Optimizing Flower Constellations for Global Coverage

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In this paper, the GDOP-optimal Flower Constellations using evolutionary methods, for a given number of satellites, or more generally, for a given set of Flower Constellation parameters is obtained. As a measure of optimality we use the maximum value of the GDOP over a large number of uniformly distributed points on the Earth during the time needed for the constellation to return to its initial pattern. The search space constellations includes eccentric orbits.

Nomenclature

N_o	number of inertial orbits
N_{so}	number of admissible locations per orbit
N_s	total number of admissible locations
N_{sat}	total number of satellites
T_p	orbital period (sec.)
T_d	period of the rotating reference frame (sec.)
N_p	number of revolution to the inertial orbit
N_d	number of revolution of the rotating reference frame
N_c	configuration number

I. Introduction

The design of optimal satellite constellations is the key problem in all kind of applications such as global navigation, global/regional coverage, telecommunications, Earth observation, radio-occultation, etc. As Draim indicates,[1] *constellation design remains more an art than a science*. To avoid proliferation of artistic solutions, general framework design methodologies have been proposed, such as Walker constellations[2] and Flower Constellations[3–5] (FCs). The key philosophical difference between Walker's and FCs is that FCs use rotating reference frames for constellation design, while Walker design is performed in the inertial frames.

The purpose of this paper is to determine the best FC for certain global coverage problems using evolutionary algorithms. In particular, we are interested in the problem of Global Positioning, with a minimum of four satellites in view from any point on the Earth at any time as a constraint. The geometry of these four or more satellites with respect to a ground station should ideally minimize the Geometric Dilution of Precision (GDOP), whose value quantifies the accuracy of the position estimation[6–8] (the lower GDOP the more accurate the estimation is). In this research the maximum value of GDOP obtained over the repetition

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time for 100 ground stations uniformly distributed on the Earth surface[9] is the metric defining our optimality. The propagation time coincides with the repetition time, which is the time that the constellation needs to return to its original position.

Evolutionary algorithms[10, 11] are a powerful optimization tool. These algorithms start with an initial set of solutions that are not necessarily optimal, and then, new generations of solutions are iteratively created from the best fitted individuals (with respect to a given fitness function) of the previous generation. Methods based on different philosophies converge with different accuracies and speeds to local or global (optimal) minimum. In our problem, this kind of algorithms are used to carry out a search among all possible orbits, to find the one that minimizes the maximum GDOP experienced in the repetition time. One of the original parts of this work is that we extend the search space of the optimization problem to include eccentric orbits, using the 2D theory of Lattice FCs.[12–14]

The Van Allen belts, whose existence was confirmed by Explorer I and Explorer II missions in early 1958, are two tori of energetic charged particles around the Earth equator with protons in the inner belt and electrons in the outer belt. The inner belt is located at an altitude around 800 km and 6,000 km, with maximum density at 3,000 km. The outer belt has maximum density at altitudes ranging between 15,000 km and 20,000 km.

The particles of the Van Allen belts may damage the electronic system of the satellites. Then, the long and also the manned missions must avoid being exposed persistently to these belts. Our constellations have satellites in orbit with semi-major axis $a = 29,655.3162$ km. The eccentricity varies between 0 and 0.2. Then, the perigee radius is $23,724.253 \leq r_p \leq 29,655.3163$ km, or considering the radius of the Earth $r_\oplus = 6,378.137$ km, the minimum altitude is $17,346.116 \leq r_m \leq 23,277.1792$ km. Meaning that, most of the time the satellites aren't in the maximum density region of the Van Allen belts.

This paper is organized as follows. The first section presents a background on Flower Constellations and the main tools that are used to solve our optimization problem, which are the GDOP and Evolutive Algorithms. The second section introduces the optimization problem in detail while the third section the results obtained are discussed.

II. Background and tools

In this section we present a background on Flower Constellations, and the main tools used in this research: Geometric Dilution of Precision and Evolutive Algorithms.

A. Flower Constellations

The original Flower Constellation theory is a set of N_{sat} satellites following the same closed trajectory with respect to a rotating reference frame. This implies the use of compatible (also called resonant) orbits. Every orbit has the same shape, inclination, and argument of perigee. According to Mortari,[3–5] the phasing of the Right Ascension of the Ascending Node (RAAN) and Mean Anomaly are defined using six integer parameters.

However, subsequent theories, such as Harmonic Flower Constellation expanded in the 2D Lattice Flower Constellation,[12] substantially improved the original theory making it independent from any reference frame, inertial or rotating, and with minimal parametrization.

The 2D Lattice Flower Constellation can be described by five integer parameters and three continuous ones. The integer parameters can be broken in two sets, the first set (N_o, N_{so}, N_c) where N_o is the number of orbital planes, N_{so} is the number of admissible position in each orbit, and N_c is the phasing parameter. The second set is (N_p, N_d), where N_p is the number of orbit revolutions and N_d the number of revolutions of the rotating reference frame, determines the orbital period and satisfy the compatibility equation:

$$N_p T_p = N_d T_d \tag{1}$$

where T_p is the orbital period and T_d is the period of the rotating reference frame.

The phasing parameters define the RAAN (Ω) and initial Mean Anomaly (M), which can be written in matrix notation as

$$\begin{bmatrix} N_o & 0 \\ N_c & N_{so} \end{bmatrix} \begin{Bmatrix} \Omega_{ij} \\ M_{ij} \end{Bmatrix} = 2\pi \begin{Bmatrix} i \\ j \end{Bmatrix} \tag{2}$$

where $i = 0, \cdots, N_o - 1$, $j = 0, \cdots, N_{so} - 1$, and $N_c \in [0, N_o - 1]$. The (i, j) satellite is the j-th satellite on the i-th orbital plane.

The condition for all the satellites to have the same repeating ground track can be easily enforced, requiring only that two coprime integers μ and λ exist such that

$$N_d = \lambda N_{so} \qquad \text{and} \qquad N_p = \mu N_o + \lambda N_c$$

where N_p and N_d are coprime.

The remaining parameters to define the constellation are continuous parameters that are the same for all orbits in the constellation: the inclination, the eccentricity, and the argument of periapsis.

B. Dilution of Precision

A particular Global Coverage problem is the Global Positioning problem that provides location and time information anywhere on the Earth up to a given altitude. As an examples of Global Coverage Systems we can mention the LOng RAnge Navigation (LORAN), Decca Navigator System, Global Positioning System (GPS), and more recently GALILEO.

