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INTERSECTION THEORY ON ABELIAN-QUOTIENT V-SURFACES AND Q-RESOLUTIONS

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Abstract. In this paper we study the intersection theory on surfaces with abelian quotient singularities and we obtain formulas for its behavior under weighted blow-ups. As applications, we extend Mumford's formulas for the intersection theory on normal divisors, we derive properties for quotients of weighted projective planes, and finally, we compute abstract **Q**-resolutions of normal surfaces using Jung's method.

Introduction

In [6], Fulton developed a general intersection theory for algebraic varieties. For the case of normal surfaces, Mumford [10] provided a more detailed description using resolution of singularities. In this work we focus on V-surfaces with abelian quotient singularities (cyclic V-surfaces for short). One of the goals is to prove that the resolution of singularities is not needed for the description of the intersection theory of these V-surfaces and in fact it can be realized following the same ideas as in the smooth case, see Definition 3.1. In the latter case, the description is based on the identification of Weil and Cartier divisors. This identification is no longer true for V-surfaces, but it becomes true using $\mathbb{Q}$-divisors. The relationship between intersection theory and Weil divisors for normal surface singularities was already studied by F. Sakai [12].

Mumford's definition is based on the formulas which relate intersection numbers before and after a blow-up, namely, the self-intersection of the exceptional component and the relationship between the intersection number of the divisors and the one of their strict transforms. The main result in this paper, Theorem 4.3, generalizes these formulas replacing smooth surfaces by cyclic V-surfaces and the standard blow-ups by *weighted blow-ups*, i.e., the result of blowing up the weight filtration for an isolated singularity with good $\mathbb{C}^*$-action. These spaces (even in higher dimension) are V-manifolds with abelian quotient singularities, see [4] (or [1] for a description closer to the language of this work).

We derive several applications of Theorem 4.3, see [2] for further applications. The first one is to provide formulas for the intersection theory of normal surfaces in terms of **Q**-resolutions instead of standard resolutions, see §2 for definitions and Theorem 4.5 for an explicit statement. These **Q**-resolutions are combinatorially less involved than standard resolutions while containing the same information; e.g., Veys [15] used them to simplify the computation of the topological zeta function.

The second one is the description of the intersection theory of the most well-known V-surfaces, namely the weighted projective planes, and more generally, their cyclic quotients, with explicit formulas, see §5.

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We finish this work with the application of cyclic quotient singularities, **Q**-resolutions and $\mathbb{Q}$-intersection theory to the implementation of Jung resolution method of normal singularities. The idea of this method is the following. Consider a finite projection π of a normal singularity S onto $(\mathbb{C}^2, 0)$ with discriminant Δ and perform an embedded resolution σ of Δ; the normalization of the pull-back of σ and π happens to be a **Q**-resolution of S; the resolution of its singularities produces a resolution of S. There are two main advantages in using the methods developed in this work with Jung resolution. For the first advantage, one can replace σ by an embedded **Q**-resolution of Δ (reducing computation time). For the second one, intersection theory gives a straightforward way to obtain the final resolution of S with the self-intersections of the exceptional divisors.

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1. V-Manifolds and Quotient Singularities

A *V-manifold* [13] (or orbifold) of dimension n is a complex analytic space which admits an open covering $\{U_i\}$ such that U_i is analytically isomorphic to B_i/G_i where $B_i \subset \mathbb{C}^n$ is an open ball and G_i is a finite subgroup of $GL(n, \mathbb{C})$. They have been classified locally by Prill [11]: it is enough to consider the so-called *small subgroups* $G \subset \mathrm{GL}(n, \mathbb{C})$, i.e., without rotations around hyperplanes other than the identity. We fix the notations when G is abelian.

For $\mathbf{d} := {}^t(d_1 \ldots d_r)$ we denote $\mu_{\mathbf{d}} := \mu_{d_1} \times \cdots \times \mu_{d_r}$ a finite abelian group written as a product of finite cyclic groups, that is, μ_{d_i} is the cyclic group of d_i-th roots of unity in $\mathbb{C}$. Consider a matrix of weight vectors

$$A := (a_{ij})_{i,j} = [\mathbf{a}_1 \,|\, \cdots \,|\, \mathbf{a}_n] \in \mathrm{Mat}(r \times n, \mathbb{Z}), \quad \mathbf{a}_j := {}^t(a_{1j} \ldots a_{rj}) \in \mathrm{Mat}(r \times 1, \mathbb{Z}),$$

and the action

$$\begin{array}{rr} (\mu_{d_1} \times \cdots \times \mu_{d_r}) \times \mathbb{C}^n \longrightarrow \mathbb{C}^n, & \boldsymbol{\xi}_{\mathbf{d}} := (\xi_{d_1}, \ldots, \xi_{d_r}), \\ \big(\boldsymbol{\xi}_{\mathbf{d}}, \mathbf{x}\big) \mapsto (\xi_{d_1}^{a_{11}} \cdot \ldots \cdot \xi_{d_r}^{a_{r1}} \, x_1, \ldots, \xi_{d_1}^{a_{1n}} \cdot \ldots \cdot \xi_{d_r}^{a_{rn}} \, x_n), & \mathbf{x} := (x_1, \ldots, x_n). \end{array} \tag{1}$$

Note that the i-th row of the matrix A can be considered modulo d_i. The set of all orbits $\mathbb{C}^n/G$ is called (*cyclic*) *quotient space of type* $(\mathbf{d}; A)$ and it is denoted by

$$X(\mathbf{d}; A) := X \left(\begin{array}{c|ccc} d_1 & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ d_r & a_{r1} & \cdots & a_{rn} \end{array} \right).$$

The orbit of an element $\mathbf{x} \in \mathbb{C}^n$ under this action is denoted by $[\mathbf{x}]_{(\mathbf{d};A)}$ and the subindex is omitted if no ambiguity seems likely to arise. Using multi-index notation the action takes the simple form

$$\begin{array}{ccc} \mu_{\mathbf{d}} \times \mathbb{C}^n & \longrightarrow & \mathbb{C}^n, \\ (\boldsymbol{\xi}_{\mathbf{d}}, \mathbf{x}) & \mapsto & \boldsymbol{\xi}_{\mathbf{d}} \cdot \mathbf{x} := (\boldsymbol{\xi}_{\mathbf{d}}^{\mathbf{a}_1} \, x_1, \ldots, \boldsymbol{\xi}_{\mathbf{d}}^{\mathbf{a}_n} \, x_n). \end{array}$$

The quotient of $\mathbb{C}^n$ by a finite abelian group is always isomorphic to a quotient space of type $(\mathbf{d}; A)$ but different types $(\mathbf{d}; A)$ can give rise to isomorphic quotient spaces. Using [1, Lemma 1.8] we can prove the following lemma which restricts the number of possible factors of the abelian group in terms of the dimension.

Lemma 1.1. *The space $X(\mathbf{d}; A) = \mathbb{C}^n/\mu_d$ can always be represented by an upper triangular matrix of dimension $(n-1) \times n$. More precisely, there exist a vector $\mathbf{e} = (e_1, \ldots, e_{n-1})$, a matrix*

$B=(b_{i,j})_{i,j}$, *and an isomorphism* $[(x_1,\ldots,x_n)]\mapsto[(x_1,\ldots,x_n^k)]$ *for some* $k\in\mathbb{N}$ *such that*

$$X(\mathbf{d};A)\cong X\left(\begin{array}{c|cccc} e_1 & b_{1,1} & \cdots & b_{1,n-1} & b_{1,n}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ e_{n-1} & 0 & \cdots & b_{n-1,n-1} & b_{n-1,n}\end{array}\right)=X(\mathbf{e};B).$$

Remark 1.2. For $n=2$ it is enough to consider cyclic quotients. Nevertheless, in order to avoid cumbersome statements, we will allow if necessary quotients of non-cyclic groups.

We say that a type $(\mathbf{d};A)$ is *normalized* if the action is free on $(\mathbb{C}^*)^n$ and $\mu_{\mathbf{d}}$ is small as subgroup of $GL(n,\mathbb{C})$, i.e., if the stabilizer subgroup of P is trivial for all $P\in\mathbb{C}^n$ with exactly $n-1$ coordinates different from zero. If $n=2$, then a normalized type is always cyclic.

In the cyclic case the stabilizer of a point as above (with exactly $n-1$ coordinates different from zero) has order $\gcd(d,a_1,\ldots,\widehat{a}_i,\ldots,a_n)$.

Definition 1.3. The *index* of a quotient $X(\mathbf{d};A)$ of $\mathbb{C}^2$ equals d for $X(\mathbf{d};A)\cong X(d;a,b)$ normalized.

Example 1.4. Following Lemma 1.1, all quotient spaces for $n=2$ are cyclic. The space $X(d;a,b)$ is written in a normalized form if and only if $\gcd(d,a)=\gcd(d,b)=1$. If this is not the case, one uses the isomorphism (assuming $\gcd(d,a,b)=1$)

$$\begin{array}{ccc} X(d;a,b) & \longrightarrow & X\left(\frac{d}{(d,a)(d,b)};\frac{a}{(d,a)},\frac{b}{(d,b)}\right),\\ \big[(x,y)\big] & \mapsto & \big[(x^{(d,b)},y^{(d,a)})\big]\end{array}$$

to convert it into a normalized one where (u,v) stands for $\gcd(u,v)$.

Weighted projective spaces are canonical examples of compact V-manifolds, see [3]. Let $\omega:=(q_0,\ldots,q_n)$ be a weight vector, that is, a finite set of coprime positive integers. There is a natural action of the multiplicative group $\mathbb{C}^*$ on $\mathbb{C}^{n+1}\setminus\{0\}$ given by

$$(x_0,\ldots,x_n)\longmapsto(t^{q_0}x_0,\ldots,t^{q_n}x_n).$$

The set of orbits $\frac{\mathbb{C}^{n+1}\setminus\{0\}}{\mathbb{C}^*}$ under this action is denoted by $\mathbb{P}^n_\omega$ (or $\mathbb{P}^n(\omega)$ in case of complicated weight vectors) and it is called the *weighted projective space* of type ω. The class of a nonzero element $(x_0,\ldots,x_n)\in\mathbb{C}^{n+1}$ is denoted by $[x_0:\ldots:x_n]_\omega$ and the weight vector is omitted if no ambiguity seems likely to arise.

Consider the decomposition $\mathbb{P}^n_\omega=U_0\cup\cdots\cup U_n$, where U_i is the open set consisting of all elements $[x_0:\ldots:x_n]_\omega$ with $x_i\neq 0$. The map

$$\widetilde{\psi}_0:\mathbb{C}^n\longrightarrow U_0,\quad \widetilde{\psi}_0(x_1,\cdots,x_n):=[1:x_1:\ldots:x_n]_\omega$$

defines an isomorphism ψ_0 if we replace $\mathbb{C}^n$ by $X(q_0;\,q_1,\ldots,q_n)$. Analogously,

$$X(q_i;\,q_0,\ldots,\widehat{q}_i,\ldots,q_n)\cong U_i$$

under the obvious analytic map.

