Electronic version of an article published as Journal of Knot Theory and Its Ramifications Vol. 21, No. 08, 1250082 (2012). DOI: 10.1142/s0218216512500824. ©World Scientific Publishing Company. https://www.worldscientific.com/worldscinet/jktr

LIFTING FILLING DEHN SPHERES

Ruben Vigara
Centro Universitario de la Defensa - Zaragoza
Academia General Militar - Ctra. Huesca s/n
C.P. 50090 Zaragoza

ABSTRACT

A Dehn sphere Σ [9] in a closed 3-manifold M is a sphere immersed in M with only double curve and triple point singularities. The Dehn sphere $\Sigma \subset M$ lifts to $M \times I$, where I is an interval, if there exists an embedded sphere in $M \times I$ that projects onto Σ. Every closed 3-manifold has a filling Dehn sphere [8], i. e. a Dehn sphere that defines a cell decomposition of M. In [12] it is shown that every closed 3-manifold M has a filling Dehn sphere that lifts to $M \times I$. In this paper it is proved that every closed 3-manifold has a filling Dehn sphere that does not lift to $M \times I$. This results solve a question of Roger Fenn.

Keywords: 3-manifold, immersed surface, filling Dehn sphere
Math. Subject Classification: 57N10, 57N35

1. Preliminary definitions and results.

Through the whole paper all 3-manifolds are assumed to be closed, that is, compact connected and without boundary, and orientable. All surfaces are assumed to be compact, orientable and without boundary. A surface may have more than one connected component. We will denote a 3-manifold by M, a surface by S, and by I any interval of real numbers.

Let M be a 3-manifold.

A subset $\Sigma \subset M$ is a *Dehn surface* in M [9] if there exists a surface S and a transverse immersion $f : S \to M$ such that $\Sigma = f(S)$. In this situation we say that f *parametrizes* Σ. If $S = S^2$ is a 2-sphere then Σ is a *Dehn sphere*. For a Dehn surface $\Sigma \subset M$, its singularities are divided into *double points*, where two sheets of Σ intersect transversely; and *triple points*, where three sheets of Σ intersect transversely; and they are arranged along *double curves* which are closed because S is compact and without boundary.

A Dehn surface $\Sigma \subset M$ *fills* M [8] if it defines a cell-decomposition of M in which the 0-skeleton is the set of triple points of Σ; the 1-skeleton is the set of double and triple points of Σ; and the 2-skeleton is Σ itself. Filling Dehn spheres of 3-manifolds are defined in [8] following ideas of W. Haken (see [7]). In [5] it is proved that every closed orientable 3-manifold has a Dehn sphere whose complement is a union of open 3-balls (these kind of Dehn spheres are called *quasi-filling* Dehn spheres in the notation of [2]). In [8] it is proved that every closed orientable 3-manifold has a filling Dehn sphere (see also [10]). Since then, some different proofs of this

Electronic version of an article published as Journal of Knot Theory and Its Ramifications Vol. 21, No. 08, 1250082 (2012). DOI: 10.1142/s0218216512500824. ©World Scientific Publishing Company. https://www.worldscientific.com/worldscinet/jktr

theorem had appeared [1,10,12], and its easy to see that this theorem holds also for non-orientable closed 3-manifolds.

Let $\Sigma \subset M$ be a Dehn surface and $f : S \to M$ a transverse immersion parametrizing Σ. The inverse image set by f in S of the set of double and triple points of Σ is the *singular set* of f. The singular set of f, together with the information of how its points are identified by f in M, is the *Johansson diagram* of Σ in the notation of [8]. Because S and M are orientable, the Johansson diagram $\mathscr{D}$ of Σ is composed by an even number of pairwise related closed curves in S. The curves of $\mathscr{D}$ intersect each other transversely at the *crossing points* of $\mathscr{D}$. The inverse image set by f of each triple point of Σ is composed of three crossing points of $\mathscr{D}$.