The three-dimensional position determination problem, consist of determining the user position (x_u, y_u, z_u) using the location of three satellites whose coordinates are well known. GPS determines the user position using the concept Time-Of-Arrival (TOA), which consists of determining the user position measuring the time-of-arrival for a signal transmitted by a satellite at a known location to reach the user location.[6] Multiplying the TOA by the speed of the signal transmitted is possible determine the user's position. Furthermore, the time offset (t_u), which represent the time offset between the receiver clock and the system time, will be another unknown. Then, four visible satellites are needed to completely determine the four unknowns; the user position and the time offset.

The pseudorange measurement is a range determined by multiplying the signal propagation velocity, c, by the time difference between two non-synchronized clocks. Then, the pseudorange measurement doesn't represent exactly the geometric distance between the satellite and the receiver. This measurement contains: (1) the geometric satellite-user range, (2) an offset attributed to the difference between system time and the user clock, (3) an offset attributed to the difference between system time and satellite clock, (4) other sources of error that corrupt a little bit more the measurements (atmospheric delay, ionospheric delay, etc).

The pseudorrange measurements ρ_j from the user position (x_u, y_u, z_u) to the j-th satellite can be described by

$$\rho_j = \sqrt{(x_j - x_u)^2 + (y_j - y_u)^2 + (z_j - z_u)^2} + ct_u \tag{3}$$

where the four unknowns are the position (x_u, y_u, z_u) and the time offset t_u.

Using an approximate user position $(\hat{x}_u, \hat{y}_u, \hat{z}_u)$ and time estimate $\hat{t}_u$. Let an approximate pseudorange be represented by

$$\hat{\rho}_j = \sqrt{(x_j - \hat{x}_u)^2 + (y_j - \hat{y}_u)^2 + (z_j - \hat{z}_u)^2} + c\hat{t}_u \tag{4}$$

The vector of the offset between the pseudorange measurements can be view as a linear combination of three terms:

$$\Delta\rho = \hat{\rho}_j - \rho_j = \rho_T - \rho_L + d\rho \tag{5}$$

where ρ_T is the vector of error-free pseudorange values, ρ_L is the vector of pseudorange values computed at the linearization point $(\hat{x}_u, \hat{y}_u, \hat{z}_u)$, and $d\rho$ represents the net error in the pseudorange values, i.e. only the error part of the pseudorange measurements.

Also, the offset between the positions can be view as a linear combination of three terms,

$$\Delta x = \begin{pmatrix} \hat{x}_u \\ \hat{y}_u \\ \hat{z}_u \\ \hat{t}_u \end{pmatrix} - \begin{pmatrix} x_u \\ y_u \\ z_u \\ t_u \end{pmatrix} = x_T - x_L + dx \tag{6}$$

where: x_T is the error-free position and time. x_L is the position and time defined as the linearization point $(\hat{x}_u, \hat{y}_u, \hat{z}_u)$ and, dx represents the error between the approximate position and time estimate with respect the real position and time.

Then, is possible arrive to the expression[8]

$$dx = (H^T H)^{-1} H^T d\rho \tag{7}$$

The previous relation, give the functional relationship between the errors in the pseudorange values ($d\rho$) and the induced errors in the computed position and time (dx). Matrix $(H^T H)^{-1} H^T$, is a $4 \times n$ matrix and depends only on the relative geometry of the user and the satellites. That's why is so important the geometry of the constellation.

The covariance of a vector is frequently of interest to asses how strongly two variables of the vector change together. Then, considering the covariance of the vectors dx and $d\rho$:

$$\mathrm{cov}(dx) = \begin{pmatrix} \sigma^2_{x_u} & \sigma_{x_u y_u} & \sigma_{x_u z_u} & \sigma_{x_u t_u} \\ \sigma_{y_u x_u} & \sigma^2_{y_u} & \sigma_{y_u z_u} & \sigma_{y_u t_u} \\ \sigma_{z_u x_u} & \sigma_{z_u y_u} & \sigma^2_{z_u} & \sigma_{z_u t_u} \\ \sigma_{t_u x_u} & \sigma_{t_u y_u} & \sigma_{t_u z_u} & \sigma^2_{t_u} \end{pmatrix} = (H^T H)^{-1} \mathrm{cov}(d\rho) = (H^T H)^{-1} \sigma^2_{UERE} \tag{8}$$

where UERE (User Equivalent Range Error) is considered to be the statistical sum of the contributions from each of the error sources associated with the satellite.

Dilution of precision parameters in GPS are defined in terms of the ratio of combinations of the components of the $\mathrm{cov}(dx)$ and σ^2_{UERE}. It is implicitly assumed in the DOP definitions that the user/satellite geometry is considered fixed. Also it is assumed that local user coordinates are being used in the specification of $\mathrm{cov}(dx)$ and dx. The positive x-axis points east, the y-axis points north, and the z-axis points up. The most general parameter is termed the geometric dilution of precision (GDOP) and is defined by the formula,

$$GDOP = \frac{\sqrt{\sigma^2_{x_u} + \sigma^2_{y_u} + \sigma^2_{z_u} + \sigma^2_{t_u}}}{\sigma_{UERE}} \tag{9}$$

A relationship for GDOP is obtained in terms of the components of $(H^T H)^{-1}$ by expressing $(H^T H)^{-1}$ in component form:

$$(H^T H)^{-1} = \begin{pmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{pmatrix} \tag{10}$$

Then, GDOP can be computed as the square root of the trace of the $(H^T H)^{-1}$ matrix

$$GDOP = \sqrt{D_{11} + D_{22} + D_{33} + D_{44}} \tag{11}$$

The above clearly shows that the geometry of the constellations has a direct role on positioning accuracies. Several tools are defined to describe the accuracy error, but Geometric Dilution of Precition (GDOP) used by GPS it is a powerful accuracy indicator. The GDOP will show how well the constellation of satellites is organized geometrically. It is a quantity varying between 1 and 99, while 1 means that the constellation presents a perfect distribution of satellites, 99 means that presents a really poor geometrical distribution. Then, the less GDOP value, the more accurate the positioning system is.

C. Evolutive Algorithms

An optimization problem consist of finding, among all possible solutions of the problem (search space), the best one. The search space is n-dimensional and depending on the problem type, the different variables can be discrete or continuous. In order to find the optimal solution, different search algorithms may be used. For a search space with only a small number of possible solutions, all of them can be examined in a reasonable amount of time. This brute-force search technique has an easy implementation, and it always find the optimal solution if it exists. However, as the dimensions of the search space increase, the exhaustive search become so expensive in running time and memory requirements, that is not practical anymore.

Instead of using the brute-force search algorithm, evolutionary algorithms has been developed.[10] These kind of algorithms abstract biological evolution or biological behaviors to search optimal solutions. Two different algorithms will be considered: the Genetic Algorithm and the Particle Swarm Optimization.

