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NECKLACE THEORY ON FLOWER CONSTELLATIONS

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Theory of Flower Constellations has been improved with the 2-D and 3-D Lattice Flower Constellations. However, placing a satellite in each admissible location is not an optimal way to design a constellation. The necklace theory considers constellations whose satellites are subsets of the satellites of a Lattice Flower Constellation, keeping all its symmetries, in order to reduce the cost of the mission. Mathematically, these subsets are parameterized by necklaces (describing which satellites in the first orbit of the underlying constellation we keep), and a shifting parameter that controls the phasing between subsequent orbits.

INTRODUCTION

Constellations of satellites have been used for a variety of space missions (e.g., global navigation systems, communications, observations, reconnaissance, etc.) and the improvement and design of new constellations are a current topic used to reduce the cost of the missions as much as possible.

General theories to design satellite constellations with symmetric distribution of satellites include the classic Walkers constellations [1] and the more recent Flower constellations [2]. The philosophic difference between Walker constellations and the original Flower Constellations is the reference frame selected where to build symmetric distributions of satellites: while Walker's choose the inertial reference frame, a generic rotating reference frame is selected in the theory of Flower Constellations.

The original theory of Flower Constellations (FCs), first presented in [2], and then expanded in details in [3, 4], was substantially improved with the 2-D Lattice FCs [5] making the theory independent from any reference frame, inertial or rotating, and with minimal parametrization. More recently, the 3-D Lattice FCs [6] extended the 2-D Lattice FCs theory to elliptical orbits subject to the J_2 effect due to the Earth oblateness.

The evolution of these theories is interesting for many reasons. First, the deep connection with Number theory mathematical tools and properties (Chinese remainder theorem, theory of Lattices, Hermite normal form, etc), second, the level of description using minimal parametrization, a property useful to ensure to include *all* the potential symmetric solution, and third, the important practical reason to include the J_2 effect, making the constellation designers free to use any inclination when selecting elliptical orbits. However, while from a mathematical point of view the theory appears

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to have reached the final level of maturity, from a practical point of view, a question arise. Since, most of these symmetric configurations involve an unpractical high number of satellites, is it possible to select a subset of them and still obtaining symmetric distributions? This paper provides a positive answer to this question and provides *all* the possible subsets keeping full symmetry in the (Ω, M)-space [7].

At the heart of FCs the (Ω, M)-space, describing the distribution of the satellites in the 2-D Lattice FCs [8]. The initial orbit is related with a necklace of N_m pearls representing the admissible locations. A number of satellites (N_{so}) less than the number of pearls are distributed in the initial necklace. The purpose is find the suitable shifting and a proper necklace to get the same initial and final orbit to see the (Ω, M)-space as a 3-D torus and reduce the number of satellites in the constellation.

FLOWER CONSTELLATIONS BACKGROUND

The evolution of the Theory of Flower Constellations

A Flower Constellation, as defined in [2] and [3, 4], is a set of N_s satellites following the same (closed) trajectory with respect to a rotating frame fixed to the Earth. This condition implies that:

1. The period of revolution, T_p, of each satellite about the Earth is a rational multiple of the period of rotation of the Earth, T_d. That is, $N_p T_p = N_d T_d$ for some positive (coprime) integers N_d and N_p.
2. The orbital parameters a, e, i and ω are the same for all the satellites.
3. The mean anomaly at epoch M_i and the right ascension of the ascending node Ω_i of the orbit of each satellite satisfy $N_p \Omega_i \equiv -N_d M_i \bmod 2\pi$.

The first item guarantees that the trajectory in the rotating frame is closed. The second and third item are necessary and sufficient conditions to have all the satellites on the same trajectory (a complete proof of this fact is given in [7]) and [8]).

Usually, when designing a Flower Constellation, the compatibility (or resonant) parameters N_d and N_p are decided first, which immediately determines the period of revolution T_p, and therefore the semimajor axis a. After that, the orbital parameters e, i and ω are selected. Finally, the angles Ω_i and M_i are computed by the recursive sequence

$$
\begin{aligned}
\Omega_1 &= M_1 = 0, \\
\Omega_{i+1} &\equiv \Omega_i + 2\pi \frac{F_n}{F_d}, \\
M_{i+1} &\equiv M_i - 2\pi \frac{N_p F_n + F_d F_h(i)}{F_d N_d},
\end{aligned}
$$

where F_n and F_d are two coprime positive integers and $F_h(i)$ is any sequence of numbers chosen in the set $\{0, 1, \ldots, N_d - 1\}$. It is easy to show that this procedure always produces pairs (Ω_i, M_i) consistent with the equation $N_p \Omega_i \equiv -N_d M_i \bmod 2\pi$. For simplicity, the parameter F_h will be considered constant. Currently, a FC is specified by the six integer parameters $(N_d, N_p, F_d, F_n, F_h, N_s)$, as well as the continuous parameters (e, i, ω). This is the approach followed so far in all the papers on Flower Constellations and also in the simulation and visualization software FCVAT [9].

