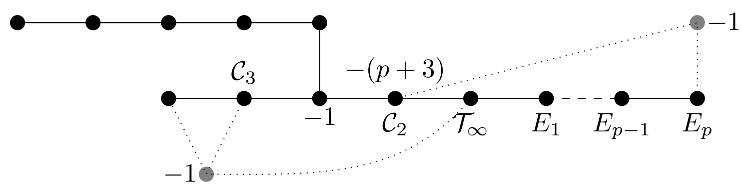


FIGURE 11. Resolution of $\mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{T}_\infty$ including $\mathcal{C}_2^3(p)$.

Corollary 2.6. *The Milnor fibre of the* $\mathbb{Q}$HD*-smoothing of* $\mathcal{C}_3^2(p)$ *is abelian.*

Proof. In Figure 11 we have depicted a (non-minimal) embedded resolution of the singularities of $\mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{T}_\infty$. Let $\pi : X \to \mathbb{P}^2$ be that resolution. Then $\mathbb{P}^2 \setminus (\mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{T}_\infty)$ is isomorphic to $X \setminus \pi^{-1}(\mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{T}_\infty)$ and then its fundamental group is abelian.

We obtain the Milnor fibre F of the $\mathbb{Q}$HD-smoothing of $\mathcal{C}_3^2(p)$ as the complement in X of all the irreducible components of $\pi^{-1}(\mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{T}_\infty)$ with the exception of the *gray* components in Figure 11. Then $\pi_1(F)$ is a quotient of $\mathbb{Z}^2$ by Proposition 2.3 and the statement follows. □

In Figure 11 one can see the dual graph of a resolution of $\mathcal{C}_3 \cup \mathcal{C}_2 \cup \mathcal{T}_\infty$, with extra blow-ups at one of the points in $\mathcal{C}_2 \cap \mathcal{T}_\infty$.

Actually, Theorem 2.5 can be proved using `SIROCCO` [11] inside `Sagemath` [14]. A simple explanation on how it works can be found in [10]. The code is very simple:

```
R.<x,y,z>=QQ[]
F=(y^2*z-x^2*(x+z))*(y^2+(x+z)*(2*x+z))*(y-x)
C=Curve(F)
C.fundamental_group()
```

We include a computer-free proof of the Theorem. The strategy is to apply birational transformations to obtain an arrangement of curves in $\mathbb{C}^2$ such that the complement of this arrangement is isomorphic to $\mathbb{P}^2 \setminus (\mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{T}_\infty \cup \mathcal{T}_0)$. This arrangement has real equations, and moreover the real picture contains all the topological information. We can compute the fundamental group using the Zariski-van Kampen method applied to the vertical projection. The first interesting property of the curve is that all the *non-transversal* vertical lines are in the real picture. Not all the real vertical lines intersect the arrangement of curves at real points, but the real part of the intersections can be tracked. As a consequence, the real picture allows one to find the braid monodromy of the curve, and so the fundamental group can be computed. In order to obtain the fundamental group of $\mathbb{P}^2 \setminus (\mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{T}_\infty)$ an extra step is needed. We can compute the meridian of $\mathcal{T}_0$ in terms on the given presentation; it is enough to *kill* this meridian.

Proof of Theorem 2.5. Let us blow-up the nodal point $[0:0:1]$ of $\mathcal{C}_3$. Let E be the exceptional component of the resulting ruled surface Σ_1, see Figure 12.

We continue with a couple of elementary transformations which yield Σ_3. We blow up $E \cap \mathcal{T}_i$ and contract the strict transforms of $\mathcal{T}_\infty, \mathcal{T}_0$, keeping the exceptional components $\mathcal{F}_\infty, \mathcal{F}_0$. The complement of $\mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{T}_\infty \cup \mathcal{T}_0$ in $\mathbb{P}^2$ is isomorphic to the complement of $\mathcal{C}_2 \cup \mathcal{C}_3 \cup E \cup \mathcal{F}_\infty \cup \mathcal{F}_0$ in Σ_3. We can provide equations. If we blow-down the (-3)-section E we obtain the weighted

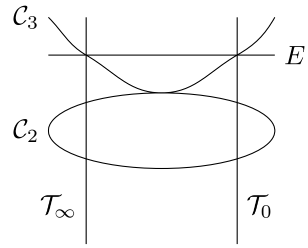


FIGURE 12. Combinatorial picture in Σ_1 (the intersections of $\mathcal{C}_2$ with $\mathcal{T}_i$ are not real).