Charles Darwin's *On the Origin of Species*, in his *Principles of Biology* (1864) proposed the idea that over several generation, biological organisms evolve based on the principle of natural selection "survival of the fittest". This idea works well in nature. An individual in a population competes with each other for different resources like food, shelter, etc. Due to the selection, the most adapted to the environment and the stronger ones have more chance to survive and reproduce, while the less adapted have less chance to survive and reproduce. Continuously improving the individual characteristics of the species, since the new generations take the good characteristics of their antecessors and will improve them at each generation. They will become more and more adapted to their environment. Notice that, sometimes in nature occur a crazy or random fact, it consists of taking random characteristics and create an individual completely new with different characteristics that sometimes are better, sometimes worse than the existing individuals.

The idea of solving different optimization problems using evolutive techniques started in 1954 with the work of Nils Aall Barricelli. However, Genetic Algorithms became popular through the work of John Holland in 1975 in his book *Adaptation in Natural and Artificial Systems*. Genetic Algorithms mimic the process of natural evolution described above. It is a search technique to find optimal solutions to a problem. Genetic Algorithm has an initial population represented by a group of individuals, each of these individuals represents a solution to the optimization problem and they are considered as the chromosomes. After evaluating all the initial population with the fitness function, to know how good the solutions are, a number of individuals are selected to create the next generation combining their genes. In the reproduction process, different reproduction operators are used, such as, recombination and mutation. The first one, consist of recombining different chromosomes of two different individuals (parents) to generate a new individual (child). The second one, is a factor that randomly generates completely new genes for the new individual. When the new generation is built, we evaluate the population with the fitness function and start again the process until the stopping criteria is reached. It can be a finite number of generations, the convergence toward the optimal solution, etc.

A flowchart of the Genetic Algorithm can be as follows

Step 1: Select randomly an initial population of n individuals from the search space, i.e. select randomly n solutions of the optimization problem.

Step 2: Evaluate the individuals of the population with the fitness function.

Step 3: Create a new population following these steps:

- **Step 3.1:** Select two individuals (parents), the better the fitness is, the bigger the chance to get selected.
- **Step 3.2:** Crossover the genes of the two parents to create a new individual (child).
- **Step 3.3:** With a mutation probability, mutate randomly the genes of the new individual (child).
- **Step 3.4:** Repeat the process until have a population of n new individuals.

Step 4: If the the stopping criteria is satisfied, evaluate the new generation and select the most suitable individual. If not, go to Step 2.

Particle Swarm Optimization (PSO) is a population based stochastic optimization method, i.e. a method that generate and use random variables to find the optimal solution. PSO was developed by Dr. Eberhart and Dr. Kennedy in 1995, inspired by the social behavior of bird flocking or fish schooling. The basic idea is to simulate these behaviors with an algorithm. In both cases, if a bird or a fish sees a good path to go (because they find food, protection or good weather), the rest of the swarm will be able to follow that path even if they were going in the opposite way. However, there is a "craziness factor" or random factor that makes some of the particles move away from the flock in order to explore new paths.

It is possible to translate this behavior into an algorithm. Each different bird or fish is considered as an initial particle in the search space. These particles are flying through the search space and have two essential capabilities: remembering their own best position (individual factor) and knowing the best position of the entire swarm (social factor). The basic idea is that individuals communicate good positions to each other and adjust their own position and velocity depending on the social and individual factors.

During the simulation each particle has a position and velocity. Additionally, each particle keeps track of the position of the best solution it has visited so far (*pbest*) and the position of the best solution visited by any other particle (*gbest*). At each step, the velocity is updated at each iteration taking into account *pbest* and *gbest*.

Changing the position and velocity of each particle at each iteration works as follows. Assume that the i-th particle has position vector $x_i(t)$ and velocity vector $v_i(t)$. Then, the updated velocity will be:

$$v_i(t+1) = \alpha v_i(t) + c_1 \cdot rand \cdot (pbest_i - x_i(t)) + c_2 \cdot rand \cdot (gbest(t) - x_i(t)) \tag{12}$$

where α is the inertia weight that controls the exploration of the search space. The constants c_1 and c_2, which in our simulation are taken between 0 and 1, determine how the individual and social factor affects the velocity of the particle. Finally, rand is a random number chosen uniformly in [0,1]. Note that without the second and third terms of the expression (12) the particle will keep in the same direction until it hits the boundary.

The position is updated as follows:

$$x_i(t+1) = x_i(t) + v_i(t+1) \tag{13}$$

This process is repeated for each particle until the best optimal solution is obtained or the stopping criteria is reached.

The PSO can be implemented as follows:

Step 1: Initialize randomly an initial swarm of n particles from the search space.

- **Step 1.1:** Initialize randomly the initial positions, i.e. the solutions of the problem, $x_i(0)$.
- **Step 1.2:** Initialize randomly the velocities of the initial particles, $v_i(0)$.
- **Step 1.3:** Update the *pbest* and *gbest* values thought the fitness function.

Step 2: Update the new velocities for the particles, $v_i(t+1)$, according to Eq. (12).

Step 3: Calculate the new positions of the particles, $x_i(t+1) = x_i(t) + v_i(t+1)$).

Step 4: Update the *pbest* and *gbest* values thought the fitness function.

Step 5: Go to step 2, and repeat until convergence or stopping criteria.

III. Problem Formulation

A. Optimization problem

Given the total number of satellites of a constellation (N_{sat}), it is possible to obtain all the different possible phasing parameters (N_o, N_{so}, N_c). Therefore, we can compute the possible distribution of satellites in the (Ω, M)-space. For example, given $N_{sat} = 27$ all the possible combinations for the phasing parameters are shown in Table 1.

N_{sat}	27	27	27	27	27	27	27	...	27	27	27	...	27
N_o	1	3	3	3	9	9	9	...	9	27	27	...	27
N_{so}	27	9	9	9	3	3	3	...	3	1	1	...	1
N_c	0	0	1	2	0	1	2	...	8	0	1	...	26

Table 1. Possible phasing parameters.

Besides, given the semi-major axis, we can determine T_p, and select integers N_p and N_d satisfying the compatibility equation $N_p T_p = N_d T_d$.

For each possible configuration, Evolution Algorithms are used to carry out a search to find the best orbital parameters (e, i, ω), which completely define the constellation, and minimize the fitness function.

In the case of the Genetic Algorithm, given an initial configuration for the satellites, an initial population of $n = 60$ individuals is taken, i.e. 60 possible values for the orbital parameters (e, i, ω). Then, each possible constellation is evaluated with the fitness function. After that, a new generation of 60 individuals is created.

