

ALGORITHMS FOR CHECKING RATIONAL ROOTS OF b -FUNCTIONS AND THEIR APPLICATIONS

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ABSTRACT. The Bernstein–Sato polynomial of a hypersurface is an important object with many applications. However, its computation is hard, as a number of open questions and challenges indicate. In this paper we propose a family of algorithms called `checkRoot` for optimized checking whether a given rational number is a root of Bernstein–Sato polynomial and in the affirmative case, computing its multiplicity. These algorithms are used in the new approach to compute the global or local Bernstein–Sato polynomial and b -function of a holonomic ideal with respect to a weight vector. They can be applied in numerous situations, where a multiple of the Bernstein–Sato polynomial can be established. Namely, a multiple can be obtained by means of embedded resolution, for topologically equivalent singularities or using the formula of A’Campo and spectral numbers. We also present approaches to the logarithmic comparison problem and the intersection homology D -module. Several applications are presented as well as solutions to some challenges which were intractable with the classical methods. One of the main applications is the computation of a stratification of affine space with the local b -function being constant on each stratum. Notably, the algorithm we propose does not employ primary decomposition. Our results can be also applied for the computation of Bernstein–Sato polynomials for varieties. The examples in the paper have been computed with our implementation of the methods described in `SINGULAR:PLURAL`.

1. INTRODUCTION

Through the article we assume \mathbb{K} to be a field of characteristic 0. By R_n we denote the ring of polynomials $\mathbb{K}[x_1, \dots, x_n]$ in n variables over \mathbb{K} and by D_n we denote the ring of \mathbb{K} -linear partial differential operators with coefficients in R_n , that is the n -th Weyl algebra [24]. The ring D_n is the associative \mathbb{K} -algebra generated by the partial differential operators ∂_i and the multiplication operators x_i subject to relations

$$\{\partial_i x_j = x_j \partial_i + \delta_{ij}, x_j x_i = x_i x_j, \partial_j \partial_i = \partial_i \partial_j \mid 1 \leq i, j \leq n\}.$$

That is, the only non-commuting pair of variables is (x_i, ∂_i) ; they satisfy the relation $\partial_i x_i = x_i \partial_i + 1$. We use the Lie bracket notation $[a, b] := ab - ba$ for operators a, b , then e.g. the latter relation can be written as $[\partial_i, x_i] = 1$. Finally, we denote by $D_n[s]$ the ring of polynomials in one variable s with coefficients in the n -th Weyl algebra, i.e. $D_n[s] = D_n \otimes_{\mathbb{K}} \mathbb{K}[s]$.

Let us recall Bernstein’s construction. Given a non-zero polynomial $f \in R_n$, we consider $M = R_n[s, \frac{1}{f}] \cdot f^s$, the free $R_n[s, \frac{1}{f}]$ -module of rank one generated by the formal symbol f^s . Then M has a natural structure of left $D_n[s]$ -module. Here the differential operators act in a natural way,

$$(1) \quad \partial_i(g(s, x) \cdot f^s) = \left(\frac{\partial g}{\partial x_i} + sg(s, x) \frac{\partial f}{\partial x_i} \frac{1}{f} \right) \cdot f^s \in M$$

Theorem 1.1 (Bernstein [4]). *For every polynomial $f \in R_n$ there exists a non-constant polynomial $b(s) \in \mathbb{K}[s]$ and a differential operator $P(s) \in D_n[s]$ such that*

$$(2) \quad P(s)f \cdot f^s = b(s) \cdot f^s \in R_n[s, \frac{1}{f}] \cdot f^s = M.$$

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The monic polynomial $b(s)$ of minimal degree, satisfying (2) is called the **Bernstein-Sato polynomial** or the **(global) b -function**. There is a more general notion of a b -function with respect to given weights, see Sect. 2.3. By saying just *b -function* we will mean the Bernstein-Sato polynomial.

This paper is organized as follows. In Section 2, the `checkRoot` family of algorithms for checking rational roots of the global and local Bernstein-Sato polynomial is developed. We also show how to compute the b -function of a holonomic ideal with respect to a certain weight vector. In Section 3.1, we show how to obtain an upper bound in various situations: by using an embedded resolution, for topologically equivalent singularities, by using A'Campo's formula and spectral numbers. In particular, we demonstrate a complicated example of (non-isolated) quasi-ordinary singularity.

In Section 4.2 we discuss the possibilities to obtain integral roots of the b -function and apply them to the computation of the minimal integral root in the context of Intersection Homology D -module and Logarithmic Comparison Theorem. In Section 5 we present a new method for computing the stratification of affine space, according to local Bernstein-Sato polynomials.

We want to stress, that Bernstein-Sato polynomials for most of the examples, presented in this paper, cannot be computed by direct methods with any computer algebra system including SINGULAR:PLURAL [15]. Indeed, these examples were known as open challenges in the community and here we present their solutions for the first time.

The examples in this paper have been computed on a PC with Intel Core i3-540 Processor (4M Cache, 3.06 GHz) equipped with 4 GB RAM running Ubuntu 10.04 LTS Linux.

2. THE CHECKROOT FAMILY OF ALGORITHMS

For the sake of completeness, some of the ideas coming from [19] and [23, Ch. VIII], as well as some results and their proofs have been included here. Several algorithms for computing the b -function associated with a polynomial are known, see e.g. [30, 31, 32], [33], [6], [29], [36], [19]. However, from the computational point of view it is quite hard to obtain this polynomial in general. Despite significant recent progress, only restricted number of examples can be actually treated. In order to enhance the computation of the Bernstein-Sato polynomial via Gröbner bases, we study the following computational problems.

- (1) Obtain an upper bound for $b_f(s)$, that is, find $B(s) \in \mathbb{K}[s]$ such that $b_f(s)$ divides $B(s)$.

$$B(s) = \prod_{i=1}^d (s - \alpha_i)^{m_i}$$

- (2) Check whether α_i is a root of $b_f(s)$.
- (3) Compute the multiplicity of α_i as a root of $b_f(s)$.

There exist some well-known methods to obtain an upper bound for the Bernstein-Sato polynomial of a hypersurface singularity once we know, for instance, an embedded resolution of such singularity [17]. However, as far as we know, there is no algorithm for computing the Bernstein-Sato polynomial from this upper bound. In this section we present algorithms for checking whether a given rational number is a root of the Bernstein-Sato polynomial and for computing its multiplicity. As a first application, using this idea, we could obtain $b_f(s)$ for some interesting non-isolated singularities, see Example 3.3 below.

From the definition of the b -function it is clear that

$$(3) \quad \langle b_f(s) \rangle = (\text{Ann}_{D_n[s]}(f^s) + \langle f \rangle) \cap \mathbb{K}[s].$$

In fact, this is another way of defining the Bernstein-Sato polynomial. This equation was used to prove the main result of this section, namely Theorem 2.1.

Theorem 2.1. *Let R be a \mathbb{K} -algebra, whose center contains $\mathbb{K}[s]$. Let $q(s) \in \mathbb{K}[s]$ be a polynomial in one variable and I a left ideal in R satisfying $I \cap \mathbb{K}[s] \neq 0$. The following equalities hold:*

- (1) $(I + R\langle q(s) \rangle) \cap \mathbb{K}[s] = I \cap \mathbb{K}[s] + \mathbb{K}[s]\langle q(s) \rangle$,
- (2) $(I : q(s)) \cap \mathbb{K}[s] = (I \cap \mathbb{K}[s]) : q(s)$,
- (3) $(I : q(s)^\infty) \cap \mathbb{K}[s] = (I \cap \mathbb{K}[s]) : q(s)^\infty$.

In particular, using $I = \text{Ann}_{D_n[s]}(f^s) + \langle f \rangle \subseteq D_n[s]$ in the previous equation (3), we have

- $[\text{Ann}_{D_n[s]}(f^s) + D_n[s]\langle f, q(s) \rangle] \cap \mathbb{K}[s] = \langle b_f(s), q(s) \rangle = \langle \text{gcd}(b_f(s), q(s)) \rangle$,
- $[(\text{Ann}_{D_n[s]}(f^s) + D_n[s]\langle f \rangle) : q(s)] \cap \mathbb{K}[s] = \langle b_f(s) \rangle : q(s) = \langle \frac{b_f(s)}{\text{gcd}(b_f(s), q(s))} \rangle$,
- $[(\text{Ann}_{D_n[s]}(f^s) + D_n[s]\langle f \rangle) : q(s)^\infty] \cap \mathbb{K}[s] = \langle b_f(s) \rangle : q(s)^\infty$.

Proof. Let $b(s) \neq 0$ be a generator of $I \cap \mathbb{K}[s]$. At first, suppose that $h(s) \in (I + R\langle q(s) \rangle) \cap \mathbb{K}[s]$. Then we have

$$(4) \quad h(s) = P(s) + Q(s)q(s)$$

where $P(s) \in I$ and $Q(s) \in R$. Let $d(s)$ be the greatest common divisor of $b(s)$ and $q(s)$. There exist $b_1(s)$ and $q_1(s)$ such that $d(s)b_1(s) = b(s)$ and $d(s)q_1(s) = q(s)$, and hence $b_1(s)q(s) = q_1(s)b(s)$. Since s commutes with all elements in R , multiplying the equation (4) by $b_1(s)$, one obtains

$$b_1(s)h(s) = b_1(s)P(s) + Q(s)q_1(s)b(s) \in I$$

Thus, $b_1(s)h(s) \in I \cap \mathbb{K}[s] = \langle b(s) \rangle$ and therefore $h(s) \in \langle b(s) \rangle : \langle b_1(s) \rangle = \langle d(s) \rangle = I \cap \mathbb{K}[s] + \langle q(s) \rangle$. The other inclusion follows obviously. The second and the third parts can be shown directly and the proof is complete. \square

Note that the second (resp. third) part of the previous theorem can be used to heuristically find an upper bound for $b_f(s)$ (resp. the roots of $b_f(s)$). Since $q(s)$ is in the center of $D_n[s]$, the quotient and saturation ideals can be computed effectively e.g. via the kernel of a module homomorphism procedure, cf. [18]. More classical but less effective approach is to use the extra commutative variable, say T , and the formula

$$I : q(s)^\infty = D_n[s, T]\langle I, 1 - Tq(s) \rangle \cap D_n[s].$$

Example 2.2. Let $f \in \mathbb{C}[x, y]$ be the polynomial $x(x^2 + y^3)$. The annihilator of f^s in $D[s]$ can be generated by the operators $P_1(s) = 3xy^2\partial_x - y^3\partial_y - 3x^2\partial_y$ and $P_2(s) = 3x\partial_x + 2y\partial_y - 9s$. Consider the univariate polynomial

$$q(s) = (s + 1)(s + 5/9)(s + 8/9)(s + 10/9)(s + 7/9)(s + 11/9)(s + 13/9).$$

Computing a Gröbner basis, one can see that the ideal in $D[s, T]$ generated by $\{P_1(s), P_2(s), f, 1 - Tq(s)\}$ is the whole ring. From Theorem 2.1 (3), one deduces that $q(s)$ contains all the roots of $b_f(s)$.

