Seeking GDOP-optimal Flower Constellations for global coverage problems through evolutionary algorithms

Daniel Casanova[a,b,*], Martín Avendaño[a,b], Daniele Mortari[c]

[a] *Centro Universitario de la Defensa Zaragoza. Spain*
[b] *Grupo de Mecánica Espacial-IUMA. Universidad de Zaragoza. Spain*
[c] *Depart. of Aerospace Engineering, Texas A&M University. College Station, TX, USA*

Abstract

In this paper, given a certain number of satellites (N_{sat}), which is limited due to the sort of mission or economical reasons, the Flower Constellation with N_{sat} satellites which has the best geometrical configuration for a certain global coverage problem is sought by using evolutionary algorithms. In particular, genetic algorithm and particle swarm optimization algorithm are used. As a measure of optimality, the Geometric Dilution Of Precision (GDOP) value over 30000 points randomly and uniformly distributed over the Earth surface during the propagation time is used. The GDOP function, which depends on the geometry of the satellites with respect to the 30000 points over the Earth surface (as ground stations), corresponds with the fitness function of the evolutionary algorithms used throughout this work. Two different techniques are shown in this paper to reduce the computational cost of the search process, one that reduces the search space and the other that reduces the propagation time. The GDOP-optimal Flower Constellations are obtained when the number of satellites varies between 18 and 40. These configurations are analysed and compared. Thanks to the Flower Constellation theory we find explicit examples where eccentric orbits outperform circular ones for a global positioning system.

Keywords: Flower Constellation Theory, Global Coverage Problem, Evolutionary Algorithms, Geometric Dilution of Precision

---

*Corresponding author
Email address: `casanov@unizar.es` (Daniel Casanova)

1. Introduction

The design of optimal satellite constellations is the key problem in all kinds of applications such as global navigation, global/regional coverage, telecommunications, Earth observation, etc. The purpose of this paper is to determine the best parameters of a 2D Lattice Flower Constellation (2D-LFC) [1] for a certain global coverage problem. This kind of satellite constellations have been previously studied [9] and they have interesting applications [10]. In particular, the problem of Global Positioning with a minimum of four satellites in view from any point on the Earth at any time as a constraint is studied. In this problem the geometry of these four or more visible satellites with respect to a ground station plays a fundamental role to know how accurate will be the computed user's position. As a measure of accuracy the concept Geometric Dilution of Precision (GDOP) [8] is used, which is a real number varying between 0 and ∞, while 0 means that the satellites have an ideal distribution to compute the user's position, a factor greater than 6 means that the precision will be rather poor.

Given the total number of satellites (N_{sat}) in the constellation, which depends on the sort of mission or economical factors, the design parameters of the Flower Constellation, which are the number of orbits (N_o), the number of satellites per orbit (N_{so}), and the configuration number (N_c) are first enumerated. Then, for each one of those configurations, the values of the eccentricity (e), inclination ($incl$) and argument of perigee (ω) which minimize the GDOP of the constellation are sought. This function is defined as the maximum value of the GDOP over the propagation time for 30000 ground stations randomly and uniformly distributed over the Earth surface [12]. In Section 2 the 2D-LFC theory and the GDOP function are presented. In section 3 an accurate formulation of the optimization problem is shown.

In this optimization problem different obstacles are dealt with. The first one is that the search space $(e, incl, \omega) \in [0, 1) \times [0, \pi] \times [0, 2\pi]$ gives too many points if it is discretized with a fine mesh, so an exhaustive search is computationally unfeasible. In order to overcome this problem, Genetic Algorithms and Particle Swarm Optimization [11, 4] are used. Those methods are explained in Section 3. Besides, one of our main results shows that it is possible to reduce the search space by a factor of two without losing any configuration (see Section 4.2). In any case, an exhaustive search on a coarse grid (in the reduced search space) will be always performed in order to have an approximate solution, and certify the solutions obtained by the evolutionary methods.

The second obstacle concerns the number of evaluations of the GDOP to find the optimal solution. Each 2D-LFC has to be propagated and the maximum value of the GDOP over the propagation time for 30000 ground stations is computed, making the number of evaluation of the GDOP function unfeasible. In Section 4.1 it is shown that it is possible to reduce the propagation time by a factor of approximately $1 : N_{sat}$ when evaluating the GDOP of a 2D-LFC, which translates into a huge computational time reduction.

Finally, the optimal configurations obtained and a comparison of the performance of the two evolutionary methods and the exhaustive search are provided in Section 5. Certain optimal 2D-LFC are also analysed in more detail, showing the evolutionary of maximum, minimum and average GDOP during the repetition time, as well as the number of visible satellites.

2. Preliminaries

In this section the tools needed for the rest of the paper: 2D-Lattice Flower Constellation theory and the three-dimensional position determination problem are introduced. Orbital elements and coordinates always refer to an Earth Center Inertial frame (ECI), whose z-axis coincides with the polar axis of the Earth, and the x-axis points the vernal equinox.

2.1. 2D Lattice Flower Constellation Theory

A 2D Lattice Flower Constellation [1] (2D-LFC) is described by three integer parameters and six continuous ones. The first set is (N_o, N_{so}, N_c) where N_o is the number of inertial orbits, N_{so} is the number of satellites per orbit, and $N_c \in [0, N_o - 1]$ is an integer parameter, named phasing parameter, which influences the initial distribution of the satellites in the constellation (see [2] for more information). The location of all the satellites of a 2D-LFC corresponds to a lattice in the (Ω, M)-space [2], where Ω represents the right ascension of the ascending node and M the mean anomaly. The (Ω, M)-space can be regarded as a 3D torus (both axes, M and Ω, are modulo 2π) and coincides with all the solutions of the following system of equations:

$$\begin{pmatrix} N_o & 0 \\ N_c & N_{so} \end{pmatrix} \begin{pmatrix} \Omega_{ij} - \Omega_{00} \\ M_{ij} - M_{00} \end{pmatrix} = 2\pi \begin{pmatrix} i \\ j \end{pmatrix}, \tag{1}$$

where $i = 0, \cdots, N_o - 1$, $j = 0, \cdots, N_{so} - 1$, and $N_c \in [0, N_o - 1]$. Satellite (i, j) is the j-th satellite on the i-th orbital plane. Consequently, Ω_{ij} and M_{ij}

represents the right ascension of the ascending node and the mean anomaly of the satellite (i, j), respectively.

The second set of parameters are Ω_{00} and M_{00}, which locate the satellite $(0, 0)$, and the remaining ones are the semi-major axis (a), the eccentricity (e), the inclination $(incl)$, and the argument of perigee (ω), which are the same for all the satellites in the constellation. This means that the orbital elements of the satellite (i, j) are $(a, e, incl, \omega, \Omega_{ij}, M_{ij})$, and its position in the ECI frame is,

$$r_{ij}(t) = Rot_z(\Omega_{ij}) Rot_x(incl) Rot_z(\omega) \begin{pmatrix} \cos\varphi_{ij}(t) \\ \sin\varphi_{ij}(t) \\ 0 \end{pmatrix} \frac{a\sqrt{1-e^2}}{1 + e\cos\varphi_{ij}(t)} \tag{2}$$

where Rot_z and Rot_x represent a rotation with respect to the z-axis and x-axis, respectively, and $\varphi_{ij}(t)$ is the true anomaly at time t.