The new individuals are created with 10 fittest ones from the previous generation, and 50 others obtained by crossover and mutation. The crossover consists of selecting a father (e_f, i_f, ω_f) and a mother (e_m, i_m, ω_m) from the previous generation at random and creating a son

$$(e_f x_1 + e_m(1 - x_1), i_f x_2 + i_m(1 - x_2), \omega_f x_3 + \omega_m(1 - x_3))$$

where $x_1, x_2, x_3 \in \{0, 1\}$ are chosen at random with 0.5 probability each. After the son is created, we decide with probability 0.05 whether it mutates or not. Mutation consists of choosing all three coordinates e, i, ω at random within their allowed ranges. The process is repeated 60 generations and, at that point, the best individual found provides the solution to the optimization process.

In the case of the Particle Swarm Optimization an initial swarm of $n = 60$ particles is taken, i.e. 60 possible values for the orbital parameters (e, i, ω) which are the positions, and 60 possible velocities for them. Both positions and velocities are chosen randomly within the search space. It should be noted that neither position or velocity correspond with the actual motion of the satellites; these quantities are unitless. Then, we evaluate each constellation with the fitness function and update the new velocities and positions according to Eq. (12) and Eq. (13). We are using an inertia factor $\alpha = 0.95$, individual factor $c_1 = 0.75$, and social factor $c_2 = 0.35$. The process is repeated 60 iterations.

The fitness function in our problem is the maximum value of the GDOP experienced among the propagation time $T = T_p N_p$, for 100 Ground Stations uniformly distributed on the Earth surface.[9] Note that, the propagation time T is the time that the constellation needs to return to its original configuration.

A formula to compute the maximum GDOP of our constellation can be implemented as:

$$\text{Max GDOP} = \max_{i,j}(dop(i,j)) \quad i \in (1, 2, \cdots, gr) \quad \text{and} \quad j \in (1, 2, \cdots, nt) \tag{14}$$

where gr indicates the number of Ground Stations, nt the number of iterations in the propagation process, and $dop(i,j)$ represents the GDOP computed from ground station i-th at time $j\delta t$, with $\delta t = 60$ seconds.

In order to compute the value of the GDOP, it is necessary to determine which satellites are visible from a ground station. Then, the concept of grazing angle or spacecraft elevation angle is defined. This is the angle between the horizon and the position vector of a satellite. Another way to refer to this angle is using the angle of incidence which is the angle between the normal vector to the surface of the Earth at the ground station and the position vector. Due to the existence of buildings, mountains, and other visibility obstacles the grazing angle is considered in the formulation of all global positioning problems. Figure 1 shows the grazing angle:

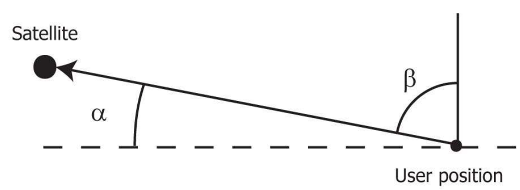

Figure 1. The grazing angle α and the angle of incidence β.

In our problem, we consider three different cases in which the grazing angle is $\alpha = 0°$, $\alpha = 5°$, and a more realistic case $\alpha = 10°$. In other words, the considered angle of incidence is $\beta = 90°$, $\beta = 85°$ and $\beta = 80°$ respectively.

The purpose of this research is to find the best parameters of a 2D LFC which minimize the maximum value of the GDOP. Also, it is possible to introduce in our search elliptic orbits obtaining interesting results proving that, in some cases, better results with eccentric orbits than with circular orbits are obtained. Finally, we compare our results with an existing satellite constellation.

IV. Results

A. Method's comparison

In this research three different algorithms have been used: a brute force search or exhaustive search to have an approximate idea of the optimal solution and two evolutive algorithms. These last two are the Genetic

Algorithm and Particle Swarm Optimization, which improve substantially the brute force search, as we show below.

For a given a number of satellites N_{sat}, according to the 2D LFC theory, the number of different constellations, is given by the following equation:

$$f(N_{sat}) = \sum_{d|N_{sat}} d. \tag{15}$$

Thus, the total number of constellations with $15 \leq N_{sat} \leq 40$ is equal to:

$$\sum_{n=15}^{40} f(n) = 1177. \tag{16}$$

Each of these 1177 cases has been analyzed to find the best parameters (e, i, ω) that minimize the GDOP with the three different methods. Figure 2 shows the number of times in which one method is better than the others, considering three different grazing angles. In all cases the PSO algorithm is the best method followed by the Genetic Algorithm. In certain configurations, it is impossible to find a constellation with GDOP better than 99. For instance, when $N_o = 1$ the satellites are always on the same orbit plane, hence the maximum GDOP is 99. Those cases have been excluded from the comparison between methods, and they are represented with a separate bar in Fig. 2. Note that, when the grazing angle is small, or equivalently, when the incidence angle is big, the cases with GDOP equal to 99 are considerably less.

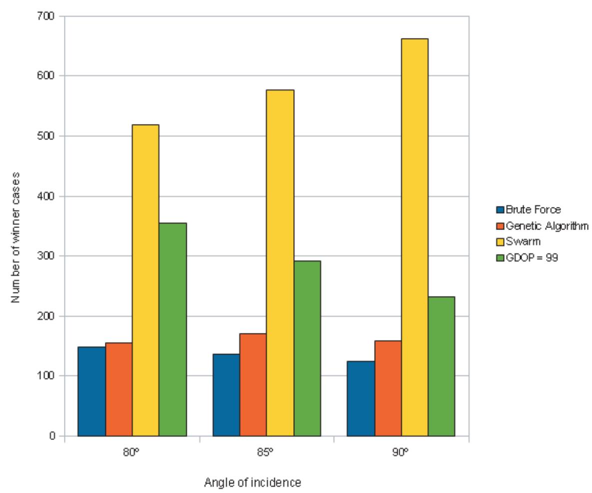

Figure 2. Comparison of methods with different angles of incidence.

Note that the comparison between the three methods is fair because they evaluate the cost function (i.e. the maximum GDOP) the same number of times.

- Genetic Algorithm has 60 generations with 60 individuals. Each individual represents a 3-tuple (e, i, ω). For each individual the maximum GDOP of the constellation is computed. In one generation the maximum GDOP is computed 60 times. Thus, in 60 generations the maximum GDOP is calculated 3,600 times.

- Particle Swarm Optimization has 60 generations of 60 particles. As the Genetic Algorithm the maximum value of the GDOP is computed 3,600 times.

- Brute Force search algorithms has 20 different values for the eccentricity, that is $e \in [0, 0.3]$ and with steps of 0.015. The inclination has 36 different possibilities, that is $i \in [0, 180°]$ with steps of 5°. Finally, the argument of perigee $\omega \in [0, 360°]$ with steps of 72°, so it assumes only 5 different values. Thus, the maximum value of the GDOP is calculated $20 \cdot 36 \cdot 5 = 3,600$ times.