It has been shown in [7, Thm 1], that the number of satellites in a Flower Constellation designed under this procedure can not exceed $N_d F_d/G$ satellites, where $G = \gcd(N_d, N_p F_n + F_d F_h)$. A constellation with the maximum number of satellites allowed by this theorem is called either a Secondary Path (as in [4]) or a Harmonic Flower Constellation (HFC)(following [7]). The location of the satellites of a HFC in the (Ω, M) space is determined [7, Thm 2] by three invariants: the number of inertial orbits F_d, the number of satellites per orbit $N_{so} = N_d/G$ and the configuration number $N_c \in [0, F_d)$, given by the formula

$$N_c = E_n \frac{N_p F_n + F_d F_h}{G} \mod F_d,$$

where E_n and E_d are integers such that $E_n F_n + E_d F_d = 1$. The numbers F_d, N_{so} and N_c are always coprime.

2-D Lattice Flower Constellations

The 2-D Lattice Flower Constellations, see [5], can be described by five integer parameters and three continuous parameters. The integer parameters can be broken into two sets, the first describing the phasing of the satellites and the second describing the orbital period (or semi-major axis). The first set is $\{N_o, N_{so}, N_c\}$ where N_o is the number of orbital planes, N_{so} is the number of satellites per orbit, and N_c is a phasing parameter. The second set is $\{N_p, N_d\}$ which satisfies the compatibility equation

$$N_p T_p = N_d T_d, \tag{1}$$

where T_p is the orbital period and T_d is the period of the rotating reference frame (e.g., the sidereal period of Earth's rotation). This definition enforces the repeating space-track requirement.

The phasing parameters define the RAAN (Ω) and initial mean anomaly (M) as

$$\begin{aligned} \Omega_{ij} &= \frac{2\pi i}{N_o}, \\ M_{ij} &= \frac{2\pi j}{N_{so}} - \frac{N_c \Omega_{ij}}{N_{so}}. \end{aligned} \tag{2}$$

These equations can be rewritten in matrix notation as

$$\begin{bmatrix} N_o & 0 \\ N_c & N_{so} \end{bmatrix} \begin{bmatrix} \Omega_{ij} \\ M_{ij} \end{bmatrix} = 2\pi \begin{bmatrix} i \\ j \end{bmatrix}, \tag{3}$$

where $i = 0, \ldots, N_o - 1$, $j = 0, \ldots, N_{so} - 1$ and $N_c \in [0, N_o - 1]$. Satellite (i, j) is the j^{th} satellite on the i^{th} orbital plane.

The remaining parameters required to define the constellation are continuous parameters that are the same for all orbits in the constellation: the inclination angle, the eccentricity, and the argument of periapsis.

Note that since the 2-D Lattice Flower Constellation separate the satellite phasing from the orbit size, non-repeating space-tracks can be used without affecting the uniformity of the satellite distribution.

Since all the satellites in a Lattice Flower Constellation have the same orbital parameters a, e, i, and ω, it is enough to use the (Ω, M)-space to represent the location of the satellites. Figure 1 shows the distribution of satellites in the Lattice Flower Constellation with $N_{so} = 6$, $N_o = 8$, and $N_c = 2$, obtained by solving Eq.(3).

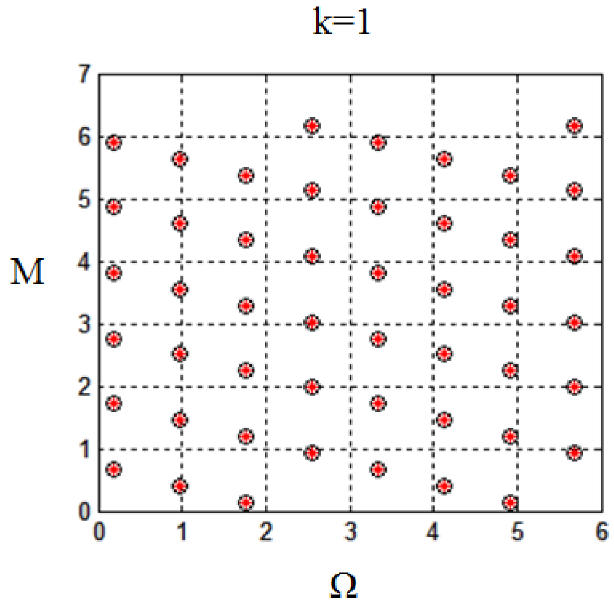

Figure 1. Distribution of satellites of a Lattice Flower Constellation in the (Ω, M)-space.