2.2. Three-dimensional position determination problem

The three-dimensional position determination problem consists of finding the user position (x_u, y_u, z_u) through the location of satellites whose coordinates are well known. The Global Positioning System (GPS) uses the concept of Time-Of-Arrival (TOA), which finds the user position by measuring the TOA of a signal transmitted by a satellite at a known location to the user location [8]. Multiplying the TOA by the speed of the signal transmitted, it is possible to determine the distance from the user to the satellite, and by triangulation, it is possible to find the position of the user. Since there is usually a the time offset (t_u) between the receiver clock and the system time, the problem has four unknowns instead of three. For that reason, four visible satellites are needed.

In this problem it is necessary to determine which satellites are visible from a ground station. A satellite will be visible from a ground station if its incidence angle (defined as the angle between the normal vector to the surface of the Earth at the ground station and the position vector) is less than a given value β. Throughout this paper $\beta = 80°$ is considered.

The geometry of the constellation plays an important role since it is possible to determine how accurate the user position is from that. Several tools are defined to describe the accuracy error [8], but Geometric Dilution of Precision (GDOP) used by GPS is the most powerful accuracy indicator since it considers all possible sources of errors (position and time). The GDOP of a set of visible satellites, located at positions $\mathbf{r}_1, \ldots, \mathbf{r}_k$ relative to the observer, is given by $\sqrt{tr((H^T H)^{-1})}$, where $H \in \mathbb{R}^{k\times 4}$ is the matrix whose rows are $(\mathbf{r}_i/|\mathbf{r}_i|, 1)$ for $i = 1, \ldots, k$ (see [8] for more details).

In other words, the GDOP value shows how well the constellation is organized geometrically. This quantity varies between 0 and ∞, while 0 means that the constellation presents an ideal distribution of satellites, a large value (greater than 6) means that it presents a really poor geometry.

For a given 2D Lattice Flower Constellation named FC, and a given ground station located at $\mathbf{r}_{gs}$, $GDOP(FC, \mathbf{r}_{gs}, t)$ represents the GDOP of the set of visible satellites of FC from the ground station at time t. The maximum of those values as $\mathbf{r}_{gs}$ sweeps all the surface of the Earth and t covers an entire orbital period (T_p) is the fitness function, written $fitness(FC)$. The goal of this paper is to find a FC that minimizes the fitness function.

$$fitness(FC) = \max_{t\in[0,T_p]} \max_{\mathbf{r}_{gs}\in Earth} GDOP(FC, \mathbf{r}_{gs}, t). \tag{3}$$

Note that, the fitness function as defined above can not be computed since it is not possible to compute the value of the GDOP at each point over the Earth surface and at each instant of time. Therefore, an accurate approximate fitness function is required. For that purpose we select 30000 ground stations uniformly and randomly distributed over the Earth surface [12] $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{30000}$ that will remain fixed throughout this research and we set a time step of $\delta_t = 60\,sec$ for propagation. The approximate fitness function will be written $\overline{fitness}(FC)$. 30000 ground station are used to keep a balance between accuracy and computational cost. A simple experiment to validate the accuracy of the approximate fitness function was performed, concluding that $|\overline{fitness}(FC) - fitness(FC)| < 10^{-2}$ for 2D-LFCs in our search space.

3. Problem formulation

3.1. Search space

Given the total number of satellites N_{sat} of a 2D-LFC, it is possible to enumerate the different triples of phasing parameters N_o, N_{so}, N_c, satisfying the constraints implied by the theory. Since $N_{sat} = N_o N_{so}$ we have to choose N_o equal to a divisor d of N_{sat} and $N_{so} = N_{sat}/d$. The parameter N_c varies freely between 0 and $N_o - 1$, i.e. for any given d there are exactly d possible triples. Consequently, the number of different configurations is the sum of the divisors of N_{sat}, which can be represented as $\sum_{d|N_{sat}} d$. A search for each of those triples will be performed. For instance, given $N_{sat} = 27$, 40 different cases will be explored.

While the original theory of FCs [9, 3] imposed a compatibility condition on the orbital period T_p and the period of rotation of the Earth T_d, namely $N_d T_d = N_p T_p$, the theory of 2D-LFC shows that this condition is not necessary. However, the original constraint is kept since in our study the condition of repetition, in both the inertial and rotating frame, is needed. This does not represent any problem, since any value for T_p can be chosen freely, or equivalently, any value for the semi-major axis a. In this paper $N_d = 10$ and $N_p = 17$ have been used, giving a value of $T_p = 10/17\,days$ and $a = 29655.3163\,km$. These values were selected in order to match as closely as possible the semi-major axis of Galileo constellation for further comparison.

The parameters Ω_{00} and M_{00} play no role in the geometry of the constellation, so they are set to zero in all our computations. The remaining three parameters e, $incl$, ω define our search space. The value of the eccentricity has been limited to 0.3 to avoid searching highly eccentric orbits which are not likely to produce good constellations.

3.2. Search algorithms

Given the total number of satellites of a 2D-LFC, and a triple of phasing parameters as explained above, a search to find the best orbital parameters $e, incl, \omega$ is carried out, using three methods: an exhaustive search, genetic algorithms and particle swarm optimization [11].

For the exhaustive search (ES), the search space is discretized as follows: 20 different values for the eccentricity are considered, that is $e \in [0, 0.3]$ with step of 0.015, the inclination has 36 different possibilities, that is $incl \in [0, 180°]$ with step of 5°, and the argument of perigee $\omega \in [0, 360°]$ with step of 72°, so it assumes only 5 different values. Thus, the fitness function is calculated $20 \cdot 36 \cdot 5 = 3600$ times.

Our first evolutionary algorithm is the Genetic Algorithm (GA) which mimics the evolution of species. In our problem, individuals correspond to vectors $(e, incl, \omega)$ representing a point in the search space. The components of these vectors are called the genes of the individual. An initial population of 60 individuals is taken, i.e. 60 possible values for the orbital parameters $e, incl, \omega$. Then, each possible constellation is evaluated with the fitness function. After that, a new generation of 60 individuals is created. The individuals of the new generation consist of the 10 fittest ones from the previous generation, and 50 others obtained by crossover and mutation. The crossover consists of selecting a father $(e_f, incl_f, \omega_f)$ and a mother $(e_m, incl_m, \omega_m)$ from the previous generation at random and creating a son

$$(e_f x_1 + e_m(1 - x_1), incl_f x_2 + incl_m(1 - x_2), \omega_f x_3 + \omega_m(1 - x_3)),$$

where $x_1, x_2, x_3 \in \{0, 1\}$ are chosen at random with 0.5 probability each. After the son is created, it is decided with probability 0.05 whether it mutates or not. Mutation consists of choosing all three coordinates e, $incl$, ω at random within their allowed ranges. The process is repeated 60 generations and, at that point, the best individual found provides the solution to the optimization process. In one generation the fitness function is computed 60 times, so in 60 generations it will be calculated 3600 times.