For a given set of phasing parameters (N_{so}, N_o, N_c), the time that PSO (60 generations of 60 particles) takes to find the optimal constellation is approximately 80 seconds. For example, if we have $N_{sat} = 27$, there are 40 possible configurations, so the total computational cost would be about $40 \cdot 80 = 3,200$ seconds. When the number of satellites is larger, not only we have more possible configurations, but also the computational time per configuration increases, since there are more satellites to evaluate.

B. The optimal configurations

Consider first a constellation with $N_{sat} = 27$ satellites. As we can see in Table 1, there are 40 possible configurations for the phasing parameters. For each of those configurations, the three Evolutive Algorithms were used to determine the best parameters (e, i, and ω) that minimize the maximum value of the GDOP along the propagation time. These optimal parameters are shown in tables 2, 3, and 4 for different grazing angles.

Method	N_{sat}	N_o	N_{so}	N_c	e	i	ω	max GDOP
BF	27	3	9	1	0.0300	125.000	0.000	3.61589
GA	27	3	9	2	0.0041	53.945	237.334	3.56904
PSO	27	3	9	2	0.0000	55.014	219.111	3.58761

Table 2. Grazing angle $\alpha = 10°$

Method	N_{sat}	N_o	N_{so}	N_c	e	i	ω	max GDOP
BF	27	27	1	2	0.0150	130.000	144.000	2.995
GA	27	27	1	2	0.0455	129.472	182.705	2.914
PSO	27	27	1	2	0.0096	130.515	140.966	2.908

Table 3. Grazing angle $\alpha = 5°$

Method	N_{sat}	N_o	N_{so}	N_c	e	i	ω	max GDOP
BF	27	27	1	15	0.0000	125.000	0.000	2.182
GA	27	27	1	15	0.0009	124.470	40.024	2.191
PSO	27	27	1	15	0.0034	124.176	239.146	2.171

Table 4. Grazing angle $\alpha = 0°$

These tables show clearly that the best constellation found depends on the method, and specially on the grazing angle. While in this paper, we keep track of the results with the three grazing angles, for practical purposes, only the case $\alpha = 10°$ is relevant. Regarding the sensitivity to the method, we decided to continue using the three methods, and use the best solution found by any of them. The solutions found by the other two are used to provide some confidence on the optimality of the GDOP.

Now we do the same for any number of satellites $15 \leq N_{sat} \leq 40$. The GDOP of the best configuration found by each of the three methods is shown in the Figs. 3, 4, and 5, when the grazing angles are $\alpha = 10°$, $\alpha = 5°$, and $\alpha = 0°$ respectively. In the case $\alpha = 10°$, we only show the configurations with more than 24 satellites, since the cases with $N_{sat} \leq 23$ have GDOP above 5. For the same reason, when $\alpha = 5°$ and $\alpha = 0°$, we only show constellations with $N_{sat} \geq 21$ and $N_{sat} \geq 17$ respectively.

Intuitively, the more satellites the constellation has, the better results for the GDOP value should be obtained. However, this is not always true, because with 26 satellites we obtained better results than with 27, 28, or 29 satellites. Also, with 33 satellites, we obtained better GDOP than with 34 satellites.

The best configurations found for $N_{sat} \in [15, 40]$ are summarized in Tables 5, 6, and 7, for grazing angles $\alpha = 10°$, $\alpha = 5°$, and $\alpha = 0°$ respectively.

