Cartier and Weil Divisors on Varieties with Quotient Singularities

Enrique ARTAL BARTOLO
Departamento de Matemáticas-IUMA, Facultad de Ciencias
Universidad de Zaragoza
C/ Pedro Cerbuna 12, 50009 Zaragoza, Spain
artal@unizar.es

Jorge MARTÍN-MORALES, Jorge ORTIGAS-GALINDO
Centro Universitario de la Defensa-IUMA
Academia General Militar
Ctra. de Huesca s/n. 50090 Zaragoza, Spain
jorge@unizar.es, jortigas@unizar.es

It is well known that the notions of Weil and Cartier $\mathbb{Q}$-divisors coincide for V-manifolds. The main goal of this paper is to give a direct constructive proof of this result providing a procedure to express explicitly a Weil divisor as a rational Cartier divisor. The theory is illustrated on weighted projective spaces and weighted blow-ups.

Keywords: Quotient singularity, Weil and Cartier divisors, embedded **Q**-resolution, weighted blow-ups

Mathematics Subject Classification 2000: 32S25, 32S45

Introduction

Singularity Theory deals with the study of complex spaces with non smooth points. The most powerful tool to study singularities is resolution, i.e. a proper analytic morphism from a smooth variety which is an isomorphism outside the singular locus. The existence of such resolutions is guaranteed by the work of Hironaka. Usually the combinatorics of the exceptional divisor of the resolution is quite complicated.

There is a class of singularities which is well understood, namely normal singularities which are obtained as the quotient of a ball in $\mathbb{C}^n$ by the linear action of a finite group. Spaces admitting only such singularities are called V-manifolds and were introduced by Baily. These varieties share several properties with manifolds, e.g. projective V-manifolds also carry a Hodge structure and a natural notion of normal crossing divisors can be defined (called $\mathbb{Q}$-normal crossing divisors). As it will be developed in the second author's Ph.D. thesis [10], the study of the so-called **Q**-resolutions (allowing quotient singularities, especially abelian) provides a better understanding of some families of singularities.

Intersection theory is needed for this study; it is also required to deal with divisors on V-manifolds. Two kinds of divisors appear in the literature: Weil and Cartier divisors. Weil divisors are formal sums of hypersurfaces and Cartier divisors are global sections of the quotient sheaf of meromorphic functions modulo non-vanishing holomorphic functions. The relationship between Cartier divisors and line bundles provides a nice way to define the intersection multiplicity of two divisors. In the smooth category, both notions coincide but this is not the case for singular varieties (not even for normal ones). The groups of Weil and Cartier $\mathbb{Q}$-divisors are isomorphic for V-manifolds and hence a rational intersection theory can be developed for Weil $\mathbb{Q}$-divisors using the theory of line bundles.

Note that standard toric geometry provides a concrete description of the embedding of the Picard Group into the Weil divisor Group and intersection numbers can be computed by toric methods as well. However, our approach is more direct and closer to the classical theory of smooth varieties, hence simplifying the computations.

This result was probably known for specialists long time ago; a non-constructive proof was given by Kollár-Mori [9] . We think that it is worthwhile to provide a constructive proof of this fact. As a consequence of this constructive proof, an algorithm to explicitly represent a Weil $\mathbb{Q}$-divisor as a Cartier $\mathbb{Q}$-divisor is provided.

We illustrate the use of this algorithm in an example living in a space obtained after a weighted blow-up. Such blow-ups can be understood from toric geometry but in this work they are presented in a more geometric way, generalizing standard blow-ups. For this definition, we need to describe abelian quotient singularities. In [1], we develop an intersection theory for surfaces with abelian quotient singularities, where the results presented in this paper are essential tools.

The paper is organized as follows. In §1 we give a general presentation of varieties with quotient singularities and list their basic properties. In §2 we deal with weighted projective spaces and in §3 we introduce weighted blow-ups. The main results about rational Weil and Cartier divisors on V-manifolds are stated and proven in §4. We finish with some conclusions and the expected future work.

1. V-Manifolds and Quotient Singularities

Definition 1.1. A V-manifold of dimension n is a complex analytic space which admits an open covering $\{U_i\}$ such that U_i is analytically isomorphic to B_i/G_i where $B_i \subset \mathbb{C}^n$ is an open ball and G_i is a finite subgroup of $GL(n,\mathbb{C})$.

The concept of V-manifolds was introduced in [12] and they have the same homological properties over $\mathbb{Q}$ as manifolds. For instance, they admit a Poincaré duality if they are compact and carry a pure Hodge structure if they are compact and Kähler, see [2]. They have been locally classified by Prill [11]. To state this local result we need the following.

Definition 1.2. A finite subgroup G of $GL(n,\mathbb{C})$ is called *small* if no element of G has 1 as an eigenvalue of multiplicity precisely $n-1$, that is, G does not contain rotations around hyperplanes other than the identity.

For every finite subgroup G of $GL(n,\mathbb{C})$ denote by G_{big} the normal subgroup of G generated by all rotations around hyperplanes. Then the G_{big}-invariant polynomials form a polynomial algebra and hence $\mathbb{C}^n/G_{\text{big}}$ is isomorphic to $\mathbb{C}^n$.

The group G/G_{big} maps isomorphically to a small subgroup of $GL(n,\mathbb{C})$, once a basis of invariant polynomials has been chosen. Hence the local classification of V-manifolds reduces to the classification of actions of small subgroups of $GL(n,\mathbb{C})$.

Theorem 1.1. *([11]). Let G_1, G_2 be small subgroups of $GL(n,\mathbb{C})$. Then $\mathbb{C}^n/G_1$ is isomorphic to $\mathbb{C}^n/G_2$ if and only if G_1 and G_2 are conjugate subgroups.* $\square$

We are interested in V-manifolds where the quotient spaces B_i/G_i are given by (finite) abelian groups. In this case the following notation is used.

Notation 1.1. For $\mathbf{d} := {}^t(d_1 \ldots d_r)$ we denote $\mu_{\mathbf{d}} := \mu_{d_1} \times \cdots \times \mu_{d_r}$ a finite abelian group written as a product of finite cyclic groups, that is, μ_{d_i} is the cyclic group of d_i-th roots of unity in $\mathbb{C}$. Consider a matrix of weight vectors

$$A := (a_{ij})_{i,j} = [\mathbf{a}_1 \mid \cdots \mid \mathbf{a}_n] \in \text{Mat}(r \times n, \mathbb{Z}), \quad \mathbf{a}_j := {}^t(a_{1j} \ldots a_{rj}) \in \text{Mat}(r \times 1, \mathbb{Z}),$$

and the action

$$\begin{array}{ll} (\mu_{d_1} \times \cdots \times \mu_{d_r}) \times \mathbb{C}^n \longrightarrow \mathbb{C}^n, & \boldsymbol{\xi}_{\mathbf{d}} := (\xi_{d_1}, \ldots, \xi_{d_r}), \\ (\boldsymbol{\xi}_{\mathbf{d}}, \mathbf{x}) \mapsto (\xi_{d_1}^{a_{11}} \cdot \ldots \cdot \xi_{d_r}^{a_{r1}} x_1, \ldots, \xi_{d_1}^{a_{1n}} \cdot \ldots \cdot \xi_{d_r}^{a_{rn}} x_n), & \mathbf{x} := (x_1, \ldots, x_n). \end{array} \tag{1.1}$$

Note that the i-th row of the matrix A can be considered modulo d_i. The set of all orbits $\mathbb{C}^n/G$ is called (*cyclic*) *quotient space of type* $(\mathbf{d}; A)$ and it is denoted by

$$X(\mathbf{d}; A) := X \left(\begin{array}{c|ccc} d_1 & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ d_r & a_{r1} & \cdots & a_{rn} \end{array} \right).$$

The orbit of an element $\mathbf{x} \in \mathbb{C}^n$ under this action is denoted by $[\mathbf{x}]_{(\mathbf{d};A)}$ and the subindex is omitted if no ambiguity seems likely to arise. Using multi-index notation the action takes the simple form

$$\begin{array}{l} \mu_{\mathbf{d}} \times \mathbb{C}^n \longrightarrow \mathbb{C}^n, \\ (\boldsymbol{\xi}_{\mathbf{d}}, \mathbf{x}) \;\mapsto\; \boldsymbol{\xi}_{\mathbf{d}} \cdot \mathbf{x} := (\boldsymbol{\xi}_{\mathbf{d}}^{\mathbf{a}_1} x_1, \ldots, \boldsymbol{\xi}_{\mathbf{d}}^{\mathbf{a}_n} x_n). \end{array}$$

The following classic result shows that the family of varieties isomorphic to $X(\mathbf{d}; A)$ is exactly the same as the family of V-manifolds with abelian quotient singularities.