Using this approach, we only have to check whether an ideal is the whole ring or not. Therefore *any* admissible monomial ordering can be chosen, hence the one, which is generically fast.

Corollary 2.3. *Suppose, that $P_1(s), \dots, P_k(s)$ generate $\text{Ann}_{D_n[s]}(f^s)$. The following conditions are equivalent:*

- (1) $\alpha \in \mathbb{Q}_{>0}$ is a root of $b_f(-s)$.
- (2) $D_n[s]\langle P_1(s), \dots, P_k(s), f, s + \alpha \rangle \neq D_n[s]$.
- (3) $D_n\langle P_1(-\alpha), \dots, P_k(-\alpha), f \rangle \neq D_n$.

Moreover, in such a case $D_n[s]\langle P_1(s), \dots, P_k(s), f, s + \alpha \rangle \cap \mathbb{K}[s] = \mathbb{K}[s]\langle s + \alpha \rangle$.

Proof. Take $J = D_n[s]\langle P_1(s), \dots, P_k(s), f, s + \alpha \rangle$ and $K = J \cap D_n = \langle P_1(-\alpha), \dots, P_k(-\alpha), f \rangle$. Since

$$J = D[s] \iff J \cap \mathbb{K}[s] = \mathbb{K}[s] \iff K = D_n,$$

and $\text{gcd}(b_f(s), s + \alpha) = 1$ if and only if $b_f(-\alpha) \neq 0$, the result follows from applying Theorem 2.1 using $q(s) = s + \alpha$. \square

Once generators of $\text{Ann}_{D[s]} f^s$ are known, the last corollary provides an algorithm for checking whether a given rational number is a root of the b -function of f , using Gröbner bases in the Weyl algebra.

Algorithm 1 CHECKROOT1 (checks whether $\alpha \in \mathbb{Q}_{>0}$ is a root of $b_f(-s)$)

Input 1: $\{P_1(s), \dots, P_k(s)\} \subseteq D_n[s]$, a system of generators of $\text{Ann}_{D_n[s]}(f^s)$;

Input 2: f , a polynomial in R_n ; α , a number in $\mathbb{Q}_{>0}$;

Output: **true**, if α is a root of $b_f(-s)$; **false**, otherwise;

$K := \langle P_1(-\alpha), \dots, P_k(-\alpha), f \rangle$;

$\triangleright K = J \cap D_n \subseteq D_n$

$G :=$ reduced Gröbner basis of K w.r.t. ANY term ordering;

return $(G \neq \{1\})$;

2.1. Multiplicities. We present two approaches of computing the multiplicity of a root.

Corollary 2.4. *Let m_α be the multiplicity of α as a root of $b_f(-s)$. Consider the ideals $J_i = \text{Ann}_{D_n[s]}(f^s) + \langle f, (s + \alpha)^{i+1} \rangle \subseteq D_n[s]$, $i = 0, \dots, n$. Then the following conditions are equivalent:*

- (1) $m_\alpha > i$.
- (2) $J_i \cap \mathbb{K}[s] = \langle (s + \alpha)^{i+1} \rangle$.
- (3) $(s + \alpha)^i \notin J_i$.

Moreover if $D_n[s] \supseteq J_0 \supseteq J_1 \supseteq \dots \supseteq J_{m-1} = J_m$, then $m_\alpha = m$. In particular, $m \leq n$ and $J_{m-1} = J_m = \dots = J_n$.

Proof. 1 \iff 2. Since $m_\alpha > i$ if and only if $\text{gcd}(b_f(s), (s + \alpha)^{i+1}) = (s + \alpha)^{i+1}$, the equivalence follows by applying Theorem 2.1 (1) using $q(s) = (s + \alpha)^{i+1}$.

2 \implies 3. If $(s + \alpha)^i \in J_i \cap \mathbb{K}[s]$, then clearly $J_i \cap \mathbb{K}[s] \supseteq \langle (s + \alpha)^{i+1} \rangle$.

3 \implies 2. Let $h(s) \in \mathbb{K}[s]$ be the monic generator of the ideal $J_i \cap \mathbb{K}[s]$. Since $(s + \alpha)^{i+1} \in J_i \cap \mathbb{K}[s] = \langle h(s) \rangle$, there exists $j \leq i + 1$ such that $h(s) = (s + \alpha)^j$. Suppose that $j \leq i$. Then $(s + \alpha)^i = (s + \alpha)^{i-j}(s + \alpha)^j = (s + \alpha)^{i-j}h(s) \in J_i$. That, however, contradicts 3 and thus $j = i + 1$.

The rest of the assertion follows by applying the previous result using $i = m$ and $i = m - 1$, since $(s + \alpha)^m \in J_m$ and $(s + \alpha)^{m-1} \notin J_{m-1}$ from the hypothesis. \square

Again, as soon as generators of $\text{Ann}_{D[s]} f^s$ are known, the last corollary provides an algorithm for checking whether a given rational number is a root of $b_f(s)$ and for computing its multiplicity, using Gröbner bases for differential operators.

Algorithm 2 CHECKROOT2 (computes the multiplicity of $\alpha \in \mathbb{Q}_{>0}$ as a root of $b_f(-s)$)

Input 1: $\{P_1(s), \dots, P_k(s)\} \subseteq D_n[s]$, a system of generators of $\text{Ann}_{D_n[s]}(f^s)$;

Input 2: f , a polynomial in R_n ; α , a number in $\mathbb{Q}_{>0}$;

Output: m_α , the multiplicity of α as a root of $b_f(-s)$;

for $i = 0$ to n **do**

$J := D_n[s] \cdot \langle P_1(s), \dots, P_k(s), f, (s + \alpha)^{i+1} \rangle$;

$\triangleright J_i$

$G :=$ Gröbner basis of J w.r.t. ANY term ordering;

$r :=$ normal form of $(s + \alpha)^i$ with respect to G ;

if $r = 0$ **then**

$m_\alpha := i$;

break

$\triangleright r = 0 \implies (s + \alpha)^i \in J_i$

\triangleright leave the **for** block

end if

end for

return m_α ;

Proof. (of Algorithm 2).

Termination: The algorithm CHECKROOT2 clearly terminates and one only has to consider the loop from 0 to n because the multiplicity of a root of $b_f(s)$ is at most n , see [35].

Correctness: Corollary 2.4 implies the correctness of the method. \square

Remark 2.5. There exists another version of checkRoot2 with just one step, due to the formula, see Corollary 2.4 above,

$$\left(\text{Ann}_{D_n[s]}(f^s) + D_n[s]\langle f, (s + \alpha)^n \rangle \right) \cap \mathbb{K}[s] = \langle (s + \alpha)^{m_\alpha} \rangle.$$

However, this method only seems to be useful when the multiplicity is close to n , otherwise checkRoot2 is more effective. The reason is that in general, the multiplicity is far lower than the number of variables.

This algorithm is much faster, than the computation of the whole Bernstein polynomial via Gröbner bases, because no elimination ordering is needed for computing a Gröbner basis of J . Also, the element $(s + \alpha)^{i+1}$, added as a generator, seems to simplify tremendously such a computation. Actually, when $i = 0$ it is possible to eliminate the variable s in advance and we can perform the whole computation in D_n , see Corollary 2.3 (3) above.

Nevertheless, Algorithm 2 meets the problem to calculate on each step a Gröbner basis G_i for an ideal of the form $I + \langle (s + \alpha)^{i+1} \rangle$ and the set G_{i-1} is not used at all for such computation. A completely new Gröbner basis has to be performed instead. The classical idea of quotient and saturation are used to solve this obstruction.

Corollary 2.6. *Let m_α be the multiplicity of α as a root of $b_f(-s)$ and $I = \text{Ann}_{D_n[s]}(f^s) + D_n[s]\langle f \rangle$. The following conditions are equivalent:*

- (1) $m_\alpha > i$.
- (2) $(I : (s + \alpha)^i) + D_n[s]\langle s + \alpha \rangle \neq D_n[s]$.
- (3) $(I : (s + \alpha)^i)|_{s=-\alpha} \neq D_n$.

Proof. Given $J \subseteq D_n[s]$ an ideal, we denote by $b_J(s)$ the monic generator of the ideal $J \cap \mathbb{K}[s]$. Then, from Theorem 2.1 (1), condition 2 is satisfied if and only if $-\alpha$ is a root of $b_{I:(s+\alpha)^i}(s)$. This univariate polynomial is nothing but $b_f(s)/\text{gcd}(b_f(s), (s + \alpha)^i)$, due to Theorem 2.1 (2). Now the claim follows from the equivalence

$$m_\alpha > i \iff (s + \alpha) \mid \frac{b_f(s)}{\text{gcd}(b_f(s), (s + \alpha)^i)}.$$

\square

Since $s + \alpha$ belongs to the center of $D_n[s]$, the ideal $I : (s + \alpha)^i$ can recursively be computed by the formulas

$$\begin{aligned} I : (s + \alpha) &= (I \cap D_n[s]\langle s + \alpha \rangle) / (s + \alpha), \\ I : (s + \alpha)^i &= (I : (s + \alpha)^{i-1}) : (s + \alpha). \end{aligned}$$

Another algorithm for computing multiplicities using quotient ideals follows. The termination and correctness follow from the previous corollary.

Algorithm 3 CHECKROOT3 (computes the multiplicity of $\alpha \in \mathbb{Q}_{>0}$ as a root of $b_f(-s)$)

Input 1: $\{P_1(s), \dots, P_k(s)\} \subseteq D_n[s]$, a system of generators of $\text{Ann}_{D_n[s]}(f^s)$;

Input 2: f , a polynomial in R_n ; α , a number in $\mathbb{Q}_{>0}$;

Output: m_α , the multiplicity of α as a root of $b_f(-s)$;

$m := 0$; $I := D_n[s]\langle P_1(s), \dots, P_k(s), f \rangle$; $J := I + D_n[s]\langle s + \alpha \rangle$;

while $G \neq \{1\}$ **do**

$m := m + 1$;

$I := I : (s + \alpha)$;

$\triangleright I : (s + \alpha)^i$

$J := I + D_n[s]\langle s + \alpha \rangle$; (or $J := I|_{s=-\alpha}$)

$G :=$ reduced Gröbner basis of J w.r.t. ANY term ordering;

end while

return m ;

Remark 2.7. Several obvious modifications of the presented algorithms can be useful depending on the context. Assume, for instance, that $q(s)$ is a known factor of the Bernstein-Sato polynomial and one is interested in computing the rest of $b_f(s)$. Then the ideal $I : q(s)$ contains such information. This easy observation can help us in some special situations.