Weights can be normalized as follows. Let $d_i:=\gcd(q_0,\ldots,\widehat{q}_i,\ldots,q_n)$ and denote

$$e_i:=d_0\cdot\ldots\cdot\widehat{d_i}\cdot\ldots\cdot d_n$$

and $p_i:=\frac{q_i}{e_i}$. The following map is an isomorphism:

$$\begin{array}{ccc}\mathbb{P}^n\big(q_0,\ldots,q_n\big) & \longrightarrow & \mathbb{P}^n(p_0,\ldots,p_n)\\ [x_0:\ldots:x_n] & \mapsto & \big[\,x_0^{d_0}:\ldots:x_n^{d_n}\,\big].\end{array}$$

Remark 1.5. One can always assume the weight vector is normalized, i.e., it satisfies

$$\gcd(q_0,\ldots,\widehat{q}_i,\ldots,q_n)=1,$$

for $i=0,\ldots,n$. In particular, $\mathbb{P}^1(q_0,q_1)\cong\mathbb{P}^1$ and for $n=2$ we can take (q_0,q_1,q_2) relatively prime numbers.

2. Abstract and Embedded **Q**-Resolutions

An embedded resolution of $\{f=0\}\subset\mathbb{C}^n$ is a proper map $\pi:X\to(\mathbb{C}^n,0)$ from a smooth variety X satisfying, among other conditions, that $\pi^{-1}(\{f=0\})$ is a normal crossing divisor. To weaken the condition on the preimage of the singularity we allow the new ambient space X to contain abelian quotient singularities and the divisor $\pi^{-1}(\{f=0\})$ to have *normal crossings* over this kind of varieties. This notion of normal crossing divisor on V-manifolds was first introduced by Steenbrink in [14].

Let $M=\mathbb{C}^{n+1}/\mu_{\mathbf{d}}$ be an abelian quotient space not necessarily cyclic or written in normalized form. Consider $H\subset M$ an analytic subvariety of codimension one.

Definition 2.1. An *embedded* ***Q****-resolution* of $(H,0)\subset(M,0)$ is a proper analytic map

$$\pi:X\to(M,0)$$

such that:

(1) X is a V-manifold with abelian quotient singularities.
(2) π is an isomorphism over $X\setminus\pi^{-1}(\mathrm{Sing}(H))$.
(3) $\pi^{-1}(H)$ is a $\mathbb{Q}$-normal crossing hypersurface on X (i.e., it has only normal crossings in the sense of Steenbrink).

In the same way we define abstract **Q**-resolutions.

Definition 2.2. Let $(X,0)$ be a germ of singular point. An *abstract good* ***Q****-resolution* is a proper birational morphism $\pi:\hat{X}\to(X,0)$ such that $\hat{X}$ is a V-manifold with abelian quotient singularities, π is an isomorphism outside $\mathrm{Sing}(X)$, and $\pi^{-1}(\mathrm{Sing}(X))$ is a $\mathbb{Q}$-normal crossing divisor.

Note that one can pass from a **Q**-resolution to a standard resolution by solving the abelian quotient singularities. These singularities were solved by Fujiki [5]; the solution of the surface was obtained much earlier and it is known as the Jung-Hirzebruch method, see [7] for an explicit description.

In the surface case, an embedded **Q**-resolution is obtained as a composition of the so-called weighted blow-ups. Let us describe this classical notion in terms of charts.

2.3. Blow-up of $X(d;a,b)$ with respect to $\omega:=(p,q)$. Let $X=X(d;a,b)$ assumed to be normalized. Let

$$\pi:=\pi_{(d;a,b),\omega}:\widehat{X(d;a,b)}_\omega\longrightarrow X(d;a,b)$$

be the weighted blow-up at the origin of $X(d;a,b)$ with respect to $\omega=(p,q)$, i.e.,

$$\widehat{X(d;a,b)}_\omega:=\{((x,y),[u:v]_\omega)\in\mathbb{C}^2\times\mathbb{P}^1_\omega\mid x^qv^p=y^pu^q\}\Big/\mu_d$$

where the action of μ_d is the natural extension of the one defining $X(d;a,b)$ and π is induced by the first projection. Then, $\widehat{X(d;a,b)}_\omega$ is covered by

$$\widehat{U}_1\cup\widehat{U}_2=X\left(\begin{array}{c|cc} p & -1 & q\\ pd & a & pb-qa\end{array}\right)\cup X\left(\begin{array}{c|cc} q & p & -1\\ qd & qa-pb & b\end{array}\right)$$

and the charts are given by

$$\begin{array}{c|lcl}
1^{\text{st}} \text{ chart} & X\left(\begin{smallmatrix} p \\ pd \end{smallmatrix}\,\middle|\,\begin{smallmatrix} -1 & q \\ a & pb-qa \end{smallmatrix}\right) \longrightarrow \widehat{U}_1 & [(x,y)] & \mapsto\ [((x^p, x^q y), [1:y]_\omega)]_{(d;a,b)}\,. \\
2^{\text{nd}} \text{ chart} & X\left(\begin{smallmatrix} q \\ qd \end{smallmatrix}\,\middle|\,\begin{smallmatrix} p & -1 \\ qa-pb & b \end{smallmatrix}\right) \longrightarrow \widehat{U}_2 & [(x,y)] & \mapsto\ [((xy^p, y^q), [x:1]_\omega)]_{(d;a,b)}\,.
\end{array}$$

The exceptional divisor $E = \pi^{-1}_{(d;a,b),\omega}(0)$ is identified with the quotient space $\mathbb{P}^1_\omega(d;a,b) := \mathbb{P}^1_\omega/\mu_d$ which is isomorphic to $\mathbb{P}^1$ under the map

$$\begin{array}{ccc}
\mathbb{P}^1_\omega(d;a,b) & \longrightarrow & \mathbb{P}^1, \\
{[x:y]_{\omega,(d;a,b)}} & \mapsto & [x^{dq/e} : y^{dp/e}],
\end{array}$$

where $e := \gcd(d, pb-qa)$.

Remark 2.4. Let us show how to convert a space of type $\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\middle|\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ into its cyclic form. By suitable multiplications of the rows, we can assume $p = q = r$: $X\left(\begin{smallmatrix} r \\ r \end{smallmatrix}\middle|\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. For the second step we add a third row by adding the first row multiplied by α and the second row multiplied by β, where $\alpha a + \beta c = m$ and $m := \gcd(a,c)$ (note that $\gcd(\alpha,\beta) = 1$):

$$X\left(\begin{array}{c|cc} r & a & b \\ r & c & d \\ r & m & \alpha b + \beta d \end{array}\right) = X\left(\begin{array}{c|cc} r & 0 & -\beta\frac{ad-bc}{m} \\ r & 0 & \alpha\frac{ad-bc}{m} \\ r & m & \alpha b + \beta d \end{array}\right) = X\left(\begin{array}{c|cc} r & 0 & \frac{ad-bc}{m} \\ r & m & \alpha b + \beta d \end{array}\right).$$

Let $t := \gcd(r, \frac{ad-bc}{m})$. Then, our space is of type $(r; m, (\alpha b + \beta d)\frac{r}{t})$ and normalization follows by taking gcd's. The isomorphism is $[(x,y)] \mapsto [(x, y^{\frac{r}{t}})]_{(r;m,(\alpha b+\beta d)\frac{r}{t})}$.

Let us apply the previous remark to the preceding charts. Assume the type $(d;a,b)$ is normalized. To normalize these quotient spaces, note that

$$e = \gcd(d, pb-qa) = \gcd(d, -q+\beta pb) = \gcd(pd, -q+\beta pb) = \gcd(qd, p-qa\mu),$$

where $\beta a \equiv \mu b \equiv 1 \mod d$. Then another expressions for the two charts are given below.

$$\begin{array}{c|lcl}
1^{\text{st}} \text{ chart} & X\left(\frac{pd}{e}; 1, \frac{-q+\beta pb}{e}\right) \longrightarrow \widehat{U}_1 & [(x^e,y)] & \mapsto\ [((x^p, x^q y), [1:y]_\omega)]_{(d;a,b)}\,. \\
2^{\text{nd}} \text{ chart} & X\left(\frac{qd}{e}; \frac{-p+\mu qa}{e}, 1\right) \longrightarrow \widehat{U}_2 & [(x,y^e)] & \mapsto\ [((xy^p, y^q), [x:1]_\omega)]_{(d;a,b)}\,.
\end{array}$$

Both quotient spaces are now written in their normalized form. The equation of the charts will be useful to compute multiplicities, see Remark 4.4.

For an irreducible germ of function in $(\mathbb{C}^2, 0)$, only a weighted blow-up is needed for each Puiseux pair in order to compute an embedded **Q**-resolution, and the weight is determined by the Puiseux pairs. In the reducible case, one has to consider the weighted blow-ups associated with the Puiseux pairs of each irreducible component and add also weighted blow-ups associated with the contact exponents for each pair of branches. There is another longer way to get this **Q**-resolution: perform a standard embedded resolution and contract any exceptional component having at most two singular points in the divisor, cf. [15].

3. Rational Intersection Number on V-Surfaces

Rational intersection multiplicity was first introduced by Mumford for normal surfaces, see [10, Pag. 17]. A general intersection theory is developed in [6]. In this section we give an alternative description of Mumford's definition restricted to V-surfaces. Moreover the use of **Q**-resolutions allows us to give an alternative description of Mumford's definition for normal surfaces which does not involve a resolution. In particular self-intersection numbers of the exceptional divisors of weighted blow-ups can be computed directly, see Theorem 4.3.

In the smooth case it is possible to define the intersection number $W \cdot D$ for divisors W, D provided $W \cap D$ is finite or W is compact; this definition can be extended to the singular case when W is a Weil divisor and D is a Cartier divisor. It is well known that V-manifolds are $\mathbb{Q}$-factorial, i.e., rational Cartier and Weil divisors coincide. Hence a $\mathbb{Q}$-divisor refer to both notions and the corresponding vector space is denoted by $\mathbb{Q}\text{-Div}(X)$.

Definition 3.1. Let X be a V-manifold of dimension 2 and consider $D_1, D_2 \in \mathbb{Q}\text{-Div}(X)$. The *intersection number* is defined as

$$D_1 \cdot D_2 := \frac{1}{k_1 k_2}(k_1 D_1 \cdot k_2 D_2) \in \mathbb{Q},$$

where $k_1, k_2 \in \mathbb{Z}$ are chosen so that $k_1 D_1$ is Weil, $k_2 D_2$ is Cartier and either the divisor D_1 is compact or $D_1 \cap D_2$ is finite [6, Ch. 2].

Analogously, it is defined the *local intersection number* at $P \in D_1 \cap D_2$, if the condition $D_1 \not\subseteq D_2$ is satisfied.

For later use we make explicit some properties of this intersection multiplicity. Their proofs are omitted since they are well known for the classical case (i.e., without tensorizing with $\mathbb{Q}$), cf. [6], and our generalization is based on extending the classical definition to rational coefficients.