Lemma 1.1. *Let G be a finite abelian subgroup of $GL(n,\mathbb{C})$. Then $\mathbb{C}^n/G$ is isomorphic to some quotient space of type $(\mathbf{d}; A)$.*

Proof. The group G can be seen as a direct product of cyclic groups generated by matrices $M_1, \ldots, M_r$ of orders $d_1, \ldots, d_r$ such that

$$G = \{M_1^{i_1} \cdots M_r^{i_r} \mid i_k = 0, \ldots, d_k - 1\}.$$

Each of these matrices M_i, $i=1,\dots,r$, is conjugated to $\mathrm{Diag}(\zeta_{d_i}^{a_{i1}},\dots,\zeta_{d_i}^{a_{in}})$ where ζ_{d_i} is a d_i-th primitive root of unity. Moreover, they are simultaneously diagonalizable because they commute. This proves $\mathbb{C}^n/G \simeq X(\mathbf{d};(a_{ij})_{i,j})$. □

Different types $(\mathbf{d};A)$ can give rise to isomorphic quotient spaces, see Remark 1.1. We shall prove that they can always be represented by an upper triangular matrix of dimension $(n-1)\times n$, see Lemmas 1.2 and 1.3. Finding a simpler type $(\mathbf{d};A)$ to represent a quotient space will lead us to the notions of normalized type and space, see Definition 1.3.

Remark 1.1. For $n=3$ the simple group automorphism on $\mu_d\times\mu_d$ given by $(\xi,\eta)\mapsto(\xi\eta^{-1},\eta)$ shows that the following two spaces are isomorphic under the identity map:

$$X\left(\begin{array}{c|ccc} d & a_{11} & a_{12} & a_{13} \\ d & a_{21} & a_{22} & a_{23}\end{array}\right) = X\left(\begin{array}{c|ccc} d & a_{11} & a_{12} & a_{13} \\ d & a_{21}-a_{11} & a_{22}-a_{12} & a_{23}-a_{13}\end{array}\right).$$

Note that the determinants of the minors of order 2 are the same in both side of the previous equation. There is an obvious generalization for higher dimensions, allowing row operations when the two rows correspond to the same cyclic group. This is not a strong condition since we can also multiply any row (of $\mathbf{d}$ and A simultaneously) by an arbitrary integer.

Example 1.1. When $n=1$ all spaces $X(\mathbf{d};A)$ are isomorphic to $\mathbb{C}$. Dividing each row by $\gcd(d_i,a_i)$ does not change the quotient and we can work with $X({}^t(d_1,\dots,d_r);{}^t(a_1,\dots,a_r))$ where $\gcd(d_i,a_i)=1$.

The map $[x]\mapsto x^{d_1}$ gives an isomorphism between $X(d_1;a_1)$ and $\mathbb{C}$. For $r=2$ one has that (we write the symbol $=$ when the isomorphism is induced by the identity map)

$$\begin{array}{ccccccc} \dfrac{\mathbb{C}}{\mu_{d_1}\times\mu_{d_2}} = \dfrac{\mathbb{C}/\mu_{d_1}}{\mu_{d_2}} & \cong & \mathbb{C}/\mu_{d_2} & \stackrel{(*)}{=} & X(d_2;a_2d_1) & \cong & \mathbb{C}, \\ & [x]\mapsto x^{d_1}, & & & & [x]\mapsto x^{\frac{d_2}{\gcd(d_1,d_2)}}. & \end{array}$$

To see the equality $(*)$ observe that $\xi_{d_2}\cdot x^{d_1}\equiv\xi_{d_2}\cdot[x]=[\xi_{d_2}^{a_2}x]\equiv\xi_{d_2}^{a_2d_1}x^{d_1}$. It follows that the corresponding quotient space is isomorphic to $\mathbb{C}$ under the map $[x]\mapsto x^{\mathrm{lcm}(d_1,d_2)}$.

From Remark 1.1 and Example 1.1 the following lemmas follow easily.

Lemma 1.2. *The following operations do not change the isomorphism type of* $X(\mathbf{d};A)$.

(1) Permutation σ of columns of A, $[\mathbf{x}]\mapsto[(x_{\sigma(1)},\dots,x_{\sigma(n)})]$.
(2) Permutation of rows of $(\mathbf{d};A)$, $[\mathbf{x}]\mapsto[\mathbf{x}]$.
(3) Multiplication of a row of $(\mathbf{d};A)$ *by a positive integer,* $[\mathbf{x}]\mapsto[\mathbf{x}]$.
(4) Multiplication of a row of A by an integer coprime with the corresponding row in $\mathbf{d}$, $[\mathbf{x}]\mapsto[\mathbf{x}]$.

(5) Replace a_{ij} by $a_{ij} + kd_j$, $[\mathbf{x}] \mapsto [\mathbf{x}]$.
(6) If e is coprime with $a_{1,n}$, and divides d_1 and $a_{1,j}$, $1 \leq j < n$, then replace, $a_{i,n} \mapsto ea_{i,n}$, $[\mathbf{x}] \mapsto [(x_1, \ldots, x_n^e)]$.
(7) If $d_r = 1$ then eliminate the last row, $[\mathbf{x}] \mapsto [\mathbf{x}]$.

Lemma 1.3. *The space $X(\mathbf{d}; A) = \mathbb{C}^n/\mu_{\mathbf{d}}$ can always be represented by an upper triangular matrix of dimension $(n-1) \times n$. More precisely, there exist a vector $\mathbf{e} := {}^t(e_1, \ldots, e_{n-1})$, a matrix $B = (b_{i,j})_{i,j}$, and an isomorphism $[(x_1, \ldots, x_n)] \mapsto [(x_1, \ldots, x_n^k)]$ for some $k \in \mathbb{N}$ such that*

$$X(\mathbf{d}; A) \cong \left(\begin{array}{c|cccc} e_1 & b_{1,1} & \cdots & b_{1,n-1} & b_{1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ e_{n-1} & 0 & \cdots & b_{n-1,n-1} & b_{n-1,n} \end{array} \right) = X(\mathbf{e}; B).$$

The action shown in Equation (1.1) of Notation 1.1 is free on $(\mathbb{C}^*)^n$, i.e $\left[\mathbf{x} \in (\mathbb{C}^*)^n,\ \boldsymbol{\xi}_{\mathbf{d}} \cdot \mathbf{x} = \mathbf{x}\right] \implies \boldsymbol{\xi}_{\mathbf{d}} = 1$, if and only if the group homomorphism $\mu_{\mathbf{d}} \to GL(n, \mathbb{C})$ given by

$$\boldsymbol{\xi}_{\mathbf{d}} = (\xi_{d_1}, \ldots, \xi_{d_r}) \longmapsto \operatorname{Diag}(\boldsymbol{\xi}_{\mathbf{d}}^{\mathbf{a}_1}, \ldots, \boldsymbol{\xi}_{\mathbf{d}}^{\mathbf{a}_r}) \tag{1.2}$$

is injective. If this is not the case, let H be the kernel of this group homomorphism. Then $\mathbb{C}^n/H \equiv \mathbb{C}^n$ and the group μ_d/H acts freely on $(\mathbb{C}^*)^n$ under the previous identification. Thus one can always assume that the free and the small conditions are satisfied. This motivates the following definition.

Definition 1.3. The type $(\mathbf{d}; A)$ is said to be *normalized* if the following two conditions hold.

(1) The action is free on $(\mathbb{C}^*)^n$.
(2) The group $\mu_{\mathbf{d}}$ is identified with a small subgroup of $GL(n, \mathbb{C})$ under the group homomorphism given in Equation (1.2).

By abuse of language we often say the space $X(\mathbf{d}; A)$ is written in a normalized form when we mean the type $(\mathbf{d}; A)$ is normalized.

Proposition 1.1. *The space $X(\mathbf{d}; A)$ is written in a normalized form if and only if the stabilizer subgroup of P is trivial for all $P \in \mathbb{C}^n$ with exactly $n-1$ coordinates different from zero.*

In the cyclic case the stabilizer of a point as above (with exactly $n-1$ coordinates different from zero) has order $\gcd(d, a_1, \ldots, \widehat{a}_i, \ldots, a_n)$.

Using Lemma 1.2 it is possible to convert general types $(\mathbf{d}; A)$ into their normalized form. Theorem 1.1 allows one to decide whether two quotient spaces are isomorphic. In particular one can use this result to compute the singular points of the space $X(\mathbf{d}; A)$ and Proposition 1.1 to decide if a given cyclic type is normalized.

In Example 1.1 we explain the previous normalization process in dimension one. The two and three-dimensional cases are treated in the following examples.

Example 1.2. Following Lemma 1.3, all quotient spaces for $n = 2$ are cyclic. The space $X(d; a, b)$ is written in a normalized form if and only if $\gcd(d, a) = \gcd(d, b) = 1$. If this is not the case, one uses the isomorphism[a] (assuming $\gcd(d, a, b) = 1$)

$$\begin{aligned} X(d; a, b) &\longrightarrow X\left(\tfrac{d}{(d,a)(d,b)}; \tfrac{a}{(d,a)}, \tfrac{b}{(d,b)}\right), \\ \big[(x, y)\big] &\mapsto \big[(x^{(d,b)}, y^{(d,a)})\big] \end{aligned}$$

to convert it into a normalized one, see also Lemma 1.2.

Example 1.3. The quotient space $X(d; a, b, c)$ is written in a normalized form if and only if $\gcd(d, a, b) = \gcd(d, a, c) = \gcd(d, b, c) = 1$. As above, isomorphisms of the form $[(x, y, z)] \mapsto [(x, y, z^k)]$ can be used to convert types $(d; a, b, c)$ into their normalized form.