Remark 2.8. Define the **reduced Bernstein-Sato polynomial** of $f \in R_n$ to be $b'_f(s) = b_f(s)/(s+1)$. Recall, that the **Jacobian ideal** of f is $J_f = \langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle \subset \mathbb{K}[x]$. It is known, that taking $\langle f \rangle + J_f$ instead of $\langle f \rangle$ has the following consequence

$$(\text{Ann}_{D[s]} f^s + \langle f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle) \cap \mathbb{K}[s] = \langle b'_f(s) \rangle = \langle \frac{b_f(s)}{s+1} \rangle.$$

Hence, all the algorithms above can be modified to this setting, resulting in more effective computations in the implementation. We decided, however, not to modify the description of algorithms in order to keep the exposition easier.

2.2. Local versus global b -functions. Here we are interested in what kind of information one can obtain from the global b -function for computing the local ones and conversely. In order to avoid theoretical problems we will assume in this paragraph that the ground field is \mathbb{C} .

Several algorithms to obtain the local b -function of a hypersurface f have been known without any Gröbner bases computation but under strong conditions on f . For instance, it was shown in [21] that the minimal polynomial of $-\partial_t t$ acting on some vector space of finite dimension coincides with the reduced local Bernstein polynomial, assuming that the singularity is isolated.

Remark 2.9. Recall, that the singular locus of $V(f)$ is $V(\langle f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle)$. One can define the **local b -function** or **local Bernstein-Sato polynomial** as follows. Let $p \in \mathbb{C}^n$ be a point and $\mathfrak{m}_p = \langle \{x_1 - p_1, \dots, x_n - p_n\} \rangle \subset R_n$ the corresponding maximal ideal. Let D_p be the local Weyl algebra at p , that is Weyl algebra with coefficients from $\mathbb{C}[x_1, \dots, x_n]_p$ instead of $R_n = \mathbb{C}[x_1, \dots, x_n]$. From the Bernstein's functional equation (2) it follows that $\exists P(s) \in D[s], b_f(s) \in \mathbb{K}[s]$, such that $P(s)f \cdot f^s = b(s) \cdot f^s$ holds. Since over $\mathbb{C}[x_1, \dots, x_n]_p$ there are invertible non-constant polynomials, there exist $P_p(s) \in D_p[s], b_{f,p}(s) \in \mathbb{K}[s]$, such that $P_p(s)f \cdot f^s = b_{f,p}(s) \cdot f^s$ holds. We define local Bernstein-Sato polynomial to be the univariate monic polynomial $b_{f,p}(s)$ of the minimal degree, such that the previous identity holds.

Theorem 2.10. (*Briançon-Maisonobe (unpublished), Mebkhout-Narváez [25]*) *Let $b_{f,p}(s)$ the local b -function of f at the point $p \in \mathbb{C}^n$ and $b_f(s)$ the global one. Then $b_f(s) = \text{lcm}_{p \in \mathbb{C}^n} b_{f,p}(s) = \text{lcm}_{p \in \Sigma(f)} b_{f,p}(s)$.*

The computation of the global b -function with Theorem 2.10 is effective, when the singular locus consists of finitely many isolated singular points. The SINGULAR library `gmssing.lib` implemented by M. Schulze [37] and based on his work [36] allows one to compute invariants related to the Gauss-Manin system of an isolated hypersurface singularity. In the non-isolated

case the situation is more complicated. For computing the local b -function in this case (which is important on its own) we suggest using the global b -function as an upper bound and a local version of the `checkRoot` algorithm, see Section 2.2.2 below.

T. Oaku presented algorithms for the local b -function in [30, 32]. In these algorithms, no knowledge of a global b -function is needed. However, these algorithms are quite hard from the computational point of view. Namely, more complicated elimination in Weyl algebra together with numerous computations of quotient ideals in a commutative ring need to be executed. An intersection of a left ideal with a principal subalgebra needs to be performed as well, and for the local case this has to be done within the localized ring.

In [26], H. Nakayama presented an algorithm for computing local b -functions. One step in his algorithm uses a bound for the multiplicity of a given rational root of the global b -function. Then the algorithm checks if this multiplicity agrees with the local one. This approach is very similar to our `checkRoot` algorithm.

2.2.1. *Localization of non-commutative rings.* We refer the reader to [13] and [24] for details on rings of fractions in non-commutative setting.

Lemma 2.11. *Let $R_1 \xrightarrow{i} R_2$ be a ring extension and $S \subset R_1$ a multiplicatively closed set. Assume $S^{-1}R_1$ and $S^{-1}R_2$ exist and consider the corresponding localization maps $\phi_1 : R_1 \rightarrow S^{-1}R_1$ and $\phi_2 : R_2 \rightarrow S^{-1}R_2$. Let $j : S^{-1}R_1 \rightarrow S^{-1}R_2$ be the map induced by i . Then j is injective and for every left ideal $I \subseteq R_2$ one has*

$$S^{-1}I \cap S^{-1}R_1 = S^{-1}(I \cap R_1).$$

Lemma 2.12. *Let R be a ring, $S \subseteq R$ a multiplicatively closed set and $I \subseteq R$ a left ideal. Assume $S^{-1}R$ exists. Then $S^{-1}I$ is not the whole ring $S^{-1}R$ if and only if $I \cap S = \emptyset$.*

Example 2.13. Let $R = D$ be the n -th Weyl algebra, $p \in \mathbb{K}^n$ an arbitrary point and $S = \mathbb{K}[\mathbf{x}] \setminus \mathfrak{m}_p$, cf. Rem. 2.9. Then S is a left and right denominator set and the Ore localization $(\mathbb{K}[\mathbf{x}] \setminus \mathfrak{m}_p)^{-1}D$ is naturally isomorphic to D_p . Analogous construction also holds for the extension $D[s] = \mathbb{K}[s] \otimes_{\mathbb{K}} D$.

2.2.2. *Local version of the `checkRoot` algorithms.* Theorem 2.1 is general enough to be applied for checking rational roots of local Bernstein-Sato polynomials. Let $\mathbb{K} \supset \mathbb{Q}$ be algebraically closed, $f \in \mathbb{K}[x_1, \dots, x_n]$, $p \in \mathbb{K}^n$ and $\alpha \in \mathbb{Q}$. Then the first part of Theorem 2.1, see also Corollary 2.3, tells us that $(s + \alpha)$ is a factor of the local b -function at p if and only if the left ideal

$$(5) \quad \text{Ann}_{D_p[s]}(f^s) + D_p[s]\langle f, s + \alpha \rangle$$

is not the whole ring $D_p[s]$. Applying Lemma 2.11 with $R_1 = D[s]$, $R_2 = D\langle t, \partial t \rangle := D \otimes_{\mathbb{K}} \mathbb{K}\langle t, \partial t \mid \partial t \cdot t = t \cdot \partial t + 1 \rangle$, $S = \mathbb{K}[\mathbf{x}] \setminus \mathfrak{m}_p$ and $I = \text{Ann}_{D\langle t, \partial t \rangle}(f^s) = I_f$, the Malgrange ideal associated with f , one obtains $\text{Ann}_{D_p[s]}(f^s) = D_p[s] \text{Ann}_{D[s]}(f^s)$.

Proposition 2.14. *Let $I = \text{Ann}_{D[s]}(f^s) + D[s]\langle f \rangle$ and α be a root of $b_f(s)$ of multiplicity m_α . Moreover, for $i = 0, \dots, m_\alpha - 1$ consider ideals $I_{\alpha,i} = (I : (s + \alpha)^i) + D[s]\langle s + \alpha \rangle$. Then*

- (1) $(s + \alpha) \mid b_{f,p}(s) \iff p \in V((I + D[s]\langle s + \alpha \rangle) \cap \mathbb{K}[\mathbf{x}])$,
- (2) $m_\alpha(p) > i \iff p \in V(I_{\alpha,i} \cap \mathbb{K}[\mathbf{x}])$.

There are several ways to check whether an ideal $I \subseteq D_p[s]$ is proper or not. However, it is an open problem to decide which one is more efficient. Mora division and standard bases techniques seem to be more suitable in this case, since otherwise a (global) elimination ordering is needed. Granger, Oaku and Takayama [14] gave algorithms both for standard basis approach and for the bi-homogenization with respect to the (F, V) -filtration¹. On the other hand, using

¹The referee pointed us, that there are implementations of these algorithms: `KAN/SM1` has a package `ecart.sm1`, implementing the full algorithm, while `RISA/ASIR` contains a library `nk_mora/localb.rr`, which implements a restricted version of the algorithm (see the Remark 2.6 in [14]). Some description of the latter

the approach above, local orderings are unavoidable for obtaining the stratification associated with local b -functions, see Sect. 5.

2.3. b -functions with respect to weights and checkRoot. The b -function associated with a holonomic ideal with respect to a weight is presented. We refer to [33] for the details. Let $0 \neq w \in \mathbb{R}_{\geq 0}^n$. Consider the V -filtration with respect to w , $\{V_m \mid m \in \mathbb{Z}\} = V$ on D , where V_m is spanned by $\{x^\alpha \partial^\beta \mid -w\alpha + w\beta \leq m\}$ over \mathbb{K} . In other words, to x_i and ∂_i the weights $-w_i$ resp. w_i are assigned; with such weights the relation $\partial_i x_i = x_i \partial_i + 1$ becomes graded of degree 0 and hence the algebra D itself becomes graded. Thus the associated graded algebra $\text{gr}^V(D) = \bigoplus_{m \in \mathbb{Z}} V_m / V_{m-1}$ is isomorphic to D , which allows us to identify them.

For a non-zero operator

$$P = \sum_{\alpha, \beta \in \mathbb{N}^n} a_{\alpha\beta} x^\alpha \partial^\beta \in D,$$

the maximum $\max_{\alpha, \beta} \{-w\alpha + w\beta \mid c_{\alpha\beta} \neq 0\} \in \mathbb{R}$ is denoted by $\text{ord}^V(P)$ and the principal symbol of P is the V -homogeneous operator given by

$$\sigma^V(P) := \sum_{-w\alpha + w\beta = \text{ord}^V(P)} a_{\alpha\beta} x^\alpha \partial^\beta.$$

Additionally, for a given ideal $I \subseteq D$, the associated graded ideal is defined as the vector space spanned by all its principal symbols, that is, $\text{gr}^V(I) := \mathbb{K} \cdot \{\sigma^V(P) \mid P \in I\}$.

Quite often the principal symbol (resp. associated graded ideal) is called the initial form (resp. initial ideal) and it is denoted by $\text{in}_{(-w, w)}(P)$ (resp. $\text{in}_{(-w, w)}(I)$).