In 2003, during the congress *Knots in Poland 2003* held in Warsaw and Bedlewo, Poland, Prof. Roger Fenn [4] asked to us the following question:

Question 1.1. Do filling Dehn spheres in M lift to embeddings in $M \times I$?

This question suggest the following definition. A Dehn surface $\Sigma \subset M$ is *liftable* if there exists a parametrization $f : S \to M$ of Σ and an embedding $\hat{f} : S \to M \times I$ such that $f = \pi \circ \hat{f}$, where $\pi : M \times I \to M$ denotes the projection onto the first factor. If there is no such embedding we will say that Σ is *non-liftable*. In [12] is proven the following Theorem.

Theorem 1.2. *Every 3-manifold M has a liftable filling Dehn sphere.*

In [6] is presented a Dehn sphere Σ_G in S^3 (Giller's sphere) which is non-liftable. This sphere Σ_G is in fact a filling Dehn sphere of S^3, and so Σ_G solves Fenn's question for S^3: in S^3 there are liftable and non-liftable filling Dehn spheres. In this paper, we will use Σ_G to solve Fenn's question in general by proving the following theorem.

Theorem 1.3. *Every 3-manifold M has a non-liftable filling Dehn sphere.*

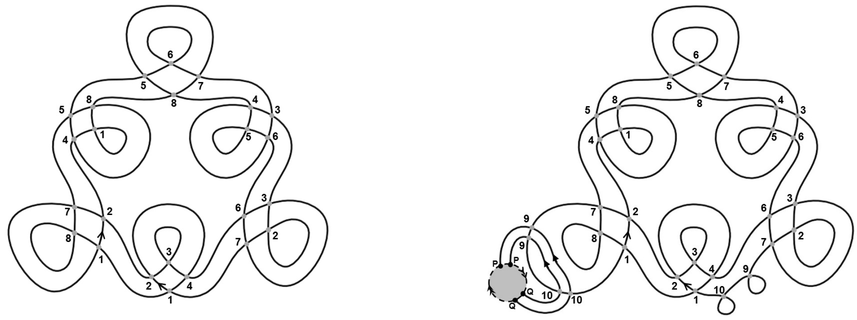
I am very grateful to Roger Fenn for asking me such an interesting question. I am very grateful also to Prof. J. M. Montesinos for his suggestions and his careful reading of this manuscript.

2. Giller's theorem and Giller's Sphere

Giller's sphere Σ_G is constructed by taking a parallel surface on "both" sides of Boy's surface. Because Boy's surface is a one-sided projective plane immersed in $\mathbb{R}^3 \subset S^3$, Σ_G is a Dehn sphere. The Johansson diagram $\mathscr{D}_G$ of Σ_G is shown in Figure 1(a), where crossing points having the same image in Σ_G are equally labelled, and the two arrows must be identified in the obvious way. By Corollary 45 of [11], Σ_G is a filling Dehn sphere of S^3.

In [6] it is given an algorithm for deciding if a Dehn surface Σ in $\mathbb{R}^3$ lifts to an embedding in $\mathbb{R}^4$ in terms of the Johansson diagram of Σ. A different such algorithm

Electronic version of an article published as Journal of Knot Theory and Its Ramifications Vol. 21, No. 08, 1250082 (2012). DOI: 10.1142/s0218216512500824. ©World Scientific Publishing Company. https://www.worldscientific.com/worldscinet/jktr



(a) (b)

Fig. 1: Johansson diagram of Giller's sphere.

is given in Theorem 3.2 of [3]. This algorithm can be generalized to an algorithm that determines when a Dehn surface in M lifts to an embedding in $M \times I$ (Theorem 2.3 below).

Let $\Sigma \subset M$ be a Dehn surface, $f : S \to M$ a parametrization of Σ, and let $\mathscr{D}$ be the Johansson diagram of Σ. Thus, $\mathscr{D}$ is composed by $2n$ closed pairwise related closed curves: $\mathscr{D} = \{\alpha_1, ..., \alpha_{2n}\}$ in the surface S.