Theorem 3.2. *Let $F : Y \to X$ be a proper morphism between two irreducible V-surfaces, and $D_1, D_2 \in \mathbb{Q}\text{-Div}(X)$.*

(1) *The cardinal of $F^{-1}(P)$, $P \in X$ being generic, is a finite constant. This number is denoted by* $\deg(F)$.
(2) *If $D_1 \cdot D_2$ is defined, then so is the number $F^*(D_1) \cdot F^*(D_2)$. In such a case*
$$F^*(D_1) \cdot F^*(D_2) = \deg(F)\,(D_1 \cdot D_2).$$
(3) *If $(D_1 \cdot D_2)_P$ is defined for some $P \in X$, then so is $(F^*(D_1) \cdot F^*(D_2))_Q$, $\forall Q \in F^{-1}(P)$, and $\sum_{Q \in F^{-1}(P)} (F^*(D_1) \cdot F^*(D_2))_Q = \deg(F)(D_1 \cdot D_2)_P$.*

4. Intersection Numbers and Weighted Blow-ups

Previously weighted blow-ups were introduced as a tool for computing embedded **Q**-resolutions. To obtain information about the corresponding embedded singularity, an intersection theory on V-manifolds has been developed. Here we calculate self-intersection numbers of exceptional divisors of weighted blow-ups on analytic varieties with abelian quotient singularities, see Theorem 4.3.

The first step in this computation is to explicitly write the exceptional divisor as a rational Cartier divisor applying the procedure described in [1, §4.3].

Let X be a surface with abelian quotient singularities. Let $\pi : \widehat{X} \to X$ be the weighted blow-up at a point of type $(d; a, b)$ with respect to $\omega = (p, q)$. In general, the exceptional divisor $E := \pi^{-1}(0) \cong \mathbb{P}^1_\omega(d; a, b)$ is a Weil divisor on $\widehat{X}$ which does not correspond to a Cartier divisor. Let us write E as an element in $\text{CaDiv}(\widehat{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

As in 2.3, assume $\pi := \pi_{(d;a,b),\omega} : \widehat{X(d; a, b)}_\omega \to X(d; a, b)$. Assume also that $\gcd(p, q) = 1$ and $(d; a, b)$ is normalized. Using the notation introduced in 2.3, the space $\widehat{X}$ is covered by $\widehat{U}_1 \cup \widehat{U}_2$ and the first chart is given by

$$\begin{array}{rccc} Q_1 := X\left(\frac{pd}{e}; 1, \frac{-q+\beta p b}{e}\right) & \longrightarrow & \widehat{U}_1, \\ \left[(x^e, y)\right] & \mapsto & \left[((x^p, x^q y), [1 : y]_\omega)\right]_{(d;a,b)}, \end{array} \tag{2}$$

where $e := \gcd(d, pb - qa)$.

In the first chart, E is the Weil divisor $\{x=0\} \subset Q_1$. Note that the type representing the space Q_1 is in a normalized form and hence the corresponding subgroup of $GL(2,\mathbb{C})$ is small.

The divisor $\{x=0\} \subset Q_1$ is written as an element in $\mathrm{CaDiv}(Q_1) \otimes_{\mathbb{Z}} \mathbb{Q}$ like $\frac{e}{pd}\{(Q_1, x^{\frac{pd}{e}})\}$, which is mapped to $\frac{e}{pd}\{(\widehat{U}_1, x^d)\} \in \mathrm{CaDiv}(\widehat{U}_1) \otimes_{\mathbb{Z}} \mathbb{Q}$ under the isomorphism (2).

Analogously E in the second chart is $\frac{e}{qd}\{(\widehat{U}_2, y^d)\}$. Finally one writes the exceptional divisor of π as claimed,

$$E = \frac{e}{dp}\big\{(\widehat{U}_1, x^d), (\widehat{U}_2, 1)\big\} + \frac{e}{dq}\big\{(\widehat{U}_1, 1), (\widehat{U}_2, y^d)\big\} = \frac{e}{dpq}\big\{(\widehat{U}_1, x^{dq}), (\widehat{U}_2, y^{dp})\big\}. \tag{3}$$

We state some preliminary lemmas separately so that the proof of the main result of this section becomes simpler.

Lemma 4.1. *Let X be an analytic surface with abelian quotient singularities and let $\pi : \widehat{X} \to X$ be a weighted blow-up at a point $P \in X$. Let C be a $\mathbb{Q}$-divisor on X and E the exceptional divisor of π. Then, $E \cdot \pi^*(C) = 0$.*

Proof. The proof uses the same ideas as in the smooth case. After multiplying by an integer we can assume that C is a Cartier divisor. In a neighborhood of P its associated line bundle is trivial, and hence it is also the case for the associated line bundle of $\pi^*(C)$ in a neighborhood of E, and hence the result follows. □

Lemma 4.2. *Let X be a V-surface locally irreducible at $P \in X$, and a $\mathbb{Q}$-divisor C_X. Consider a weighted blow-up $\pi_X : \widehat{X} \to X$ at P. Denote by E_X the exceptional divisor of π_X, and $\widehat{C}_X$ the strict transform of C_X.*

Let Y be another V-surface locally irreducible at $Q \in Y$ and a proper morphism $h : Y \to X$ such that $h^{-1}(P) = Q$ and the map π_Y in the diagram

$$\begin{tikzcd} \widehat{Y} \arrow[r, "H"] \arrow[d, "\pi_Y"'] \arrow[dr, phantom, "\#"] & \widehat{X} \arrow[d, "\pi_X"] \\ Y \arrow[r, "h"'] & X \end{tikzcd}$$

is a weighted blow-up at Q; the exceptional divisor of π_Y is denoted by E_Y. Let us suppose that there exist two rational numbers, e and ν such that

(a) $H^*(E_X) = eE_Y$, (b) $\pi_Y^*(h^*(C_X)) = H^*(\widehat{C}_X) + \nu E_Y$.

Then the following equalities hold:

(1) $\pi_X^*(C_X) = \widehat{C}_X + \frac{\nu}{e} E_X$, (2) $E_X \cdot \widehat{C}_X = \frac{-e\,\nu}{\deg(h)} E_Y^2$, (3) $E_X^2 = \frac{e^2}{\deg(h)} E_Y^2$.

Proof. For (1) note the total transform $\pi_X^*(C_X)$ can always be written as $\widehat{C}_X + mE_X$ for some $m \in \mathbb{Q}$. Considering its pull-back under H^* one obtains two expressions for the same $\mathbb{Q}$-divisor on $\widehat{Y}$,

$$H^*(\pi_X^*(C_X)) \overset{\text{diagram}}{=} \pi_Y^*(h^*(C_X)) \overset{\text{(b)}}{=} H^*(\widehat{C}_X) + \nu E_Y,$$
$$H^*(\widehat{C}_X + mE_X) = H^*(\widehat{C}_X) + mH^*(E_X) \overset{\text{(a)}}{=} H^*(\widehat{C}_X) + meE_Y.$$

It follows that $m = \frac{\nu}{e}$.

For (2) first note that $\deg(H)=\deg(h)$. From Lemma 4.1, one has that $E_Y\cdot\pi_Y^*(h^*(C_X))=0$. On the other hand, H being proper, Theorem 3.2(2) can be applied thus obtaining

$$\deg(h)(E_X\cdot\widehat{C}_X)=H^*(E_X)\cdot H^*(\widehat{C}_X)\overset{\text{(a)-(b)}}{=}eE_Y\cdot\left[\pi_Y^*(h^*(C_X))-\nu E_Y\right]=-e\nu E_Y^2.$$

Analogously $\deg(h)E_X^2=H^*(E_X)^2=e^2E_Y^2$ and (3) follows. ☐

Now we are ready to present the main result of this section.

Theorem 4.3. *Let X be an analytic surface with abelian quotient singularities and let $\pi:\widehat{X}\to X$ be the (p,q)-weighted blow-up at a point $P\in X$ of type $(d;a,b)$. Assume $\gcd(p,q)=1$ and $(d;a,b)$ is normalized, i.e., $\gcd(d,a)=\gcd(d,b)=1$. Also write $e=\gcd(d,pb-qa)$.*

Consider two $\mathbb{Q}$-divisors C and D on X. As usual, denote by E the exceptional divisor of π, and by $\widehat{C}$ (resp. $\widehat{D}$) the strict transform of C (resp. D). Let ν and μ be the (p,q)-multiplicities of C and D at P, i.e., x (resp. y) has (p,q)-multiplicity p (resp. q). Then there are the following equalities:

(1) $\pi^*(C)=\widehat{C}+\dfrac{\nu}{e}E$.

(2) $E\cdot\widehat{C}=\dfrac{e\nu}{dpq}$.

(3) $E^2=-\dfrac{e^2}{dpq}$.

(4) $\widehat{C}\cdot\widehat{D}=C\cdot D-\dfrac{\nu\mu}{dpq}$.

In addition, if D has compact support then $\widehat{D}^2=D^2-\dfrac{\mu^2}{dpq}$.

Proof. The item (4), and final conclusion, are an easy consequence of (1)-(3) and the fact that $\pi^*(C)\cdot\pi^*(D)=C\cdot D$.

For the rest of the proof, one assumes that $\pi:=\pi_X:\widehat{X(d;a,b)}_\omega\longrightarrow X(d;a,b)$ is the weighted blow-up at the origin of $X(d;a,b)$ with respect to $\omega=(p,q)$. Now the idea is to apply Lemma 4.2 to the commutative diagram

$$\begin{tikzcd}
\widehat{Y}:=\widehat{\mathbb{C}}^2 \arrow[r, "H"] \arrow[d, "\pi_Y"'] \arrow[dr, phantom, "\#"] & \widehat{X(d;a,b)}_\omega=:\widehat{X} \arrow[d, "\pi_X"] \\
Y:=\mathbb{C}^2 \arrow[r, "h"'] & X(d;a,b)=:X
\end{tikzcd}$$

where H and h are the morphisms defined by

$$\begin{array}{ccc}
((x,y),[u:v]) & \overset{H}{\longmapsto} & [((x^p,y^q),[u^p:v^q])_\omega]_{(d;a,b)};\\
(x,y) & \overset{h}{\longmapsto} & [(x^p,y^q)]_{(d;a,b)},
\end{array}$$

and π_Y is the classical blowing-up at the origin. In this situation $E_Y^2=-1$. The claim is reduced to the calculation of $\deg(h)$ and the verification of the conditions (a)-(b) of Lemma 4.2.