In [4] Fujiki computes resolutions of these cyclic quotient singularities and also studies, among others, the properties shown above.

2. Weighted Projective Spaces

Weighted projective spaces are the main examples of V-manifolds. The main reference that has been used in this section is [3], see also [8]. Here we concentrate our attention on describing the analytic structure and singularities.

Let $\omega := (q_0, \ldots, q_n)$ be a weight vector, that is, a finite set of coprime positive integers. There is a natural action of the multiplicative group $\mathbb{C}^*$ on $\mathbb{C}^{n+1} \setminus \{0\}$ given by

$$(x_0, \ldots, x_n) \longmapsto (t^{q_0} x_0, \ldots, t^{q_n} x_n).$$

The set of orbits $\frac{\mathbb{C}^{n+1}\setminus\{0\}}{\mathbb{C}^*}$ under this action is denoted by $\mathbb{P}^n_\omega$ (or $\mathbb{P}^n(\omega)$ in case of complicated weight vectors) and is called the *weighted projective space* of type ω. The class of a nonzero element $(x_0, \ldots, x_n) \in \mathbb{C}^{n+1}$ is denoted by $[x_0 : \ldots : x_n]_\omega$ and the weight vector is omitted if no ambiguity seems likely to arise. When $(q_0, \ldots, q_n) = (1, \ldots, 1)$ one obtains the usual projective space and the weight vector is always omitted. For $\mathbf{x} \in \mathbb{C}^{n+1} \setminus \{0\}$, the closure of $[\mathbf{x}]_\omega$ in $\mathbb{C}^{n+1}$ is obtained by adding the origin and it is an algebraic curve.

As in the classical case, weighted projective spaces can be endowed with an **analytic structure**. However, in general they contain cyclic quotient singularities. To understand this structure, consider the decomposition $\mathbb{P}^n_\omega = U_0 \cup \cdots \cup U_n$, where U_i is the open set consisting of all elements $[x_0 : \ldots : x_n]_\omega$ with $x_i \neq 0$. The map

$$\widetilde{\psi}_0 : \mathbb{C}^n \longrightarrow U_0, \quad \widetilde{\psi}_0(x_1, \cdots, x_n) := [1 : x_1 : \ldots : x_n]_\omega$$

is clearly a surjective analytic map but it is not a chart since injectivity fails. In fact, $[1 : x_1 : \ldots : x_n]_\omega = [1 : x'_1 : \ldots , x'_n]_\omega$ if and only if there exists $\xi \in \mu_{q_0}$ such

[a]The notation $(i_1, \ldots, i_k) = \gcd(i_1, \ldots, i_k)$ is used in case of complicated or long formulas.

that $x'_i = \xi^{q_i} x_i$ for all $i = 1, \ldots, n$. Hence the map above induces the isomorphism

$$\begin{array}{rcl} \psi_0 \ : \ X(q_0; q_1, \ldots, q_n) & \longrightarrow & U_0, \\ {[(x_1, \ldots, x_n)]} & \mapsto & [1 : x_1 : \ldots : x_n]_\omega. \end{array} \tag{2.1}$$

Analogously, $X(q_i; q_0, \ldots, \widehat{q_i}, \ldots, q_n) \cong U_i$ under the obvious analytic map. Since the *transition maps* are analytic, $\mathbb{P}^n_\omega$ is an analytic space with cyclic quotient singularities as claimed.

For different weight vectors ω and ω' the corresponding spaces $\mathbb{P}^n_\omega$ and $\mathbb{P}^n_{\omega'}$ can be isomorphic, i.e., **weights can be simplified**. Consider

$$d_i := \gcd(q_0, \ldots, \widehat{q_i}, \ldots, q_n),$$
$$e_i := \operatorname{lcm}(d_0, \ldots, \widehat{d_i}, \ldots, d_n).$$

Note that $e_i | q_i$, $\gcd(d_i, d_j) = 1$ for $i \neq j$, and $\gcd(e_i, d_i) = 1$.

Proposition 2.1. *Using the notation above, the following map is an isomorphism:*

$$\begin{array}{rcl} \mathbb{P}^n\big(q_0, \ldots, q_n\big) & \longrightarrow & \mathbb{P}^n\big(\frac{q_0}{e_0}, \ldots, \frac{q_n}{e_n}\big), \\ {[x_0 : \ldots : x_n]} & \mapsto & \big[x_0^{d_0} : \ldots : x_n^{d_n}\big]. \end{array}$$

Proof. Since $\gcd(q_i, d_i) = 1$ we have $e_i = d_0 \cdot \ldots \cdot \widehat{d_i} \cdot \ldots \cdot d_n$. Now from Lemma 1.2 one has the following sequence of isomorphisms of analytic spaces:

$$X(q_0; q_1, \ldots, q_n) \overset{\text{id}}{=} X(q_0; \tfrac{q_1}{d_0}, \tfrac{q_2}{d_0}, \ldots, \tfrac{q_n}{d_0}) \overset{\text{1st}}{\cong} X(\tfrac{q_0}{d_1}; \tfrac{q_1}{d_0}, \tfrac{q_2}{d_0 d_1}, \ldots, \tfrac{q_n}{d_0 d_1})$$
$$\overset{\text{2nd}}{\cong} X(\tfrac{q_0}{d_1 d_2}; \tfrac{q_1}{d_0 d_2}, \tfrac{q_2}{d_0 d_1}, \tfrac{q_3}{d_0 d_1 d_2}, \ldots, \tfrac{q_n}{d_0 d_1 d_2}) \overset{\text{3rd}}{\cong} \cdots \overset{n\text{th}}{\cong} X(\tfrac{q_0}{e_0}; \tfrac{q_1}{e_1}, \ldots, \tfrac{q_n}{e_n})$$

Observe that in the i-th step, we divide the corresponding weight vector by d_i except the i-th coordinate and hence the associated map is

$$[(x_1, \ldots, x_i, \ldots, x_n)] \longmapsto [(x_1, \ldots, x_i^{d_i}, \ldots, x_n)].$$

Therefore $[1 : x_1 : \ldots : x_n]_\omega \longmapsto [1 : x_1^{d_1} : \ldots : x_n^{d_n}]_{\omega'}$ is an isomorphism by composition. Analogously one proceeds with the other charts. $\square$

Remark 2.1. Note that, due to the preceding proposition, one can always assume the weight vector satisfies $\gcd(q_0, \ldots, \widehat{q_i}, \ldots, q_n) = 1$, for $i = 0, \ldots, n$. This notion is called *well-formedness* in [8] . In particular, $\mathbb{P}^1(q_0, q_1) \cong \mathbb{P}^1$ and for $n = 2$ we can take (q_0, q_1, q_2) relatively prime numbers. In higher dimension the situation is a bit more complicated.

3. Weighted Blow-ups and Embedded Q-Resolutions

Classically an embedded resolution of $\{f = 0\} \subset \mathbb{C}^n$ is a proper map $\pi : X \to (\mathbb{C}^n, 0)$ from a smooth variety X satisfying, among other conditions, that $\pi^{-1}(\{f = 0\})$ is a normal crossing divisor. To weaken the condition on the preimage of the singularity we allow the new ambient space X to contain abelian quotient singularities and the divisor $\pi^{-1}(\{f=0\})$ to have *normal crossings* over this kind of varieties. This notion of normal crossing divisor on V-manifolds was first introduced by Steenbrink in [13].

Definition 3.1. Let X be a V-manifold with abelian quotient singularities. A hypersurface D on X is said to be with $\mathbb{Q}$-*normal crossings* if it is locally isomorphic to the quotient of a union of coordinate hyperplanes under a group action of type $(\mathbf{d};A)$. That is, given $x\in X$, there is an isomorphism of germs $(X,x)\simeq(X(\mathbf{d};A),[0])$ such that $(D,x)\subset(X,x)$ is identified under this morphism with a germ of the form

$$\big(\{[\mathbf{x}]\in X(\mathbf{d};A)\mid x_1^{m_1}\cdot\ldots\cdot x_k^{m_k}=0\},[(0,\ldots,0)]\big).$$

Let $M=\mathbb{C}^{n+1}/\mu_{\mathbf{d}}$ be an abelian quotient space not necessarily cyclic or written in normalized form. Consider $H\subset M$ an analytic subvariety of codimension one.

Definition 3.2. An *embedded* ***Q****-resolution* of $(H,0)\subset(M,0)$ is a proper analytic map $\pi:X\to(M,0)$ such that:

(1) X is a V-manifold with abelian quotient singularities.
(2) π is an isomorphism over $X\setminus\pi^{-1}(\mathrm{Sing}(H))$.
(3) $\pi^{-1}(H)$ is a hypersurface with $\mathbb{Q}$-normal crossings on X.