Take a map $\sigma : \mathscr{D} \to \{-, +\}$. Let P be a crossing point of $\mathscr{D}$, where two curves α, β of $\mathscr{D}$ intersect. It can be $\sigma(\alpha) = \sigma(\beta) = -$, $\sigma(\alpha) = \sigma(\beta) = +$, or $\sigma(\alpha) \neq \sigma(\beta)$. Then, we will say that the *signature* $s(P, \sigma)$ of the crossing point P with respect to σ is $(--)$ in the former case, $(++)$ in the second case, and $(-+)$ in the latter case.

Definition 2.1. The map $\sigma : \mathscr{D} \to \{-, +\}$ is a *coloration* when the following holds:

(1) if $\alpha_i, \alpha_j \in \mathscr{D}$ verify $f(\alpha_i) = f(\alpha_j)$, then $\sigma(\alpha_i) \neq \sigma(\alpha_j)$;
(2) let P_1, P_2, P_3 be three different crossing points of $\mathscr{D}$ such that $f(P_1) = f(P_2) = f(P_3)$; then, the set of signatures: $\{s(P_1, \sigma), s(P_2, \sigma), s(P_3, \sigma)\}$ equals the set $\{(--), (-+), (++)\}$.

Definition 2.2. The diagram $\mathscr{D}$ is *colourable* if it admits a coloration.

Theorem 2.3 (adapted from [3], see [12]). *The Dehn surface Σ is liftable if and only if $\mathscr{D}$ is colourable.*

It is easy to see that $\mathscr{D}_G$ is not colourable, and thus Giller's sphere Σ_G cannot be liftable.

3. Proof of theorem 1.3.

Proof. [Proof of Theorem 1.3] Assume that Σ is a filling Dehn sphere of M, and take a copy Σ_G of Giller's sphere inside the complementary set of Σ in M, that is:

Electronic version of an article published as Journal of Knot Theory and Its Ramifications Vol. 21, No. 08, 1250082 (2012). DOI: 10.1142/s0218216512500824. ©World Scientific Publishing Company. https://www.worldscientific.com/worldscinet/jktr

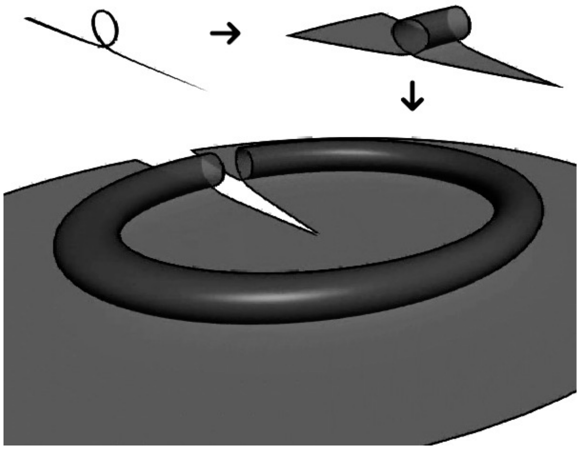
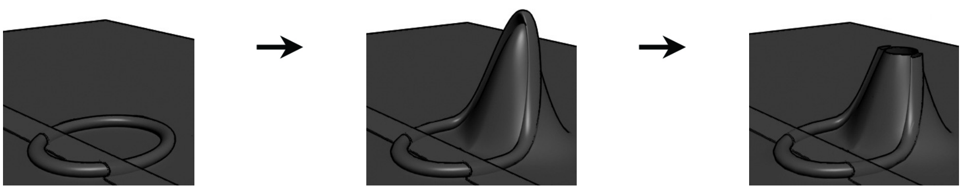


Fig. 2: rotating a loop

$\Sigma_G \subset M - \Sigma$. We will connect Σ with Σ_G in such a way that the resulting surface inherits the filling property from Σ and the non-liftable property from Σ_G.