The degree is $\deg(h)=pq\cdot\deg\left[\mathrm{pr}:\mathbb{C}^2\to X(d;a,b)\right]=dpq$. For (a), first recall the decompositions

$$\widehat{X(d;a,b)}_\omega=\widehat{U}_1\cup\widehat{U}_2,\qquad \widehat{\mathbb{C}}^2=U_1\cup U_2. \tag{4}$$

One has already written in (3) the exceptional divisor of π_X as

$$E_X=\frac{e}{dpq}\left\{(\widehat{U}_1,x^{dq}),(\widehat{U}_2,y^{dp})\right\}.$$

Hence its pull-back under H, computed by pulling back the local equations, is

$$H^*(E_X) = \frac{e}{dpq}\Big\{(U_1, x^{dpq}), (U_2, y^{dpq})\Big\} = e\Big\{(U_1, x), (U_2, y)\Big\} = eE_Y.$$

Finally for (b) one uses local equations to check $\pi_Y^*(h^*(C)) = H^*(\widehat{C}) + \nu E_Y$. Suppose the divisor C is locally given by a meromorphic function $f(x,y)$ defined on a neighborhood of the origin of $X(d;a,b)$; note that $\nu = \operatorname{ord}_{(p,q)}(f)$. The charts associated with the decompositions (4) are described in detail in 2.3. As a summary we recall here the first chart of each blowing-up:

$$\begin{array}{lrcll} \pi_X: & Q_1 := X\left(\begin{smallmatrix} p \\ pd \end{smallmatrix}\middle|\begin{smallmatrix} -1 & q \\ a & pb-qa \end{smallmatrix}\right) & \longrightarrow & \widehat{U}_1, & \big[(x,y)\big] \mapsto \big[((x^p, x^q y), [1:y]_\omega)\big]. \\ \pi_Y: & \mathbb{C}^2 & \longrightarrow & U_1, & (x,y) \mapsto ((x,xy),[1:y]). \end{array}$$

Note that H respects the decompositions and takes the form $(x,y) \mapsto [(x, y^q)]$ in the first chart. Then one has the following local equations for the divisors involved:

Divisor	**Equation**	**Ambient space**
$h^*(C)$	$f(x^p, y^q) = 0$	$\mathbb{C}^2$
$\pi_Y^*(h^*(C))$	$f(x^p, x^q y^q) = 0$	$\mathbb{C}^2 \cong U_1$
$\widehat{C}$	$\dfrac{f(x^p, x^q y)}{x^\nu} = 0$	$Q_1 \cong \widehat{U}_1$
$H^*(\widehat{C})$	$\dfrac{f(x^p, x^q y^q)}{x^\nu} = 0$	$\mathbb{C}^2 \cong U_1$
E_Y	$x = 0$	$\mathbb{C}^2 \cong U_1$

From these local equations (b) is satisfied and now the proof is complete. □

Remark 4.4. In order to compute multiplicities when looking at *multicharts* (for quotient spaces) we must be careful with the expressions in coordinates in case the space is represented by a non-normalized type. For instance, if a divisor is locally given by the function $x^{md} : X\left(\begin{smallmatrix} p \\ d \end{smallmatrix}\middle|\begin{smallmatrix} -1 & q \\ 1 & 0 \end{smallmatrix}\right) \to \mathbb{C}$, its multiplicity is m.

For a sequence of weighted blow-ups we can adapt Mumford's approach [10]. Let us fix $X := X(d;a,b)$ and let us consider $\pi : \hat{X} \to X$ a sequence of weighted blow-ups. Let $\mathcal{E}_1, \ldots, \mathcal{E}_r$ be the set of exceptional components in $\hat{X}$ and let $A := (\mathcal{E}_i \cdot \mathcal{E}_j)_{1 \le i,j \le r}$ be the intersection matrix in $\hat{X}$, which is a negative definite matrix with rational coefficients. We restrict X to a small neighborhood of the origin. An $\hat{X}$-curvette γ_i of $\mathcal{E}_i$ is a Weil divisor obtained by considering a disk transversal to a point of $\mathcal{E}_i \setminus \bigcup_{j \neq i} \mathcal{E}_j$ and $\delta_i = \pi(\gamma_i)$ is called an X-curvette of $\mathcal{E}_i$; the index $d(\gamma_i) := d(\delta_i)$ is the order of the cyclic group associated with $\gamma_i \cap \mathcal{E}_i$. We say that (γ_i, γ'_j) form a pair of $\hat{X}$-curvettes for $(\mathcal{E}_i, \mathcal{E}_j)$ if they are disjoint curvettes for each divisor; in that case their images in X form a pair (δ_i, δ'_j) X-curvettes.

Theorem 4.5. *Let $B := -A^{-1} = (b_{ij})_{1 \le i,j \le r}$. Let (δ_i, δ'_j) be a pair of X-curvettes for $(\mathcal{E}_i, \mathcal{E}_j)$. Then, $\delta_i \cdot \delta'_j = \frac{b_{ij}}{d(\delta_i) d(\delta'_j)}$.*

Proof. Let γ'_i be a generic $\hat{X}$-curvette. Since γ'_i and $d(\gamma_i)\gamma_i$ are equivalent Weil divisors, we can assume that $d(\gamma_i) = 1$. We have $\pi^*(\delta_i) = \gamma_i + \sum_{j=1}^n c_{ij} \mathcal{E}_j$. Note that $\gamma_i \cdot \mathcal{E}_j = \delta_{ij}$ (δ_{ij} being the Kronecker delta).

For a generic γ'_j we have $\delta'_j \cdot \delta_i = \pi^*(\delta'_j) \cdot \pi^*(\delta_i) = \gamma'_j \cdot \pi^*(\delta_i) = c_{ij}$. Since

$$\delta_{ik} = \gamma_i \cdot \mathcal{E}_k = \big(\pi^*(\delta_i) - \sum_{j=1}^{n} c_{ij}\mathcal{E}_j\big) \cdot \mathcal{E}_k = -\sum_{j=1}^{n} (\delta_i \cdot \delta'_j)(\mathcal{E}_j \cdot \mathcal{E}_k),$$

we deduce the result. □

Example 4.6. Assume $\gcd(p,q) = \gcd(r,s) = 1$ and $\frac{p}{q} < \frac{r}{s}$. Let $f = (x^p + y^q)(x^r + y^s)$ and consider $\mathbf{C}_1 = \{x^p + y^q = 0\}$ and $\mathbf{C}_2 = \{x^r + y^s = 0\}$. After a sequence of two weighted blow-ups one obtains Figure 1 representing an embedded $\mathbf{Q}$-resolution of $\{f = 0\} \subset \mathbb{C}^2$. We start with a (q,p)-blow-up over a smooth point; the exceptional divisor $\tilde{\mathcal{E}}_1$ has self-intersection $\frac{-1}{pq}$. We continue with an $(s, qr - ps)$-blow-up over a point of type $(q; -1, p)$. We denote by $\mathcal{E}_1$ the strict transform of $\tilde{\mathcal{E}}_1$ and by $\mathcal{E}_2$ the exceptional divisor.

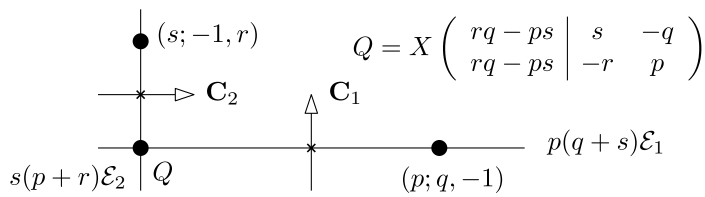

FIGURE 1. Embedded $\mathbf{Q}$-resolution of $\{(x^p + y^q)(x^r + y^s) = 0\} \subset \mathbb{C}^2$.

The point Q is also of type $(rq - ps; ar + bs, -1)$ where $ap + bq = 1$. In fact, it is in normalized form, since $\gcd(rq - ps, ar + bs) = 1$. Since $\tilde{\mathcal{E}}_1$ has multiplicity s the self-intersection of $\mathcal{E}_1$ is $\frac{-1}{pq} - \frac{s}{q(rq-ps)} = -\frac{r}{p(rq-ps)}$. The self-intersection of $\mathcal{E}_2$ is $-\frac{q}{s(rq-ps)}$. The intersection matrix A and its opposite inverse B are

$$A = \frac{1}{rq - ps}\begin{pmatrix} -\frac{r}{p} & 1 \\ 1 & -\frac{q}{s} \end{pmatrix}, \quad B = \begin{pmatrix} pq & ps \\ ps & sr \end{pmatrix}.$$

Example 4.7. Let us consider the following divisors on $\mathbb{C}^2$,

$$C_1 = \{((x^3 - y^2)^2 - x^4 y^3) = 0\}, \quad C_2 = \{x^3 - y^2 = 0\},$$

$$C_3 = \{x^3 + y^2 = 0\}, \quad C_4 = \{x = 0\}, \quad C_5 = \{y = 0\}.$$

We shall see that the local intersection numbers $(C_i \cdot C_j)_0$, $i, j \in \{1, \ldots, 5\}$, $i \neq j$, are encoded in the intersection matrix associated with any embedded $\mathbf{Q}$-resolution of $C = \bigcup_{i=1}^{5} C_i$.

Let $\pi_1 : \mathbb{C}^2_{(2,3)} \to \mathbb{C}^2$ be the $(2,3)$-weighted blow-up at the origin. The new space has two cyclic quotient singular points of type $(2;1,1)$ and $(3;1,1)$ located at the exceptional divisor $\mathcal{E}_1$. The local equation of the total transform in the first chart is given by the function

$$x^{29}\ ((1 - y^2)^2 - x^5 y^3)\ (1 - y^2)\ (1 + y^2)\ y : X(2;1,1) \longrightarrow \mathbb{C},$$

where $x = 0$ is the equation of the exceptional divisor and the other factors correspond in the same order to the strict transform of C_1, C_2, C_3, C_5 (denoted again by the same symbol). To study the strict transform of C_4 one needs the second chart, the details are left to the reader.

Hence $\mathcal{E}_1$ has multiplicity 29 and self-intersection number $-\frac{1}{6}$; the divisor intersects transversally C_3, C_4 and C_5 at three different points, while it intersects C_1 and C_2 at the same smooth point P, different from the other three. The local equation of the divisor $\mathcal{E}_1 \cup C_2 \cup C_1$ at this point P is $x^{29}\, y\, (x^5 - y^2) = 0$, see Figure 2 below.

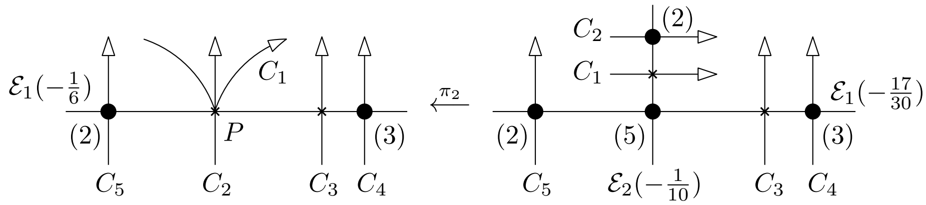

FIGURE 2. Embedded **Q**-resolution of $C = \bigcup_{i=1}^{5} C_i \subset \mathbb{C}^2$.

Let π_2 be the $(2,5)$-weighted blow-up at the point P above. The new ambient space has two singular points of type $(2;1,1)$ and $(5;1,2)$. The local equations of the total transform of $\mathcal{E}_1 \cup C_2 \cup C_1$ are given by the functions in Table 1.