Remark 3.1. Let $f:(M,0)\to(\mathbb{C},0)$ be a non-constant analytic function germ. Consider $(H,0)$ the hypersurface defined by f on $(M,0)$. Let $\pi:X\to(M,0)$ be an embedded **Q**-resolution of $(H,0)\subset(M,0)$. Then $\pi^{-1}(H)=(f\circ\pi)^{-1}(0)$ is locally given by a function of the form $x_1^{m_1}\cdot\ldots\cdot x_k^{m_k}:X(\mathbf{d};A)\to\mathbb{C}$ for some type $(\mathbf{d};A)$.

Classical blow-up of $\mathbb{C}^{n+1}$. Using multi-index notation we consider

$$\widehat{\mathbb{C}}^{n+1}:=\big\{(\mathbf{x},[\mathbf{u}])\in\mathbb{C}^{n+1}\times\mathbb{P}^n\mid\mathbf{x}\in\overline{[\mathbf{u}]}\big\}.$$

Then $\pi:\widehat{\mathbb{C}}^{n+1}\to\mathbb{C}^{n+1}$ is an isomorphism over $\widehat{\mathbb{C}}^{n+1}\setminus\pi^{-1}(0)$. The *exceptional divisor* $E:=\pi^{-1}(0)$ is identified with $\mathbb{P}^n$. The space $\widehat{\mathbb{C}}^{n+1}=U_0\cup\cdots\cup U_n$ can be covered with $n+1$ charts each of them isomorphic to $\mathbb{C}^{n+1}$. For instance, the following map defines an isomorphism:

$$\begin{array}{rcl}\mathbb{C}^{n+1} & \longrightarrow & U_0=\{u_0\neq0\}\ \subset\ \widehat{\mathbb{C}}^{n+1},\\ \mathbf{x} & \mapsto & \big((x_0,x_0x_1,\ldots,x_0x_n),[1:x_1:\ldots:x_n]\big).\end{array}$$

Weighted $(p_0,\ldots,p_n)$-blow-up of $\mathbb{C}^{n+1}$. Let $\omega=(p_0,\ldots,p_n)$ be a weight vector. As above, consider the space

$$\widehat{\mathbb{C}}^{n+1}(\omega):=\big\{(\mathbf{x},[\mathbf{u}]_\omega)\in\mathbb{C}^{n+1}\times\mathbb{P}^n_\omega\mid\mathbf{x}\in\overline{[\mathbf{u}]}_\omega\big\}.$$

Here the condition about the closure means that $\exists t\in\mathbb{C},\quad x_i=t^{p_i}u_i,\quad i=0,\ldots,n.$ Then the natural projection $\pi:\widehat{\mathbb{C}}^{n+1}(\omega)\to\mathbb{C}^{n+1}$ is an isomorphism over $\widehat{\mathbb{C}}^{n+1}(\omega)\setminus\pi^{-1}(0)$ and the exceptional divisor $E:=\pi^{-1}(0)$ is identified with $\mathbb{P}^n_\omega$. Again the space $\widehat{\mathbb{C}}^{n+1}(\omega) = U_0 \cup \cdots \cup U_n$ can be covered with $n+1$ charts. However, $\varphi_0 : \mathbb{C}^{n+1} \to U_0$ given by

$$\begin{array}{rcl} \mathbb{C}^{n+1} & \xrightarrow{\varphi_0} & U_0 = \{u_0 \neq 0\} \;\subset\; \widehat{\mathbb{C}}^{n+1}(\omega), \\ \mathbf{x} & \mapsto & \big((x_0^{p_0}, x_0^{p_1} x_1, \ldots, x_0^{p_n} x_n), [1 : x_1 : \ldots : x_n]_\omega\big), \end{array}$$

is surjective but not injective. In fact $\varphi_0(\mathbf{x}) = \varphi_0(\mathbf{y})$ if and only if

$$\exists \xi \in \mu_{p_0} : \begin{cases} y_0 = \xi^{-1} x_0, \\ y_i = \xi^{p_i} x_i, \qquad i = 1, \ldots, n. \end{cases}$$

Hence the previous map φ_0 induces an isomorphism

$$X(p_0; -1, p_1, \ldots, p_n) \longrightarrow U_0.$$

Note that these charts are compatible with the ones described in (2.1) for the weighted projective space. In U_0 the exceptional divisor is $\{x_0 = 0\}$ and the first chart of $\mathbb{P}^n_\omega$ is the quotient space $X(p_0; p_1, \ldots, p_n)$.

Weighted ω-blow-up of $X(\mathbf{d}; A)$. The action $\mu_{\mathbf{d}}$ on $\mathbb{C}^{n+1}$ extends naturally to an action on $\widehat{\mathbb{C}}^{n+1}(\omega)$ as follows,

$$\boldsymbol{\xi}_{\mathbf{d}} \cdot \big(\mathbf{x}, [\mathbf{u}]_\omega\big) \xmapsto{\mu_d} \left((\boldsymbol{\xi}_{\mathbf{d}}^{\mathbf{a}_0} x_0, \ldots, \boldsymbol{\xi}_{\mathbf{d}}^{\mathbf{a}_n} x_n), [\boldsymbol{\xi}_{\mathbf{d}}^{\mathbf{a}_0} u_0 : \ldots : \boldsymbol{\xi}_{\mathbf{d}}^{\mathbf{a}_n} u_n]_\omega\right).$$

Let $\widehat{X(\mathbf{d}; A)}(\omega) := \widehat{\mathbb{C}}^{n+1}(\omega)/\mu_{\mathbf{d}}$ denote the quotient space under this action. Then the induced projection

$$\pi : \widehat{X(\mathbf{d}; A)}(\omega) \longrightarrow X(\mathbf{d}; A), \quad \big[(\mathbf{x}, [\mathbf{u}]_\omega)\big]_{(\mathbf{d};A)} \mapsto [\mathbf{x}]_{(\mathbf{d};A)}$$

is an isomorphism over $\widehat{X(\mathbf{d}; A)}(\omega) \setminus \pi^{-1}([0])$ and the exceptional divisor $E := \pi^{-1}([0])$ is identified with $\mathbb{P}^n_\omega/\mu_{\mathbf{d}}$.

The action $\mu_{\mathbf{d}}$ above respects the charts of $\widehat{\mathbb{C}}^{n+1}(\omega)$ so that the new ambient space can be covered as

$$\widehat{X(\mathbf{d}; A)}(\omega) = \widehat{U}_0 \cup \cdots \cup \widehat{U}_n,$$

where $\widehat{U}_i := U_i/\mu_{\mathbf{d}} = \{u_i \neq 0\}$.

Let us study for instance the first chart. By using φ_0 one identifies U_0 with

$$X(p_0; -1, p_1, \ldots, p_n).$$

Let us consider the space $X(p_0\mathbf{d}; (\mathbf{a}_0 | p_0\mathbf{a}_1 - p_1\mathbf{a}_0 | \ldots | p_0\mathbf{a}_n - p_n\mathbf{a}_0))$. Since the following diagram commutes:

$$\begin{array}{ccc} (x_0, x_1, \ldots, x_n) & \mapsto & (\boldsymbol{\xi}_{p_0\mathbf{d}}^{\mathbf{a}_0} x_0, \boldsymbol{\xi}_{p_0\mathbf{d}}^{p_0\mathbf{a}_1 - p_1\mathbf{a}_0} x_1, \ldots, \boldsymbol{\xi}_{p_0\mathbf{d}}^{p_0\mathbf{a}_n - p_n\mathbf{a}_0} x_n) \\ \downarrow & & \downarrow \\ (x_0^{p_0}, x_0^{p_1} x_1, \ldots, x_0^{p_n} x_n) & \mapsto & (\boldsymbol{\xi}_{p_0\mathbf{d}}^{p_0\mathbf{a}_0} x_0^{p_0}, \boldsymbol{\xi}_{p_0\mathbf{d}}^{p_0\mathbf{a}_1} x_0^{p_1} x_1, \ldots, \boldsymbol{\xi}_{p_0\mathbf{d}}^{p_0\mathbf{a}_n} x_0^{p_n} x_n) \end{array}$$

where the vertical arrows are the chart morphisms, the upper arrow corresponds to the action defining $X(p_0\mathbf{d}; (\mathbf{a}_0 | p_0\mathbf{a}_1 - p_1\mathbf{a}_0 | \ldots | p_0\mathbf{a}_n - p_n\mathbf{a}_0))$ and the lower arrow to the one defining $X(p_0\mathbf{d}; p_0 A) = X(\mathbf{d}; A)$. Since the chart is an isomorphism for $X(p_0; -1, p_1, \ldots, p_n)$ we conclude that

$$X\left(\left(\frac{p_0}{p_0\mathbf{d}}\right); \left(\frac{-1}{\mathbf{a}_0}\left|\frac{p_1}{p_0\mathbf{a}_1 - p_1\mathbf{a}_0}\right|\cdots\left|\frac{p_n}{p_0\mathbf{a}_n - p_n\mathbf{a}_0}\right.\right)\right)$$

is isomorphic to $\widehat{U}_0$ and the isomorphism is defined by

$$[\mathbf{x}] \overset{\widehat{\varphi}_0}{\longmapsto} \big[\big((x_0^{p_0}, x_0^{p_1}x_1, \ldots, x_0^{p_n}x_n), [1 : x_1 : \ldots : x_n]_\omega\big)\big].$$