THE NECKLACE PROBLEM

The Necklace Theory

The necklace problem is a combinatorial problem which answer the following question: in how many different arrangements of n pearls in a circular loop are there, assuming that each pearl come in one of k different colors? Two arrangements that differ only by a rotation of the loop, are consider the to be identical. The mathematical solution to this problem (see [10]) is a simple application of Burnside's counting theorem, and summarized by the following formula:

$$N_k(n) = \frac{1}{n}\sum_{d|n}\varphi(d)k^{n/d},$$

where the sum is taken over all the divisors d of n, and $\varphi(d)$ is the number of integers in the interval $[1, d]$ that have no common prime factor with d*.

Mathematically, each configuration will be represented as a subset $\mathcal{G} \subseteq \{1, \ldots, n\}$. The set of all possible necklaces with n pearls and two colors will be written $K(n)$. Figure 2 shows all possible necklaces using three pearls of two colors, i.e. the elements of $K(3)$.

Algorithm 1 (provided in the appendix in pseudo-code), computes all possible necklaces involving a total of n pearls, of which w are white and $n - w$ are black. In order to obtain all possible necklaces with n pearls, it is necessary to call the algorithm with $w = 0, \ldots, n$.

*The function $\varphi(d)$ is called Euler's totient function. A simple computation shows that $\varphi(1) = \varphi(2) = 1$, $\varphi(3) = \varphi(4) = 2$, $\varphi(5) = 4$, $\varphi(6) = 2$, $\varphi(7) = 6$, etc.

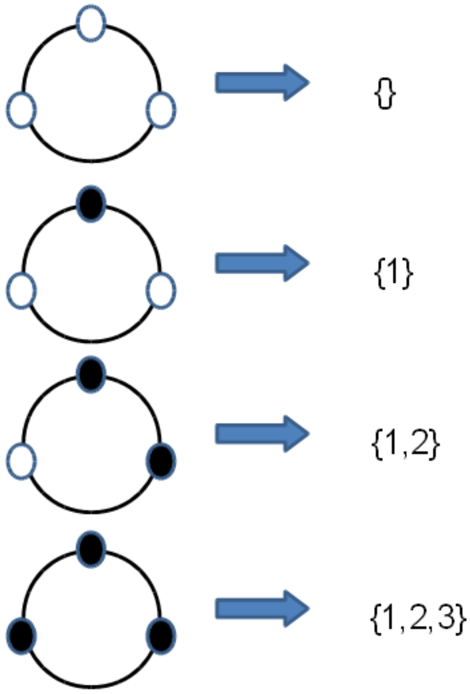

Figure 2. Mathematical representation of necklaces.

Symmetries of the necklaces

Let $\mathcal{G}$ be a necklace such as $\mathcal{G} \in K(n)$. We say that $\mathcal{G}$ has a symmetry of length r if $\mathcal{G}$ and $\mathcal{G} + r$ coincide modulo n.

As an example, consider the necklace $\mathcal{G} = \{1, 3, 5, 7\} \in K(8)$. What symmetries does it have?

- $r = 2$ is a symmetry, since $\mathcal{G} + 2 = \{3, 5, 7, 9\}$ is equivalent to $\mathcal{G}$ modulo 8.
- $r = 4$ and $r = 8$ are also symmetries, since $\{5, 7, 9, 11\}$ and $\{7, 9, 11, 13\}$ reduce to $\{1, 3, 5, 7\}$ modulo 8.
- $r = 1$ is not a symmetry, since $\{2, 4, 6, 8\}$ and $\{1, 3, 5, 7\}$ do not coincide modulo 8.

From the example it is easy to see that if r is a symmetry of a necklace, then any multiple of r is also a symmetry. This remark motivates our following definition: for each necklace $\mathcal{G} \in K(n)$, the symmetry number of $\mathcal{G}$, denoted $\mathrm{Sym}(\mathcal{G})$, is the shortest of the symmetries of $\mathcal{G}$. Note that $\mathrm{Sym}(\mathcal{G})$ always divides n.

$$sym(\mathcal{G}) = \min\{1 \leq r \leq n : \mathcal{G} + r \equiv \mathcal{G} \pmod{n}\} \tag{4}$$

Algorithm 2 (provided in the appendix) can be used to find all the symmetries and the symmetry number of a given necklace.