Finally, Particle Swarm Optimization Algorithm (PSO) simulates the social behaviour of bird flocking or fish schooling. Each different bird or fish is considered as an initial particle in the search space. These particles are flying through the search space and have two essential capabilities: remembering their own best position and knowing the best position of the entire swarm. The basic idea is that individuals communicate good positions to each other and adjust their own position and velocity depending on the social and individual factors.

During the simulation, each particle has a position and velocity. Additionally, each particle keeps track of the position of the best solution it has visited so far (*pbest*) and the position of the best solution visited by any other particle (*gbest*). At each step, the velocity is updated at each iteration taking into account *pbest* and *gbest*. Changing the position and velocity of each particle at each iteration works as follows. Assume that the l-th particle has position vector $x_l(t)$ and velocity vector $v_l(t)$ [5]. Then, the updated velocity $v_l(t+1)$ will be:

$$\alpha v_l(t) + c_1 \cdot rand_1 \cdot (pbest_l - x_l(t)) + c_2 \cdot rand_2 \cdot (gbest(t) - x_l(t)), \quad (4)$$

where α is the inertia weight that controls the exploration of the search space. The constants c_1 and c_2, which in our simulation are taken between 0 and 1, determine how the individual and social factor affects the velocity of the particle. Finally, $rand_1$, $rand_2$ are random numbers chosen uniformly in $[0, 1]$. Note that without the second and third terms of the expression (4) the particle will keep the same direction until it hits the boundary.

The position is updated as follows:

$$x_l(t+1) = x_l(t) + v_l(t+1). \tag{5}$$

This process is repeated for each particle until the best optimal solution is obtained or the stopping criteria is reached. Our approach takes an initial swarm of $n = 60$ particles, i.e. 60 possible values for the orbital parameters $(e, incl, \omega)$ which are the positions, and 60 possible velocities for them. Both positions and velocities are chosen randomly within the search space. It should be noted that neither position or velocity correspond with the actual motion of the satellites; these quantities are unitless. Then, each constellation is evaluated with the fitness function and the new velocities and positions are updated according to Eq. (4) and Eq. (5). An inertia factor $\alpha = 0.95$, individual factor $c_1 = 0.75$, and social factor $c_2 = 0.35$ are used. The process is repeated 60 iterations. As in the Genetic Algorithm, in one generation the maximum GDOP is computed 60 times. Thus, in 60 generations the fitness function is calculated 3600 times.

For example, if a Flower Constellation with 27 satellites ($N_{sat} = 27$) is considered, the time that PSO (60 generations of 60 particles) takes to find the optimal constellation with one core is approximately 3200 seconds. There are 40 possible configurations for the phasing parameters N_{so}, N_o, N_c, so the total computational cost would be about $40 \cdot 3200 = 128000$ seconds, which is around 1.5 days. When the number of satellites is larger, not only we have more possible configurations, but also the computational time per configuration increases, since there are more satellites to evaluate.

To deal with this computationally intensive problem some parallelization techniques are applied, which consist of performing each search with a different core (if available). This means that each core will handle a search with for a different triple (N_{so}, N_o, N_c). Doing so, a speed up by a factor of $1 : N_{cores}$ is obtained. In addition to that, two reductions explained in the following section are applied.

4. Reductions

Two results that allow us to reduce the propagation time, and also the range of some variables in the search space are presented. The first reduction translates into an approximate $1 : N_{sat}$ reduction in the computational cost, and the other into a $1 : 2$ speed up. This two reductions, combined with parallelization of the search, give a total speed-up of $1 : 2N_{cores}N_{sat}$.

4.1. Propagation time reduction

The algorithms presented in the previous section require evaluation of the fitness function 3600 times. Each of these evaluations compute the GDOP of the constellation every $\delta_t = 60\,sec$ (until an orbital period is completed) in 30000 ground stations. The number of ground stations can not be reduced without hurting the accuracy of the value of the fitness. However, the total propagation time, can be reduced by a factor of $N_{sat}/\gcd(N_c, N_o)$, where gcd is the greatest common divisor, without changing the value of the fitness function, as proven below.

A 2D Lattice Flower Constellation FC is considered, and the amount $\Delta t = \frac{T_p}{N_o N_{so}} \gcd(N_c, N_o)$ is defined. It will be illustrated how the position of the satellites of the FC at time t and at time $t + \Delta t$ are related. More precisely, it will be shown that the previous relation is just a rotation with respect to the z-axis. If so, automatically it is concluded that $fitness(FC)$ can be computed propagating in the range $t \in [0, \Delta t]$, instead of $t \in [0, T_p]$, since rotations do not affect the geometry of the constellation. The same is valid for $\overline{fitness}(FC)$ if it is considered that the 30000 ground stations are uniformly distributed. Thus,

$$\overline{fitness}(FC) = \max_{i=1,\dots,30000} \max_{t\in[0,\delta t, 2\delta t,\dots,(\frac{\Delta t}{\delta t})\Delta t]} GDOP(FC, \mathbf{r}_i, t), \qquad (6)$$

where $\mathbf{r}_i$ represent the position vector of the 30000 ground stations, and $\delta_t = 60\,sec$ is the time step. This propagation time reduction translates into an speed-up of $N_{sat}/\gcd(N_c, N_o)$, which in most cases is N_{sat}.

Figure 1 relates the position of the satellites at time t and t'. The mean anomaly of each satellite has increased $\Delta M = \frac{2\pi}{T_p}(t' - t)$. The difference of the mean anomaly between satellite $(0, 0)$ and satellite (i, j) is $M_{ij} - M_{00} = \frac{2\pi}{N_{sat}}(jN_o - iN_c)$ from equation 1. If it is selected t' such as $\frac{2\pi}{T_p}(t' - t) = \frac{2\pi}{N_{sat}}(jN_o - iN_c)$, the orbit 0 at t' is a rotation of the orbit i at time t. The same happens between the orbit 1 at time t' and the orbit $i + 1$ at time t, and so on. Then, the FC at time t and t' is the same under a rotation.