Consider a closed arc in which we have performed a loop, introducing a transverse self-intersection. We make this arc rotate around one of its endpoints obtaining an immersed closed disk (Figure 2). We will say that this immersed disk is a *loop-disk* and we denote it by D_∞. Note that the closed 2-disk bounded by the loop will describe a solid torus during the rotation.



(a) (b) (c)

Fig. 3: pushing and cutting a loop-disk

Let B be the unique connected component of $M - (\Sigma \cup \Sigma_G)$ such that ∂B

Electronic version of an article published as Journal of Knot Theory and Its Ramifications Vol. 21, No. 08, 1250082 (2012). DOI: 10.1142/s0218216512500824. ©World Scientific Publishing Company. https://www.worldscientific.com/worldscinet/jktr

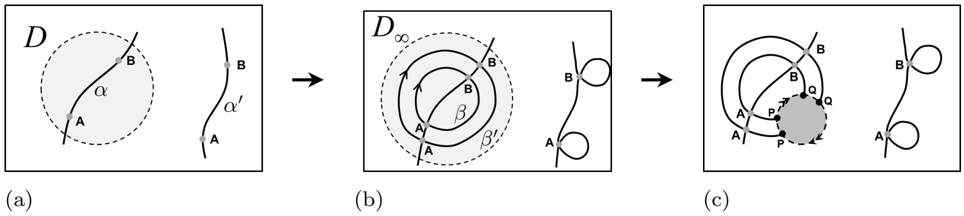


Fig. 4: Diagram version of Figure 3

intersects both Dehn surfaces Σ and Σ_G. This connected component B is the only one which is not an open 3-ball, it is homeomorphic to $S^2 \times (0,1)$. The rest of connected components $B_1, ..., B_n$ of $M - \Sigma$ different from B are open 3-balls. For $i = 1, ..., n$, the 3-ball B_i either verify $\partial B_i \subset \Sigma$, or $\partial B_i \subset \Sigma_G$. We will say that B_i *is bounded by* Σ in the former case and that B_i *is bounded by* Σ_G in the latter case.

Let α be an arc inside a double curve of Σ such that α belongs to the closure of B, and consider a double point $P \in \alpha$. There are two sheets of Σ intersecting transversely at P. Consider a small embedded 2-disk $D \subset \Sigma$ contained in one of these sheets and such that α intersects the interior of D in an open arc. If we substitute D with a loop-disk D_∞ as in Figure 3(a), we obtain a new filling Dehn surface Σ' with two more triple points than Σ. Now, we push a region of D_∞ inside B as in Figure 3(b) and we excise a closed 2-disk from D_∞ as in Figure 3(c). After this operation, we obtain an immersed surface with boundary $\tilde{\Sigma}$ whose boundary $\partial\tilde{\Sigma} \subset B$ is an immersed S^1 with two self-intersections. The effect of these modifications in the diagram $\mathscr{D}$ is represented in Figure 4.

We perform the same operation in Σ_G around an arc β inside a double curve of Σ_G such that β belongs to the closure of B. We substitute a closed 2-disk with a loop-disk to obtain a new Dehn sphere Σ'_G and after this we excise an immersed 2-disk from Σ'_G. We call $\tilde{\Sigma}_G$ the resulting surface, and we assume also that $\partial\tilde{\Sigma}_G \subset B$. Thus, we can connect $\tilde{\Sigma}$ and $\tilde{\Sigma}_G$ as in Figure 5. The Johansson diagram $\mathscr{D}\#\mathscr{D}_G$ of $\tilde{\Sigma}\#\tilde{\Sigma}_G$ is obtained by pasting the diagrams of $\tilde{\Sigma}$ and $\tilde{\Sigma}_G$ represented in Figures 4(c) and 1(b) respectively by identifying the boundaries of the removed disks (depicted in grey in the mentioned figures).