NECKLACES AND 2-D FLOWER CONSTELLATIONS

The basic idea is the following: we start with a standard Lattice Flower Constellations (with parameters N_{so}, N_o, and N_c), and instead of placing satellites in each admissible location (as given by Eq. 2), we choose a subset of admissible locations $\mathcal{G} \subseteq \{1, 2, \ldots, N_{so}\}$ for the satellites in the first orbit, and then we duplicate this configuration for each subsequent orbit using a shifting parameter (an integer $k \in \{1, \ldots, N_{so}\}$). The subset $\mathcal{G}$ can be any necklace. Once $\mathcal{G}$ and the shifting parameters are given, the constellation is automatically determined. Figure 3 shows how the satellite in the second orbit corresponding to one given satellite in the first orbit, for different values of the shifting parameter k.

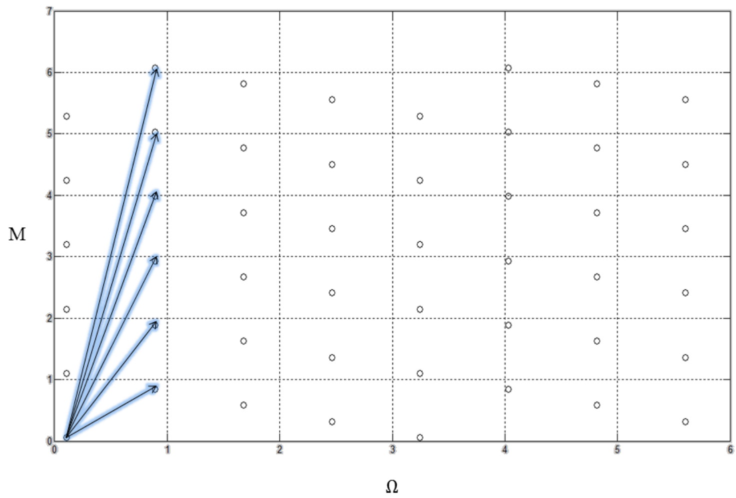

Figure 3. The shifting depends on the value of k.

There are two simple details that have to be taken into account:

Consistency Due to the modular nature of the parameter Ω, the shifting has to be chosen in such a way that the satellites in the orbit with $\Omega = 0$ coincide with the satellites in the orbit with $\Omega = 2\pi$. This problem will be discussed in detail in the next subsection.

Minimality Sometimes, for the same $\mathcal{G}$, there are two values of the shifting parameter generate the same distribution of satellites in the (Ω, M)-space. This is solved by simple taking $1 \leq k \leq \mathrm{Sym}(\mathcal{G})$.

The constellations obtained by these procedure will be called "Necklace Flower Constellations".

ΔM-Shifting Between Subsequent Orbits

The first satellite ($j = 0$) in the zero or initial orbit ($i = 0$) is chosen, without loss of generality $M_{00} = 0$ and $\Omega_{00} = 0$. Taking into account (2) the mean anomaly of our satellite in the next orbit will be:

$$M_{10} = \frac{-2\pi N_c}{N_o N_{so}}. \tag{5}$$

Then, the amount ΔM, called ΔM-Shifting between subsequent orbits, will be:

$$\Delta M = \frac{-2\pi N_c}{N_o N_{so}} + k\frac{2\pi}{N_{so}}. \tag{6}$$

This means that the mean anomalies of the satellites in the second orbit can be obtained by adding ΔM to the mean anomalies of the satellites of the first orbit. Similarly, the mean anomalies on the third orbit are the mean anomalies on the second plus ΔM, and so on.

After a rotation of 360° of the initial orbit, the mean anomaly of the satellite will increase by:

$$N_o \Delta M = N_o \left(\frac{-2\pi N_c}{N_o N_{so}} + k\frac{2\pi}{N_{so}} \right) = \frac{2\pi}{N_{so}}(kN_o - N_c). \tag{7}$$

Admissible pair $(\mathcal{G}, k)$

Let $\mathcal{G}$ be a necklace such as $\mathcal{G} \in K(N_{so})$ and a shifting parameter $k \in \{1, \ldots, N_{so}\}$, the pair $(\mathcal{G}, k)$ is called admissible if the distribution of satellites in the initial orbit is invariant by the adding $N_o \Delta M$ to the mean anomaly of each satellite. By the definition of symmetry number, this condition translates into $\mathrm{Sym}(\mathcal{G}) | kN_o - N_c$. This equation represents the solution to the consistency problem.

