Computing Volumes of Solids of Revolution with Double Integrals

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The computation of the volume of solids of revolution in calculus courses is usually presented by two methods, namely:

1. The disk method, which consists roughly of decomposing the solid into slices that are perpendicular to the axis of revolution.
2. The shell method, which considers the solid as a series of concentric cylindrical shells wrapping the axis.

From a geometrical point of view, these two methods look quite different, and it is the shape of the solid that motivates the choice between one or another. Nevertheless, from an analytical point of view, both methods produce the same result, as is shown using integration by parts [**1**], inverse functions [**3**], and even Rolle's theorem [**2**].

We wondered if there was an even deeper relation between the above methods, and we looked at computing the volume of a solid of revolution as a double integral in a very intuitive way. We show that the classical methods (disks and shells) are recovered if this double integral is computed by each of the two possible applications of Fubini's theorem. Furthermore, we can also obtain Pappus' volume theorem from the formula.

Let S be a bounded and closed region in the yz-plane, and let ℓ be any straight line in the same plane such that ℓ is exterior to S. For every point $P = (x, y) \in S$, let $d_\ell(x, y)$ be the distance from P to ℓ. Let us denote by $V(S, \ell)$ the volume of the solid obtained by rotating the region S around the line ℓ; see Figure 1.

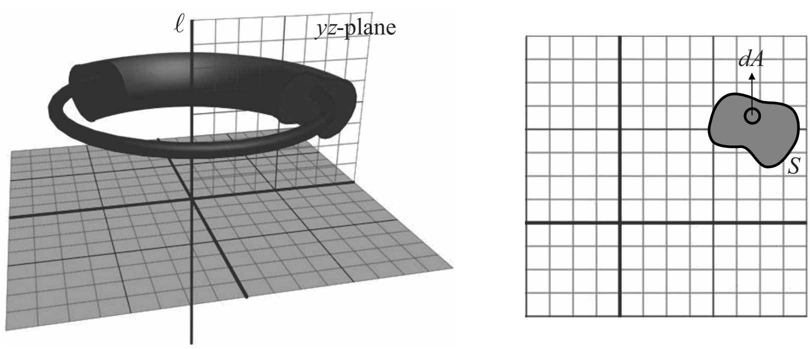

Figure 1. Solid of revolution together with the rotating section.

We claim that

$$V(S, \ell) = \iint_S 2\pi d_\ell(x, y)\, dA. \qquad (\ddagger)$$

Although a rigorous proof can be obtained, the underlying idea is, in fact, very intuitive. For every point $P(x, y) \in S$, consider a tiny circle with center P and area dA (see the right-hand side of Figure 1). When this circle rotates around the line ℓ,

http://dx.doi.org/10.4169/college.math.j.45.3.219
MSC: 26B15

it generates a torus of volume $2\pi d_\ell(x, y)dA$. Then it is enough to sum up all these volumes, i.e., to integrate over S.

Let us show how the classical disk and shell methods can be obtained from this double integral (‡).

Assume, without loss of generality, that the axis of revolution is the y-axis, OY. First, consider a region S bounded by continuous functions $y = f_1(x)$ and $y = f_2(x)$ between $x = a$ and $x = b$, as in Figure 2. By Fubini's theorem,

$$\iint_S 2\pi x\, dA = \int_a^b \left(\int_{f_1(x)}^{f_2(x)} 2\pi x\, dy \right) dx = \int_a^b 2\pi x\, (f_2(x) - f_1(x))\, dx,$$

which is precisely the well-known formula for the shell method.

Assume now that S is bounded by continuous functions $x = g_1(y)$ and $x = g_2(y)$ between $y = c$ and $y = d$, as in Figure 2. Another application of Fubini's theorem gives us

$$\iint_S 2\pi x\, dA = \int_c^d \left(\int_{g_1(y)}^{g_2(y)} 2\pi x\, dx \right) dy = \int_c^d \pi \left(g_2(y)^2 - g_1(y)^2 \right) dy,$$

which is the formula for the disk method.