Definition 2.15. Let $I \subset D$ be a holonomic ideal. Consider $0 \neq w \in \mathbb{R}_{\geq 0}^n$ and $s := \sum_{i=1}^n w_i x_i \partial_i$. Then $\text{gr}^V(I) \cap \mathbb{K}[s] \neq 0$ is a principal ideal in $\mathbb{K}[s]$. Its monic generator is called the *global b -function of I with respect to the weight w* .

Although Theorem 2.1 can not be applied in this setting, since $s = \sum_i w_i x_i \partial_i$ does not belong to the center of the algebra, a similar result still holds, due to the properties of the V -filtration, see Proposition 2.16 below. Also Corollaries 2.3, 2.4 and 2.6 can be established using initial parts instead of annihilators.

Proposition 2.16. $(\text{gr}^V(I) + \text{gr}^V(D)\langle q(s) \rangle) \cap \mathbb{K}[s] = \text{gr}^V(I) \cap \mathbb{K}[s] + \mathbb{K}[s]\langle q(s) \rangle$.

Proof. Consider $h(s) = Q + R \cdot q(s)$, where $Q \in \text{gr}^V(I)$ and $R \in \text{gr}^V(D)$. Taking V -homogeneous parts in the preceding expression, one finds $Q_0 \in \text{gr}^V(I)$ and $R_0 \in \text{gr}^V(D)$ of degree 0 such that $h(s) = Q_0 + R_0 \cdot q(s)$. Now, since $q(s)$ commutes with Q_0 , one can proceed as in the proof of Theorem 2.1 (1). \square

Many algorithms in the realm of D -modules are based on the computation of such b -functions. For some applications like localization, integration and restriction, only the maximal and the minimal integral roots have to be computed. However, the previous proposition can not be used to find the set of *all* integral roots, since neither upper nor lower bound is known in advance (N. Takayama suggested the following example: for $k \in \mathbb{Z}$, $\text{in}_{(-1, 1)}(\langle t\partial_t + k \rangle) \cap \mathbb{K}[t\partial_t] = \langle t\partial_t + k \rangle$). Nevertheless, there is the natural possibility to check a particular root of a b -function with respect to the non-negative weight w .

3. COMPUTING b -FUNCTIONS VIA UPPER BOUNDS

As different possible ways to find upper bounds, we present embedded resolutions, topologically equivalent singularities and A'Campo's formula. Depending on the context local or global version of our algorithm is used.

library is contained in [26]. However, both mentioned packages are not sufficiently documented, what hopefully will be improved in the future.

3.1. Embedded resolutions. In this part of the paper we set $\mathbb{K} = \mathbb{C}$. However, in actual computation we can assume that the ground field is generated by a finite number of (algebraic or transcendental) elements over the field \mathbb{Q} and the algebraic relations among these elements are specified.

Definition 3.1. Let $h : Y \rightarrow \mathbb{C}^n$ be a proper birational morphism. We say that h is a *global embedded resolution* of the hypersurface defined by a polynomial $f \in \mathbb{C}[x]$, $X = V(f)$, if the following conditions are satisfied:

- (1) Y is a non-singular variety.
- (2) $h : Y \setminus h^{-1}(X) \rightarrow \mathbb{C}^n \setminus X$ is an isomorphism.
- (3) $h^{-1}(X)$ is a normal crossing divisor.

Since $h^{-1}(X)$ is a normal crossing divisor, the morphism $F = f \circ h : Y \rightarrow \mathbb{C}$ is locally given by a monomial. Hence, we can define the b -function of F as the least common multiple of the local ones. If F is locally given by the monomial $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ at the point p , then

$$b_{F,p}(s) = \prod_{i=1}^{\alpha_1} \left(s + \frac{i}{\alpha_1} \right) \cdots \prod_{i=1}^{\alpha_n} \left(s + \frac{i}{\alpha_n} \right) = \prod_{1 \leq i_j \leq \alpha_j} \prod_{1 \leq k \leq n} \left(s + \frac{i_k}{s_k} \right).$$

The following is the global version of the classical result by Kashiwara [17]. The upper bound statement is due to Varchenko ([41]) and Saito ([34, 35]).

Theorem 3.2. For $f \in R_n$, there exists an integer k such that $b_f(s)$ is a divisor of the product $b_F(s)b_F(s+1) \cdots b_F(s+k)$. Moreover $0 \leq k \leq n-1$.

Proof. Since h is a global embedded resolution of $X = V(f)$, h induces a local embedded resolution of the germ (X, p) at every point $p \in X$. Now, the existence of $k \geq 0$ with the divisibility property follows from the theorem by Kashiwara [17] and from the fact that the global b -function is the least common multiple of the local ones, see Theorem 2.10. The proof for the upper bound can be found in the references above. \square

This theorem allows one to find upper bounds also for the global case. Let us apply the algorithm `checkRoot` for computing the b -function.

Example 3.3. Let $f = (xz + y)(x^4 + y^5 + xy^4) \in \mathbb{Q}[x, y, z]$ and $B_1(s) = b_{x^5}(s)b_{y^{18}}(s)b_{z^{24}}(s)$. Since every root of $b_f(-s)$ belongs to the real interval $(0, 3)$, see Theorem 3.2, computing an embedded resolution of the singularity and using Kashiwara's result [17], we obtain that $B(s) = B_1(s)B_1(s+1)B_1(s+2)$ is an upper bound for $b_f(s)$. Once we know a system of generators of $\text{Ann}_{D_n[s]} f^s$, checking whether each root of the upper bound is a root of the Bernstein-Sato polynomial was simple. It took less than 5 seconds except for those ones which appear in the table below. We also observe that when a candidate is not a root indeed, the computation is very fast. To the best of our knowledge, this example (first appeared in [8]) is intractable by any computer algebra system.

$$\begin{aligned} b_f(s) = & (s+1)^2(s+17/24)(s+5/4)(s+11/24)(s+5/8)(s+31/24)(s+13/24) \\ & (s+13/12)(s+7/12)(s+23/24)(s+5/12)(s+3/8)(s+11/12)(s+9/8) \\ & (s+7/8)(s+19/24)(s+3/4)(s+29/24)(s+25/24) \end{aligned}$$

The running time is given in the format *minutes:seconds*.

Root of $B(-s)$	Running time		Root of $b_f(-s)$?
	<code>checkRoot2</code>	<code>checkRoot1</code>	
5/4	18:47	12:42	Yes
31/24	47:31	31:05	Yes
9/8	0:56	0:24	Yes
29/24	17:41	7:57	Yes

Remark 3.4. Choosing the lexicographical ordering with $\partial_x > x$ in D , when using the `checkRoot` algorithm reduced the running time to just 25 sec.

Let us give a brief indication for computing a global embedded resolution of f . Denote by $V_1 := V(xz + y)$ and $V_2 := V(x^4 + y^5 + xy^4)$ the two components of $V(f)$. First note that $\text{Sing}(V_2) \subset V_1 \cap V_2$ and the singular locus $\text{Sing}(f) = V_1 \cap V_2$ can be decomposed into two disjoint algebraic sets as

$$\text{Sing}(f) = V(xz + y, yz^4 - yz^3 + 1) \cup V(x, y) =: Y \sqcup Z.$$

The varieties V_1 and V_2 intersect transversely at every point of Y . Indeed, let us consider $P = (a, b, c) \in Y$. Then V_1 and V_2 are smooth at P and their tangent spaces $\{cx + y + az = 0\}$, $\{(4a^3 + b^4)x + (5b^4 + 4ab^3)y = 0\}$ can not be the same ($a \neq 0$ holds).

Consider $\pi : \widehat{\mathbb{C}}^3 \rightarrow \mathbb{C}^3$, the blow-up of \mathbb{C}^3 with center in Z . Denote by \widehat{V}_1 and \widehat{V}_2 the corresponding strict transforms of V_1 and V_2 . The exceptional divisor E_1 has multiplicity 5 and, \widehat{V}_1 and \widehat{V}_2 do not meet in a small neighborhood of E_1 . Moreover \widehat{V}_1 and E_1 intersect transversely. The local equation of $\widehat{V}_2 \cup E$ is given by the polynomial $y^5(x^4 + y + xy)$.

Now, one proceeds as in the case of plane curves, since the local equation involves just two variables. Finally, we obtain seven divisors with normal crossings, see Figure 3.3. This method can also be applied to the family $(xz + y)g(x, y)$ under some extra conditions on $g(x, y)$.

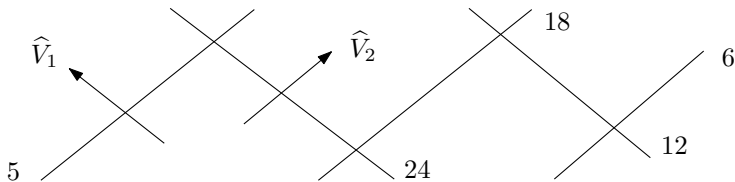


FIGURE 1. Embedded resolution of $V((xz + y)(x^4 + y^5 + xy^4))$

Remark 3.5. To the best of our knowledge, resolution of singularities has never been used before for computing Bernstein-Sato polynomials in an algorithmic way. Recall that an embedded resolution can be computed algorithmically in any dimension and for any affine algebraic variety [5], [12, 11]. There is a sophisticated implementation by A. Frühbis-Krüger and G. Pfister [9, 10] in SINGULAR.

One can find upper bounds for the case of hyperplane arrangements by computing an embedded resolution. This allows one among other to test formulas for Bernstein-Sato polynomials of non-generic arrangements. A formula for the Bernstein-Sato polynomial of a generic hyperplane arrangement was given by Walther in [42].

3.2. Topologically equivalent singularities. Let f, g be two topologically equivalent singularities and assume that $b_f(s)$ is known. Since the set $E_f = \{e^{2\pi i\alpha} \mid b_{f,0}(\alpha) = 0\}$ is a topological invariant of the singularity $\{f = 0\}$ at the origin [21, 22] and every root belongs to $(-n, 0)$ (Theorem 3.2), one can find an upper bound for $b_g(s)$ from the roots of $b_f(s)$ and use our algorithms for computing $b_g(s)$. The upper bound is constructed as $\prod_{\beta \in E} (s - \beta)$, where $E = \{\alpha + k \mid \alpha \in E_f, k \in \mathbb{Z}, \alpha + k \in (-n, 0)\}$.

In general it is complicated to check, whether two singularities are equivalent. However, there are some special families for which this can be done. This is the case of quasi-ordinary singularities, see e.g. [20]. Consider an example of a non-isolated one.

Example 3.6. Let $f = z^4 + x^6y^5$ and $g = f + x^5y^4z$. Since the corresponding discriminants with respect to z are normal crossing divisors, the associated germs at the origin define quasi-ordinary singularities. Moreover the characteristic exponents are in both cases the same and hence they are topologically equivalent, see e.g. [20].