projective plane $\mathbb{P}^2_\omega$, $\omega := (1,1,3)$, where the curves are defined by weighted homogeneous polynomials in the variables $x_\omega, y_\omega, z_\omega$. In fact, the birational transformation is

$$\begin{array}{ccc} \mathbb{P}^2 & \overset{\psi}{\dashrightarrow} & \mathbb{P}^2_\omega \\ [x:y:z] & \longmapsto & \left[y-x : x+y : 8(y^2z - x^2(x+z))\right]_\omega . \end{array}$$

The curves $\mathcal{F}_\infty, \mathcal{F}_0$ have equations $x_\omega = 0$, $y_\omega = 0$, respectively, while $\mathcal{C}_3$ has equation $z_\omega = 0$. By writing down the inverse of the map ψ, a long but straightforward calculation yields that the equation of $\mathcal{C}_2$ is

$$z_\omega^2 - 2(x_\omega^3 + 3x_\omega^2 y_\omega - 3x_\omega y_\omega^2 - y_\omega^3)z_\omega + (x_\omega + y_\omega)^6 = 0, \tag{2.2}$$

with weighted degree 6. The intersection point of $\mathcal{C}_2$ and $\mathcal{C}_3$ is $[1:-1:0]_\omega$ and $\mathcal{C}_2$ has a nodal point on $\mathcal{F}_0$ at $[1:0:1]_\omega$; in particular, $\mathcal{C}_2 \cup \mathcal{F}_0$ has an ordinary triple point there. The curve $\mathcal{C}_2$ has another double point $[0:-1:1]_\omega$ (in $\mathcal{F}_\infty$).

Among the pencil of *lines* through $[0:0:1]_\omega$, those with equations $x_\omega = 0$, $y_\omega = 0$ and $x_\omega + y_\omega = 0$ intersect $\mathcal{C}_2 \cup \mathcal{C}_3$ in two points. A calculation shows that the other lines with this property are the tangent lines to $\mathcal{C}_2 \cup \mathcal{C}_3$, with equations $x_\omega + 3y_\omega = 0$ and $3x_\omega + y_\omega = 0$, and the tangency points have quasi-homogeneous coordinates $[3:-1:-8]_\omega$ and $[1:-3:-8]_\omega$. These lines would be the vertical tangent lines to $\mathcal{C}_2$ in Figure 13.

The local equation of $\mathcal{C}_2$ at the singular point $[1:0:1]_\omega$ can be described in local coordinates $(u,v) \mapsto [1:u:v+1]_\omega$, and one finds the tangent cone is $0 = 21u^2 - 6uv + v^2$. These two lines are not real, so in the real picture one has an isolated point and can see $\mathcal{F}_0$ but not $\mathcal{C}_2$.

We can consider the affine chart $\mathbb{C}^2 \equiv \Sigma_3 \setminus (E \cup \mathcal{F}_\infty)$, or equivalently, the affine chart $(y_\omega, z_\omega) \mapsto [1:y_\omega:z_\omega]_\omega$ of of $\mathbb{P}^2_\omega$. Figure 13 shows a real picture of this affine chart $x_\omega = 1$.

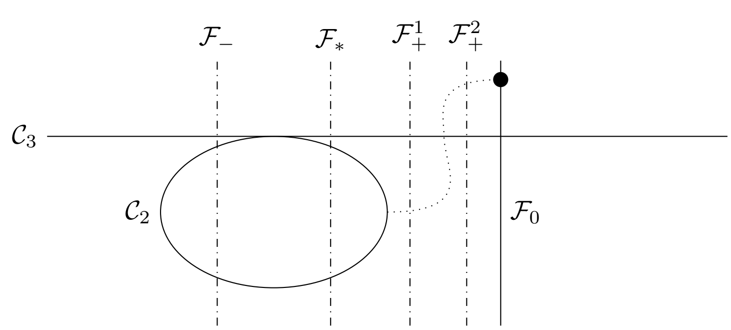


FIGURE 13. Affine chart (y_ω, z_ω) of $\mathbb{P}^2_\omega$. The dotted line represents real parts of the y_ω-coordinates of the strict transform of the conic.

The base point for the fundamental group is in $\mathcal{F}_* = \{y_\omega = y_*\}$, with z_ω-coordinate a real number $z_* \gg 1$. The geometric basis c, q_1, q_2 in this fibre, plus a meridian of $\mathcal{F}_0$ lying on the horizontal line $z_\omega = z_*$, together generate the fundamental group of

$$\mathbb{P}^2 \setminus (\mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{T}_\infty \cup \mathcal{T}_0) \cong \Sigma_3 \setminus (\mathcal{C}_2 \cup \mathcal{C}_3 \cup E \cup \mathcal{F}_\infty \cup \mathcal{F}_0) \cong \mathbb{C}^2 \setminus (\mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{F}_0).$$

Let $y_-, y_+^1, y_+^2 \in \mathbb{R}$ such that $\mathcal{F}_- = \{y_\omega = y_-\}$, $\mathcal{F}_+^1 = \{y_\omega = y_+^1\}$, and $\mathcal{F}_+^2 = \{y_\omega = y_+^2\}$. We consider geometric bases in these fibres which can be considered to have (y_*, z_*) as base points. This is done if we take in the horizontal line $z_\omega = z_*$ paths connecting (y_*, z_*) with the base points in each vertical fiber, namely (y_-, z_*), (y_+^1, z_*), and (y_+^2, z_*), see the upper part of Figure 15.