1st chart	2nd chart
$\underbrace{x^{73}}_{\mathcal{E}_2} \cdot \underbrace{y}_{C_2} \cdot \underbrace{(1-y^2)}_{C_1} : X(2;1,1) \longrightarrow \mathbb{C}$	$\underbrace{x^{29}}_{\mathcal{E}_1} \cdot \underbrace{y^{73}}_{\mathcal{E}_2} \cdot \underbrace{(x^5-1)}_{C_1} : X(2;1,1) \longrightarrow \mathbb{C}$

TABLE 1. Equations of the total transform.

Thus the new exceptional divisor $\mathcal{E}_2$ has multiplicity 73 and it intersects transversally the strict transform of C_1, C_2 and $\mathcal{E}_1$. Hence the composition $\pi_2 \circ \pi_1$ is an embedded **Q**-resolution of $C = \bigcup_{i=1}^{5} C_i \subset \mathbb{C}^2$. Figure 2 above illustrates the whole process.

As for the self-intersection numbers, $\mathcal{E}_2^2 = -\frac{1}{10}$ and $\mathcal{E}_1^2 = -\frac{1}{6} - \frac{2^2}{1\cdot 2\cdot 5} = -\frac{17}{30}$. The intersection matrix associated with the embedded **Q**-resolution obtained and its opposite inverse are

$$A = \begin{pmatrix} -17/30 & 1/5 \\ 1/5 & -1/10 \end{pmatrix}, \qquad B = -A^{-1} = \begin{pmatrix} 6 & 12 \\ 12 & 34 \end{pmatrix}.$$

Now one observes that the intersection number is encoded in B as follows. For $i = 1, \ldots, 5$, set $k_i \in \{1, \ldots, 5\}$ such that $\emptyset \neq C_i \cap \mathcal{E}_{k_i} =: \{P_i\}$. Denote by $d(C_i)$ the index of P_i, see Definition 1.3. Then,

$$(C_i \cdot C_j)_0 = \frac{b_{k_i,k_j}}{d(C_i)\, d(C_j)}.$$

One has $(k_1, \ldots, k_5) = (2,2,1,1,1)$ and $(d(C_1), \ldots, d(C_5)) = (1,2,1,3,2)$. Hence, for instance,

$$(C_1 \cdot C_2)_0 = \frac{b_{k_1,k_2}}{d(C_1)\, d(C_2)} = \frac{b_{22}}{1\cdot 2} = \frac{34}{2} = 17,$$

which is indeed the intersection multiplicity at the origin of C_1 and C_2. Analogously for the other indices.

Example 4.8. Consider the group action of type $(5;2,3)$ on $\mathbb{C}^2$. The previous plane curve C is invariant under this action and then it makes sense to compute an embedded **Q**-resolution of $\overline{C} := C/\mu_5 \subset X(5;2,3)$. Similar calculations as in the previous example, lead to a figure as the one obtained above with the following relevant differences:

- $\mathcal{E}_1 \cap \mathcal{E}_2$ is a smooth point.
- $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) has self-intersection number $-\frac{17}{6}$ (resp. $-\frac{1}{2}$).

- The intersection matrix is $A' = \begin{pmatrix} -17/6 & 1 \\ 1 & -1/2 \end{pmatrix}$ and its opposite inverse is

$$B' = -(A')^{-1} = \begin{pmatrix} 6/5 & 12/5 \\ 12/5 & 34/5 \end{pmatrix}.$$

Hence, for instance, $(\overline{C}_1 \cdot \overline{C}_2)_0 = \frac{b'_{22}}{1\cdot 2} = \frac{34/5}{2} = \frac{17}{5}$, which is exactly the intersection number of the two curves, since that local number can be also computed as $(\overline{C}_1 \cdot \overline{C}_2)_0 = \frac{1}{5}(C_1 \cdot C_2)_0$.

5. Applications for Weighted Projective Planes

For a given weight vector $\omega = (p,q,r) \in \mathbb{N}^3$ and an action on $\mathbb{C}^3$ of type $(d;a,b,c)$, consider the quotient weighted projective plane $\mathbb{P}^2_\omega(d;a,b,c) := \mathbb{P}^2_\omega/\mu_d$ and the projection morphism $\tau_{(d;a,b,c),\omega} : \mathbb{P}^2 \to \mathbb{P}^2_\omega(d;a,b,c)$ defined by

$$\tau_{(d;a,b,c),\omega}([x:y:z]) = [x^p : y^q : z^r]_{\omega,d}. \tag{5}$$

The space $\mathbb{P}^2_\omega(d;a,b,c)$ is a variety with abelian quotient singularities. The *degree of a $\mathbb{Q}$-divisor on* $\mathbb{P}^2_\omega(d;a,b,c)$ is the degree of its pull-back under the map $\tau_{(d;a,b,c),\omega}$, that is, by definition,

$$D \in \mathbb{Q}\text{-Div}\left(\mathbb{P}^2_\omega(d;a,b,c)\right), \quad \deg_\omega(D) := \deg\left(\tau^*_{(d;a,b,c),\omega}(D)\right).$$

Thus if $D = \{F=0\}$ is a $\mathbb{Q}$-divisor on $\mathbb{P}^2_\omega(d;a,b,c)$ given by a ω-homogeneous polynomial that indeed defines a zero set on the quotient projective space, then $\deg_\omega(D)$ is the classical degree, denoted by $\deg_\omega(F)$, of the quasi-homogeneous polynomial.

The following result can be stated in a more general setting. However, it is presented in this way to keep the exposition as simple as possible.

Lemma 5.1. *The degree of the projection* $\mathrm{pr} : \mathbb{C}^2 \longrightarrow X\left(\begin{smallmatrix} d; & a & b \\ e; & r & s \end{smallmatrix}\right)$ *is given by the formula*

$$\frac{d \cdot e}{\gcd\left[d \cdot \gcd(e,r,s),\ e \cdot \gcd(d,a,b),\ as - br\right]}.$$

Proof. Assume $\gcd(d,a,b) = \gcd(e,r,s) = 1$; the general formula is obtained easily from this one.

The degree of the required projection $\mathbb{C}^2 \to X\left(\begin{smallmatrix} d; & a & b \\ e; & r & s \end{smallmatrix}\right)$ is $\frac{de}{\ell}$, where ℓ is the order of the abelian group

$$H = \left\{(\xi,\eta) \in \mu_d \times \mu_e \mid \xi^a\eta^r = 1,\ \xi^b\eta^s = 1\right\} \lhd (\mu_d \times \mu_e).$$

To calculate ℓ, consider $(\xi,\eta) \in \mu_d \times \mu_e$ and solve the system $\xi^a\eta^r = 1$, $\xi^b\eta^s = 1$. Raising both equations to the e-th power, one obtains $\xi^{ae} = 1$ and $\xi^{be} = 1$. Hence,

$$\xi \in \mu_d \cap \mu_{ae} \cap \mu_{be} = \mu_{\gcd(d,ae,be)} = \mu_{\gcd(d,e)}.$$

Note that the assumption $\gcd(d,a,b) = 1$ was used in the last equality. Analogously, it follows that $\eta \in \mu_{\gcd(d,e)}$, provided that $\gcd(e,r,s) = 1$.

Thus, there exist $i,j \in \{0,1,\ldots,\gcd(d,e)-1\}$ such that $\xi = \zeta^i$ and $\eta = \zeta^j$, where ζ is a fixed (d,e)-th primitive root of unity. Now the claim is reduced to finding the number of solutions of the system of congruences

$$\left\{\begin{array}{rcl} ai + rj & \equiv & 0 \\ bi + sj & \equiv & 0 \end{array}\right. \pmod{\gcd(d,e)}.$$

This is known to be $\gcd(d,e,as-br)$ and the proof is complete. □

Proposition 5.2. *Using the notation above, let us denote by m_1, m_2, m_3 the determinants of the three minors of order 2 of the matrix $\left(\begin{smallmatrix} p & q & r \\ a & b & c \end{smallmatrix}\right)$. Denote $e := \gcd(d, m_1, m_2, m_3)$.*

Then the intersection number of two $\mathbb{Q}$-divisors on $\mathbb{P}^2_\omega(d; a, b, c)$ is

$$D_1 \cdot D_2 = \frac{e}{dpqr} \deg_\omega(D_1) \deg_\omega(D_2) \in \mathbb{Q}.$$

In particular, the self-intersection number of a $\mathbb{Q}$-divisor is given by $D^2 = \frac{e}{dpqr} \deg_\omega(D)^2$. Moreover, if $|D_1| \nsubseteq |D_2|$, then $|D_1| \cap |D_2|$ is a finite set of points and

$$\frac{e}{dpqr} \deg_\omega(D_1) \deg_\omega(D_2) = \sum_{P \in |D_1| \cap |D_2|} (D_1 \cdot D_2)_P. \tag{6}$$

Proof. For simplicity, let us just write τ for the map defined in (5) omitting the subindex. Note that τ is a proper morphism between two irreducible V-manifolds of dimension 2. Thus by Theorem 3.2(2) and the classical Bézout's theorem on $\mathbb{P}^2$ one has the following sequence of equalities,

$$\deg(\tau)(D_1 \cdot D_2) = \tau^*(D_1) \cdot \tau^*(D_2) = \deg\left(\tau^*(D_1)\right) \deg\left(\tau^*(D_2)\right) = \deg_\omega(D_1) \deg_\omega(D_2).$$

The rest of the proof is the computation of $\deg(\tau)$.

In the first chart τ takes the form $\mathbb{C}^2 \to X\left(\begin{smallmatrix} p & ; & q & r \\ pd & ; & m_1 & m_2 \end{smallmatrix}\right)$, $(y, z) \mapsto [(y^q, z^r)]$. By decomposing this morphism into $\mathbb{C}^2 \to \mathbb{C}^2$, $(y, z) \mapsto (y^q, z^r)$ and the natural projection $\mathbb{C}^2 \to X\left(\begin{smallmatrix} p & ; & q & r \\ pd & ; & m_1 & m_2 \end{smallmatrix}\right)$, $(y, z) \mapsto [(y, z)]$, one obtains

$$\deg(\tau) = qr \cdot \deg\left[\mathbb{C}^2 \xrightarrow{\text{pr}} X\left(\begin{smallmatrix} p & ; & q & r \\ pd & ; & m_1 & m_2 \end{smallmatrix}\right)\right].$$

The determinant of the corresponding matrix is $qm_2 - rm_1 = pm_3$. From Lemma 5.1 the latter degree is

$$\frac{p \cdot pd}{\gcd\left(p \cdot \gcd(pd, m_1, m_2), pd, pm_3\right)} = \frac{dp}{\gcd\left(d, m_1, m_2, m_3\right)},$$

and hence the proof is complete. ∎

Corollary 5.3. *Let X, Y, Z be the Weil divisors on $\mathbb{P}^2_\omega(d; a, b, c)$ given by $\{x = 0\}$, $\{y = 0\}$ and $\{z = 0\}$, respectively. Using the notation of Proposition 5.2 one has:*

(1) $X^2 = \dfrac{ep}{dqr}$, $\quad Y^2 = \dfrac{eq}{dpr}$, $\quad Z^2 = \dfrac{er}{dpq}$.