The Bernstein-Sato polynomial of f at the origin has 27 roots, all of them with multiplicity one except for $\alpha = -1$ which has multiplicity two. Here is the list in positive format.

$$1, \frac{5}{6}, \frac{9}{10}, \frac{4}{3}, \frac{13}{10}, \frac{2}{3}, \frac{3}{4}, \frac{19}{20}, \frac{5}{12}, \frac{11}{10}, \frac{17}{12}, \frac{17}{20}, \frac{11}{12}, \frac{7}{10}, \boxed{\frac{19}{12}}, \frac{13}{20}, \frac{27}{20}, \frac{7}{6}, \frac{21}{20}, \frac{9}{20}, \frac{13}{12}, \frac{5}{4}, \mathbf{\frac{3}{2}}, \frac{7}{12}, \mathbf{\frac{31}{20}}, \boxed{\frac{7}{4}}, \frac{23}{20}$$

The exponential of the previous set has 24 elements. Each of them gives three candidates for $b_{g,0}(-s)$ except for $-\alpha = 1$ which gives just two. For instance $-\alpha = 1/2$ gives the following three possible roots.

$$\frac{1}{2} \rightarrow \left\{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \right\}$$

There are 71 possible roots in total. Note that using this approach we do not have any information about the multiplicities. Finally one obtains the roots for $b_{g,0}(-s)$.

$$1, \frac{5}{6}, \frac{9}{10}, \frac{4}{3}, \frac{13}{10}, \frac{2}{3}, \frac{3}{4}, \frac{19}{20}, \frac{5}{12}, \frac{11}{10}, \frac{17}{12}, \frac{17}{20}, \frac{11}{12}, \frac{7}{10}, \frac{13}{20}, \frac{27}{20}, \frac{7}{6}, \frac{21}{20}, \frac{9}{20}, \frac{13}{12}, \frac{5}{4}, \mathbf{\frac{1}{2}}, \frac{7}{12}, \mathbf{\frac{11}{20}}, \frac{23}{20}$$

Observe that the Bernstein-Sato polynomials are very similar. The roots of $b_{f,0}(-s)$ marked with a box have disappeared in $b_{g,0}(-s)$ and the ones in bold $3/2, 31/20$ have become $1/2, 11/20$.

We have selected this example to show the topologically equivalent approach to keep the exposition as simple as possible. However, there is a family of examples depending on three indices

$$f_{m,p,q} = x^m + x^p y^q, \quad g_{m,p,q} = x^m + x^p y^q + x^{p-1} y^{q-1} z$$

where the polynomials define topologically equivalent singularities if $m \leq p, m \leq q$ and at least one of the two inequalities is strict.

In the table we put the information on timings in `[hours:]minutes:seconds` format for the computation of the Bernstein-Sato polynomial of g . The symbol “-” means that the computation did not terminate (or full memory) after 5 hours.

(m, p, q)	SINGULAR			Risa/Asir		$\deg b_g(s)$
	checkRoot	bfct	bfctAnn	bfct	bfunction	
$(4, 6, 5)$	0:27	3:19	0:18	1:32	1:03	26
$(5, 7, 6)$	7:22	-	12:32	-	28:27	49
$(6, 8, 7)$	51:15	-	1:33:28	-	2:34:11	57

Observe that although `bfctAnn` and `bfunction` are competitive in this family of examples we notice a better control of the memory due to the fact that many “small” Gröbner bases were needed for the `checkRoot` approach while a “big” Gröbner basis is performed for the other methods. That is why our new algorithm is specially useful for extreme examples.

3.3. A’Campo’s formula. The Jordan form of the local Picard-Lefschetz monodromy of superisolated surface singularities was calculated by Artal-Bartolo in [2]. The main step in this computation was to present explicitly an embedded resolution for this family and to study the mixed Hodge structure of the Milnor fibration.

Since every root of the Bernstein-Sato polynomial belongs to the real interval $(-n, 0)$ (Theorem 3.2) and the characteristic polynomial is a topological invariant, using the results by Malgrange [21, 22], one can eventually provide an upper bound for the b -function. Let us see an example that was not feasible even with the powerful specialized implementation by Schulze [37].

Example 3.7. Let V be the superisolated singularity defined by $f = z^6 + (x^4 z + y^5 + xy^4)$. The characteristic polynomial is

$$\Delta(t) = \frac{(t^5 - 1)(t^6 - 1)(t^{120} - 1)}{(t - 1)(t^{30} - 1)(t^{24} - 1)}.$$

This polynomial has 76 different roots modulo \mathbb{Z} and thus we know in advance that the Bernstein-Sato polynomial (resp. the reduced one) has at least 77 (resp. 76) different roots.

Using the results above in 230 possible candidates, only 77 of them are roots of the b -function indeed, all of them with multiplicity one.

$$\begin{aligned}
& 1, \frac{27}{40}, \frac{101}{120}, \frac{41}{60}, \frac{17}{20}, \frac{83}{120}, \frac{103}{120}, \frac{43}{60}, \frac{53}{60}, \frac{29}{40}, \frac{107}{120}, \frac{23}{40}, \frac{89}{120}, \frac{109}{120}, \frac{71}{120}, \frac{91}{120}, \frac{37}{40}, \frac{73}{120}, \frac{31}{40}, \\
& \frac{113}{120}, \frac{37}{60}, \frac{47}{60}, \frac{19}{20}, \frac{77}{120}, \frac{97}{120}, \frac{39}{40}, \frac{13}{20}, \frac{49}{60}, \frac{59}{60}, \frac{79}{120}, \frac{33}{40}, \frac{119}{120}, \frac{3}{5}, \frac{4}{5}, \frac{121}{120}, \frac{47}{120}, \frac{161}{120}, \mathbf{181}, \frac{61}{60}, \\
& \frac{71}{60}, \frac{27}{20}, \mathbf{60}, \frac{41}{40}, \frac{143}{120}, \frac{163}{120}, \mathbf{61}, \frac{21}{20}, \frac{73}{60}, \frac{83}{60}, \mathbf{20}, \frac{127}{120}, \frac{49}{40}, \frac{167}{120}, \frac{187}{120}, \frac{43}{40}, \frac{149}{120}, \frac{169}{120}, \frac{131}{120}, \\
& \frac{151}{120}, \frac{57}{40}, \frac{133}{120}, \frac{51}{40}, \frac{173}{120}, \frac{67}{60}, \frac{77}{60}, \frac{29}{20}, \frac{137}{120}, \frac{157}{120}, \frac{59}{40}, \frac{23}{20}, \frac{79}{60}, \frac{89}{60}, \frac{139}{120}, \frac{53}{40}, \frac{179}{120}, \frac{6}{5}, \frac{7}{5}
\end{aligned}$$

The total running time was 41.5 minutes. In the table below we show the candidates for roots, for which computation ran more than 2 minutes. Again we observe that the detection of a non-root is very fast indeed.

Candidate	Running time	Root of $b_f(-s)$?
181/20	2:52	Yes
91/60	6:01	Yes
61/40	4:53	Yes
31/20	4:21	Yes

Remark 3.8. Spectral numbers are defined using the semi-simple part of the action of the monodromy on the mixed Hodge structure on the cohomology of the Milnor fiber [38], [41]. In [16, Th. 3.3], [34, Th. 0.7] it is proved, that some roots of the Bernstein-Sato polynomial of a germ with an isolated critical point at the origin could be obtained from the knowledge of the spectral numbers of the germ. Since spectral numbers do not change under μ -constant deformations, this also gives a set of common roots of the Bernstein-Sato polynomials, associated with the members of a μ -constant deformation of a germ. Therefore, they provide a lower bound for $b_f(s)$, as well as an upper bound.

4. INTEGRAL ROOTS OF b -FUNCTIONS

For several applications only integral roots of the b -function are needed, e.g. [33]. We present here problems related to the so-called Logarithmic Comparison Theorem and Intersection Homology D -module. Depending on the context local or global version of our algorithm is used.

4.1. Upper bounds from different ideals. Consider a left ideal $I \subseteq \text{Ann}_{D[s]} f^s$. Then $I + \langle f \rangle \subseteq \text{Ann}_{D[s]} f^s + \langle f \rangle \subsetneq D[s]$, that is the former is a proper ideal. Then define the **relative b -polynomial** $b_f^I(s) \in \mathbb{K}[s]$ to be the monic generator of $(I + \langle f \rangle) \cap \mathbb{K}[s]$, then $b_f(s) \mid b_f^I(s)$. Note, that quite often $b_f^I(s) = 0$. But if $b_f^I(s) \neq 0$, it gives us an upper bound for $b_f(s)$. In particular, one can take I , giving rise to a holonomic $D[s]$ -module, that is $\text{GK.dim} D[s]/I = \text{GK.dim} D[s]/\text{Ann}_{D[s]} f^s = n + 1$.

Since $(s + 1) \mid b_f(s) \mid b_f^I(s)$, one can consider the **reduced relative b -polynomial** $\widetilde{b}_f^I(s) \in \mathbb{K}[s]$ to be the monic generator of $(I + \langle f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle) \cap \mathbb{K}[s]$.

A prominent example of I as above is the logarithmic annihilator. Let $I = \text{Ann}_{D[s]}^{(1)}(f^s)$ be the ideal in $D[s]$ generated by the operators $P(s) \in \text{Ann}_{D[s]}(f^s)$ of total degree at most one in ∂_i . Let us define $b_f^{(1)}(s) := b^I(f^s)_f(s) = (\text{Ann}_{D[s]}^{(1)}(f^s) + D[s]\langle f \rangle) \cap \mathbb{K}[s]$. The reduced $\widetilde{b}_f^{(1)}(s)$ is useful as well.

4.2. Minimal integral root of $b_f(s)$ and the logarithmic comparison problem. Since every root of $b_f(s)$ belongs to the real interval $(-n, 0)$, integral roots are bounded and therefore the whole Bernstein-Sato polynomial is not needed. Let us see an example that could not be treated before with the classical methods.

Example 4.1. Let A be the matrix given by

$$A = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \\ x_9 & x_{10} & x_{11} & x_{12} \end{pmatrix}.$$

Let us denote by Δ_i , $i = 1, 2, 3, 4$, the determinant of the minor resulting from deleting the i -th column of A , and consider $f = \Delta_1\Delta_2\Delta_3\Delta_4$. The polynomial f defines a non-isolated hypersurface in \mathbb{C}^{12} . Following Theorem 3.2, the set of all possible integral roots of $b_f(-s)$ is $\{11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1\}$.

Using the algorithm `checkRoot` with the logarithmic annihilator, see Section 4.1 above, instead of the classical one, we have proved for $\alpha = 2, \dots, 11$ that

$$\text{Ann}_{D_n[s]}^{(1)}(f^s) + D_n[s]\langle f, s + \alpha \rangle = D_n[s],$$

and hence -1 is the minimal integral root of $b_f(s)$. The following is the timing information of the whole procedure. Of course, -1 is always a root, but it is interesting to compare the timings of confirming this fact.