Figure 4 shows an example of the constellation generated by an admissible pair $(\mathcal{G}, k)$. In this case, the underlying Lattice Flower Constellation has parameters $N_{so} = 9$, $N_o = 6$, and $N_c = 3$. The necklace is $\mathcal{G} = \{1, 4, 6\}$ that has symmetry number $\mathrm{Sym}(\mathcal{G}) = 9$, and the shifting parameter is $k = 2$. Note that in this example we have $9 | 2 \cdot 6 - 3$.

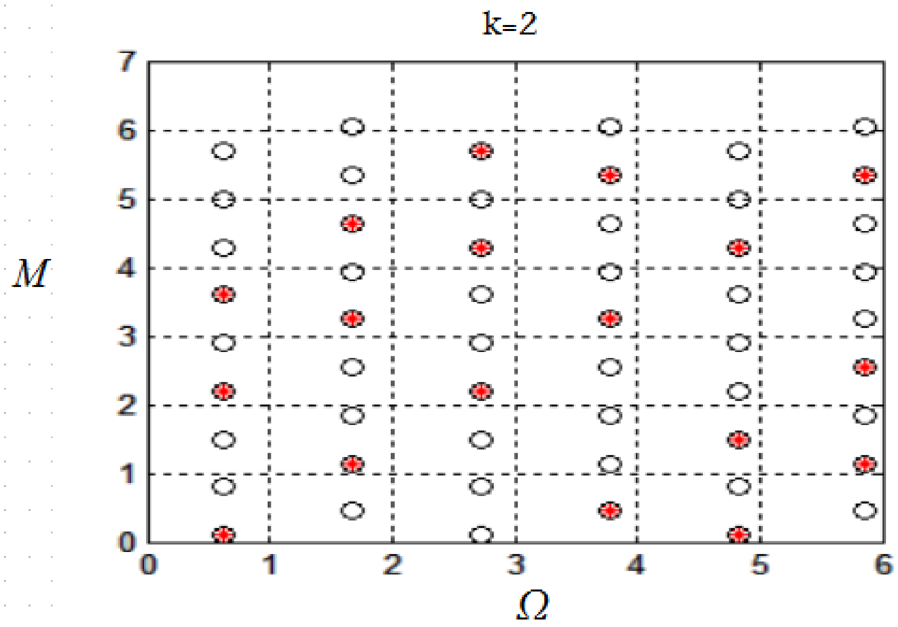

Figure 4. A necklace flower constellation generated by an admissible pair.

As we mention before, the minimality problem is solved by restricting the range of values of k to the interval $[1, \mathrm{Sym}(\mathcal{G})]$. It is clear that $(\mathcal{G}, k)$ and $(\mathcal{G}, k')$ will generate the same constellation if and only if $k - k'$ is an integer multiple of $\mathrm{Sym}(\mathcal{G})$. This is impossible for two values in the proposed interval.

Figure 5 shows an example of this situation: in the Lattice Flower Constellation $N_{so} = 9$, $N_o = 6$, and $N_c = 3$, the necklace $\mathcal{G} = \{1, 4, 7\}$, which has $\mathrm{Sym}(\mathcal{G}) = 3$, generates the same configuration for $k = 2$, $k = 5$, and $k = 8$.

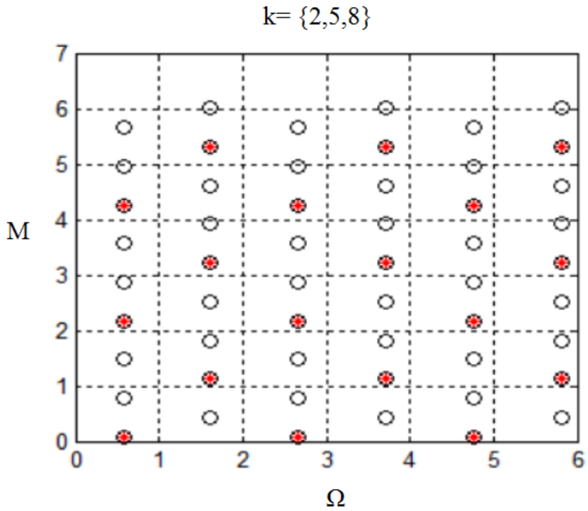

Figure 5. Different values of k can generate the same configuration.