The conclusion is valid for any value (i, j) selected. If (i, j) is selected such that the expression $(jN_o - iN_c)$ is minimized, we obtain that,

$$\Delta t = t' - t = \frac{T_p \gcd(N_o, N_c)}{N_{sat}} \qquad (7)$$

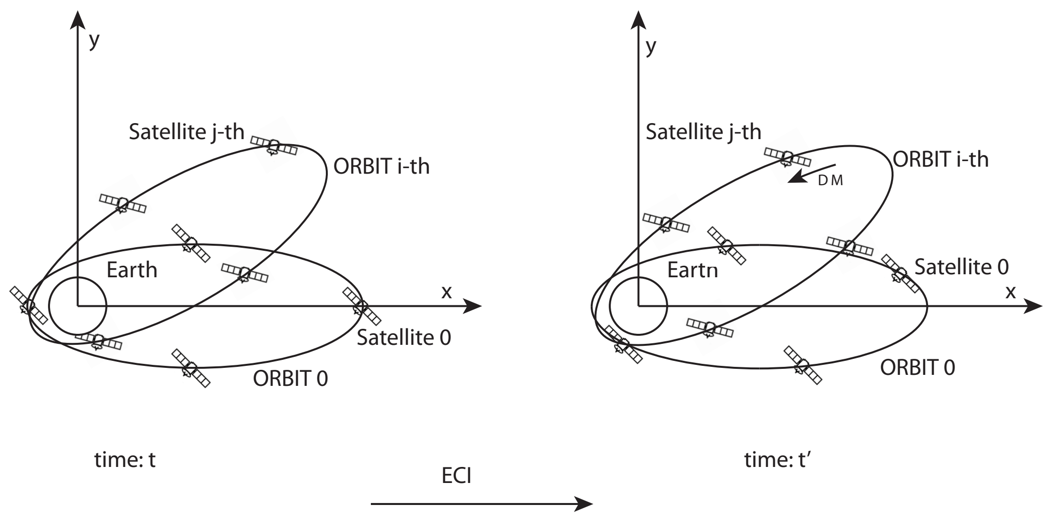

Figure 1: First orbit of the FC at time t, and at time t'. We observe how the satellites have been displaced an amount ΔM.

4.2. Search Space Reduction

The computational cost has been decreased by reducing the propagation time to compute the GDOP of the constellation. Another way to decrease the computational cost is by reducing the search space. By selecting the inclination in a range $0° \leq incl \leq 90°$ (instead of $0° \leq incl \leq 180°$) or choosing the parameter N_c in a range $[0, \ldots, \frac{N_o}{2}]$ (instead of $[0, \ldots, N_o - 1]$) it is possible to reduce considerably the computational cost. The following theorems show that either of these two reductions of the search space do not miss any possible configuration.

The values of the configuration number, N_c, and the index i are considered modulo N_o, i.e. it is always reduce to the representative value in the interval $[0, N_o - 1]$. Similarly, the index j is considered modulo N_{so}. Thus, the value $-N_c$ represents $-N_c \mod (N_o)$, and the value $-j$ represents $-j \mod (N_{so})$.

A 2D Lattice Flower Constellation FC is considered with parameters $(N_o, N_{so}, N_c, a, e, incl, \omega, \Omega_{00}, M_{00})$ and it is defined a FC' with parameters $(N_o, N_{so}, -N_c, a, e, \pi - incl, -\omega, \Omega_{00} + \pi, -M_{00})$. It will be shown how the position of the satellites of FC at time t and the satellites of FC' at time $-t$ are related. More precisely, it will be shown that the previous relation is just a symmetry with respect to the center of the Earth. If so, we conclude automatically that $fitness(FC) = fitness(FC')$, since neither the symmetry nor the time reversal affect the geometry of the constellation. The same is valid for $\overline{fitness}(FC)$ if it is considered that the 30000 ground stations are uniformly distributed. Thus,

$$\overline{fitness}(FC) = \overline{fitness}(FC') \tag{8}$$

This fact translates into two possible reductions in the search space, both of them translates into a speed up of 1 : 2. The first one is a 1 : 2 reduction in the inclination, and the second one a 1 : 2 reduction in the parameter N_c.

- $N_c \in \{0, 1, \ldots, N_o - 1\}$ and $(e, incl, \omega) \in [0, 1) \times [0, \frac{\pi}{2}] \times [0, 2\pi]$.
- $N_c \in \{0, 1, \ldots, \lfloor \frac{N_o}{2} \rfloor\}$ and $(e, incl, \omega) \in [0, 1) \times [0, \pi] \times [0, 2\pi]$.

The position of the satellites of FC at time t and the satellites of FC' at time $-t$ are related by,

$$\mathbf{r}_{ij}(t) = -\mathbf{r}'_{i(-j)}(-t), \tag{9}$$

where $\mathbf{r}_{ij}(t)$ represents the position of the satellite (i, j) at time t of the Flower Constellation FC, and $\mathbf{r}'_{i(-j)}(-t)$ represents the position of the satellite $(i, (-j))$ at time $-t$ of the Flower Constellation FC'.

From Eq. (1) we relate the right ascension of the ascending node and the mean anomaly of the Flower Constellations FC and FC',

$$\Omega'_{i(-j)} = \Omega_{00} + \pi + \frac{2\pi i}{N_o} = \Omega_{ij} + \pi, \tag{10}$$

$$M'_{i(-j)}(-t) = -M_{00} + \frac{2\pi}{N_o N_{so}}(-jN_o - i(-N_c)) + \frac{2\pi}{T_p}(-t) = -M_{ij}(t). \tag{11}$$

Eq. (11) also implies that,

$$\varphi'_{i(-j)}(-t) = -\varphi_{ij}(t) \tag{12}$$

By replacing relations (10) and (12) in the position equation given in Eq. (2) we obtain Eq. (9). Then, the position of the satellites of FC at time t and the satellites of FC' at time $-t$ are related by the Eq. (9).

5. Results

5.1. Method comparison

In this research three different algorithms have been used: an exhaustive search algorithm, and two evolutionary algorithms, which improve substantially the exhaustive search algorithm, as it is shown below.

For a given number of satellites N_{sat}, according to Section 3.1, the number of different constellations is given by the sum of the divisors of N_{sat}, given by $\sum_{d|N_{sat}} d$ as explained in section 3.1. Thus, if it is considered a constellation with the number of satellites varying between 18 and 40 ($18 \leq N_{sat} \leq 40$) due to design limitations (i.e. economical cost of the satellites), the total number of configurations is equal to:

$$\sum_{\substack{d|N_{sat} \\ 18\leq N_{sat}\leq 40}} d = 1104. \quad (13)$$

Each of these 1104 cases has been analysed to find the best parameters $(e, incl, \omega)$ that minimize the GDOP with the three methods. Figure 2 shows the number of times in which one method is better than the others. The PSO algorithm is the best method followed by the Genetic Algorithm and the exhaustive search algorithm. In certain configurations, it is impossible to find a constellation with GDOP better than 99. For instance, when $N_o = 1$ the satellites are always on the same orbit plane, hence the maximum GDOP is 99. Those cases have been excluded from the comparison between methods, and they are represented with a separate bar in Figure 2.

Note that the comparison between the three methods is fair because they evaluate the fitness function (i.e. the maximum GDOP) the same number of times in each method, as it is shown in Section 3.2.