For $i = 1, \ldots, n$, one proceeds analogously. As for the the exceptional divisor $E = \pi^{-1}(0) = \mathbb{P}^n_\omega/\mu_{\mathbf{d}}$, it is as usual covered by $\widehat{V}_0 \cup \cdots \cup \widehat{V}_n$ so that these charts are compatible with the ones of $\widehat{X(\mathbf{d}; A)}(\omega)$ in the sense that $\widehat{V}_i = \widehat{U}_i|_{\{x_i=0\}}$, $i = 0, \ldots, n$. Hence, for example,

$$\widehat{V}_0 \cong X\left(\begin{array}{c|ccc} p_0 & p_1 & \cdots & p_n \\ p_0\mathbf{d} & p_0\mathbf{a}_1 - p_1\mathbf{a}_0 & \cdots & p_0\mathbf{a}_n - p_n\mathbf{a}_0 \end{array}\right).$$

Example 3.1. We study the weighted blow-up

$$\pi := \pi_{(d;a,b),\omega} : \widehat{X(d; a, b)}_\omega \longrightarrow X(d; a, b)$$

of the origin of a normalized space $X(d; a, b)$ with respect to $\omega = (p, q)$. The new space is covered by

$$\widehat{U}_1 \cup \widehat{U}_2 = X\left(\begin{array}{c|cc} p & -1 & q \\ pd & a & pb - qa \end{array}\right) \cup X\left(\begin{array}{c|cc} q & p & -1 \\ qd & qa - pb & b \end{array}\right)$$

and the charts are given by

$$\text{First chart}\quad\left|\begin{array}{ccc} X\left(\begin{array}{c|cc} p & -1 & q \\ pd & a & pb - qa \end{array}\right) & \longrightarrow & \widehat{U}_1, \\ \big[(x, y)\big] \mapsto \big[\big((x^p, x^q y), [1 : y]_\omega\big)\big]_{(d;a,b)}. \end{array}\right.$$

$$\text{Second chart}\quad\left|\begin{array}{ccc} X\left(\begin{array}{c|cc} q & p & -1 \\ qd & qa - pb & b \end{array}\right) & \longrightarrow & \widehat{U}_2, \\ \big[(x, y)\big] \mapsto \big[\big((xy^p, y^q), [x : 1]_\omega\big)\big]_{(d;a,b)}. \end{array}\right.$$

The exceptional divisor $E = \pi^{-1}_{(d;a,b),\omega}(0)$ is identified with the quotient space $\mathbb{P}^1_\omega(d; a, b) := \mathbb{P}^1_\omega/\mu_d$ which is isomorphic to $\mathbb{P}^1$ under the map

$$\begin{array}{rcl} \mathbb{P}^1_\omega(d; a, b) & \longrightarrow & \mathbb{P}^1, \\ [x : y]_\omega & \mapsto & [x^{dq/e} : y^{dp/e}], \end{array}$$

where $e := \gcd(d, pb - qa)$. Again the singular points are cyclic and correspond to the origins. To normalize these quotient spaces, note that $e = \gcd(d, pb - qa) = \gcd(d, -q + \beta pb) = \gcd(pd, -q + \beta pb) = \gcd(qd, p - qa\mu)$, where $\beta a \equiv \mu b \equiv 1 \mod d$.

Then another expression for the two charts are given below.

$$
\begin{array}{r|l}
\text{First chart} & X\left(\dfrac{pd}{e};1,\dfrac{-q+\beta pb}{e}\right) \longrightarrow \widehat{U}_1, \\
& \big[(x^e,y)\big] \mapsto \big[((x^p,x^q y),[1:y]_\omega)\big]_{(d;a,b)}. \\
\text{Second chart} & X\left(\dfrac{qd}{e};\dfrac{-p+\mu qa}{e},1\right) \longrightarrow \widehat{U}_2, \\
& \big[(x,y^e)\big] \mapsto \big[((xy^p,y^q),[x:1]_\omega)\big]_{(d;a,b)}.
\end{array}
\tag{3.1}
$$

Both quotient spaces are now written in their normalized form.

Example 3.2. Let $\pi := \pi_\omega : \widehat{\mathbb{C}}^3_\omega \to \mathbb{C}^3$ be the weighted blow-up at the origin with respect to $\omega = (p,q,r)$, $\gcd(\omega) = 1$. The new space is covered as

$$\widehat{\mathbb{C}}^3_\omega = U_1 \cup U_2 \cup U_3 = X(p;-1,q,r) \cup X(q;p,-1,r) \cup X(r;p,q,-1),$$

and the charts are given by

$$
\begin{aligned}
X(p;-1,q,r) &\longrightarrow U_1 : [(x,y,z)] \mapsto ((x^p,x^q y,x^r z),[1:y:z]_\omega),\\
X(q;p,-1,r) &\longrightarrow U_2 : [(x,y,z)] \mapsto ((xy^p,y^q,y^r z),[x:1:z]_\omega),\\
X(r;p,q,-1) &\longrightarrow U_3 : [(x,y,z)] \mapsto ((xz^p,yz^q,z^r),[x:y:1]_\omega).
\end{aligned}
\tag{3.2}
$$

In general $\widehat{\mathbb{C}}^3_\omega$ has three lines of (cyclic quotient) singular points located at the three axes of the exceptional divisor $\pi_\omega^{-1}(0) \simeq \mathbb{P}^2_\omega$. Namely, a generic point in $x = 0$ is a cyclic point of type $\mathbb{C} \times X(\gcd(q,r);p,-1)$.

Note that although the quotient spaces are written in their normalized form, the exceptional divisor can be simplified:

$$
\begin{aligned}
\mathbb{P}^2(p,q,r) &\longrightarrow \mathbb{P}^2\left(\frac{p}{(p,r)\cdot(p,q)},\frac{q}{(q,p)\cdot(q,r)},\frac{r}{(r,p)\cdot(r,q)}\right),\\
[x:y:z] &\mapsto [x^{\gcd(q,r)}:y^{\gcd(p,r)}:z^{\gcd(p,q)}].
\end{aligned}
$$

Using just a weighted blow-up of this kind, one can find an embedded $\mathbf{Q}$-resolution for Brieskorn-Pham surfaces singularities, i.e. $x^a + y^b + z^c = 0$. For simplicity, let us assume a,b,c are pairwise coprime and consider the weight $\omega := (p,q,r)$ where $p := bc$, $q := ac$ and $r = ab$. The blow-up π_ω gives a $\mathbf{Q}$-resolution of this singularity. The exceptional divisor E is isomorphic to $\mathbb{P}^2_\omega \cong \mathbb{P}^2$. The intersection of the strict transform with E is a generic line in $\mathbb{P}^2$. Note that the space $\widehat{\mathbb{C}}^3_\omega$ has three singular points of type $(bc;-1,ac,ab)$, $(ac;bc,-1,ab)$ and $(ab;bc,ac,-1)$, and three strata (isomorphic to $\mathbb{C}^*$) with transversal singularities of type $X(a;bc,-1)$, $X(b;ac,-1)$ and $X(c;ab,-1)$.

4. Cartier and Weil Divisors on V-Manifolds: $\mathbb{Q}$-Divisors

The aim of this section is to show that when X is a V-manifold there is an isomorphism of $\mathbb{Q}$-vector spaces between Cartier and Weil divisors, see Theorem 4.2 below. It is explained in Summary 4.1 how to write explicitly a $\mathbb{Q}$-Weil divisor as a $\mathbb{Q}$-Cartier divisor. Also, the case of the exceptional divisor of a weighted blow-up is treated in Example 4.3.

4.1. *Divisors on complex analytic varieties*

Let X be an irreducible complex analytic variety. As usual, consider $\mathcal{O}_X$ the structure sheaf of X and $\mathcal{K}_X$ the sheaf of total quotient rings of $\mathcal{O}_X$. Denote by $\mathcal{K}_X^*$ the (multiplicative) sheaf of invertible elements in $\mathcal{K}_X$. Similarly $\mathcal{O}_X^*$ is the sheaf of invertible elements in $\mathcal{O}_X$. Note that $\mathcal{O}_X$ (resp. $\mathcal{K}_X$) is the sheaf of holomorphic (resp. meromorphic) functions and $\mathcal{O}_X^*$ (resp. $\mathcal{K}_X^*$) is the sheaf of nowhere-vanishing holomorphic (resp. non-zero meromorphic) functions.

Remark 4.1. By a *complex analytic variety* we mean a reduced complex space. A *subvariety* V of X is a reduced closed complex subspace of X, or equivalently, an analytic set in X, cf. [6]. An irreducible subvariety V corresponds to a prime ideal in the ring of sections of any local complex model space meeting V.