At this point we can state our main result: each Necklace Flower Constellation correspond with one (and only one) pair $(\mathcal{G}, k)$ with $\mathcal{G} \in K(N_{so})$, $1 \leq k \leq \mathrm{Sym}(\mathcal{G})$, and $\mathrm{Sym}(\mathcal{G}) | kN_o - N_c$.

Figure 6 shows the only three possible constellations (according to our main result) induced by the necklace $\mathcal{G} = \{1, 4, 7, 10\} \in K(12)$, which has symmetry number 3. The underlying Lattice Flower Constellation has parameters $N_{so} = 12$, $N_o = 9$, and $N_c = 3$, so the three possible values of $k \in \{1, 2, 3\}$ are admissible.

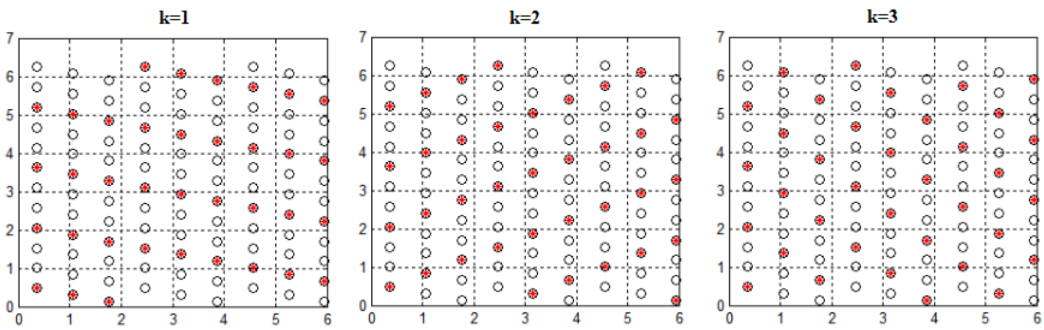

Figure 6. All the possible different configurations with $\mathcal{G} = \{1, 4, 7, 10\}$.

The Diophantine Equation for the Shifting parameter

The admissibility condition for a pair $(\mathcal{G}, k)$, motivates us to study the diophantine equation $d|ak-b$, where a, b, d are given (positive) integers and the unknown k takes integer values in the range $[1, d]$. All the solutions can be obtained trivially by trial and error (since there are finitely many possibilities for k), but we would like a more efficient procedure.

The number of solutions of this diophantine equation will be denoted $Y(d, a, b)$, which can be proven to be

$$Y(d,a,b) = \begin{cases} 0 & \text{if } \gcd(d,a) \nmid b \\ \gcd(d,a) & \text{otherwise.} \end{cases} \tag{8}$$

The idea is that, independently of the value of k, the product ak is always divisible by $\gcd(d, a)$, so when $\gcd(d, a) \nmid b$, it is impossible to have $\gcd(d, a)|ak-b$, and therefore we will never have $d|ak-b$. In the case where $\gcd(d, a)|b$, we can divide a, b, and d by $\gcd(d, a)$, and reduce the problem to the equation $d'|a'k - b'$ where $a' = a/\gcd(d, a)$, $b' = b/\gcd(d, a$, and $d' = d/\gcd(d, a)$. This problem has only one solution in the interval $[1, d']$, since a' and d' have no common factor, and therefore has $d/d' = \gcd(d, a)$ solutions in the $[1, d]$.

An efficient algorithm that implements this idea to compute the actual solutions of the equation $d|ak - b$ is given in the appendix (see algorithm 3).

THE TOTAL NUMBER OF NECKLACE FLOWER CONSTELLATIONS

From a mathematical point of view, it is interesting to compute the total number of Necklace Flower Constellation that can be obtained from a Lattice Flower Constellation with parameters N_{so}, N_o, and N_c. This amount, denoted $W(N_{so}, N_o, N_c)$, is exactly the number of admissible pairs, i.e.

$$W(N_{so}, N_o, N_c) = \#\{(\mathcal{G}, k) : \mathcal{G} \in K(N_{so}), 1 \le k \le sym(\mathcal{K}), kN_o \equiv N_c \mod (sym(\mathcal{G}))\}. \tag{9}$$

If we denote $X(d)$ the number of necklaces with symmetry number equal to d, then we can rewrite the previous formula as:

$$W(N_{so}, N_o, N_c) = \sum_{d|N_{so}} X(d) Y(d, N_o, N_c). \tag{10}$$

It is apparent that we should write $X(d, N_{so})$ instead of $X(d)$, since we are considering necklaces in $K(N_{so})$. However, it is clear that the number of necklaces with symmetry number d in $K(N_{so})$ correspond one-to-one with the necklaces in $K(d)$ with symmetry number d. This shows that $X(d, N_{so})$ does not depend on N_{so}, as long as $d|N_{so}$. For practical purposes we can define $X(d) = X(d, d)$, i.e. the number of necklaces in $K(d)$ with no symmetry of length smaller than d. A simple corollary of this discussion is the formula

$$\sum_{d|n} X(d) = N_2(n), \tag{11}$$

that follows from the fact that $X(d) = X(d, n)$ for any $d|n$, and that any necklace in $K(n)$ has a symmetry number that divides n.