5.2. Optimal configurations

Consider first a constellation with $N_{sat} = 27$ satellites and $N_p = 17$, $N_d = 10$ (to have the same semi-major axis as Galileo constellation satellites). There are 40 possible configurations for the phasing parameters. For each of those configurations, the three algorithms were used to determine the best parameters (e, $incl$, and ω) that minimize the maximum value of the GDOP along the propagation time. These optimal parameters are shown in Table 1.

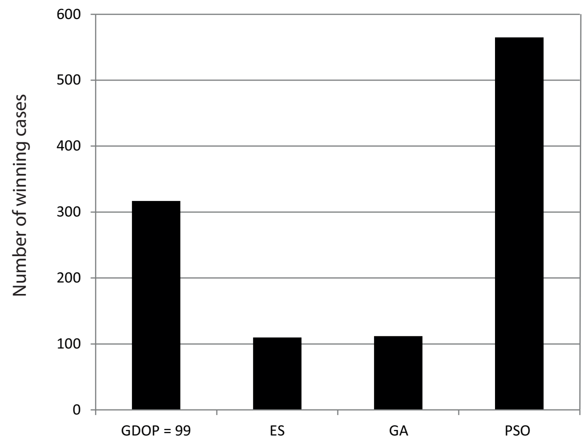

Figure 2: Comparison of the three methods.

Table 1: Optimal configurations with three different methods.

Meth.	N_{sat}	N_o	N_{so}	N_c	e	$incl[deg]$	$\omega[deg]$	$fitness(FC)$
ES	27	3	9	2	0.03	55.00	0.00	3.64
GA	27	3	9	2	0.04	55.59	177.94	3.65
PSO	27	3	9	2	0.00	54.06	173.71	3.61

It can be clearly seen that the best constellation found depends on the method. Regarding the sensitivity to the method, the three methods are going to be considered, and the best solution found by any of them is selected. The solutions found by the other two are used to provide some confidence on the optimality of the GDOP.

Now, the same procedure for any number of satellites $18 \leq N_{sat} \leq 40$ is performed. The GDOP of the best configuration found by each of the three methods is shown in Figure 3. Only the configurations with more than 23 satellites are shown, since the cases with $N_{sat} \leq 23$ have GDOP above 5, which is considered not good for solving a global positioning problem.

Intuitively, the more satellites the constellation has, the better results for the GDOP value should be obtained. However, this is not always true, because with 27 satellites better results than with 28 satellites are obtained. A similar behaviour is observed with 29 and 30 satellites and also with 38 and 39 satellites.

It seems that the number of configurations is a potential factor to find

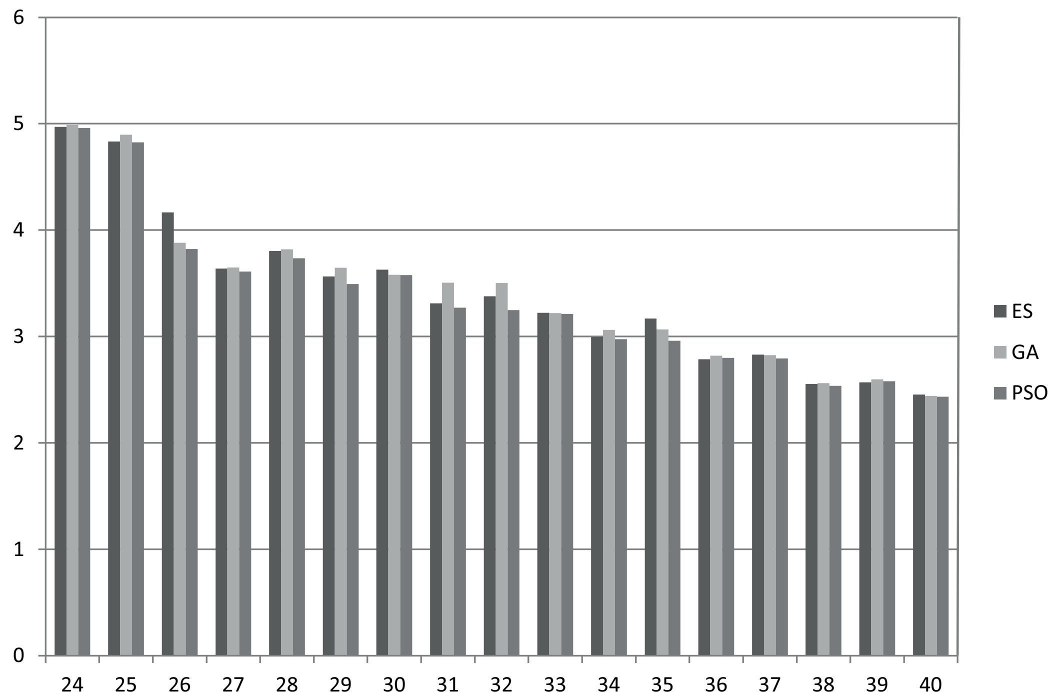

Figure 3: Maximum GDOP experienced with three different methods for constellation with $N_{sat} \in [24, 40]$.

Table 2: Optimal configurations.

N_{sat}	N_o	N_{so}	N_c	e	$incl\,[deg]$	$\omega\,[deg]$	$fitness(FC)$
24	24	1	2	0.000	125.187	88.61	4.96074
25	25	1	2	0.000	127.492	236.48	4.82628
26	26	1	10	0.000	61.104	492.41	3.82216
27	3	9	2	0.000	54.057	173.71	3.61023
28	7	4	2	0.000	127.535	150.96	3.73561
29	29	1	11	0.023	61.518	100.86	3.49341
30	10	3	4	0.036	57.836	263.91	3.57843
31	31	1	4	0.000	71.774	256.26	3.27212
32	16	2	7	0.253	63.514	179.55	3.24969
33	11	3	4	0.006	59.795	94.01	3.21361
34	34	1	12	0.000	120.478	229.41	2.97527
35	35	1	8	0.300	63.005	0.08	2.95912
36	12	3	4	0.075	60.000	0.00	2.78647
37	37	1	5	0.000	60.637	82.59	2.79373
38	38	1	14	0.000	59.039	184.67	2.53557
39	13	3	4	0.065	60.000	0.00	2.57115
40	10	4	7	0.000	58.009	25.72	2.43542

good constellations, i.e. the more configurations are possible, the more possibilities to find a good constellation for global coverage. However this is not always true as it can be observed with 29 and 30 satellites, because the 29 satellites constellation has fewer configurations than the 30 satellites constellation and better results are obtained.

The best configurations found for $N_{sat} \in [24, 40]$ are summarized in Table 2.

While our algorithms compare constellations based on the worst GDOP value seen by any of the ground stations at any instant of time, it would be interesting to see the evolution in time of the maximum GDOP, average GDOP, and minimum GDOP experienced by the 30000 ground stations. These three values of the GDOP are shown in Fig. 4 for our optimal constellation with 27 satellites. For clarity, Fig. 5 shows only the evolution of the maximum value of the GDOP over time.