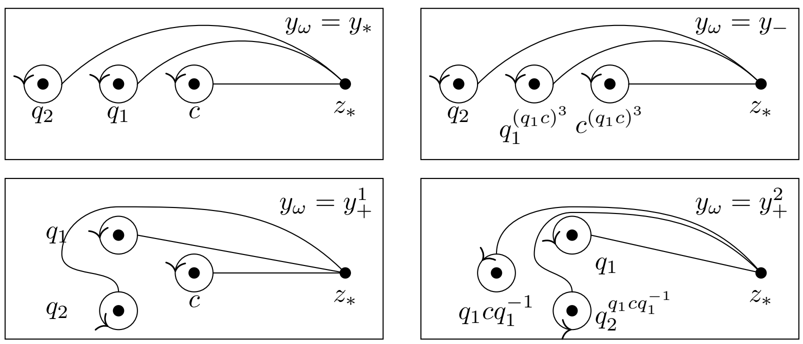


FIGURE 14. Geometric bases at the fibres; superindices stand for conjugation.

The elements of the geometric bases in each vertical line (Figure 14) can be expressed in terms of the generators in $\mathcal{F}_*$. The expression of each of this element in terms of the generators c, q_1, q_2 in $\mathcal{F}_*$ is shown in Figure 14. These equalities are obtained by the action of the connecting braids in the lower part of Figure 15 which defined isomorphisms of the fundamental group of the punctured line $\mathcal{F}_*$ with the fundamental group of the punctured lines $\mathcal{F}_-$, $\mathcal{F}_-$, $\mathcal{F}_+^1$ and $\mathcal{F}_+^2$. In order to draw the connecting braids, when two points have the same real part, we put the one with positive imaginary part to the left of the one with negative imaginary part.

The fundamental group is generated by c, q_1, q_2 in $\mathcal{F}_*$ (Figure 14) and f (Figure 15). The first relation is obtained by turning around $y_\omega = -\frac{1}{3}$ (vertical tangency) and it is $q_1 = q_2$. For the sake of brevity, we set $q := q_1 = q_2$.

Turning around $y_\omega = -1$, as the singular point is simple of type A_{11}, then the relation is $[q, (q \cdot c)^6] = 1$.

The next relation comes from turning around $y_\omega = -3$. This is a again a vertical ordinary tangency, and we obtain the equality of the second and third meridians if $\mathcal{F}_-$, i.e.,

$$q_2 = (q_1 \cdot c)^{-3} \cdot q_1 \cdot (q_1 \cdot c)^3,$$

which can be expressed as $[q, (q \cdot c)^3] = 1$. The previous relation becomes a consequence of this one.

Since $\mathcal{F}_0$ is part of the curve, the relations obtained by turning around $y_\omega = 0$ involve also the generator f together with the meridians in $\mathcal{F}_+^2$. We obtain:

$$f^{-1} \cdot (q_1 \cdot c \cdot q_1^{-1}) \cdot f = (q_1 \cdot c \cdot q_1^{-1}), \text{ i.e. } [f, q \cdot c \cdot q^{-1}] = 1,$$

and

$$[f, q_1 \cdot c^{-1} \cdot q_1^{-1} \cdot q_2 \cdot q_1 \cdot c \cdot q_1^{-1}, q_1] = 1 \Longleftrightarrow [f, q \cdot c^{-1} \cdot q \cdot c \cdot q^{-1}, q] = 1 \Longleftrightarrow [f, q, c^{-1} \cdot q \cdot c] = 1.$$

Recall that this relation means that $f \cdot q \cdot c^{-1} \cdot q \cdot c$ commute with the three factors.