Consider two positive integers n and m. Denote $(n : m^\infty)$ the integer obtained by removing all the prime factors corresponding to the primes that appear in m. For instance, $(120 : 70^\infty) = 3$, since $60 = 2^3 \cdot 3 \cdot 5$ and the primes 2 and 5 appear in $70 = 2 \cdot 5 \cdot 7$.

Now we have all the tools needed to state our main counting result:

Theorem 1. *Assume* $\gcd(N_{so}, N_o, N_c) = 1$. *Then,*

$$W(N_{so}, N_o, N_c) = N_2(N_{so} : N_o^\infty),$$

regardless of the value of N_c.

Proof. We will use Eq. 10 to compute the value of $W(N_{so}, N_o, N_c)$. In this equation, we have a sum ranging over all divisors d of N_{so}. However, if the divisor d has a common factor with N_o, then it can not have any common factor with N_c by our assumption $\gcd(N_{so}, N_o, N_c) = 1$, and therefore $Y(d, N_o, N_c) = 0$ according to Eq. 8. This means that it is enough to consider divisors of $(N_{so} : N_o^\infty)$. For any of these divisors, we have $Y(d, N_o, N_c) = 1$, since $\gcd(d, N_o) = 1$. All together this means that

$$W(N_{so}, N_o, N_c) = \sum_{d|(N_{so}:N_o^\infty)} X(d),$$

which is equal to $N_2(N_{so} : N_o^\infty)$ by Eq. 11. □

We derive from Theorem 1, two particular cases of independent interest:

Theorem 2. *If* $\gcd(N_{so}, N_o) = 1$*, then* $W(N_{so}, N_o, N_c) = N_2(N_{so})$.

Proof. When N_{so} and N_o have no common factors, then $(N_{so} : N_o^\infty) = N_{so}$, since there are no primes to remove from N_{so}. Knowing this, the result follows immediately from Theorem 1. □

Theorem 3. *If* $N_{so}|N_o$ *and* $\gcd(N_c, N_{so}) = 1$*, then* $W(N_{so}, N_o, N_c) = 2$.

Proof. The assumption $N_{so}|N_o$, implies that all the primes in N_{so} appear in N_o, and therefore $(N_{so} : N_o^\infty) = 1$. By Theorem 1, we conclude $W(N_{so}, N_o, N_c) = N_2(1) = 2$. □

While Theorem 1 is enough to deal with any Harmonic Flower Constellation (which are Lattice Flower Constellations with the additional constrain $\gcd(N_{so}, N_o, N_c) = 1$ as shown in [5]), it would be nice to have a simple closed formula for $W(N_{so}, N_o, N_c)$ that works in general. The following two results represent one positive step in that direction, but are clearly not enough.

We start with a formula for $X(d)$.

Theorem 4. *For any positive integer d, we have*

$$X(d) = \frac{1}{d} \sum_{e|d} \mu(e) 2^{d/e},$$

where μ *is Moebius' function*†.

†The function $\mu(n)$ is zero when the factorization of n contains a prime number to a power greater than 1, and otherwise, when n is the product of r different primes, is equal to $(-1)^r$.

Proof. The idea is to invert Eq. 11 using Moebius' inversion formula:

$$X(d) = \sum_{e|d} \mu(d/e) N_2(e) = \sum_{e|d} \sum_{f|e} \mu(d/e) \frac{\varphi(f)}{e} 2^{e/f}.$$

Writing $r = e/f$, and changing the order of summation, we get:

$$X(d) = \sum_{r|d} \frac{2^r}{r} \sum_{f|\frac{d}{r}} \mu\left(\frac{d}{rf}\right) \frac{\varphi(f)}{f}.$$

Finally, the theorem of multiplicative arithmetic functions show that the second sum reduces to $\mu(d/r)/(d/r)$, and therefore

$$X(d) = \sum_{r|d} \frac{2^r}{r} \frac{\mu(d/r)}{d/r} = \frac{1}{d} \sum_{r|d} \mu(d/r) 2^r,$$

as stated. □

Now we can give a formula for $W(N_{so}, N_o, N_c)$ in cases not included in Theorem 1 or any of its corollaries.