Definition 4.1. A *Cartier divisor* on X is a global section of the sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$, that is, an element in $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*) = H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. Any Cartier divisor can be represented by giving an open covering $\{U_i\}_{i\in I}$ of X and, for all $i \in I$, an element $f_i \in \Gamma(U_i, \mathcal{K}_X^*)$ such that

$$\frac{f_i}{f_j} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*), \quad \forall i, j \in I.$$

Two systems $\{(U_i, f_i)\}_{i\in I}$, $\{(V_j, g_j)\}_{j\in J}$ represent the same Cartier divisor if and only if on $U_i \cap V_j$, f_i and g_j differ by a multiplicative factor in $\mathcal{O}_X(U_i \cap V_j)^*$. The abelian group of Cartier divisors on X is denoted by $\operatorname{CaDiv}(X)$. If $D := \{(U_i, f_i)\}_{i\in I}$ and $E := \{(V_j, g_j)\}_{j\in J}$ then $D + E = \{(U_i \cap V_j, f_i g_j)\}_{i\in I, j\in J}$.

The functions f_i above are called *local equations* of the divisor on U_i. A Cartier divisor on X is *effective* if it can be represented by $\{(U_i, f_i)\}_i$ with all local equations $f_i \in \Gamma(U_i, \mathcal{O}_X)$.

Any global section $f \in \Gamma(X, \mathcal{K}_X^*)$ determines a *principal* Cartier divisor $(f)_X := \{(X, f)\}$ by taking all local equations equal to f. That is, a Cartier divisor is principal if it is in the image of the natural map $\Gamma(X, \mathcal{K}_X^*) \to \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. Two Cartier divisors D and E are *linearly equivalent*, denoted by $D \sim E$, if they differ by a principal divisor. The *Picard group* $\operatorname{Pic}(X)$ denotes the group of linear equivalence classes of Cartier divisors.

The *support* of a Cartier divisor D, denoted by $\mathrm{Supp}(D)$ or $|D|$, is the subset of X consisting of all points x such that a local equation for D is not in $\mathcal{O}^*_{X,x}$. The support of D is a closed subset of X.

Definition 4.2. A *Weil divisor* on X is a locally finite linear combination with integral coefficients of irreducible subvarieties of codimension one. The abelian group of Weil divisors on X is denoted by $\mathrm{WeDiv}(X)$. If all coefficients appearing in the sum are non-negative, the Weil divisor is called *effective*.

Remark 4.2. In the algebraic category the locally finite sum of Definition 4.2 is automatically finite. Therefore $\mathrm{WeDiv}(X)$ is the free abelian group on the codimension one irreducible algebraic subvarieties of X. Similar considerations hold if X is a compact analytic variety.

Given a Cartier divisor there is a Weil divisor associated with it. To see this, the notion of order of a divisor along an irreducible subvariety of codimension one is needed.

Order function. Let $V \subset X$ be an irreducible subvariety of codimension one. It corresponds to a prime ideal in the ring of sections of any local complex model space meeting V. The *local ring of X along V*, denoted by $\mathcal{O}_{X,V}$, is the localization of such ring of sections at the corresponding prime ideal; it is a one-dimensional local domain.

For a given $f \in \mathcal{O}_{X,V}$ define $\mathrm{ord}_V(f)$ to be

$$\mathrm{ord}_V(f) := \mathrm{length}_{\mathcal{O}_{X,V}} \left(\frac{\mathcal{O}_{X,V}}{\langle f \rangle} \right).$$

This determines a well-defined group homomorphism $\mathrm{ord}_V : \Gamma(X, \mathcal{K}^*_X) \to \mathbb{Z}$ satisfying, for a given $f \in \Gamma(X, \mathcal{K}^*_X)$, the following local finiteness property:

$$\forall x \in X,\ \exists U_x \subset X \text{ open neighborhood of } x \mid \#\{\mathrm{ord}_V(f) \neq 0 \mid V \cap U_x \neq \emptyset\} < +\infty.$$

The previous length, X being a complex analytic variety of dimension $n \geq 2$, can be computed as follows. Choose $x \in V$ such that x is smooth in X and (V, x) defines an irreducible germ. This germ is the zero set of an irreducible $g \in \mathcal{O}_{X,x}$. Then

$$\mathrm{ord}_V(f) = \mathrm{ord}_{V,x}(f), \tag{4.1}$$

where $\mathrm{ord}_{V,x}(f)$ is the classical order of a meromorphic function at a smooth point with respect to an irreducible subvariety of codimension one; it is known to be given by the equality $f = g^{\mathrm{ord}} \cdot h \in \mathcal{O}_{X,x}$ where $h \nmid g$. The same applies if X is 1-dimensional and smooth.

Remark 4.3. The order $\mathrm{ord}_{V,x}(f)$ does not depend on the defining equation g, as long as we choose g irreducible. In fact, two irreducible $g, g' \in \mathcal{O}_{X,x}$ with $V(g) = V(g')$ only differ by a unit in $\mathcal{O}_{X,x}$. Moreover, $\mathrm{ord}_{V,x}(f)$ does not depend on x, since the set V_{red} of regular points is connected if V is irreducible.

Now if D is a Cartier divisor on X, one writes $\operatorname{ord}_V(D) = \operatorname{ord}_V(f_i)$ where f_i is a local equation of D on any open set U_i with $U_i \cap V \neq \emptyset$. This is well defined since f_i is uniquely determined up to multiplication by units and the order function is a homomorphism. Define the *associated Weil divisor* of a Cartier divisor D by

$$\begin{aligned} T_X : \operatorname{CaDiv}(X) &\longrightarrow \operatorname{WeDiv}(X), \\ D \qquad &\mapsto \sum_{V \subset X} \operatorname{ord}_V(D) \cdot [V], \end{aligned}$$

where the sum is taken over all codimension one irreducible subvarieties V of X. The previous sum is locally finite, i.e. for any $x \in X$ there exists an open neighborhood U such that the set $\{\operatorname{ord}_V(D) \neq 0 \mid V \cap U \neq \emptyset\}$ is finite. By the additivity of the order function, the mapping T_X is a homomorphism of abelian groups.

A Weil divisor is *principal* if it is the image of a principal Cartier divisor under T_X; they form a subgroup of $\operatorname{WeDiv}(X)$. If $\operatorname{Cl}(X)$ denotes the quotient group of their equivalence classes, then T_X induces a morphism $\operatorname{Pic}(X) \to \operatorname{Cl}(X)$.

These two homomorphisms (T_X and the induced one) are in general neither injective nor surjective. In this sense one has the following result.

Theorem 4.1. (cf. [7]). *If X is normal (resp. locally factorial) then the previous maps* $\operatorname{CaDiv}(X) \to \operatorname{WeDiv}(X)$ *and* $\operatorname{Pic}(X) \to \operatorname{Cl}(X)$ *are injective (resp. bijective). The image of the first map is the subgroup of locally principal*[b] *Weil divisors.*

Remark 4.4. Locally factorial essentially means that every local ring $\mathcal{O}_{X,x}$ is a unique factorization domain. In particular, every smooth analytic variety is locally factorial. In such a case, Cartier and Weil divisors are identified and denoted by $\operatorname{Div}(X) := \operatorname{CaDiv}(X) = \operatorname{WeDiv}(X)$. Their equivalence classes coincide under this identification and we often write $\operatorname{Pic}(X) = \operatorname{Cl}(X)$.

Example 4.1. Let X be the surface in $\mathbb{C}^3$ defined by the equation $z^2 = xy$. The line $V = \{x = z = 0\}$ defines a Weil divisor which is not a Cartier divisor. In this case $\operatorname{Pic}(X) = 0$ and $\operatorname{Cl}(X) = \mathbb{Z}/(2)$. Note that X is normal but not locally factorial. However, the associated Weil divisor of $\{(X, x)\}$ is

$$T_X\big(\{(X,x)\}\big) = \sum_{\substack{Z \subset X,\ \text{irred} \\ \operatorname{codim}(Z)=1}} \operatorname{ord}_Z(x) \cdot [Z] = 2[V].$$

Thus $[V]$ is principal as an element in $\operatorname{WeDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and corresponds to the $\mathbb{Q}$-Cartier divisor $\frac{1}{2}\{(X, x)\}$.

This fact can be interpreted as follows. First note that identifying our surface X with $X(2; 1, 1)$ under $[(x, y)] \mapsto (x^2, y^2, xy)$, the previous Weil divisor corresponds to $D = \{x = 0\}$. Although $f = x$ defines a zero set on $X(2; 1, 1)$, it does not induce a function on the quotient space. However, $x^2 : X(2; 1, 1) \to \mathbb{C}$ is a well-defined

[b] A Weil divisor D on X is said to be *locally principal* if X can be covered by open sets U such that $D|_U$ is principal for each U.

function and gives rise to the same zero set as f. Hence as $\mathbb{Q}$-Cartier divisors one writes $D = \frac{1}{2}\{(X(2;1,1), x^2)\}$.

4.2. *Divisors on V-manifolds*

Example 4.1 above illustrates the general behavior of Cartier and Weil divisors on V-manifolds, namely Weil divisors are all locally principal over $\mathbb{Q}$. To prove it we need some preliminaries.

Lemma 4.1. *Let $B \subset \mathbb{C}^n$ be an open ball and let G be a finite group acting on B. Then one has $\mathrm{Cl}(B/G) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.*

Proof. Let $V \subset B/G =: U$ be an irreducible subvariety of codimension one. We shall prove that there exists $k \geq 1$ such that $k[V] \in \mathrm{WeDiv}(U)$ is principal.