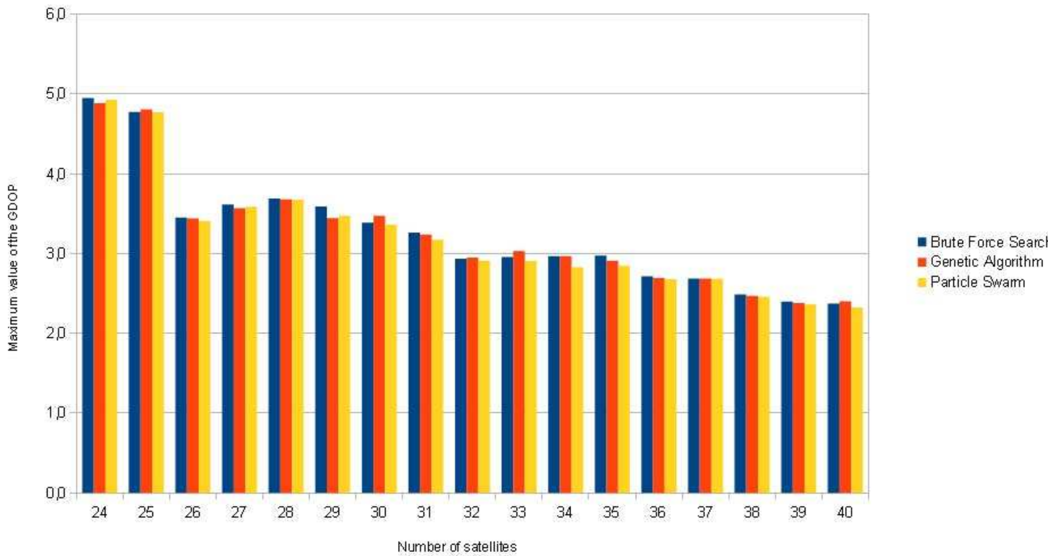

Figure 3. Maximum GDOP with a grazing angle of $10°$**.**

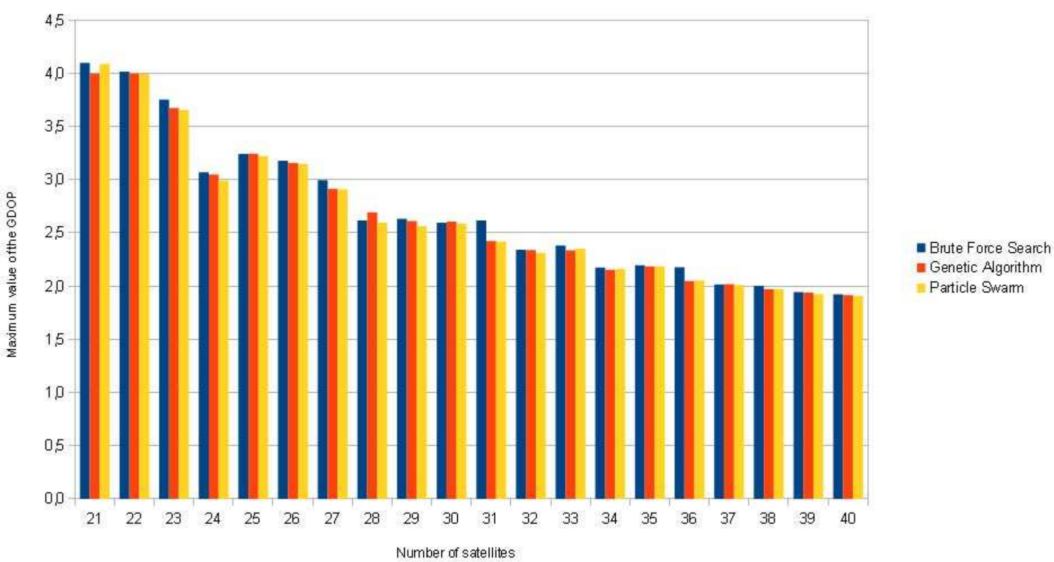

Figure 4. Maximum GDOP with a grazing angle of $5°$**.**

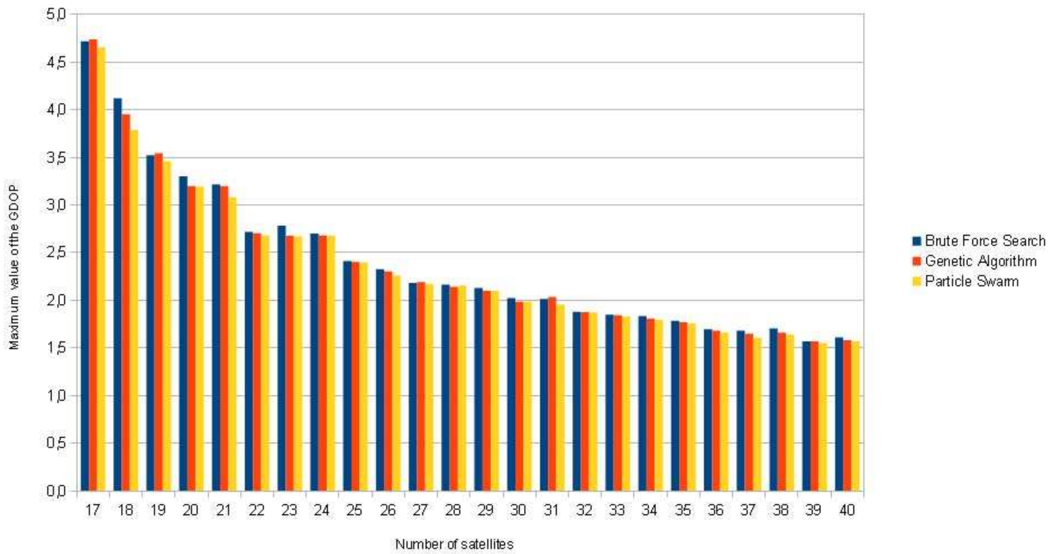

Figure 5. Maximum GDOP with a grazing angle of $0°$.

Table 5. Grazing angle with $\alpha = 10°$

N_{sat}	N_o	N_{so}	N_c	e	i	ω	max GDOP
20	20	1	18	0.1884	302.957	177.961	8.828
21	21	1	2	0.0000	129.765	134.656	6.558
22	22	1	6	0.1138	125.804	187.001	5.574
23	23	1	2	0.0000	118.348	252.455	5.135
24	24	1	2	0.0062	126.236	293.370	4.892
25	25	1	19	0.1985	114.352	179.710	4.773
26	26	1	10	0.0016	58.929	305.336	3.408
27	3	9	2	0.0041	53.945	237.334	3.569
28	7	4	2	0.0000	129.598	59.658	3.674
29	29	1	11	0.0267	61.368	90.356	3.445
30	30	1	2	0.0000	129.502	207.473	3.362
31	31	1	27	0.0000	105.420	102.427	3.175
32	32	1	20	0.0062	123.059	90.918	2.910
33	3	11	0	0.0000	52.891	286.820	2.908
34	34	1	22	0.0000	60.000	216.000	2.966
35	35	1	29	0.2971	57.557	179.898	2.846
36	12	3	4	0.0728	62.842	357.765	2.679
37	37	1	11	0.0718	122.418	145.593	2.684
38	38	1	14	0.0115	60.536	85.848	2.459
39	39	1	15	0.0000	60.529	244.292	2.364
40	10	4	7	0.0000	60.725	66.539	2.322

Table 6. Grazing angle with $\alpha = 5°$

N_{sat}	N_o	N_{so}	N_c	e	i	ω	max GDOP
18	6	3	4	0.0413	59.639	169.143	6.440
19	19	1	3	0.0086	46.857	350.621	5.894
20	10	2	7	0.2196	112.996	185.550	5.417
21	21	1	2	0.0178	126.355	304.679	3.996
22	22	1	2	0.0000	124.234	314.356	3.992
23	23	1	2	0.0181	125.295	87.359	3.655
24	3	8	2	0.0000	58.309	206.622	2.992
25	25	1	14	0.0046	69.779	127.550	3.221
26	26	1	20	0.3000	113.260	180.182	3.145
27	27	1	2	0.0096	130.515	140.966	2.908
28	28	1	4	0.0007	74.962	300.987	2.596
29	29	1	27	0.0000	53.633	270.361	2.563
30	10	3	6	0.0736	123.719	359.414	2.588
31	31	1	14	0.0000	110.692	131.599	2.416
32	32	1	20	0.0100	120.872	159.225	2.313
33	33	1	29	0.1003	103.876	359.971	2.335
34	34	1	10	0.0047	118.012	294.027	2.152
35	35	1	13	0.0038	300.180	114.586	2.184
36	12	3	8	0.0041	123.967	334.521	2.048
37	37	1	11	0.0000	118.727	106.899	2.011
38	19	2	5	0.0000	68.668	275.952	1.971
39	3	13	2	0.0101	53.611	130.555	1.924
40	40	1	12	0.0000	117.188	123.586	1.906