Possible integral roots	1	2 ... 11
Root of $b_f^{(1)}(s)$?	Yes	No
Running time	1:19:31	\approx 0:03:24

This example was suggested by F. Castro-Jiménez and J.-M. Ucha for testing the Logarithmic Comparison Theorem, see e.g. [39]. The use of logarithmic annihilator allowed us to reduce the computation time. However, for f from this example it is known, that $\text{Ann}_{D_n[s]}(f^s) = \text{Ann}_{D_n[s]}^{(1)}(f^s)$ and this fact together with some homogeneous properties were used to compute other roots of $b_f(s)$, see Example 4.4 below.

4.2.1. Quasi-homogeneous polynomials. Assume $F \in R_n$ is a w -quasi-homogeneous polynomial with $w_i \neq 0$, that is, there are $w_1, \dots, w_n \in \mathbb{K}$ such that $F = \sum_{i=1}^n w_i x_i \partial_i(F)$. Take $c \in \mathbb{K}^*$ and let us denote $f = F|_{x_k=1}$ for some fixed k . We are interested in studying the relationship between the Bernstein-Sato polynomials of f and F .

Proposition 4.2. *Let $F \in R_n$ be a quasi-homogeneous polynomial with respect to the weight vector $w = (w_1, \dots, w_n)$. Assume $w_k \neq 0$ for some $k \in \{1, \dots, n\}$ and define $f = F|_{x_k=c}$ for $c \in \mathbb{K}^*$. Then $b_f(s)$ divides $b_F(s)$.*

Proof. Consider the V -filtration on D_n given by the variable x_k . Let $P(s) \in D_n[s]$ a differential operator satisfying the functional equation for F . There exists $d \geq 0$ such that $x_k^d P(s) \in \sum_{i \geq 0} x_k^i \cdot V_0$. From the quasi-homogeneity of F one can deduce that

$$x_k \partial_k \bullet F^{s+1} = \frac{1}{w_k} \left(s + 1 - \sum_{i \neq k} w_i x_i \partial_i \right) \bullet F^{s+1}.$$

Let D' be the $(n-1)$ -th Weyl algebra in the variables $x_1, \dots, \hat{x}_k, \dots, x_n$. Thus $V_0 = D'[x_k \partial_k]$ and $x_k^d P(s) \bullet F^{s+1}$ can be written in the form $Q(s) \bullet F^{s+1}$ where the operator ∂_k does not appear in $Q(s) \in D_n[s]$. The functional equation for F has been converted in the following one:

$$x_k^d P(s) \bullet F^{s+1} = Q(s) \bullet F^{s+1} = x_k^d b_F(s) \bullet F^s$$

Now the substitution $x_k = c \in \mathbb{K}^*$ can be made and the claim follows. \square

Example 4.3. The Bernstein-Sato polynomials of $F = x^2 z + y^3$ and $f = F|_{z=1} = x^2 + y^3$ are

$$b_F(s) = \underbrace{(s+1) \left(s + \frac{5}{6}\right) \left(s + \frac{7}{6}\right)}_{b_f(s)} \left(s + \frac{4}{3}\right) \left(s + \frac{5}{3}\right).$$

From the result by Kashiwara [17] one can see, blowing up the origin of F , that the last two factors are related to the b -function of $\{z^3 = 0\}$. This is a general fact.

Example 4.4. Now, we continue with Example 4.1. Let g be the polynomial, resulting from f by substituting $x_1, x_2, x_3, x_4, x_5, x_9$ with 1. Using Proposition 4.2 several times, one can easily see that $b_g(s)$ divides $b_f(s)$. Finally, the `checkRoot` algorithm is used to obtain that

$$(s+1)^4(s+1/2)(s+3/2)(s+3/4)(s+5/4)$$

is a factor of $b_g(s)$ and therefore a factor of $b_f(s)$.

Factor of $b_g(s)$	$(s+1/2)$	$(s+3/4)$	$(s+3/2)$	$(s+1)^4$	$(s+5/4)$
Running time	0:02	0:04	0:10	3:45	4:46

4.3. Intersection homology \mathcal{D} -module. In this part of the paper we introduce some new notation. We refer to [40] for further details. Let X be a complex analytic manifold of dimension $n \geq 2$, \mathcal{O}_X the sheaf of holomorphic function on X and \mathcal{D}_X the sheaf of differential operators with holomorphic coefficients. At a point $x \in X$, we identify the stalks $\mathcal{O}_{X,x}$ with the ring $\mathcal{O} = \mathbb{C}\langle x_1, \dots, x_n \rangle$ of converging power series and $\mathcal{D}_{X,x}$ with $\mathcal{D} = \mathcal{O}\langle \partial_1, \dots, \partial_n \rangle$.

Given a closed subspace $Y \subset X$ of pure codimension $p \geq 1$, we denote by $H_{[Y]}^p(\mathcal{O}_X)$ the sheaf of local algebraic cohomology with support in Y . Let $\mathcal{L}(Y, X) \subset H_{[Y]}^p(\mathcal{O}_X)$ be the intersection homology \mathcal{D}_X -Module of Brylinski-Kashiwara. This is the smallest \mathcal{D}_X -submodule of $H_{[Y]}^p(\mathcal{O}_X)$ which coincides with $H_{[Y]}^p(\mathcal{O}_X)$ at the generic points of Y .

A natural problem is to characterize the subspaces Y such that $\mathcal{L}(Y, X)$ coincides with $H_{[Y]}^p(\mathcal{O}_X)$. Indeed, from the Riemann-Hilbert correspondence of Kashiwara-Mebkhout, the regular holonomic \mathcal{D}_X -module $H_{[Y]}^p(\mathcal{O}_X)$ corresponds to the perverse sheaf $\mathbf{C}_Y[p]$, while $\mathcal{L}(Y, X)$ corresponds to the intersection complex IC_Y^\bullet . This way, the condition $\mathcal{L}(Y, X) = H_{[Y]}^p(\mathcal{O}_X)$ is equivalent to the following one: the real link of Y at a point $x \in Y$ is a rational homology sphere. Torrelli proved, that the following connection to local Bernstein-Sato polynomial exists.

Theorem 4.5 (Theorem 1.2 in [40]). *Let $Y \subset X$ be a hypersurface and $h \in \mathcal{O}_{X,x}$ a local equation of Y at a point $y \in Y$. The following conditions are equivalent:*

- (1) $\mathcal{L}(Y, X)_y$ coincides with $H_{[Y]}^p(\mathcal{O}_X)_y$.
- (2) The reduced local Bernstein-Sato polynomial of h has no integral root.

The proof of the theorem is based on a natural generalization of a classical result due to Kashiwara which links the roots of the b -function to some generators of $\mathcal{O}[\frac{1}{f}]f^\alpha$, $\alpha \in \mathbb{C}$.

Example 4.6. Let Y be the affine variety in $X = \mathbb{C}^3$ defined by the polynomial $f = z^7 + (x^2z + y^3)(x^3 + y^2z)$. The surface Y has the origin as its only singular point and thus the local b -function and the global one coincide. The only possible integral roots for the b -function are -2 and -1 . Now consider J_f , the Jacobian ideal of f , cf. Rem. 2.8. Since the reduced Bernstein-Sato polynomial is required, the ideal

$$\text{Ann}_{D[s]}(f^s) + D[s]\langle f, J_f, s + \alpha \rangle$$

is used for checking rational roots, compare with Corollary 2.3 (2). We see that the previous ideal is not the whole ring for $\alpha = 1$ and hence the set of points $x \in Y$ such that $\mathcal{L}(Y, X)_x = H_{[Y]}^p(\mathcal{O}_X)_x$ is $Y \setminus \{0\}$.

Using the implementation by Schulze [37] (based on Gauss-Manin connection), the computation of the whole Bernstein-Sato polynomial took 123 sec., while with our approach only 11 sec. were needed.

Remark 4.7. Given Y as above, the set of points $x \in Y$ for which the condition $\mathcal{L}(Y, X)_x = H_{[Y]}^p(\mathcal{O}_X)_x$ is satisfied, defines an open set in Y that can be effectively computed with the stratification associated with the integral roots of the reduced local b -functions, see the sequence of varieties (6) below. For instance, in Example 5.1, the open set is $V(f) \setminus V_1$.

5. STRATIFICATION ASSOCIATED WITH LOCAL b -FUNCTIONS

From Theorem 2.10, one can find a stratification of \mathbb{C}^n so that $b_{f,p}(s)$ is constant on each stratum. The first method for computing such stratification was suggested by Oaku [31] (see also [30], [32] and [3] for further information). However, this method relies on the primary decomposition of commutative ideals. Following the ideas started in Section 2.2.2, we propose a new natural algorithm for computing such a stratification. At first, a stratification for each root of the global b -function is computed. Then one obtains a stratification, associated with the local b -function, notably without any primary ideal decomposition, see Example 5.1, 5.3, and 5.4 below. We have created an experimental implementation, which was used for presented examples. The substitution of primary decomposition with elementary operations clearly decreases the total complexity of this algorithm.

Recall Prop. 2.14: for the ideal $I = \text{Ann}_{D[s]}(f^s) + D[s]\langle f \rangle$ and for the root α of $b_f(s)$ with the multiplicity m_α , we define ideals $I_{\alpha,i} = (I : (s + \alpha)^i) + D[s]\langle s + \alpha \rangle$ for $i = 0, \dots, m_\alpha - 1$. Let $V_{\alpha,i}$ be the affine variety corresponding to the ideal $I_{\alpha,i} \cap \mathbb{C}[\mathbf{x}]$. Then

$$(6) \quad \emptyset =: V_{\alpha,m_\alpha} \subset V_{\alpha,m_\alpha-1} \subset \dots \subset V_{\alpha,0} \subset V_{\alpha,-1} := \mathbb{C}^n,$$

and $m_\alpha(p) = i$ if and only if $p \in V_{\alpha,i-1} \setminus V_{\alpha,i}$. We call this sequence the *stratification associated with the root α* . Let us see some examples² to show how this result can be use to compute a stratification associated with local b -functions.