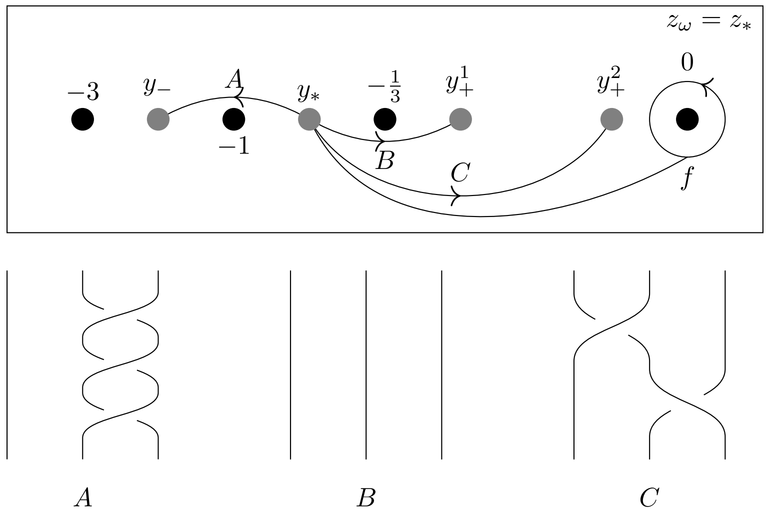


FIGURE 15. Paths in $y_\omega = y_*$ avoiding the non transversal vertical lines and associated braids

We are interested in the fundamental group of $\mathbb{P}^2 \setminus (\mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{T}_\infty)$. Let $\hat{\Sigma}_3$ be the space obtained by blowing up; it turns out that the exceptional component is the strict transform of $\mathcal{T}_0$ and that $\mathbb{P}^2 \setminus (\mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{T}_\infty)$ is isomorphic to the complement of $\mathcal{C}_2 \cup \mathcal{C}_3 \cup E \cup \mathcal{F}_\infty \cup \mathcal{F}_0$ in $\hat{\Sigma}_3$. Considering the meridian of $\mathcal{T}_0$, by Proposition 2.3 this means that $f \cdot q \cdot c^{-1} \cdot q \cdot c = 1$. Summarizing the group is generated by q, c, f with relations

$$[q, (q \cdot c)^3] = 1, \quad [f, q \cdot c \cdot q^{-1}] = 1, \quad f \cdot q \cdot c^{-1} \cdot q \cdot c = 1.$$

The third relation allows one to solve for f; inserting the value of f into the second relation, one deduces that $[q, cqc] = 1$; then, applying this new relation to the first relation written out, one sees that $qc = cq$. Thus, the group is abelian, isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. ☐

2.4. Fundamental group of the complement of the Milnor fiber of a smoothing of $\mathcal{C}_3^3(p)$.

In this subsection we consider the curve $\mathcal{C}_2 \cup \mathcal{C}_3$, to be used for the construction of the $\mathbb{Q}$HD-Milnor fiber for the family $\mathcal{C}_3^3(p)$ [13, (8.6)]. In Figure 16 we have a resolution of the singularities of $\mathcal{C}_2 \cup \mathcal{C}_3$ with extra blowing-ups at one of the branches of the node. The $\mathbb{Q}$HD-Milnor fiber for the family $\mathcal{C}_3^3(p)$ is obtained by forgetting the last exceptional component (gray vertex).

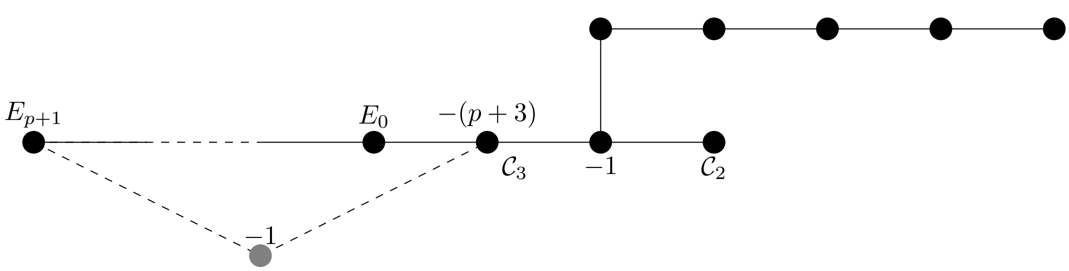


FIGURE 16. Graph at infinity for $\mathcal{C}_3^3(p)$.

Corollary 2.7. *The Milnor fibre of the $\mathbb{Q}$HD-smoothing of $\mathcal{C}_3^3(p)$ is abelian.*

Proof. We work as in the proof of Corollary 2.6. The first step is to use Proposition 2.3 to prove that the fundamental group of $\mathbb{P}^2 \setminus (\mathcal{C}_2 \cup \mathcal{C}_3)$ is abelian (in fact, isomorphic to $\mathbb{Z}$). Let $\pi : X \to \mathbb{P}^2$ be the (non-minimal) embedded resolution of the singularities of $\mathcal{C}_2 \cup \mathcal{C}_3$ depicted in Figure 16; we have that the fundamental group of $X \setminus \pi^{-1}(\mathcal{C}_2 \cup \mathcal{C}_3)$ is abelian and we proceed as in the proof of Corollary 2.6. □

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