In the first of these figures, it is observed that the maximum GDOP experienced by the 30000 stations is around 3.6 at any time, meaning that there is always a ground station where the GDOP is about 3.6, and that no ground station has a GDOP worse than that. Similarly, it is observed that the minimum GDOP is approximately 1.5, so there is always a point on the Earth where the GDOP is as good as 1.5. Finally, the average moves around 2.3, so it can be expected that half of the ground stations have a GDOP

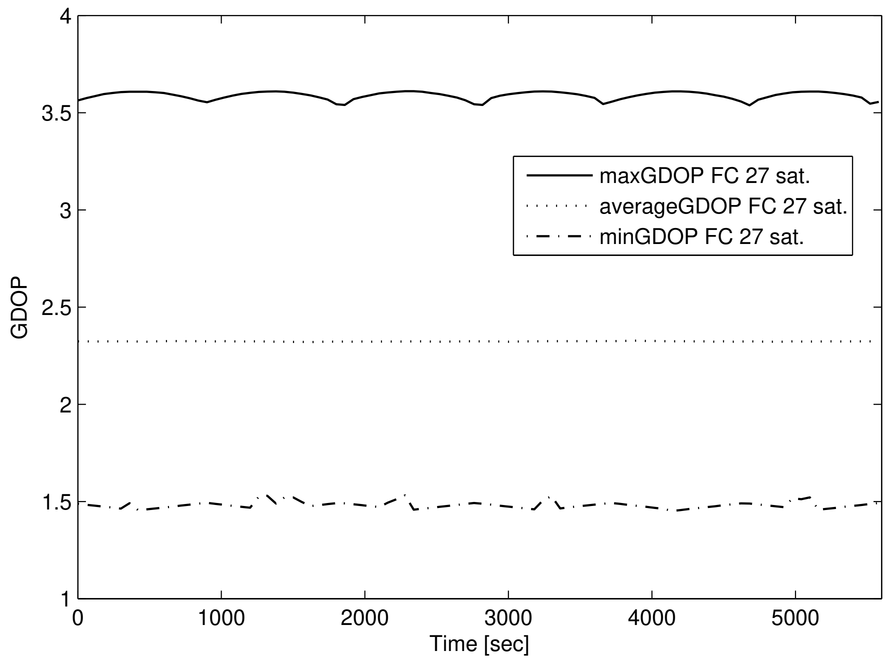

Figure 4: Maximum, minimum and average GDOP value of our 27 satellite constellation.

between 1.5 and 2.3, and the other half in the interval $[2.3, 3.6]$. Intuitively, this means that about half of the surface of the Earth would experience a GDOP better than 2.3.

In the next figure, it is observed that the maximum GDOP oscillates between 3.58 ± 0.04. The deviation from the center value is less than 1.2%. This indicates that the performance of the constellation remains almost constant over time.

Fig. 6 shows the number of visible satellites from a Ground Station chosen at random for the optimal 27 satellite constellation shown in Table 2. More than four satellites are always visible at any instant of time. Fig. 7 shows the average of visible satellites considering all ground stations at any instant of time. For each instant of time the visible satellites for each of the 30000 ground stations are computed. In this particular case, the visible satellites at any ground station will be an integer varying between 4 and 27. However,

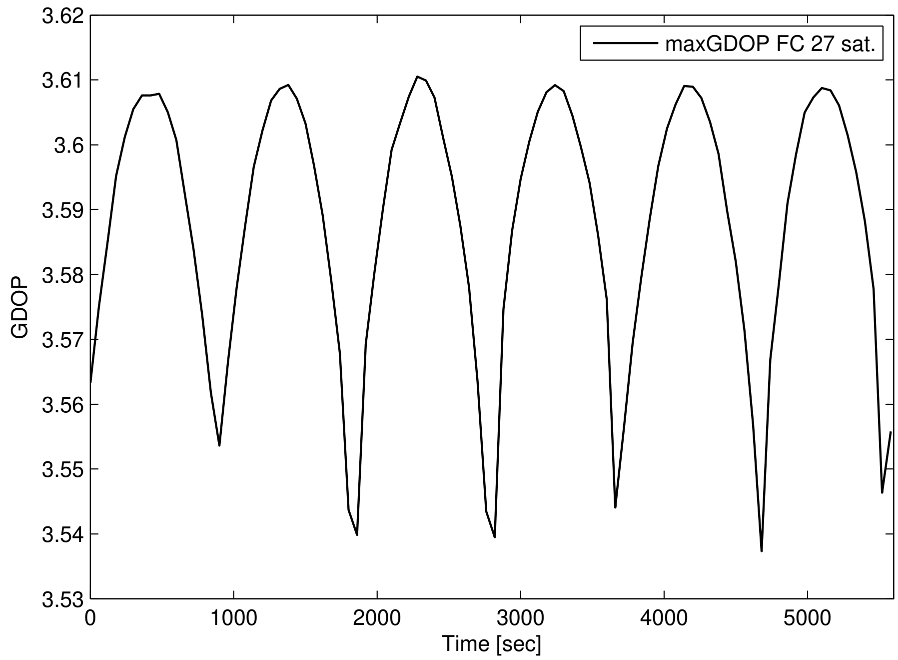

Figure 5: Maximum GDOP value of our 27 satellite constellation over time.

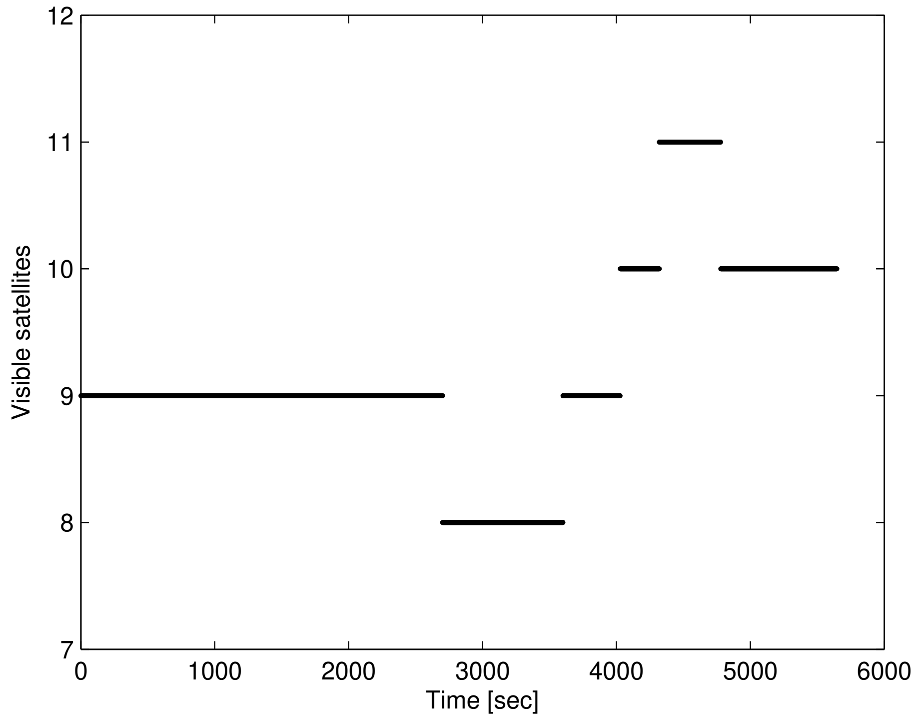

Figure 6: Number of visible satellites at any instant of time.

the average of the number of visible satellites considering the 30000 ground stations, gives a rational number which oscillates between 8.3899 and 8.4009.