(2) $X \cdot Y = \dfrac{e}{dr}$, $\quad X \cdot Z = \dfrac{e}{dq}$, $\quad Y \cdot Z = \dfrac{e}{dp}$. ∎

Remark 5.4. If $d = 1$ then $e = 1$ too and the formulas above become a bit simpler. In particular, one obtains the classical Bézout's theorem on weighted projective planes, (the last equality holds if $|D_1| \nsubseteq |D_2|$ only)

$$D_1 \cdot D_2 = \frac{1}{pqr} \deg_\omega(D_1) \deg_\omega(D_2) = \sum_{P \in |D_1| \cap |D_2|} (D_1 \cdot D_2)_P.$$

Example 5.5. Let us consider $X := \mathbb{P}^2_\omega$, for $\omega = (p, q, r)$. We recall that $P := [0 : 1 : 0]_\omega$ is a singular point of type $(q; p, r)$. We are going to perform the (p, r)-blow-up at this point. The new surface $\hat{X}_P$ admits a map onto $\pi : \hat{X}_P \to \mathbb{P}^1_{(p,r)} \cong \mathbb{P}^1$ with rational fibers. This surface has (at most) four singular points; two of them come from X and they are of type $(p; q, r)$, $Q := [1 : 0 : 0]_\omega$, and $(r; p, q)$, $R := [0 : 0 : 1]_\omega$. The other two points are in the exceptional divisor E and they are of type $(p; -q, r)$ and $(r; p, -q)$; the singular points which are quotient by μ_p are in the same fiber for π and the same happens for μ_r. The map has two relevant sections, E and the transform of $y = 0$.

The exceptional component E has self-intersection $-\frac{q}{pr}$. Since the curve $y=0$ does not pass through P its self-intersection is the one in $\mathbb{P}^2_\omega$, i.e., $\frac{q}{pr}$. The fibers F of π have self-intersection 0; a generic fiber is obtained as follows. Consider a curve L of equation $x^r - z^p = 0$ in $\mathbb{P}^2_\omega$. Then $\pi^*L = F + \frac{pr}{q}E$ and $F^2 = 0$. The surface $\hat{X}_P$ looks like a Hirzebruch surface of index $\frac{q}{pr}$.

6. Application to Jung resolution method

One of the main reasons to work with **Q**-resolutions of singularities is the fact that they are much simpler from the combinatorial point of view and they essentially provide the same information as classical resolutions. In the case of embedded resolutions, there are two main applications. One of them is concerned with the study of the Mixed Hodge Structure and the topology of the Milnor fibration, see [14]. The other one is the Jung method to find abstract resolutions, see [8] and a modern exposition [9] by Laufer.

The study of the Mixed Hodge Structure is related to a process called the semistable resolution which introduces abelian quotient singularities and $\mathbb{Q}$-normal crossing divisors. The work of the second author in his thesis guarantees that one can substitute embedded resolutions by embedded **Q**-resolutions obtaining the same results. As for the Jung method, we will explain the usefulness of **Q**-resolutions at the time they are presented.

6.1. Classical Jung Method. Let $H \subset (\mathbb{C}^{d+1}, 0)$ be a hypersuface singularity defined by a Weiestraß polynomial $f(x_0, x_1, \ldots, x_d) \in \mathbb{C}\{x_1, \ldots, x_d\}[x_0]$. Let $\Delta \in \mathbb{C}\{x_1, \ldots, x_d\}$ be the discriminant of f. We consider the projection $\pi : H \to (\mathbb{C}^d, 0)$ which is an n-fold covering ramified along Δ. Let $\sigma : X \to (\mathbb{C}^d, 0)$ be an embedded resolution of the singularities of Δ. Let $\hat{X}$ be the pull-back of σ and π. In general, this space has non-normal singularities. Denote by $\nu : \tilde{X} \to \hat{X}$ its normalization.

$$
\begin{tikzcd}
\bar{X} \arrow[r, "\tau"] & \tilde{X} \arrow[r, "\nu"] \arrow[d, "\tilde{\pi}"'] \arrow[rr, bend left, "\tilde{\sigma}"] \arrow[dr, phantom, "\#"] & \hat{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\#"] & H \arrow[d, "\pi"] \\
& X \arrow[r, equal] & X \arrow[r, "\sigma"'] & \mathbb{C}^d
\end{tikzcd}
$$

There are two mappings issued from $\tilde{X}$: $\tilde{\pi} : \tilde{X} \to X$ and $\tilde{\sigma} : \tilde{X} \to H$. The map $\tilde{\pi}$ is an n-fold covering whose branch locus is contained in $\sigma^{-1}(\Delta)$. In general, $\tilde{X}$ is not smooth, it has abelian quotient singular points over the (normal-crossing) singular points of $\sigma^{-1}(\Delta)$. Consider $\tau : \bar{X} \to \tilde{X}$ the resolution of $\tilde{X}$, see [5]. Then $\tilde{\sigma} \circ \tau : \bar{X} \to H$ is a good abstract resolution of the singularities of H.

6.2. Jung Q-method. In the previous method, $\tilde{\sigma}$ is a **Q**-resolution of H. This is why replacing σ by an embedded **Q**-resolution is a good idea. First, the process to obtain an embedded **Q**-resolution is much shorter; we can reproduce the process above and the space $\tilde{X}$ obtained only has abelian quotient singularities and the exceptional divisor has $\mathbb{Q}$-normal crossings, i.e., $\tilde{\sigma}$ is an abstract **Q**-resolution of H, usually simpler than the one obtained by the classical method.

If anyway, one is really interested in a standard resolution of H, the most direct way to find the intersection properties of the exceptional divisor of $\tilde{\sigma} \circ \tau$ is to study the $\mathbb{Q}$-intersection properties of the exceptional divisor of $\tilde{\sigma}$ and construct τ as a composition of weighted-blow ups.

We explain this process more explicitly in the case $d = 2$. After the pull-back and the normalization process, the preimage of each irreducible divisor $\mathcal{E}$ of Δ is a (possibly non-connected) ramified covering of $\mathcal{E}$. In order to avoid technicalities to describe these coverings, we restrict our attention to the cyclic case, i.e., $H = \{z^n = f(x, y)\}$.

In this case the reduced structure of Δ is the one of $f(x,y)=0$. We consider the minimal **Q**-resolution of Δ, which is obtained as a composition of weighted blow-ups following the Newton process.

Let $\mathcal{E}$ be an irreducible component of $\sigma^{-1}(\Delta)$ with multiplicity $s:=m_{\mathcal{E}}$.

6.3. Generic points of $\mathcal{E}$. Consider a generic point $p\in\mathcal{E}$ with local coordinates (u,v) such that $v=0$ is $\mathcal{E}$ and $(f\circ\sigma)(u,v)=v^s$. Note that p has only one preimage in $\hat{X}$; $\bar{X}$ looks in a neighborhood of this preimage like $\{(u,v,z)\in\mathbb{C}^3\mid z^n=v^s\}$. The normalization of this space produces $\gcd(s,n)$ points which are smooth. Then, the preimage of $\mathcal{E}$ in $\tilde{X}$ is (possibly non-connected) $\gcd(s,n)$-sheeted cyclic covering ramified on the singular points of $\mathcal{E}$ in $\sigma^{-1}(\Delta)$; the number of connected components and their genus will be described later. Note also that $\tilde{X}$ is smooth over the smooth part of $\mathcal{E}$ in $\sigma^{-1}(\Delta)$.

Remark 6.4. In the general (non-cyclic) case, the local equations can be more complicated but we always have that the preimage of $\mathcal{E}$ in $\tilde{X}$ is a possibly non-connected covering ramified on the double points of $\mathcal{E}$ in $\sigma^{-1}(\Delta)$ and $\tilde{X}$ is smooth over the non-ramified part of $\mathcal{E}$.

Let $p\in\operatorname{Sing}^0(\mathcal{E})$ of normalized type $(d;a,b)$. Since d divides s, let us denote:

$$s_0:=\frac{s}{d}, g:=\gcd(n,s_0), n_1:=\frac{n}{g}, s_1:=\frac{s_0}{g}, e:=\gcd(n_1,d), n_2:=\frac{n_1}{e}, d_1:=\frac{d}{e}.$$

Lemma 6.5. *The preimage of p under $\tilde{\pi}$ consists of g points of type $(d_1;an_2,b)$.*

Proof. The local model of $\hat{X}$ around the preimage of p is of the type

$$\{([(u,v)],z)\in X(d;a,b)\times\mathbb{C}\mid z^n=v^s\}.$$

Consider

$$z^n-v^s=\prod_{\zeta_g\in\mu_g}(z^{n_1}-\zeta_g v^{s_1 d}).$$

Note that each factor is well defined in $X(d;a,b)\times\mathbb{C}$, and hence the normalization is composed by g copies of the normalization of $z^{n_1}=v^{s_1 d}$.

In $\mathbb{C}^3$ the space $z^{n_1}=v^{s_1 d}$ has e irreducible components and the action of μ_d permutes cyclically these components. Hence the quotient of this space by μ_d is the same as the quotient of $z^{n_2}=v^{s_1 d_1}$ by the action of μ_{d_1} defined by $\zeta_{d_1}\cdot(u,v,z)\mapsto(\zeta_{d_1}^a u,\zeta_{d_1}^b v,z)$. The normalization of $z^{n_2}=v^{s_1 d_1}$ is given by

$$(u,t)\mapsto(u,t^{n_2},t^{s_1 d_1})$$

and the induced action of μ_{d_1} is defined by

$$\zeta_{d_1}\cdot(u,t)\mapsto(\zeta_{d_1}^a u,\zeta_{d_1}^{b\alpha}t),\quad \alpha n_2\equiv 1\mod d_1.$$

The result follows since $(d_1;a,b\alpha)=(d_1;an_2,b)$. ∎

Let us consider now a double point p of type $X(d;a,b)$ (normalized), $(\mathcal{E}_1,\mathcal{E}_2)$ and let r,s be the multiplicities of $\mathcal{E}_1,\mathcal{E}_2$. Some notation is needed:

$$m_0:=\frac{ar+bs}{d}, n_1:=\frac{n}{g}, r_1:=\frac{r}{g}, s_1:=\frac{s}{g}, m_1:=\frac{m_0}{g}.$$

Note that $ar_1+bs_1=m_1 d$. We complete the notation:

$$e:=\gcd(n_1,r_1,s_1), n_2:=\frac{n_1}{e}, r_2:=\frac{r_1}{e}, s_2:=\frac{s_1}{e}, d_1:=\frac{d}{e}.$$

Since $\gcd(m_1, e) = 1$, e divides d and then $d_1 \in \mathbb{Z}$. Note that $ar_2 + bs_2 = m_1 d_1$. Since $\gcd(n_2, r_2, s_2) = 1$, one fixes $k, l \in \mathbb{Z}$ such that $m_1 + kr_2 + ls_2 \equiv 0 \mod n_2$ and denote:

$$a' := a + kd_1, \quad b' := b + ld_1.$$

Lemma 6.6. *The preimage of p under $\tilde{\pi}$ consists of g points of type*

$$X\left(\begin{array}{c|cc} d_1 n_2 & a' & b' \\ n_2 & s_2 & -r_2 \end{array}\right).$$

The type is not normalized.