Consider the natural projection $\pi : B \to U$. Then $W := \pi^{-1}(V)$ gives rise to an effective Weil divisor on the open ball B. There exists $f : B \to \mathbb{C}$ a holomorphic function such that $W = \{f = 0\} \subset B$. Thus,

$$V = \pi(W) = \{[\mathbf{x}] \mid \mathbf{x} \in B,\ f(\mathbf{x}) = 0\} = \{f = 0\} \subset U.$$

Moreover, by construction the holomorphic function f satisfies the following property

$$\forall P \in U,\ \big[f(P) = 0 \implies f(\sigma \cdot P) = 0, \forall \sigma \in G\big]. \tag{4.2}$$

Note that f does not necessarily defines an analytic function on U. This reflects the fact that, although V is given by just one equation, $[V] \in \mathrm{WeDiv}(U)$ is not principal, see Example 4.1. Now the main idea is to change f by another holomorphic function F such that $V = \{F = 0\}$ but now with $F \in \Gamma(U, \mathcal{O}_U)$.

Let us consider $F = \prod_{\sigma \in G} f^\sigma$ where $f^\sigma(\mathbf{x}) = f(\sigma \cdot \mathbf{x})$; clearly it verifies the previous conditions. Then $\{(U, F)\}$ is a principal Cartier divisor and its associated Weil divisor is

$$T_U\big(\{(U,F)\}\big) = \sum_{\substack{Z \subset U, \text{ irred} \\ \mathrm{codim}(Z)=1}} \mathrm{ord}_Z(F) \cdot [Z] = \mathrm{ord}_V(F) \cdot [V].$$

Note that $\mathrm{ord}_Z(F) \neq 0$ implies $Z = V$, since V is irreducible. □

Remark 4.5. Note that the proof of this result is based on an idea extracted from [5] .

Theorem 4.2. *Let X be a V-manifold. Then the notion of Cartier and Weil divisor coincide over $\mathbb{Q}$. More precisely, the linear map*

$$T_X \otimes 1 : \mathrm{CaDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathrm{WeDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism of $\mathbb{Q}$-vector spaces. In particular, for a given Weil divisor D on X there always exists $k \in \mathbb{Z}$ such that $kD \in \mathrm{CaDiv}(X)$.

Proof. The variety X is normal and then Theorem 4.1 applies. Therefore the linear map $T_X \otimes 1$ is injective and its image is the $\mathbb{Q}$-vector space generated by the locally principal Weil divisors on X.

Let $V \subset X$ be an irreducible subvariety of codimension one. Consider $\{U_i\}_i$ an open covering of X such that U_i is analytically isomorphic to B_i/G_i where $B_i \subset \mathbb{C}^n$ is an open ball and G_i is a finite subgroup of $GL(n,\mathbb{C})$. By Lemma 4.1, $\mathrm{Cl}(U_i) \otimes \mathbb{Q} = 0$ for all i.

Thus $[V|_{U_i}]$ is principal as an element in $\mathrm{WeDiv}(U_i) \otimes_{\mathbb{Z}} \mathbb{Q}$ which implies that V is locally principal over $\mathbb{Q}$ and hence belongs to the image of $T_X \otimes 1$. $\square$

Definition 4.3. Let X be a V-manifold. The vector space of $\mathbb{Q}$-Cartier divisors is identified under T_X with the vector space of $\mathbb{Q}$-Weil divisors. A $\mathbb{Q}$-*divisor* on X is an element in $\mathrm{CaDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{WeDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. The set of all $\mathbb{Q}$-divisors on X is denoted by $\mathbb{Q}\text{-Div}(X)$.

4.3. *Writing a Weil divisor as a $\mathbb{Q}$-Cartier divisor*

Following the proofs of Lemma 4.1 and Theorem 4.2, every Weil divisor on X can locally be written as $\mathbb{Q}$-Cartier divisor like

$$[V|_U] = \frac{1}{\mathrm{ord}_V(F)}\{(U,F)\},$$

where $F = \prod_{\sigma \in G} f^\sigma$ and $V \cap U = \{f = 0\}$ with $f : B \to \mathbb{C}$ being holomorphic on an open ball and satisfying (4.2).

The rest of this section is devoted to explicitly calculating $\mathrm{ord}_V(F)$. First, in Proposition 4.1 it is shown that F is essentially a power of f, if the latter is chosen properly. Then, $\mathrm{ord}_V(F)$ is computed in Proposition 4.2.

Proposition 4.1. *Let $f : B \to \mathbb{C}$ be a non-zero holomorphic function on an open ball $B \subset \mathbb{C}^n$ such that the germ $f_x \in \mathcal{O}_{B,x}$ is reduced for all $x \in B$. Let G be a finite subgroup of $GL(n,\mathbb{C})$ acting on B. As above consider $F = \prod_{\sigma \in G} f^\sigma$ where $f^\sigma(\mathbf{x}) = f(\sigma \cdot \mathbf{x})$. The following conditions are equivalent:*

(1) $\forall P \in B$, $\big[f(P) = 0 \implies f(\sigma \cdot P) = 0, \forall \sigma \in G\big]$.
(2) $\forall \sigma \in G$, $\exists h_\sigma \in \Gamma(B, \mathcal{O}_B^*)$ *such that* $f^\sigma = h_\sigma f$.
(3) $\exists h \in \Gamma(B, \mathcal{O}_B^*)$ *such that* $F = h f^{|G|}$.
(4) $\exists k \geq 1$, $\exists h \in \Gamma(B, \mathcal{O}_B^*)$ *such that* $h f^k \in \Gamma(B/G, \mathcal{O}_{B/G})$.

Proof. For (1) $\Rightarrow$ (2), first note that $f^\sigma \in IV(f)$ (the ideal of the zero locus of f). Now fix $x \in B$. Since f_x is reduced, there exists a holomorphic function h on a small enough open neighborhood of x such that as germs $(f^\sigma)_x = h_x f_x$. The order of the converging power series $(f^\sigma)_x$ and f_x are equal because the action is linear. Thus h_x is a unit in $\mathcal{O}_{B,x}$. In particular $\frac{f^\sigma}{f}$ is holomorphic and does not vanish at $x \in B$.

For (2) $\Rightarrow$ (3), consider $h = \prod_{\sigma \in G} h_\sigma$. Then one has

$$F = \prod_{\sigma \in G} f^\sigma = \prod_{\sigma \in G} (h_\sigma f) = \left(\prod_{\sigma \in G} h_\sigma \right) \cdot f^{|G|} = h f^{|G|}.$$

For (3) $\Rightarrow$ (4), since $F : B/G \to \mathbb{C}$ is analytic, take $k = |G|$. Finally, note that $\forall P \in B$, $f(P) = 0 \iff (hf^k)(P) = (hf^k)(\sigma \cdot P) = 0 \iff f(\sigma \cdot P) = 0$. Hence (4) $\Rightarrow$ (1) follows and the proof is complete. $\square$

This example shows that the reduceness condition in the statement of the previous result is necessary.

Example 4.2. Let $f = (x^2 + y)(x^2 - y)^3 \in \mathbb{C}[x, y]$ and consider the cyclic quotient space $M = X(2; 1, 1)$. Then $\{f = 0\} \subset M$ defines a zero set, i.e. condition (1) holds, but there are no $k \geq 1$ and $h \in \Gamma(B, \mathcal{O}_B^*)$ such that hf^k is a well-defined function over M.

Proposition 4.2. *Let $B \subset \mathbb{C}^n$ be an open ball and G a finite subgroup of $GL(n, \mathbb{C})$ acting on B. Let $V \subset B/G =: U$ be an irreducible subvariety of codimension one and consider*

$$F = \prod_{\sigma \in G} f^\sigma$$

where $f : B \to \mathbb{C}$ is a holomorphic function defining V.

If G is small and f is chosen so that $f_x \in \mathcal{O}_{B,x}$ is reduced $\forall x \in B$, then $\operatorname{ord}_V(F : U \to \mathbb{C}) = |G|$.

Proof. Choose $[P] \in V$ such that $[P]$ is smooth in U and $(V, [P])$ defines an irreducible germ; then $\operatorname{ord}_V(F) = \operatorname{ord}_{V,[P]}(F)$, see (4.1).

By Theorem 1.1, since G is small and $[P] \in U$ is smooth, using the covering $\pi : B \to U$, one finds an isomorphism of germs $(U, [P]) \cong (B/G_P, [P]) = (B, P)$ induced by the identity map. The germ $(V, [P])$ is converted under this isomorphism into (W, P) where W is the zero set of $f_P \in \mathcal{O}_{B,P}$.

By Proposition 4.1, there exists $h \in \Gamma(B, \mathcal{O}_B^*)$ such that $F = hf^{|G|}$. Then the required order is $\operatorname{ord}_{V,[P]}(F : U \to \mathbb{C}) = \operatorname{ord}_{V(f_P),P}(hf^{|G|} : B \to \mathbb{C}) = |G|$ as claimed. $\square$

Summary 4.1. Here we summarize how to write a Weil divisor as a $\mathbb{Q}$-Cartier divisor where X is an algebraic V-manifold.