Theorem 5. *If $N_{so}|N_o$ and $N_{so}|N_c$ then,* $W(N_{so}, N_o, N_c) = 2^{N_{so}}$.

Proof. The key observation here is that for any divisor d of N_{so}, we have $Y(d, N_o, N_c) = d$, since d also divides N_o and N_c. Using Eq. 10 and Theorem 4, we obtain:

$$W(N_{so}, N_o, N_c) = \sum_{d|N_{so}} X(d)d = \sum_{d|N_{so}} \sum_{e|d} \mu(e) 2^{d/e}.$$

Writing $d = ek$, and changing the order of summation, the formula above reduces to:

$$W(N_{so}, N_o, N_c) = \sum_{k|N_{so}} \sum_{e|\frac{N_{so}}{k}} \mu(e) 2^k.$$

The sum $\sum_{e|r} \mu(e)$ is equal to 1 when $r = 1$ and 0 otherwise. In particular, the sum above (the one depending on e) will vanish unless $k = N_{so}$. This shows that $W(N_{so}, N_o, N_c) = 2^{N_{so}}$, as claimed. □

CONCLUSIONS

The cost of the missions is one of the most important factors to account when building a Constellations of satellites. The theory of necklaces allows us to reduce the number of satellites in a Flower Constellation without losing their symmetric character. Throughout the paper we have shown what parameters are needed to define one of these objects (basically, a pair $(\mathcal{G}, k)$ consisting of a necklace and a positive integer), and which constrains have to be imposed on these parameters (a simple diophantine equation). We have also provided algorithms in pseudo-code, but ready to be implemented, that enumerate all the possible Necklace Constellations that can be extracted from a Lattice Constellation.

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APPENDIX: PSEUDOCODE OF THE PROPOSED ALGORITHMS

Algorithm 1 Find all the necklaces of N_m spots and N_{so} satellites

$a = zeros(1, Nm)$
$b = []$
$neckrek(1, 1, 0, N_m, N_{so})$ % Call to the recursive function neckrek(t,p,ones,N_m,N_{so})
if $ones <= N_{so}$ **then**
  if $t > N_m$ **then**
    if $\mod(N_m, p) == 0$ **then**
      if $sum(a) == N_{so}$ **then**
        $b(size(b, 1) + 1, :) = a(2, : end)$
      end if
    end if
  else
    $a(t + 1) = a(t - p + 1)$
    if $a(t + 1) > 0$ **then**
      $neckrec(t + 1, p, ones + 1, N_m, N_{so})$
    else
      $neckrec(t + 1, p, ones, N_m, N_{so})$
    end if
    for $j = a(t + 1 - p) + 1 : 1$ **do**
      $a(t + 1) = j$
      $neckrec(t + 1, t, ones + 1, N_m, N_{so})$
    end for
  end if
end if
$M = fliplr(b)$

Algorithm 2 Finds the symmetries of a necklaces matrix

$[roll, Nm] = size(M)$
$[nod, d] = divisors(Nm)$
$N = zeros(roll, 1)$
$S = zeros(roll, nod)$
for $i = 0, \ldots, roll$ **do**
    $cs = 0$
    for $j = 1, \ldots, nod$ **do**
        $sym = true$
        $I = 1 : d(j)$
        for $k = 1, \ldots, ((Nm/d(j)) - 1)$ **do**
            $I_{-} = I + k * d(j)$
            if $isequal(M(i, I), M(i, I_{-})) == 0$ **then**
                $sym = false$
                $break$
            end if
        end for
        if $sym == true$ **then**
            $cs = cs + 1$
            $S(i, cs) = d(j)$
        end if
    end for
    $N(i) = cs$
end for

Algorithm 3 All the solutions of the diophantine equation $Ak + B \equiv 0 \mod C$

$[d, x1, k1] = \gcd(C, -A)$
$w = zeros(1, C + 1)$
$symmetry\_counter = 0$
if $\mod (B, d) == 0$ **then**
    for $lambda = -C : C$ **do**
        $k = (k1\frac{B}{d}) + (lambda - 1)(\frac{C}{d})$
        if $k \geq 0$ && $k < C$ **then**
            $symmetry\_counter = symmetry\_counter + 1$
            $w(symmetry\_counter) = k$
        end if
    end for
end if
$w(symmetry\_counter + 1) = -1$