Table 7. Grazing angle with $\alpha = 0°$

N_{sat}	N_o	N_{so}	N_c	e	i	ω	max GDOP
17	17	1	4	0.1500	74.939	0.043	6.647
18	18	1	16	0.0000	63.732	165.832	3.788
19	19	1	2	0.0000	129.198	323.969	3.461
20	20	1	2	0.0000	121.435	356.140	3.193
21	3	7	0	0.0000	51.640	323.739	3.080
22	22	1	6	0.0000	121.204	6.460	2.681
23	23	1	9	0.0000	58.709	129.965	2.671
24	24	1	20	0.2452	58.522	179.649	2.676
25	25	1	18	0.0000	58.647	190.444	2.398
26	13	2	10	0.0000	110.571	227.220	2.258
27	27	1	15	0.0034	124.176	239.146	2.171
28	28	1	8	0.0264	122.840	64.085	2.141
29	29	1	9	0.0000	51.279	219.439	2.099
30	30	1	26	0.0035	56.198	294.761	1.984
31	31	1	27	0.0000	53.009	92.531	1.955
32	32	1	14	0.0000	63.166	42.629	1.873
33	33	1	9	0.0000	118.704	327.201	1.827
34	34	1	10	0.0080	67.232	191.544	1.795
35	35	1	15	0.0000	63.319	311.235	1.759
36	36	1	31	0.0000	59.414	91.516	1.660
37	37	1	11	0.0000	56.425	187.868	1.608
38	38	1	33	0.0000	58.669	9.203	1.640
39	39	1	6	0.0000	56.679	300.869	1.551
40	8	5	2	0.0005	122.323	232.913	1.567

C. Eccentric Orbits

One of the innovative results, thanks to the 2D LFC theory, is that eccentric orbits are considered in the searching process. As we can see in Tables 5, 6, and 7, in many occasions the optimal configuration has a highly eccentric orbit. For instance, when $N_{sat} = 25$ and the grazing angle is $\alpha = 10°$, the optimal constellation has $e = 0.1985$. This case is shown in Fig. 6.

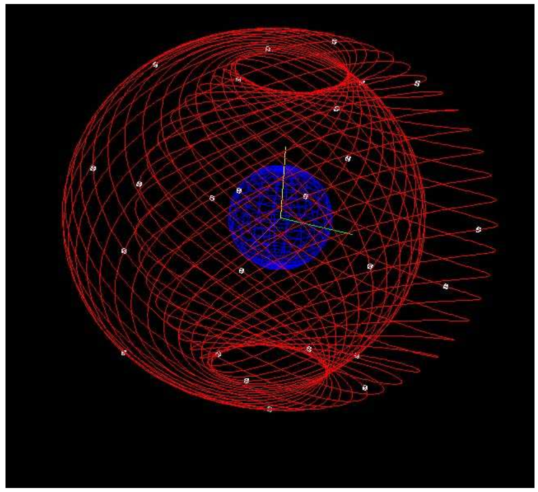

Figure 6. A $(25, 1, 19, 17, 10, 0.1985, 114.352, 179.710)$ **2D LFC with** $\alpha = 10°$**.**

D. Comparison with Galileo Constellation

Galileo Constellation is currently being built by the European Union to have an alternative navigation system to the existing GPS System (US), the GLONASS (Russian), and the Chinese Compass System. This constellation has 27 satellites moving in three circular orbits with an inclination of 56°. This corresponds to the 2D LFC with parameters $N_o = 3$, $N_{so} = 9$, $N_c = 2$, $e = 0$, and $i = 56°$. The semimajor axis is determined by the compatibility ratio $N_p/N_d = 17/10$.

Using a grazing angle $\alpha = 10°$ and our algorithms, the original Galileo Constellation has a $GDOP = 3.775$. Table 2 shows that the three methods were able to find constellations with $N_{sat} = 27$ that are marginally better than Galileo. The best of these three constellations, which was found by the Genetic Algorithm, is also shown in Table 5 and it has $GDOP = 3.569$.

E. Time-evolution of the GDOP

While our algorithms compare constellations based on the worst GDOP value seen by any of the ground stations at any instant of time, it would be interesting to see the evolution in time of the maximum GDOP, average GDOP, and minimum GDOP experienced by the 100 ground stations. These three values of the GDOP are shown in Fig. 7 for our optimal constellation with 27 satellites. For clarity, Fig. 8 shows only the evolution of the maximum value of the GDOP over time.

In the first of these figures, we can see that the maximum GDOP experienced by the 100 stations is around 3.5 at any time, meaning that there is always a ground station where the GDOP is about 3.5, and that no ground station has a GDOP worse than that. Similarly, we can see that the minimum GDOP is aproximately 1.6, so there is always a point on the Earth where the GDOP is as good as 1.6. Finally, the average moves around 2.4, so we can expect half of the ground stations to have a GDOP between 1.6 and 2.4, and the other half in the interval [2.4, 3.5]. Intuitively, this means that about half of the surface of the Earth would experience a GDOP better than 2.4.

In the next figure, we can see that the maximum GDOP oscillates between 3.52 ± 0.05. The deviation from the center value is less than 2%. This indicates that the performance of the constellation remains almost constant over time.

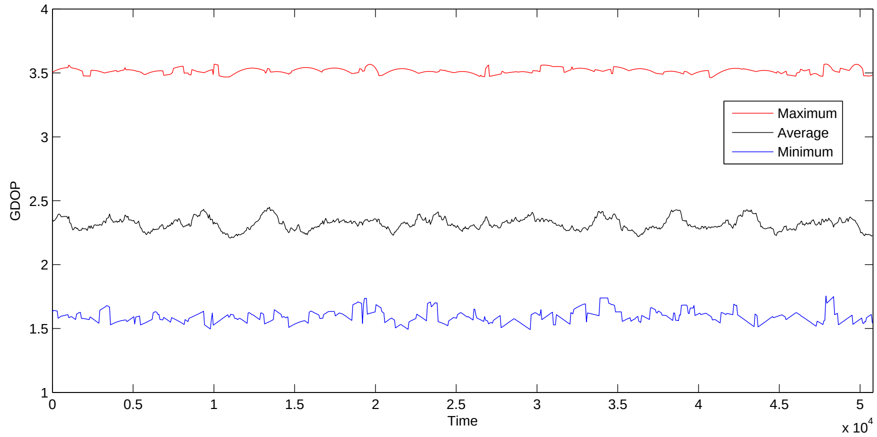

Figure 7. Maximum, minimum and average GDOP value of our 27 satellite constellation.

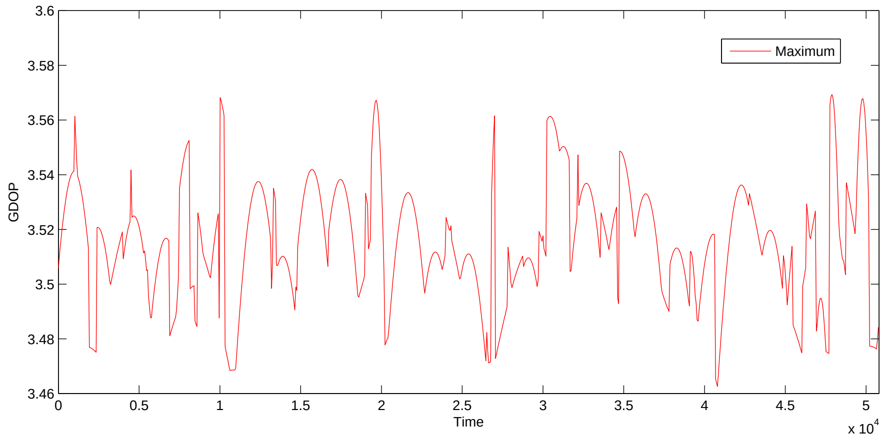

Figure 8. Maximum GDOP value of our 27 satellite constellation over time.