5.3. Other remarkable facts

One of the innovative results, thanks to the 2D-LFC theory, is that eccentric orbits are considered in the searching process. Table 2 illustrates that in many occasions the optimal configuration has a highly eccentric orbit. For instance, when $N_{sat} = 35$, the optimal constellation has $e = 0.3$. This case is shown in Figure 8.

It is observed in Table 2 that there exist some configurations that obtain better results with less satellites. For example with 27 satellites it is obtained better results than with 28 satellites. The same thing occurs with 29 and 30 satellites, and also with 38 and 39 satellites. Figure 9 and Figure 10 show the

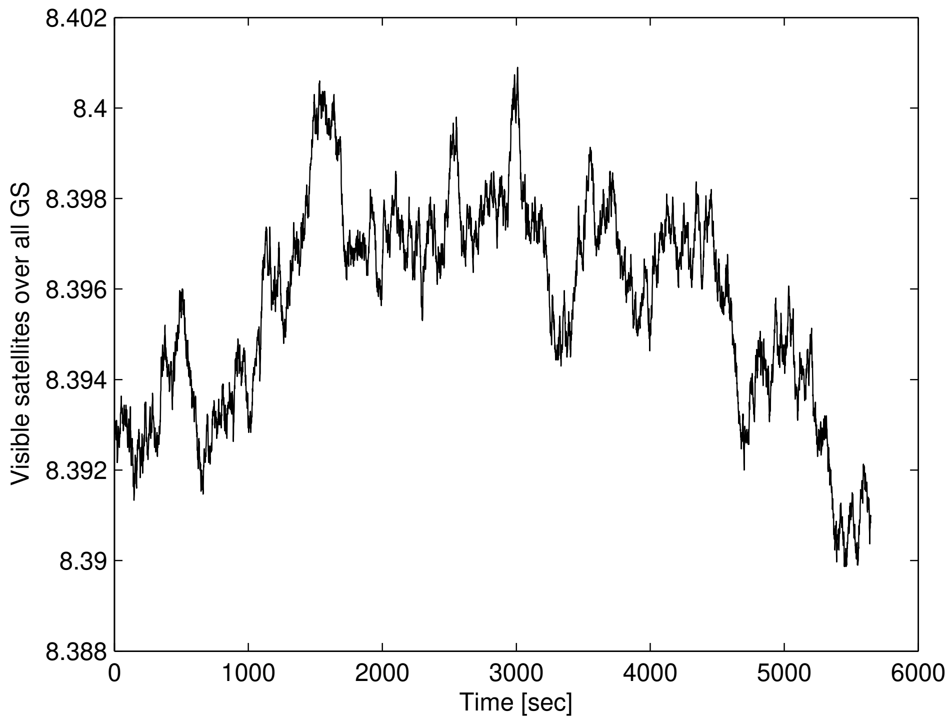

Figure 7: Average visible satellites over all Ground Stations.

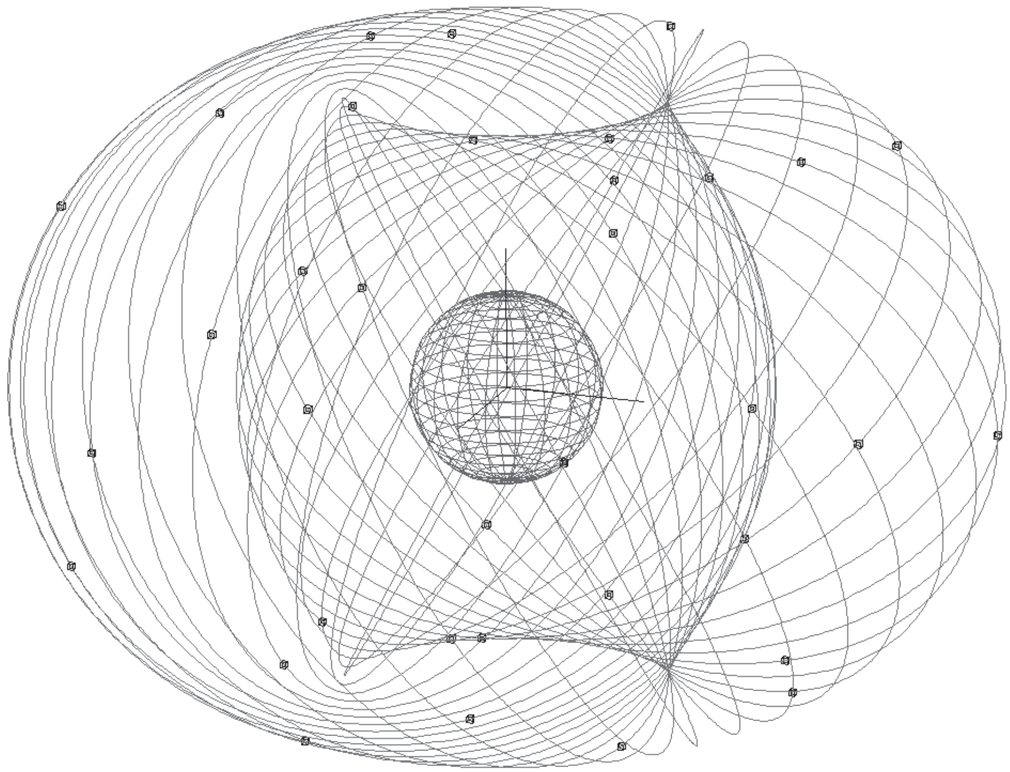

Figure 8: A ($N_o = 35, N_{so} = 1, N_c = 8, N_p = 17, N_d = 10, e = 0.3, incl = 63.005, \omega = 0.084$) 2D-LFC in the ECI frame. This Flower Constellation has a suitable distribution of satellites that result into a good GDOP value. Furthermore, the value of the eccentricity is considerable hight ($e = 0.3$)

maximum GDOP of the constellations experienced over time that confirms that sometimes with less satellites it is possible to obtain better results.