Proof. The local model of $\hat{X}$ over p is $\{([(u,v)], z) \in X(d; a, b) \times \mathbb{C} \mid z^n = u^r v^s\}$. We have

$$z^n - u^r v^s = \prod_{\zeta_g \in \mu_g} (z^{n_1} - \zeta_g u^{r_1} v^{s_1}).$$

Since each factor is well-defined in $X(d; a, b) \times \mathbb{C}$, the normalization is composed by g copies of the normalization of $z^{n_1} = u^{r_1} v^{s_1}$.

In $\mathbb{C}^3$ the space $z^{n_1} = u^{r_1} v^{s_1}$ has e irreducible components and the action of μ_d permutes cyclically these components. Hence the quotient of this space by μ_d is the same as the quotient of $z^{n_2} = u^{r_2} v^{s_2}$ by the action of μ_{d_1} defined by $\zeta_{d_1} \cdot (u, v, z) \mapsto (\zeta_{d_1}^a u, \zeta_{d_1}^b v, z)$.

Note that a, b can be replaced by a', b' in the action of μ_{d_1}. Moreover, $D := a' r_2 + b' s_2 \equiv 0 \mod n_2$. The map

$$(t, w) \mapsto (t^{n_2}, w^{n_2}, t^{r_2} w^{s_2})$$

parametrizes (not in a biunivocal way) the space $z^{n_2} = u^{r_2} v^{s_2}$. The action of $\mu_{n_2 d_1}$ defined by

$$\zeta_{n_2 d_1} \cdot (t, w) \mapsto (\zeta_{n_2 d_1}^{a'} t, \zeta_{n_2 d_1}^{b'} w)$$

lifts the former action of ζ_{d_1}. The normalization of the quotient of $z^{n_2} = u^{r_2} v^{s_2}$ by the action of μ_{d_1} is deduced to be of (non-normalized) type

$$X\left(\begin{array}{c|cc} d_1 n_2 & a' & b' \\ n_2 & s_2 & -r_2 \end{array}\right). \qquad \square$$

Remark 6.7. It is easier to normalize this type case by case, but at least a method to present it as a cyclic type is shown here. Let $\mu := \gcd(a', d_1 s_2)$ and let $\beta, \gamma \in \mathbb{Z}$ such that $\mu = \beta a' + \gamma d_1 s_2$. Note that μ divides D. Then the preceding type is isomorphic (via the identity) to

$$X\left(\begin{array}{c|cc} n_2 & 0 & \gamma \frac{D}{\mu} \\ n_2 & 0 & -\beta \frac{D}{\mu} \\ d_1 n_2 & \mu & \beta b' - \gamma d_1 r_2 \end{array}\right) = X\left(\begin{array}{c|cc} n_2 & 0 & \frac{D}{\mu} \\ d_1 n_2 & \mu & \beta b' - \gamma d_1 r_2 \end{array}\right)$$

since $\gcd(\beta, \gamma) = 1$. Let $h := \gcd(n_2, \frac{D}{\mu})$. Then, this space is isomorphic to $X(d_1 n_2; \alpha, (\beta b' - \gamma d_1 r_2)\frac{n_2}{h})$. If $j := \gcd(\mu, \frac{n_2}{h})$, then it is isomorphic to the space $X(d_1 h; \frac{\mu}{j}, \beta b' - \gamma d_1 r_2)$ (maybe non normalized).

The following statement summarizes the results for each irreducible component of the divisor.

Lemma 6.8. *Let $\mathcal{E}$ be an exceptional component of σ with multiplicity s, $m := \gcd(s, n)$. Let $\operatorname{Sing}(\mathcal{E})$ be the union of $\operatorname{Sing}^0(\mathcal{E})$ with the double points of $\sigma^{-1}(\Delta)$ in $\mathcal{E}$. Let ν be the* gcd *of s and the values g_P for each $P \in \operatorname{Sing}(\mathcal{E})$ obtained in Lemmas 6.5 and 6.6. Then, $\tilde{\pi}^{-1}(\mathcal{E})$ consists of ν connected components. Each component is an $(\frac{s}{\nu})$-fold cyclic covering whose genus is computed using Riemann-Hurwitz formula and the self-intersection of each component is $\frac{m^2 \eta}{n\nu}$ if $\eta = \mathcal{E}^2$.*

Proof. Only the self-intersection statement needs a proof. Let $\tilde{\mathcal{E}} := \tilde{\pi}^{-1}(\mathcal{E})$. Then $\tilde{\pi}^*(\mathcal{E}) = \frac{n}{m}\tilde{\mathcal{E}}$. Hence:

$$\tilde{\mathcal{E}}^2 = \frac{m^2}{n^2}\tilde{\pi}^*(\mathcal{E})^2 = \frac{m^2}{n}(\mathcal{E})^2 = \frac{m^2\eta}{n}.$$

Since $\tilde{\mathcal{E}}$ has ν disjoint components related by an automorphism of $\tilde{X}$, the result follows. $\square$

Example 6.9. Let us consider the singularity $z^n - (x^2+y^3)(x^3+y^2) = 0$, $n > 1$. As it was shown in Example 4.6, the minimal **Q**-embedded resolution of $(x^2+y^3)(x^3+y^2) = 0$ has two exceptional components $\mathcal{E}_1, \mathcal{E}_2$. Each component has multiplicity 10, self-intersection $-\frac{3}{10}$, intersects the strict transform at a smooth point and has one singular point of type $(2;1,1)$. The two components intersect at a double point of type $(5;2,-3) = (5;1,1)$. Let us denote $g_p(n)$ the previous numbers for a given n. The computations are of four types depending on $\gcd(n,10) = 1,2,5,10$.

Let us fix one of the exceptional components, say $\mathcal{E}_1$, since they are symmetric. Before studying separately each case, let p_0 be the intersection point of $\mathcal{E}_1$ with the strict transform, then its preimage is the normalization of $z^n - xy^{10} = 0$ which is of type $(n;-10,1)$. In particular $g_p(n) = 1$ and $\nu_{\mathcal{E}_1} = 1$, i.e., $\tilde{\mathcal{E}_1} := \tilde{\pi}^{-1}(\mathcal{E}_1)$ is irreducible. Let us denote $p_1 := \mathcal{E}_1 \cap \mathcal{E}_2$.

Case 1. $\gcd(n,10) = 1$.

Let us study first the preimage over a generic point of $\mathcal{E}_1$, which will be the normalization of $z^n - y^{10} = 0$, i.e., one point. By Lemma 6.8, $\tilde{\mathcal{E}_1}^2 = -\frac{3}{10n}$ and $\tilde{\mathcal{E}_1}$ is rational. The preimage of p_0 is of reduced type $(n;-10,1)$.

Let $p \in \operatorname{Sing}^0(\mathcal{E}_1)$. It is of type $(2;1,1)$. Applying Lemma 6.5, one obtains that it is of type $(2;11,1) = (2;1,1)$.

One has $g_{p_1}(n) = e = 1$. Following the notation previous to Lemma 6.6, we choose $k = l \in \mathbb{Z}$ such that $5k+1 \equiv 0 \mod n$. A type

$$\left(\begin{array}{c|cc} 5n & 1+5k & 1+5k \\ n & 10 & -10 \end{array}\right) = \left(\begin{array}{c|cc} 5n & 1+5k & 1+5k \\ n & 1 & -1 \end{array}\right) = \left(\begin{array}{c|cc} 5n & 1+5k & 1+5k \\ 5n & 5 & -5 \end{array}\right),$$

is obtained, which is of type $(5n;1,10k+1)$; since $(10k+1)^2 \equiv 1 \mod 5n$, this type is symmetric and normalized. Then, the minimal embedded **Q**-resolution of the surface singularity consists of two rational divisors of self-intersection $-\frac{3}{10n}$, with a unique double point of type $X(5n;1;1+10k)$ and each divisor has two other singular points, one double and the other one of type $X(n;-10,1)$.

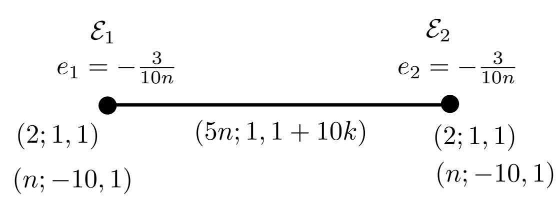

FIGURE 3. Dual graph for $z^n = (x^2+y^3)(x^3+y^2)$, $\gcd(n,10) = 1$, $5k+1 \equiv 0 \mod n$.

Case 2. $\gcd(n,10) = 2$.

The preimage over a generic point of $\mathcal{E}_1$, which will be the normalization of $z^n - y^{10} = 0$, i.e., $\tilde{\mathcal{E}_1} := \tilde{\pi}^{-1}(\mathcal{E}_1)$ is a 2-fold covering of $\mathcal{E}_1$. The point p_0 is a ramification point of the covering (with one preimage) and it is of type $(n;-10,1) = (\frac{n}{2};-5,1)$.

Let $p \in \operatorname{Sing}^0(\mathcal{E}_1)$. Since $s_0 = 2$, $g_p(n) = 1$ and $e = 2$, applying Lemma 6.5, one has $d_1 = 1$. There is only one preimage and it is a smooth point.

Let us finish with p_1. In this case, $g_{p_1}(n) = 2$, $n_1 := \frac{n}{2}$, and $e = 1$. It can be chosen $k = l \in \mathbb{Z}$ such that $5k + 1 \equiv 0 \mod n_1$. Using the same computations as in the previous case, two points of type $X(5n_1; 1, 10k + 1)$ are obtained.

Using Riemann-Hurwitz formula $\tilde{\mathcal{E}}_1$ is irreducible and rational; since $\tilde{\pi}^*(\mathcal{E}_1) = 5\tilde{\mathcal{E}}_1$ one has that $\tilde{\mathcal{E}}_1^{\,2} = -\frac{3}{5n_1}$. Then, the minimal embedded $\mathbf{Q}$-resolution of the surface singularity consists of two rational divisors, with two double points of type $X(5n_1; 1, 1 + 10k)$ and each divisor has another singular point of type $X(n_1; -5, 1)$. Note that the graph is not a tree.