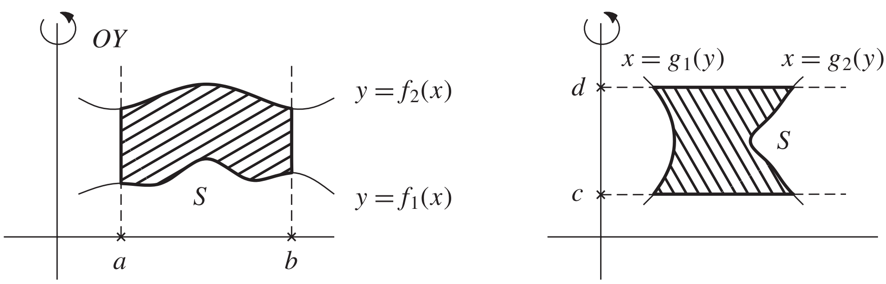

Figure 2. Two possible plane regions.

Example. Consider S, the plane region bounded by the unit circle $x^2 + y^2 = 1$ and the lines $y = x$ and $y = -\sqrt{3}x$, as in Figure 3. We want to compute the volume of the

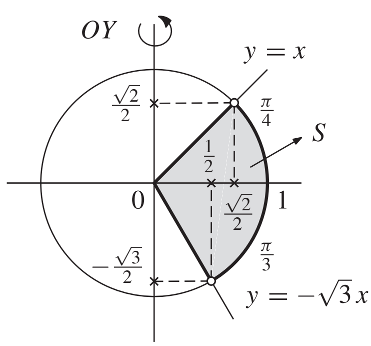

Figure 3. Area of the rotating region about the y-axis.

solid obtained by rotating S about the y-axis. Although the shell and disc methods can be used, they lead to complicated integration.

Using (‡), we compute $V = \iint_S 2\pi x \, dA$ by changing to polar coordinates:

$$V = 2\pi \int_0^1 \rho^2 d\rho \int_{-\pi/3}^{\pi/4} \cos\theta d\theta = 2\pi \frac{\rho^3}{3}\bigg]_0^1 \sin\theta \bigg]_{-\pi/3}^{\pi/4} = \frac{\pi(1+\sqrt{2})}{3}.$$

Pappus' volume theorem. Consider a region S of the yz-plane and the solid obtained by rotating it around the axis ℓ. Let C denote the centroid of S, $\mathcal{A}$ the area of S, and d the distance from the centroid C to the line ℓ. Then Pappus' volume theorem states, in its classical form, that the volume of this solid is given by

$$V(S, \ell) = 2\pi d \mathcal{A}.$$

Let us see how to obtain Pappus' theorem from (‡). Recall that if ℓ has equation $ax + by + c = 0$, then $d_\ell(x_0, y_0) = |ax_0 + by_0 + c|/\sqrt{a^2 + b^2}$. It is also well known that if $C = (x_C, y_C)$, then

$$x_C = \frac{\iint_S x \, dA}{\mathcal{A}}, \quad y_C = \frac{\iint_S y \, dA}{\mathcal{A}}.$$

Since S can be assumed to lie on the semiplane determined by $ax + by + c > 0$ (hence $d_\ell(x, y)$ is a polynomial in x and y), we have

$$\begin{aligned} V(S, \ell) &= \iint_S 2\pi d_\ell(x, y) \, dA \\ &= 2\pi \iint_S \frac{ax + by + c}{\sqrt{a^2 + b^2}} \, dA \\ &= 2\pi \frac{ax_C + by_C + c}{\sqrt{a^2 + b^2}} \mathcal{A} = 2\pi d \mathcal{A}. \end{aligned}$$

Acknowledgment. We would like to thank Rubén Vigara for his help with Figure 1. The first author is partially supported by the projects MTM2010-21740-C02-02, "E15 Grupo Consolidado Geometría" from the government of Aragón, FQM-333 from "Junta de Andalucía," and PRI-AIBDE-2011-0986 Acción Integrada hispano-alemana. The second author is partially supported by the group "S119 Investigación en Educación Matemática" from the government of Aragón.

Summary. We show that the disk and shell methods for computing volume are connected by Fubini's theorem. The same double integral allows us to derive Pappus' volume theorem.

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