Giller's diagram remains essentially the same as a sub-diagram of $\mathscr{D}\#\mathscr{D}_G$. Let γ_1, γ_2 be the two curves of $\mathscr{D}\#\mathscr{D}_G$ coming from $\mathscr{D}_G$, and let $\sigma : \mathscr{D}\#\mathscr{D}_G \to \{-,+\}$ be a map such that $\sigma(\gamma_1) \neq \sigma(\gamma_2)$. For any such σ, all the crossing points denoted by **4** or **7** in Figure 1 have signature $(-,+)$. According to Theorem 2.3, the Dehn sphere $\tilde{\Sigma}\#\tilde{\Sigma}_G$ is non-liftable.

In order to prove that $\tilde{\Sigma}\#\tilde{\Sigma}_G$ is a filling Dehn sphere of M, it is necessary to check that:

(i) {double and triple points of $\tilde{\Sigma}\#\tilde{\Sigma}_G$}−{triple points of $\tilde{\Sigma}\#\tilde{\Sigma}_G$} is a union of

Electronic version of an article published as Journal of Knot Theory and Its Ramifications
Vol. 21, No. 08, 1250082 (2012). DOI: 10.1142/s0218216512500824. ©World Scientific
Publishing Company. https://www.worldscientific.com/worldscinet/jktr

open arcs;

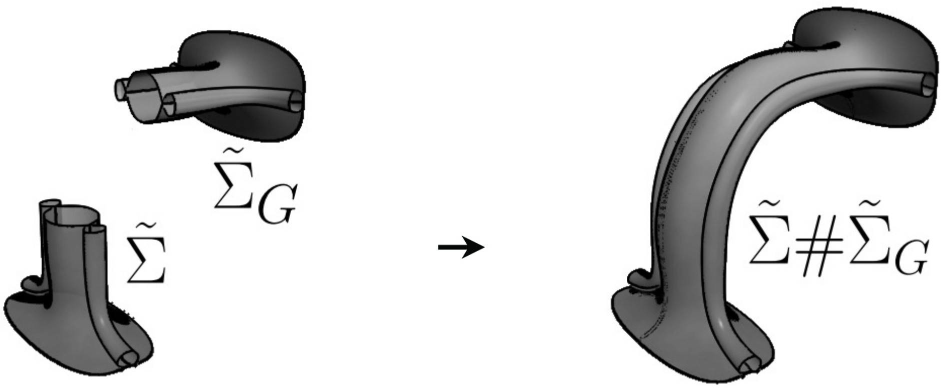
(ii) $\tilde{\Sigma}\#\tilde{\Sigma}_G-\{$double and triple points of $\tilde{\Sigma}\#\tilde{\Sigma}_G\}$ is a union of open 2-disks; and

(iii) $M-\left(\tilde{\Sigma}\#\tilde{\Sigma}_G\right)$ is a union of open 3-balls.

Because $\mathscr{D}\#\mathscr{D}_G$ is a diagram in the 2-sphere S^2, both properties (i) and (ii) of $\tilde{\Sigma}\#\tilde{\Sigma}_G$ of previous paragraph can be derived from the fact that the diagram $\mathscr{D}\#\mathscr{D}_G$ is connected when considered as a subset of S^2 (the diagram $\mathscr{D}$ is connected because Σ is a filling Dehn sphere of M, and it's obvious that $\mathscr{D}_G$ is connected too, therefore $\mathscr{D}\#\mathscr{D}_G$ is also connected).