Example 5.1. Consider $f = (x^2 + 9/4y^2 + z^2 - 1)^3 - x^2z^3 - 9/80y^2z^3 \in \mathbb{C}[x, y, z]$. The global b -function is

$$b_f(s) = (s + 1)^2(s + 4/3)(s + 5/3)(s + 2/3).$$

Take $V_1 = V(x^2 + 9/4y^2 - 1, z)$, $V_2 = V(x, y, z^2 - 1)$ and $V_3 = V(19x^2 + 1, 171y^2 - 80, z)$. Then V_2 (resp. V_3) consists of two (resp. four) different points and $V_3 \subset V_1$, $V_1 \cap V_3 = \emptyset$. The singular locus of f is union of V_1 and V_2 . The stratification associated with each root of $b_f(s)$ is given by

$$\begin{array}{lll} \alpha = -1, & \emptyset \subset V_1 \subset V(f) \subset \mathbb{C}^3; \\ \alpha = -4/3, & \emptyset \subset V_1 \cup V_2 \subset \mathbb{C}^3; \\ \alpha = -5/3, & \emptyset \subset V_2 \cup V_3 \subset \mathbb{C}^3; \\ \alpha = -2/3, & \emptyset \subset V_1 \subset \mathbb{C}^3. \end{array}$$

From this, one can easily find a stratification of \mathbb{C}^3 into constructible sets such that $b_{f,p}(s)$ is constant on each stratum.

$$b_{f,p}(s) = \begin{cases} 1 & p \in \mathbb{C}^3 \setminus V(f), \\ s + 1 & p \in V(f) \setminus (V_1 \cup V_2), \\ (s + 1)^2(s + 4/3)(s + 2/3) & p \in V_1 \setminus V_3, \\ (s + 1)^2(s + 4/3)(s + 5/3)(s + 2/3) & p \in V_3, \\ (s + 1)(s + 4/3)(s + 5/3) & p \in V_2. \end{cases}$$

The total running time including the computation of the global Bernstein-Sato polynomial was 8 min. The system Risa/Asir, which uses the algorithms from [28] by Nishiyama and Noro, needed more than 7 hours to obtain the same stratification.

Remark 5.2. Note that one can define a stratification associated with the roots of the local b -functions, that is taking no multiplicities into account. We have observed that our algorithm is especially useful and very fast for computing this stratification. In particular, this is the case when each root has multiplicity one. Finally, also observe that in any case the global b -function is not actually needed, if a set containing the roots of $b_f(s)$ is used instead.

Consider more interesting examples.

²The hypersurfaces for examples have been taken from <http://www.freigeist.cc/gallery.html>

Example 5.3. Let us proceed with Example 3.3. The stratification associated with every root of $b_f(s)$ except for $\alpha = 1$ is given by the sequence $\emptyset \subset Z \subset \mathbb{C}^3$. For $\alpha = 1$ of multiplicity 2, the corresponding sequence is $\emptyset \subset Y \sqcup Z \subset V(f) \subset \mathbb{C}^3$. Hence the local b -function at $p \in \mathbb{C}^3$ is

$$b_{f,p}(s) = \begin{cases} 1 & p \in \mathbb{C}^3 \setminus V(f), \\ s+1 & p \in V(f) \setminus (Y \sqcup Z), \\ (s+1)^2 & p \in Y, \\ b_f(s) & p \in Z. \end{cases}$$

Using the lexicographical ordering with $\partial_x > x$ on D during the computation of the intersection with $\mathbb{C}[\mathbf{x}]$, cf. Rem. 3.4 reduced the total running time to just 38 sec.

Example 5.4. Let us compute here the stratification associated with local b -functions of Example 3.6. Denote by V_1, V_2 the two axes $V_1 := V(x, z), V_2 := V(y, z)$. The singular locus is in both cases the union of these varieties.

$$b_{f,p}(s) = \begin{cases} 1 & p \in \mathbb{C}^3 \setminus V(f), \\ s+1 & p \in V(f) \setminus (V_1 \cup V_2), \\ b_{z^4+x^6}(s) & p \in V_1 \setminus \{0\}, \\ b_{z^4+y^5}(s) & p \in V_2 \setminus \{0\}, \\ (s+3/2)(s+7/4) \operatorname{lcm}(b_{z^4+x^6}(s), b_{z^4+y^5}(s)) & p = 0. \end{cases}$$

The stratification given by the singularity $\{g = 0\}$ is the same as above. The local b -functions in each stratum is obtained performing the replacements

$$\begin{aligned} z^4 + x^6 &\longmapsto z^4 + x^6 + x^5z, \\ z^4 + y^5 &\longmapsto z^4 + y^5 + y^4z, \\ (s+3/2)(s+7/4) &\longmapsto (s+1/2). \end{aligned}$$

This can be interpreted as follows. Let $P = (a, 0, 0) \in V_1 \setminus \{0\}$. The local equation of f (resp. g) is $z^4 + x^6 = 0$ (resp. $z^4 + x^6 + x^5z$). By the semi-continuity of the Bernstein-Sato polynomial the local b -function at P divides the Bernstein-Sato polynomial at the origin. Analogous considerations hold for $P \in V_2 \setminus \{0\}$. The system Risa/Asir, which apparently uses the algorithms from [28] by Nishiyama and Noro, did not finish the computation of the stratification after more than 40 hours.

Remark 5.5. We see some common properties between the factorization of a Bernstein-Sato polynomial with the so-called *central character decomposition* by Levandovskyy [18]. In particular, for $b_f(s) = \prod_{\alpha \in A} (s - \alpha)^{m_\alpha}$, where $A \subset \mathbb{Q}$ is the set of roots of $b_f(s)$, there is an algorithm for computing the following direct sum decomposition of the module

$$D[s]/(\operatorname{Ann}_{D[s]}(f^s) + \langle f \rangle) \cong \bigoplus_{\alpha \in A} D[s]/(\operatorname{Ann}_{D[s]}(f^s) + \langle f \rangle) : J(\alpha)^\infty,$$

where $J(\alpha) = \langle b_f(s)/(s - \alpha)^{m_\alpha} \rangle$. We plan to investigate this topic further and provide cyclic $D[s]$ -modules, corresponding to different strata.

As mentioned above, there is a very recent paper [28] by Nishiyama and Noro, where the authors build a stratification without using primary decomposition. The authors use initial ideals with respect to weight vectors in computations, which is a classical (cf. [33]) alternative to the methods, utilizing annihilators $\operatorname{Ann}_{D[s]}(f^s)$. In [1] there is a comparison of performance of both approaches for the computation of Bernstein-Sato polynomials. Notably, no method is clearly superior over another. Rather there are classes of examples, where the difference is very distinct. In particular, initial-based method scores better results on hyperplane arrangements, while annihilator-based methods are better at complicated singularities, which are not hyperplane arrangements. A comparison of two methods for stratification is very interesting and it is an important task for the future. However, it seems to us that the method we presented

will allow more thorough analysis of the algebraic situation due to the applicability of central character decomposition. At the moment it is not clear, whether such a decomposition exists for initial ideals.

6. OTHER APPLICATIONS

6.1. Bernstein-Sato Polynomials for Varieties. Let $f = (f_1, \dots, f_r)$ be an r -tuple in $\mathbb{K}[x]^r$. Denote by $\mathbb{K}\langle S \rangle$ the universal enveloping algebra $U(\mathfrak{gl}_r)$, generated by the set of variables $S = (s_{ij})$, $i, j = 1, \dots, r$ subject to relations:

$$[s_{ij}, s_{kl}] = \delta_{jk}s_{il} - \delta_{il}s_{kj}.$$

Then, we denote by $D_n\langle S \rangle := D_n \otimes_{\mathbb{K}} \mathbb{K}\langle S \rangle$. Consider a free $\mathbb{K}[x, s, \frac{1}{f}]$ -module of rank one generated by the formal symbol f^s and denote it by $M = \mathbb{K}[x, s_{11}, \dots, s_{rr}, \frac{1}{f_1 \dots f_r}] \cdot f^s$, where $f^s = f_1^{s_{11}} \dots f_r^{s_{rr}}$. The module M has a natural structure of left $D_n\langle S \rangle$ -module. Denote by $\text{Ann}_{D_n\langle S \rangle}(f^s)$ the left ideal of all elements $P(S) \in D_n\langle S \rangle$ such that $P(S) \bullet f^s = 0$, that is the *annihilator* of f^s in $D_n\langle S \rangle$.

Theorem 6.1 (Budur, Mustața, Saito [7]). *For every r -tuple $f = (f_1, \dots, f_r) \in \mathbb{K}[x]^r$ there exists a non-zero polynomial in one variable $b(s) \in \mathbb{K}[s]$ and r differential operators $P_1(S), \dots, P_r(S) \in D_n\langle S \rangle$ such that*

$$(7) \quad \sum_{k=1}^r P_k(S) f_k \cdot f^s = b(s_{11} + \dots + s_{rr}) \cdot f^s \in M.$$

The *Bernstein-Sato polynomial* $b_f(s)$ of $f = (f_1, \dots, f_r)$ is defined to be the monic polynomial of the lowest degree in the variable s satisfying the equation (7). It can be verified that $b_f(s)$ is independent of the choice of a system of generators of $\langle f_1, \dots, f_r \rangle$. Then the Bernstein-Sato polynomial of f can be computed as follows

$$(\text{Ann}_{D_n\langle S \rangle}(f^s) + \langle f_1, \dots, f_r \rangle) \cap \mathbb{K}[s_{11} + \dots + s_{rr}] = \langle b_f(s_{11} + \dots + s_{rr}) \rangle.$$

In [1] an algorithm to find a system of generators of $\text{Ann}_{D\langle S \rangle}(f^s)$ was given. Moreover, in computing the intersection of an ideal with the univariate subalgebra an optimized algorithm (which avoids elimination with Gröbner basis) was used.

The preceding formula together with Theorem 2.1 can be used to check rational roots of Bernstein-Sato polynomials also for affine algebraic varieties. Hence, following Corollary 2.6, a stratification associated with the local b -functions can be computed.

6.2. A remark in Narváez's paper. In [27], Narváez introduces a polynomial denoted by $\beta(s)$ verifying $\beta(s) \text{Ann}_{D[s]}(f^s) \subseteq \text{Ann}_{D[s]}^{(1)}(f^s)$. For all the examples treated in [27], he was able to compute an operator $P'(s) \in D[s]$ such that $b_f(s) - P'(s)f \in \text{Ann}_{D[s]}^{(1)}(f^s)$. The last example in the paper is quite involved and could not be computed by using any computer algebra system directly. An iterated process for finding approximations of involutive bases was used instead.

Indeed, for this propose the operator is not really needed, since

$$b_f(s) - P(s)f \in \text{Ann}_{D[s]}^{(1)}(f^s) \iff b_f^{(1)}(s) = b_f(s) \iff b_f^{(1)}(s) \mid b_f(s),$$

and thus after computing $b_f^{(1)}(s)$, one only has to check whether each root of the latter polynomial is indeed a root of the b -function and the same with the multiplicities.

By definition the following inclusions hold

$$\beta(s) (\text{Ann}_{D[s]}(f^s) + \langle f \rangle) \subset \text{Ann}_{D[s]}^{(1)}(f^s) + \langle f \rangle \subset \text{Ann}_{D[s]}(f^s) + \langle f \rangle.$$

This implies that $b_f(s) \mid b_f^{(1)}(s) \mid \beta(s)b_f(s)$. Additionally, if $\beta(s)$ divides $b_f(s)$, then the polynomials $b_f^{(1)}(s)$ and $b_f(s)$ both have the same roots and the previous condition is equivalent to $m_\alpha(b_f^{(1)}(s)) = m_\alpha(b_f(s))$ for every root α of $\beta(s)$.