6. Conclusions

The present contribution focuses on the search of GDOP-optimal 2D-Lattice Flower Constellation for solving a global positioning problem. Thanks to the evolutionary algorithms and some reduction techniques the computational time cost to find the optimal solution is substantially reduced. It is possible to conclude that any constellation with less than 23 satellites has a poor GDOP, hence not useful for a global positioning system. A constellation

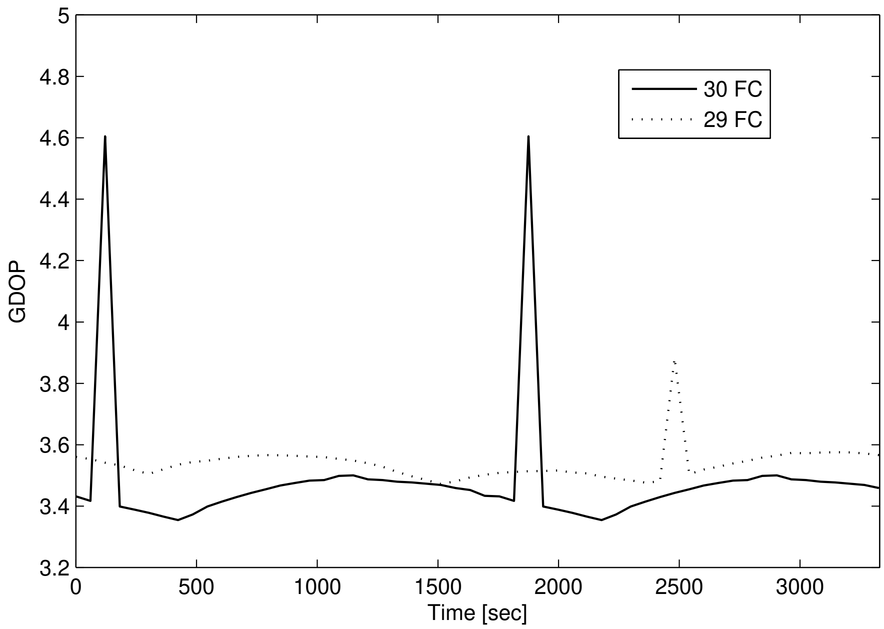

Figure 9: Maximum GDOP experienced over time of our 29 satellite 2D-LFC and our 30 satellite 2D-LFC.

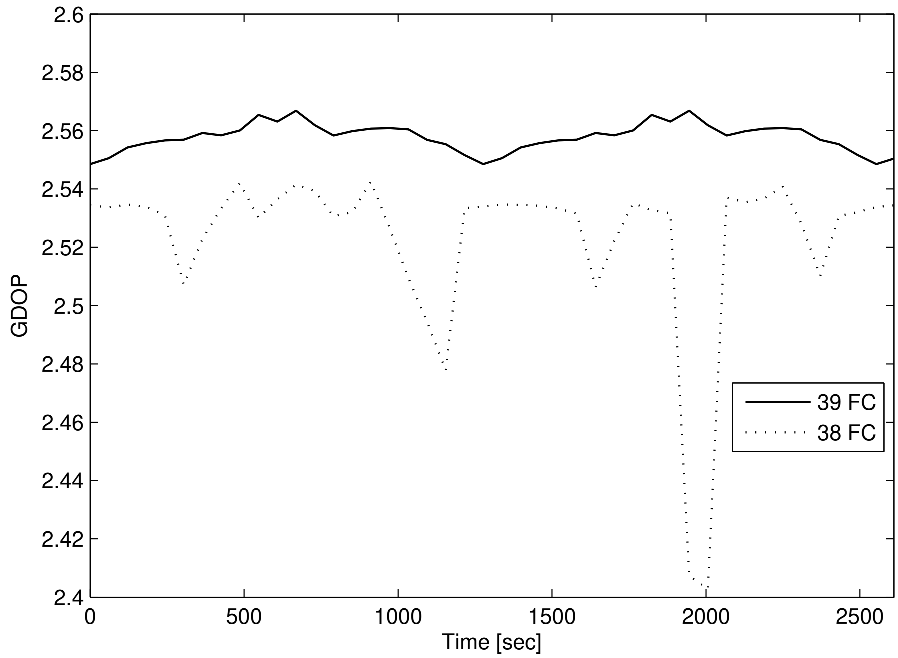

Figure 10: Maximum GDOP experienced over time of our 38 satellite 2D-LFC and our 39 satellite 2D-LFC.

with 27 satellites was found whose GDOP value is lower than a constellation with 28 satellites. Something similar happens with 29 and 30 satellites, and also with 38 and 39 satellites. It is shown the evolution of the GDOP in time, and the number of visible satellites at any instant of time from a certain ground station. Thanks to the 2D-LFC theory it is possible to include eccentric orbits in the search space, and explicit examples where eccentric orbits outperform circular ones were found.

As a future work, the study of the GDOP of a constellation can be expanded with the Necklace Flower Constellation theory [6], which decreases the cost of the mission by reducing the number of satellites in each orbit while keeping the symmetries in the (Ω, M)-space.

Acknowledgments

This work was supported by the research project with reference CUD13-15 in the organism *Centro Universitario de la Defensa de Zaragoza*, and the research project with reference ESP2013–44217-R supported by the *Spanish Ministry of Economy and Competitiveness*.

References

[1] M.E. Avendaño, J.J. Davis, D. Mortari, The 2D Lattice Theory of Flower Constellations, Celest. Mech. Dyn. Astr. 116 (2013) 325-337.

[2] M.E. Avendaño, D. Mortari, Rotating Symmetries in Space: The Flower Constellations, AAS/AIAA Space Flight Mechanics Meeting Conference. Paper AAS 19-189, Savannah, GA, February 2009.

[3] M.E. Avendaño, D. Mortari, New insights on Flower Constellations theory, IEEE T. Aero. Elec. Sys. 48(2) (2012) 1018-1030.

[4] T. Back, Evolutionary algorithms in theory and practice : evolution strategies, evolutionary programming, genetic algorithms, New York, NY: Oxford Univ. Press (1996)

[5] L. Blasi, S. Barbato, M. Mattei, A particle swarm approach for flight path optimization in a constrained environment, Aerosp. Sci. Technol. 26 (2013) 128-137

[6] D. Casanova, M.E. Avendaño, D. Mortari, Design of Flower Constellations using Necklaces, IEEE T. Aero. Elec. Sys. 50(2) (2014) 1347-1358

[7] D. Casanova, Flower Constellations: Optimization and Applications, PhD Thesis, http://zaguan.unizar.es/record/11733 (2013).

[8] E.D. Kaplan, C.J. Hegarty, Understanding GPS: principles and applications, second ed., Artech House Publishers, 2006.

[9] D. Mortari, M.P. Wilkins, C. Bruccoleri, The Flower Constellations, J. Astronaut. Sci. 52 (2004) 107-127.

[10] D. Mortari, M. Sanctis, M. Lucente, Design of Flower Constellations for Telecommunication Services, Proceedings of the IEEE, 99(11) (2011)

[11] S.N. Sivanandam, S.N. Deepa, Introduction to Genetic Algorithms, Springer, 2008.

[12] E.W. Weisstein, Sphere Point Picking, MathWorld-A Wolfram Web Resource (2013), http://mathworld.wolfram.com/SpherePointPicking.html