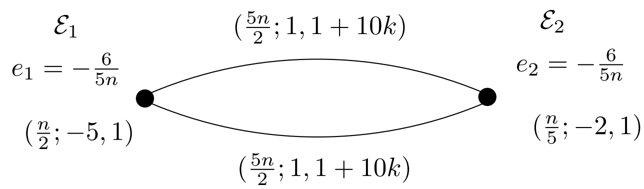

FIGURE 4. Dual graph for $z^n = (x^2 + y^3)(x^3 + y^2)$, $\gcd(n, 10) = 2$, $5k + 1 \equiv 0 \mod \frac{n}{2}$.

Case 3. $\gcd(n, 10) = 5$.

The preimage over a generic point of $\mathcal{E}_1$, which will be the normalization of $z^n - y^{10} = 0$, i.e., $\tilde{\mathcal{E}}_1 := \tilde{\pi}^{-1}(\mathcal{E}_1)$ is a 5-fold covering of $\mathcal{E}_1$. As above, p_0 is a ramification point of the covering (with one preimage) and it is of type $(n; -10, 1) = (\frac{n}{5} : -2, 1)$.

Let $p \in \operatorname{Sing}^0(\mathcal{E}_1)$. One has $g_p(n) = 5$ and $d_1 = 2$. Hence the covering does not ramify at p and its preimage consists of 5 points of type $(2; 1, 1)$.

In the case of p_1 we have $g_{p_1}(n) = 1$, $e = 5$, $n_2 = \frac{n}{5}$ and $d_1 = 1$. Hence a point of type $X(n_2; 1, -1)$ is obtained.

As a consequence, $\tilde{\mathcal{E}}_1$ is rational and $\tilde{\mathcal{E}}_1^{\,2} = -\frac{3}{2n_2}$. Then, the minimal embedded $\mathbf{Q}$-resolution of the surface singularity consists of two rational divisors, with a single double point of type $X(n_2; 1; -1)$ and each divisor has another singular point of type $X(n_2; -2, 1)$ and five double points.

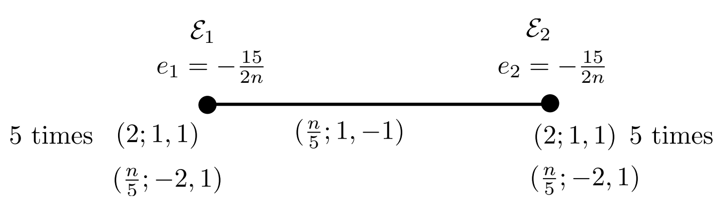

FIGURE 5. Dual graph for $z^n = (x^2 + y^3)(x^3 + y^2)$, $\gcd(n, 10) = 5$.

Case 4. $\gcd(n, 10) = 10$.

The preimage over a generic point of $\mathcal{E}_1$, which will be the normalization of $z^n - y^{10} = 0$, i.e., $\tilde{\mathcal{E}}_1 := \tilde{\pi}^{-1}(\mathcal{E}_1)$ is a 10-fold covering of $\mathcal{E}_1$. The point p_0 is a ramification point of the covering (with one preimage) and it is of type $(n; -10, 1) = (\frac{n}{10} : -1, 1)$.

Let $p \in \operatorname{Sing}^0(\mathcal{E}_1)$. One has $g_p(n) = 5$ and $d_1 = 1$. Hence the preimage of p consists of 5 smooth points.

Finally one has $g_{p_1}(n) = 2$, $e = 5$, $n_2 = \frac{n}{10}$ and $d_1 = 1$. Hence a point of type $X(n_2; 1, -1)$ is obtained.

Using Riemann-Hurwitz, $\tilde{\mathcal{E}_1}$ has genus 2; since $\tilde{\pi}^*(\mathcal{E}_1) = \tilde{\mathcal{E}_1}$, then $\tilde{\mathcal{E}_1}^2 = -\frac{3}{n_2}$. Then, the minimal embedded **Q**-resolution of the surface singularity consists of two divisors of genus 2, with one double point of type $X(n_2;1;-1)$ and each divisor has another singular point of type $X(n_2;-1,1)$.

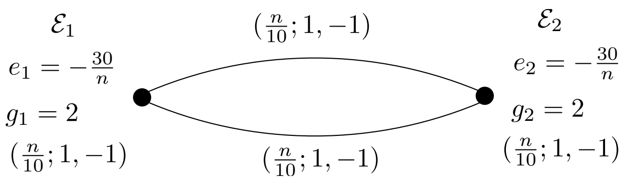

FIGURE 6. Dual graph for $z^n = (x^2+y^3)(x^3+y^2)$, $\gcd(n,10)=10$.

As a final application, intersection theory and weighted blow-ups are essential tools to construct a resolution from a **Q**-resolution. Note that even when one uses the classical Jung method, this step is needed. The resolution of cyclic quotient singularities for surfaces is known, see §6 for references.

This resolution process uses the theory of continuous fractions. We illustrate the use of weighted blow-ups to solve these singularities in two ways.

First, let $X := X(d;a,b)$, where d,a,b are pairwise coprime, $d>1$, and $1 \le a,b < d$. Then the (a,b)-blow-up of X produces a new space with an exceptional divisor (of self-intersection $-\frac{d}{ab}$) and two singular points of type $(a;-d,b)$ and $(b;a-d)$. Since the index of these singularities is less than d we finish by induction. Note that if we have a compact divisor passing through the singular point, it is possible to compute the self-intersection multiplicity of the strict transform, see Theorem 4.3.

The second way allows us to recover the Jung-Hirzebruch resolution. Recall briefly the notion of continuous fraction. Let $s \in \mathbb{Q}$, $r>1$. The continuous fraction associated with s is a tuple of integers $cf(s) := [q_1,\dots,q_n]$, $q_j > 1$, defined inductively as follows:

- If $s \in \mathbb{Z}$ then $cf(s) := [s]$.
- If $s \notin \mathbb{Z}$, write $s = \frac{d}{k}$ in reduced form. Consider the *excess division algorithm* $d = qk - r$, $q,r \in \mathbb{Z}$, $0 < r < k$. Then, $cf(s) := [q, cf(\frac{k}{r})]$.

Hence, for instance, $[q_1,q_2,q_3] = q_1 - \frac{1}{q_2 - \frac{1}{q_3}}$.

With this technique we recover the well-known Jung-Hirzebruch resolution, see e.g. [7].

Proposition 6.10. *Let $X := X(d;1,k)$ be a normalized type with $1 \le k < d$ and let*

$$cf(\frac{d}{k}) := [q_1,\dots,q_n].$$

Then the exceptional locus of the resolution of X consists of a linear chain of rational curves with self-intersections $-q_1,\dots,-q_n$.

Proof. As stated above, we perform the $(1,k)$-blow-up of X. We obtain an exceptional divisor $\mathcal{E}_1$ such that $\mathcal{E}_1^2 = -\frac{d}{k}$. If $k=1$, we are done. If $k>1$ then $\mathcal{E}_1$ contains a singular point $Y := X(k;1,-d)$. We know that $d = q_1k - r$, $1 < r < k$, and $cf(\frac{k}{r}) = [q_2,\dots,q_n]$. Since $r = q_1k - d$, then $Y = X(k;1,r)$. We may apply induction hypothesis (in the length of cf) and the result follows if we obtain the right self-intersection multiplicity of the first divisor.

The next blow-up is with respect to $(1,r)$. Following Theorem 4.3, the self-intersection of the strict transform of $\mathcal{E}_1$ equals $-\frac{d}{k} - \frac{r^2}{k\cdot 1\cdot r} = -\frac{d}{k} - \frac{r}{k} = -q_1$, since the divisor $\mathcal{E}_1$ is given by $\{y=0\} \subset X(k;1,r)$. ◻

Remark 6.11. The last part of the proof allows us to give the right way to pass from a **Q**-resolution to a resolution. The Jung-Hirzebruch method gives the resolution of the cyclic singularities. Let $\mathcal{E}$ be an irreducible component of the **Q**-resolution with self-intersection $-s \in \mathbb{Q}$, and let $\operatorname{Sing}(\mathcal{E}) := \{P_1, \ldots, P_r\}$, with P_i of type $(d_i; 1, k_i)$, $1 \leq k_i < d_i$ and $\gcd(d_i, k_i) = 1$. Then, the self-intersection number of $\mathcal{E}$, assuming its local equation is $y = 0$, can be computed as above. That is, one performs the weighted blow-ups of type $(1, k_i)$ at each point, obtaining $-s - \sum_{i=1}^{r} \frac{k_i}{d_i}$, which must be an integer.

References

[1] E. Artal, J. Martín-Morales, and J. Ortigas-Galindo, *Cartier and Weil divisors on varieties with quotient singularities*, Preprint available at `arXiv:1104.5628 [math.AG]`, 2011.

[2] J.I. Cogolludo-Agustín, J. Martín-Morales, and J. Ortigas-Galindo, *Local invariants on quotient singularities and a genus formula for weighted plane curves*, Int. Math. Res. Not. IMRN (2013). DOI: 10.1093/imrn/rnt052

[3] I. Dolgachev, *Weighted projective varieties*, Group actions and vector fields (Vancouver, B.C., 1981), Lecture Notes in Math., vol. 956, Springer, Berlin, 1982, pp. 34–71.

[4] H. Flenner, *Divisorenklassengruppen quasihomogener Singularitäten*, J. Reine Angew. Math. **328** (1981), 128–160. DOI: 10.1515/crll.1981.328.128

[5] A. Fujiki, *On resolutions of cyclic quotient singularities*, Publ. Res. Inst. Math. Sci. **10** (1974/75), no. 1, 293–328.

[6] W. Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 2, Springer-Verlag, Berlin, 1998.

[7] F. Hirzebruch, W.D. Neumann, and S.S. Koh, *Differentiable manifolds and quadratic forms*, Marcel Dekker Inc., New York, 1971, Appendix II by W. Scharlau, Lecture Notes in Pure and Applied Mathematics, Vol. 4.

[8] H.W.E. Jung, *Einführung in die Theorie der algebraischen Funktionen zweier Veränderlicher*, Akademie Verlag, Berlin, 1951.

[9] H.B. Laufer, *Normal two-dimensional singularities*, Princeton University Press, Princeton, N.J., 1971, Annals of Mathematics Studies, No. 71.

[10] D. Mumford, *The topology of normal singularities of an algebraic surface and a criterion for simplicity*, Inst. Hautes Études Sci. Publ. Math. (1961), no. 9, 5–22.

[11] D. Prill, *Local classification of quotients of complex manifolds by discontinuous groups*, Duke Math. J. **34** (1967), 375–386. DOI: 10.1215/S0012-7094-67-03441-2

[12] F. Sakai, *Weil divisors on normal surfaces*, Duke Math. J. **51** (1984), no. 4, 877–887.

[13] I. Satake, *On a generalization of the notion of manifold*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 359–363. DOI: 10.1073/pnas.42.6.359

[14] J.H.M. Steenbrink, *Mixed Hodge structure on the vanishing cohomology*, Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), Sijthoff and Noordhoff, Alphen aan den Rijn, 1977, pp. 525–563.

[15] W. Veys, *Zeta functions for curves and log canonical models*, Proc. London Math. Soc. (3) **74** (1997), no. 2, 360–378.

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