The 26 satellite constellation has better GDOP value than the constellations with 27, 28, or 29 satellites. Figure 9 shows a comparison between the maximum and average GDOP of our optimal 26 and 27 satellite constellations. With respect to the maximum GDOP metric, the 26 satellite constellation has a better GDOP than the 27 satellite at any instant of time. With respect to the average metric, again 26 satellites are better than 27, except during some small intervals of time.

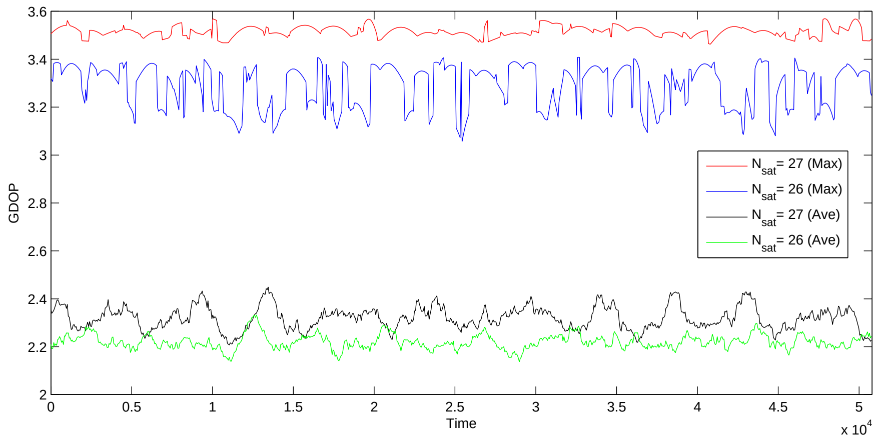

Figure 9. Comparison of the maximum and average GDOP of the 26 and 27 satellite constellation.

Finally, we provide in Figs. 10 and 11 a comparison between Galileo and the 26 satellite optimal constellation, which we already know has better maximum GDOP. The figures show that both maximum and average GDOP are better at any time.

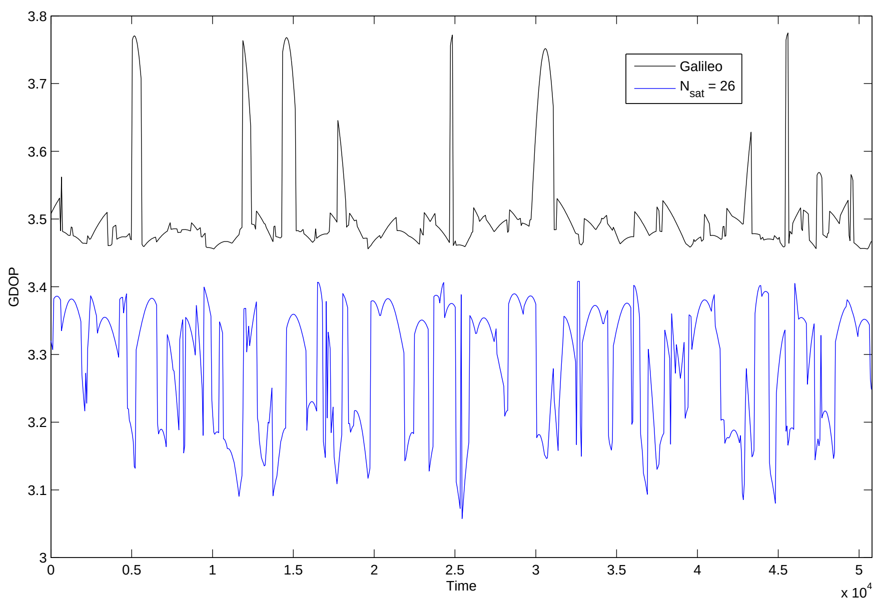

Figure 10. Maximum GDOP of Galileo Constellation and our 26 satellite constellation.

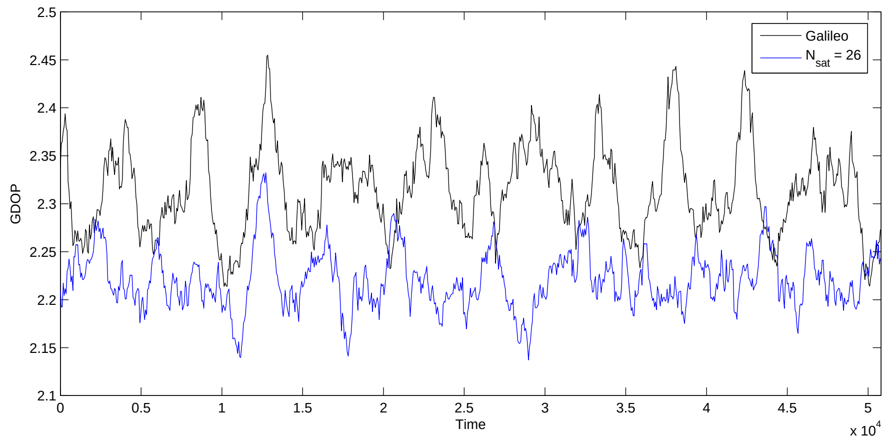

Figure 11. Average GDOP of Galileo Constellation and our 26 satellite constellation.

V. Conclusions and future work

From this study it is possible to conclude that any constellation with less than 20 satellites has a poor GDOP, hence not useful for a global positioning system. A constellation with 26 satellites was found whose GDOP value is lower than a constellation with 27, 28, and 29 satellites. Thanks to the 2D LFC theory it is possible to include eccentric orbits in the search space. We found explicit examples where eccentric orbits outperform circular ones. Finally, our results are compared with the existing Galileo Constellation. Both of them have good qualities, but our 26 satellite constellation seems to be better in all our tests.

As a future work, the study of the GDOP of a constellation can be expanded with the Necklace Flower Constellation theory,[15] which decreases the cost of the mission by reducing the number of satellites in each orbit while keeping the symmetries in the (Ω, M)-space,[13,14] that describes the distribution of the satellites in the 2D Lattice FCs.

Acknowledgments

D. Casanova acknowledges financial support from the Spanish Ministry of Science through Project #AYA2008-05572.

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