(1) Write $D = \sum_{i \in I} a_i [V_i] \in \operatorname{WeDiv}(X)$, where $a_i \in \mathbb{Z}$ and $V_i \subset X$ irreducible. Also choose $\{U_j\}_{j \in J}$ an open covering of X such that $U_j = B_j/G_j$ where $B_j \subset \mathbb{C}^n$ is an open ball and G_j is a **small** finite subgroup of $GL(n, \mathbb{C})$.

(2) For each $(i, j) \in I \times J$ choose a reduce polynomial $f_{i,j} : U_j \to \mathbb{C}$ such that $V_i \cap U_j = \{f_{i,j} = 0\}$, then

$$[V_i|_{U_j}] = \frac{1}{|G_j|}\{(U_j, f_{i,j}^{|G_j|})\}.$$

(3) Identifying $\{(U_j, f_{i,j}^{|G_j|})\}$ with its image $\mathrm{CaDiv}(U_j) \hookrightarrow \mathrm{CaDiv}(X)$, one finally writes D as a sum of locally principal Cartier divisors over $\mathbb{Q}$,

$$D = \sum_{(i,j)\in I\times J} \frac{a_i}{|G_j|}\{(U_j, f_{i,j}^{|G_j|})\}.$$

We finish this section with an example where the exceptional divisor of a weighted blow-up (which is in general just a Weil divisor) is explicitly written as a $\mathbb{Q}$-Cartier divisor.

Example 4.3. Let X be a surface with abelian quotient singularities. Let $\pi : \widehat{X} \to X$ be the weighted blow-up at a point of type $(d; a, b)$ with respect to $\omega = (p, q)$. In general, the exceptional divisor $E := \pi^{-1}(0) \cong \mathbb{P}^1_\omega(d; a, b)$ is a Weil divisor on $\widehat{X}$ which does not correspond to a Cartier divisor. Let us write E as an element in $\mathrm{CaDiv}(\widehat{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

As in 3.1, assume $\pi := \pi_{(d;a,b),\omega} : \widehat{X(d; a, b)}_\omega \to X(d; a, b)$. Assume also that $\gcd(p, q) = 1$ and $(d; a, b)$ is normalized. Using the notation introduced in 3.1, the space $\widehat{X}$ is covered by $\widehat{U}_1 \cup \widehat{U}_2$ and the first chart is given by

$$\begin{array}{rcl} Q_1 := X\left(\frac{pd}{e}; 1, \frac{-q+\beta pb}{e}\right) & \longrightarrow & \widehat{U}_1, \\ \big[(x^e, y)\big] & \mapsto & \big[((x^p, x^q y), [1 : y]_\omega)\big]_{(d;a,b)}, \end{array} \tag{4.3}$$

where $e := \gcd(d, pb - qa)$, see 3.1 for details.

In the first chart, E is the Weil divisor $\{x = 0\} \subset Q_1$. Note that the type representing the space Q_1 is in a normalized form and hence the corresponding subgroup of $GL(2, \mathbb{C})$ is small.

Following Summary 4.1, the divisor $\{x = 0\} \subset Q_1$ is written as an element of the vector space $\mathrm{CaDiv}(Q_1) \otimes_{\mathbb{Z}} \mathbb{Q}$ like $\frac{e}{pd}\{(Q_1, x^{\frac{pd}{e}})\}$, which is mapped to $\frac{e}{pd}\{(\widehat{U}_1, x^d)\} \in \mathrm{CaDiv}(\widehat{U}_1) \otimes_{\mathbb{Z}} \mathbb{Q}$ under the isomorphism (4.3).

Analogously E in the second chart is $\frac{e}{qd}\{(\widehat{U}_2, y^d)\}$. Finally one writes the exceptional divisor of π as claimed,

$$E = \frac{e}{dp}\big\{(\widehat{U}_1, x^d), (\widehat{U}_2, 1)\big\} + \frac{e}{dq}\big\{(\widehat{U}_1, 1), (\widehat{U}_2, y^d)\big\} = \frac{e}{dpq}\big\{(\widehat{U}_1, x^{dq}), (\widehat{U}_2, y^{dp})\big\}.$$

5. Conclusions and Future Work

In order to define the intersection number between two divisors the following fact is essential. Let X be a compact connected Riemann surface and let $\pi : E \to X$ be a complex line bundle. Such a line bundle admits non-trivial meromorphic sections and the degree of such a section (the sum of the orders of the zeroes minus the sum of the orders of the poles) only depends on E. In order to define the intersection number between two divisors on a V-surface (one of them with compact support), we can assume (up to multiplication by an integer) that they are Cartier divisors. Then, it is possible to define the intersection number as the degree of the pull-back in the normalization of the compact divisor of the line bundle associated with the other divisor. These ideas will be developed in [1]. The following is an illustrative example.

Example 5.1. Let $X = \mathbb{P}^2_{(2,1,1)}$ and consider the Weil divisors $D_1 = \{x = 0\}$ and $D_2 = \{y = 0\}$. Let us compute the Weil divisor associated with $j^*_{D_1} D_2$, where $j_{D_1} : D_1 \hookrightarrow X$ is the inclusion. Following Summary 4.1, the divisor D_2 can be written as

$$D_2 = \frac{1}{2}(2D_2) = \frac{1}{2}\Big\{\big(U_1, \frac{y^2}{x^2}\big), \big(U_2, 1\big), \big(U_3, \frac{y^2}{z}\big)\Big\} \in \mathrm{CaDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where the covering is the one given in Section 2. Recall that $U_3 \cong X(2;1,1)$. By definition, since $D_1 \not\subseteq D_2$, the pull-back is

$$j^*_{D_1} D_2 = \frac{1}{2}\{(U_2 \cap D_1, 1), (U_3 \cap D_1, \frac{y^2}{z})\} = \frac{1}{2} E$$

and its associated Weil divisor on D_1 is

$$T_{D_1}(j^*_{D_1} D_2) = \frac{1}{2} \sum_{P \in E} \mathrm{ord}_P(E) \cdot [P] = \frac{1}{2} \mathrm{ord}_{[0:0:1]}(E) \cdot [0:0:1] = \frac{1}{2} \cdot [0:0:1].$$

Note that there is an isomorphism $U_3 \cap D_1 = X(2;1) \simeq \mathbb{C}$, $[y] \mapsto y^2$, and the function $y^2 : X(2;1) \to \mathbb{C}$ is converted into the identity map $\mathbb{C} \to \mathbb{C}$ under this isomorphism. Hence, using the third chart, $\mathrm{ord}_{[0:0:1]}(E) = 1$. It is natural to define the (global and local) intersection multiplicity as $D_1 \cdot D_2 = (D_1 \cdot D_2)_{[0:0:1]} = \frac{1}{2}$.

Acknowledgements.

We wish to express our gratitude to J.I. Cogolludo for his support. All authors are partially supported by the Spanish projects MTM2010-2010-21740-C02-02 and "E15 Grupo Consolidado Geometría" from the government of Aragón. Also, the second author, J. Martín-Morales, is partially supported by FQM-333, Junta de Andalucía.

References

[1] E. Artal, J. Martín-Morales, and J. Ortigas-Galindo, *Intersection theory on abelian-quotient V-surfaces and* **Q***-resolutions*, J. Singul. **8** (2014), 11–30.

[2] W.L. Baily, Jr., *The decomposition theorem for V-manifolds*, Amer. J. Math. **78** (1956), 862–888.

[3] I. Dolgachev, *Weighted projective varieties*, Group actions and vector fields (Vancouver, B.C., 1981), Lecture Notes in Math., vol. 956, Springer, Berlin, 1982, pp. 34–71.

[4] A. Fujiki, *On resolutions of cyclic quotient singularities*, Publ. Res. Inst. Math. Sci. **10** (1974/75), no. 1, 293–328.

[5] W. Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 2, Springer-Verlag, Berlin, 1998.

[6] H. Grauert and R. Remmert, *Coherent analytic sheaves*, Grundlehren der Mathematischen Wissenschaften, vol. 265, Springer-Verlag, Berlin, 1984.

[7] A. Grothendieck and J. Dieudonné, *Éléments de Géométrie Algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. **32** (1967), 361.

[8] A.R. Iano-Fletcher, *Working with weighted complete intersections*, Explicit birational geometry of 3-folds, London Math. Soc. Lecture Note Ser., vol. 281, Cambridge Univ. Press, Cambridge, 2000, pp. 101–173.

[9] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998.

[10] J. Martín-Morales, *Embedded* $\mathbf{Q}$*-Resolutions and Yomdin-Lê Surface Singularities*, PhD dissertation, IUMA-University of Zaragoza, December 2011, URL: `http://cud.unizar.es/martin`.

[11] D. Prill, *Local classification of quotients of complex manifolds by discontinuous groups*, Duke Math. J. **34** (1967), 375–386.

[12] I. Satake, *On a generalization of the notion of manifold*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 359–363.

[13] J.H.M. Steenbrink, *Mixed Hodge structure on the vanishing cohomology*, Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), Sijthoff and Noordhoff, Alphen aan den Rijn, 1977, pp. 525–563.