The set of connected components of $M-(\Sigma\cup\Sigma_G)$ is composed by B and the collection of open 3-balls $B_1,...,B_n$. When we introduce the loop-disks in Σ and Σ_G, in $M-(\Sigma'\cup\Sigma'_G)$ there appear four new small connected components B'_1,B'_2,B'_3,B'_4 which are open 3-balls. Assume that B'_1,B'_2 are bounded by Σ' and that B'_3,B'_4 are bounded by Σ'_G, in the sense explained before ($\partial B'_i\subset\Sigma'$ for $i=1,2$ and $\partial B'_i\subset\Sigma'_G$ for $i=3,4$), and that $B_2,B_3\subset B$. The closure of $B'_1\cup B'_2$ is the solid torus of the loop-disk introduced in Σ' and the closure of $B'_3\cup B'_4$ is the solid torus of the loop-disk introduced in Σ'_G. When we connect $\tilde{\Sigma}$ with $\tilde{\Sigma}_G$ to obtain $\tilde{\Sigma}\#\tilde{\Sigma}_G$, the connected components of $M-(\Sigma'\cup\Sigma'_G)$ are modified as follows: (1) each of the open 3-balls B'_2,B'_3 is splitted into two cylinder-shaped open 3-balls and the four resulting open 3-balls become connected in pairs producing two open 3-balls; (2) two more open 3-balls, say B_1 and B_2, bounded by Σ' and Σ'_G respectively, become connected producing another open 3-ball; (3) finally, an "unknotted hole" is introduced in B connecting $\partial B\cap\Sigma$ with $\partial B\cap\Sigma_G$ in such a way that B becomes an open 3-ball. This proves that $M-\left(\tilde{\Sigma}\#\tilde{\Sigma}_G\right)$ is a union of open 3-balls.

Thus, $\tilde{\Sigma}\#\tilde{\Sigma}_G$ is a non-liftable filling Dehn sphere of M. □



(a) (b)

Fig. 5: filling piping

Electronic version of an article published as Journal of Knot Theory and Its Ramifications Vol. 21, No. 08, 1250082 (2012). DOI: 10.1142/s0218216512500824. ©World Scientific Publishing Company. https://www.worldscientific.com/worldscinet/jktr

References

[1] G. Amendola, *A local calculus for nullhomotopic filling Dehn spheres*, Algebr. Geom. Topol. 9 (2009), no. 2, 903–933

[2] G. Amendola, *A 3-manifold complexity via immersed surfaces*, J. Knot Theory Ramifications 19 (2010), no. 12, 1549–1569.

[3] J. S. Carter and M. Saito, *Surfaces in 3-space that do not lift to embeddings in 4-space*, Knot theory (Warsaw, 1995), 29-47, Banach Center Publ., 42, Polish Acad. Sci., Warsaw, 1998.

[4] R. Fenn, private communication (2003).

[5] R. Fenn and C. Rourke, *Nice spines on 3-manifolds*, Topology of low-dimensional manifolds, LNM 722 (1977), 31-36.

[6] C. A. Giller, *Towards a classical knot theory for surfaces in* $\mathbb{R}^4$, Illinois J. Math. 26 (1982), no. 4, 591-631.

[7] W. Haken, *Some special presentations of homotopy 3-spheres*, Topology Conference, LNM 375 (1973), 97-107.

[8] J. M. Montesinos-Amilibia, *Representing 3-manifolds by Dehn spheres*, Contribuciones Matemáticas: Homenaje a Joaquín Arregui Fernández, Editorial Complutense (2000), 239-247.

[9] C. D. Papakyriakopoulos, *On Dehn's Lemma and the asphericity of knots*, Ann. Math. (2) 66 (1957), 1-26.

[10] R. Vigara, *A new proof of a theorem of J. M. Montesinos*, J. Math. Sci. Univ. Tokyo 11 (2004), no. 3, 325-351.

[11] R. Vigara, *A set of moves for Johansson representation of 3-manifolds*, Fund. Math. 190 (2006), 245-288.

[12] R. Vigara, *Representación de 3-variedades por esferas de Dehn rellenantes*, PhD Thesis, UNED, Madrid, 2006. English translation in preparation.