Example 6.2. Let $f = (x_1x_3 + x_2)(x_1^7 - x_2^7)$ be the last example from [27]. The Bernstein-Sato polynomial and the polynomial $\beta(s)$ are respectively

$$\begin{aligned} b_f(s) &= (s+1)^3(s+3/4)(s+3/8)(s+9/8)(s+1/4)(s+7/8)(s+1/2)(s+5/8), \\ \beta(s) &= (s+3/4)(s+5/8)(s+1/2)(s+3/8)(s+1/4). \end{aligned}$$

Now one only has to check that all roots of $\beta(s)$ have multiplicity 1 as a root of $b_f^{(1)}(s)$. This can be done using Theorem 2.1 with $I = \text{Ann}_{D[s]}^{(1)}(f^s) + \langle f \rangle$. Using this approach, the computations become very easy (less than 5 seconds in this example).

7. CONCLUSION AND FURTHER WORK

As we have demonstrated, the family of `checkRoot` algorithms (implemented in the library `dmod.lib` of SINGULAR) has many useful applications in the realm of D -modules. Nowadays, it is the only method that allows one to obtain some roots of the b -function without computing the whole Bernstein-Sato polynomial. The latter is often infeasible despite all the recent progress in computational D -module theory.

We emphasize, that presented techniques are elementary (by utilizing the principal ideal domain of the center $\mathbb{K}[s]$ of $D_n[s]$) but very powerful from computational point of view. Many intractable examples and conjectures could be treated with this new method, as we have partially illustrated. Moreover, a stratification associated with the local b -functions can be obtained without primary decomposition [31] as in the very recent [28]. It is very interesting to study these algorithms further and compare our approach with the one of [28].

Unfortunately, these techniques cannot be generalized for Bernstein-Sato ideals, since such ideals lie in $\mathbb{K}[s_1, \dots, s_m]$ for $m \geq 2$.

We have demonstrated that one can use the idea of `checkRoot` for checking rational roots of b -function of a holonomic ideal with respect to a weight vector [33]. This gives an easier method for computing, among other, integral roots of such b -functions. In this context, it would be very interesting to have a version of Kashiwara's result for some holonomic ideals and certain weights, since many algorithms in D -modules theory are based on integrations and restrictions which need minimal/maximal roots.

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REFERENCES

- [1] D. Andres, V. Levandovskyy, and J. Martín-Morales. Principal intersection and Bernstein-Sato polynomial of an affine variety. In *Proc. of the International Symposium on Symbolic and Algebraic Computation (ISSAC'09)*. ACM Press, 2009.
- [2] Enrique Artal-Bartolo. Forme de Jordan de la monodromie des singularités superisolées de surfaces. *Mem. Amer. Math. Soc.*, 109(525):x+84, 1994.
- [3] Rouchdi Bahloul and Toshinori Oaku. Local Bernstein-Sato ideals: algorithm and examples. *J. Symbolic Computation*, 45(1):46–59, 2010.
- [4] I. N. Bernstein. Analytic continuation of generalized functions with respect to a parameter. *Funkcional. Anal. i Priložen.*, 6(4):26–40, 1972.

- [5] Ana María Bravo, Santiago Encinas, and Orlando Villamayor U. A simplified proof of desingularization and applications. *Rev. Mat. Iberoamericana*, 21(2):349–458, 2005.
- [6] J. Briançon and Ph. Maisonobe. Remarques sur l’idéal de Bernstein associé à des polynômes. *Preprint no. 650, Univ. Nice Sophia-Antipolis*, 2002.
- [7] N. Budur, M. Mustață, and M. Saito. Bernstein-Sato polynomials of arbitrary varieties. *Compos. Math.*, 142(3):779–797, 2006.
- [8] F. J. Castro-Jiménez and J. M. Ucha-Enríquez. Logarithmic comparison theorem and some Euler homogeneous free divisors. *Proc. Amer. Math. Soc.*, 133(5):1417–1422 (electronic), 2005.
- [9] A. Frühbis-Krüger and G. Pfister. A SINGULAR 3.0 library for resolution of singularities (desingularization) with the algorithm of villamayor `resolve.lib`, 2005.
- [10] A. Frühbis-Krüger and G. Pfister. A SINGULAR 3.0 library for topological zeta-function and some other applications of desingularization `reszeta.lib`, 2005.
- [11] Anne Frühbis-Krüger. An application of resolution of singularities: Computing the topological ζ -function of isolated surface singularities in $(3,0)$. In D. Cheniot and N. Dutertre, editors, *Singularity Theory*. World Scientific Publishing, 2007.
- [12] Anne Frühbis-Krüger. Computational aspects of singularities. In J.-P. Brasselet and J. Damon, editors, *Singularities in Geometry and Topology*, pages 253–327. World Scientific Publishing, 2007.
- [13] K. R. Goodearl and R. B. Warfield, Jr. *An introduction to noncommutative Noetherian rings*, volume 61 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, second edition, 2004.
- [14] M. Granger, T. Oaku, and N. Takayama. Tangent cone algorithm for homogenized differential operators. *J. Symbolic Computation*, 39(3–4):417–431, 2005.
- [15] G.-M. Greuel, V. Levandovskyy, O. Motsak, and H. Schönemann. PLURAL. A SINGULAR 3.0 Subsystem for Computations with Non-commutative Polynomial Algebras. Centre for Computer Algebra, University of Kaiserslautern, 2006.
- [16] Andréa G. Guimarães and Abramo Hefez. Bernstein-Sato polynomials and spectral numbers. *Ann. Inst. Fourier (Grenoble)*, 57(6):2031–2040, 2007.
- [17] M. Kashiwara. B -functions and holonomic systems. Rationality of roots of B -functions. *Invent. Math.*, 38(1):33–53, 1976/77.
- [18] V. Levandovskyy. On preimages of ideals in certain non-commutative algebras. In G. Pfister, S. Cojocaru, and V. Ufnarovski, editors, *Computational Commutative and Non-Commutative Algebraic Geometry*. IOS Press, 2005.
- [19] V. Levandovskyy and J. Martín-Morales. Computational D -module theory with SINGULAR, comparison with other systems and two new algorithms. In *Proc. of the International Symposium on Symbolic and Algebraic Computation (ISSAC’08)*. ACM Press, 2008.
- [20] Joseph Lipman. Topological invariants of quasi-ordinary singularities. *Mem. Amer. Math. Soc.*, 74(388):1–107, 1988.
- [21] B. Malgrange. Le polynôme de Bernstein d’une singularité isolée. In *Fourier integral operators and partial differential equations (Colloq. Internat., Univ. Nice, Nice, 1974)*, pages 98–119. Lecture Notes in Math., Vol. 459. Springer, Berlin, 1975.
- [22] B. Malgrange. Polynômes de Bernstein-Sato et cohomologie évanescence. In *Analysis and topology on singular spaces, II, III (Luminy, 1981)*, volume 101 of *Astérisque*, pages 243–267. Soc. Math. France, Paris, 1983.
- [23] Jorge Martín-Morales. *Embedded \mathbf{Q} -Resolutions and Yomdin-Lê Surface Singularities*. PhD dissertation, IUMA-University of Zaragoza, December 2011. URL: <http://cud.unizar.es/martin>.
- [24] J. C. McConnell and J. C. Robson. *Noncommutative Noetherian rings*, volume 30 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, revised edition, 2001. With the cooperation of L. W. Small.
- [25] Z. Mebkhout and L. Narváez-Macarro. La théorie du polynôme de Bernstein-Sato pour les algèbres de Tate et de Dwork-Monsky-Washnitzer. *Ann. Sci. École Norm. Sup. (4)*, 24(2):227–256, 1991.
- [26] Hiromasa Nakayama. Algorithm computing the local b -function by an approximate division algorithm in \hat{D} . *J. Symbolic Comput.*, 44(5):449–462, 2009.
- [27] L. Narváez-Macarro. Linearity conditions on the Jacobian ideal and logarithmic-meromorphic comparison for free divisors. In *Singularities I*, volume 474 of *Contemp. Math.*, pages 245–269. Amer. Math. Soc., Providence, RI, 2008.
- [28] K. Nishiyama and M. Noro. Stratification associated with local b -functions. *J. Symbolic Comput.*, 45(4):462–480, 2010.
- [29] Masayuki Noro. An efficient modular algorithm for computing the global b -function. In *Mathematical software (Beijing, 2002)*, pages 147–157. World Sci. Publ., River Edge, NJ, 2002.
- [30] Toshinori Oaku. An algorithm of computing b -functions. *Duke Math. J.*, 87(1):115–132, 1997.

- [31] Toshinori Oaku. Algorithms for b -functions, restrictions, and algebraic local cohomology groups of D -modules. *Adv. in Appl. Math.*, 19(1):61–105, 1997.
- [32] Toshinori Oaku. Algorithms for the b -function and D -modules associated with a polynomial. *J. Pure Appl. Algebra*, 117/118:495–518, 1997. Algorithms for algebra (Eindhoven, 1996).
- [33] M. Saito, B. Sturmfels, and N. Takayama. *Gröbner deformations of hypergeometric differential equations*, volume 6 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin, 2000.
- [34] Morihiko Saito. On b -function, spectrum and rational singularity. *Math. Ann.*, 295(1):51–74, 1993.
- [35] Morihiko Saito. On microlocal b -function. *Bull. Soc. Math. France*, 122(2):163–184, 1994.
- [36] Mathias Schulze. A normal form algorithm for the Brieskorn lattice. *J. Symbolic Comput.*, 38(4):1207–1225, 2004.
- [37] Mathias Schulze. A SINGULAR 3.0 library to compute invariants related to the Gauss-Manin system of an isolated hypersurface singularity `gmssing.lib`. 2004.
- [38] J. H. M. Steenbrink. Mixed Hodge structure on the vanishing cohomology. In *Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976)*, pages 525–563. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.
- [39] Tristan Torrelli. Logarithmic comparison theorem and \mathcal{D} -modules: an overview. In *Singularity theory*, pages 995–1009. World Sci. Publ., Hackensack, NJ, 2007.
- [40] Tristan Torrelli. Intersection homology \mathcal{D} -module and Bernstein polynomials associated with a complete intersection. *Publ. Res. Inst. Math. Sci.*, 45(2):645–660, 2009.
- [41] A. N. Varchenko. Asymptotic Hodge structure on vanishing cohomology. *Izv. Akad. Nauk SSSR Ser. Mat.*, 45(3):540–591, 1981.
- [42] Uli Walther. Bernstein-Sato polynomial versus cohomology of the Milnor fiber for generic hyperplane arrangements. *Compos. Math.*, 141(1):121–145, 2005.

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