Preservation of the log concavity by Bernstein operator with an application to ageing properties of a coherent system

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Abstract

In this paper, we provide a new proof of the preservation of the log concavity by Bernstein operator. It is based on the bivariate characterization of the likelihood ratio order. In addition, we give new conditions under which the previous property leads to preservation of ageing properties in coherent systems with independent and identically distributed components.

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1 Preservation of log concavity property

The classical Bernstein operators B_n are defined for a bounded real function on $[0,1]$ f and a natural number n as:

$$
\begin{aligned}
B_n(f,x) &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k} = E\left[f\left(\frac{B(n,x)}{n}\right)\right] && (1)\\
&= xE\left[f\left(\frac{B(n-1,x)+1}{n}\right)\right] + (1-x)E\left[f\left(\frac{B(n-1,x)}{n}\right)\right] \\
&= xE\left[\Delta^1_{\frac{1}{n}} f\left(\frac{B(n-1,x)}{n}\right)\right] + E\left[f\left(\frac{B(n-1,x)}{n}\right)\right] && (2)\\
&= \sum_{k=0}^{n}\binom{n}{k}\Delta^k_{\frac{1}{n}} f(0)x^k, \quad x \in [0,1], && (3)
\end{aligned}
$$

where $E[.]$ denotes the expectation operator, Δ_h^k is the difference operator of order k defined as

$$\Delta_h^1 f(x) = f(x+h) - f(x), \quad h \geq 0,$$

for $k = 1$ and inductively for $k \geq 2$

$$\Delta_h^k f(x) = \Delta_h^1 \Delta_h^{k-1} f(x),$$

and $B(n,x)$ is a binomial random variable with parameters n and x. Expression (2) is a rearrangement of previous equation which follows since $B(n,x) \stackrel{\mathrm{d}}{=} \sum_{i=1}^n Y_i(x)$, where $\stackrel{\mathrm{d}}{=}$ means equality in distribution and the $Y_i(x)$ are independent and identically distributed random variables such that $P(Y_i(x) = 1) = 1 - P(Y_i(x) = 0) = x$. Equation (3) can be found in [9].

These polynomials are very useful for the shape preserving approximation. It is well known that they preserve convex, concave functions and extensions of both types of functions. Shape preservation properties of Bernstein and analogous operators as Szàsz can be found in [1, 2]. The log-concavity preserving property of Bernstein operators was first established in 1988 by T.N.T. Goodman in an article [7] devoted to computer aided geometric design. A. Komisarski [8] got a similar result by strengthening of Goodman's result.Finally, Bieniek el al. [6] gave a different proof, as an application of shape properties of the lifetime of a reliability system. On the other hand, Vinogradov and Ulitskaya [22] gave an example showing that Bernstein Polynomials do not preserve log-convexity. For further results and details about Bernstein polynomials, see the monography [9]. The preservation of log convexity by some Bernstein-type operators including for instance Szàsz and Baskakov operators using probabilistic tools was addressed in [4]. The preservation of log concavity was explored in [5] for operators admitting a probabilistic representation through a stochastic process with non negative independent increments. Previous property is obtained based on the bivariate characterization of likelihood ratio order, hazard rate order and reversed hazard rate order (see [23] for an alternative proof of the results in [4, 5]). In the case of Bernstein operator, since the stochastic process $B(n,x)$ has dependent increments, previous approach fails for log concavity preservation. In [8], a probabilistic proof of Bernstein log concavity preservation was conjectured and, in some sense, obtained in [6]. In this paper we provide a short proof of it based on the bivariate characterization of likelihood ratio stochastic order. As consequence of latter preservation property of Bernstein operator an application to ageing classes of coherent systems is derived. Numerical applications of shape preserving properties of Bernstein operator are discussed in [14].

Recall that a non negative real function f on a convex interval I is log concave if $f(\lambda x+(1-\lambda)y)\geq f^{\lambda}(x)f^{1-\lambda}(y)$ for $x,y\in I$ and $0\leq\lambda\leq 1$, thus $\ln f$ is concave on I. A function f on I twice derivable is log concave if

$$(f'(x))^2\geq f(x)f''(x) \tag{4}$$

or, equivalently, if $f>0$, $f'(x)/f(x)$ is non increasing in $x\in I$.

Recall that for X and Y random variables with probability density functions f_X and f_Y, respectively, X is said less than or equal than Y in likelihood ratio written $X\leq_{\mathrm{lr}}Y$ if $\frac{f_Y(x)}{f_X(x)}$ is non decreasing in x either on the support of X or Y being $\frac{a}{0}=\infty$. Latter definition is analogous for discrete random variables interchanging probability density function with probability mass function. Bivariate characterization of likelihood ratio stochastic order says that $X\leq_{\mathrm{lr}}Y$ iff for all bivariate function g such that $g(x,y)-g(y,x)\geq 0$ for $x\leq y$, it holds that $E[g(X^*,Y^*)]\geq E[g(Y^*,X^*)]$ for X^* and Y^* independent random variables such that $X\stackrel{\mathrm{d}}{=}X^*$ and $Y\stackrel{\mathrm{d}}{=}Y^*$. This characterization was stated for the first time in [19].

It is well known that in the case of binomial distribution, it holds that $B(n,x)\leq_{\mathrm{lr}}B(n,y)$ for $x\leq y$.

The derivative of the Bernstein polynomial can be written as (see [9])

$$\begin{aligned} B'_n(f,x) &= n\sum_{k=0}^{n-1}(f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right))\binom{n-1}{k}x^k(1-x)^{n-1-k} \\ &= nE\left[\Delta^1_{\frac{1}{n}} f\left(\frac{B(n-1,x)}{n}\right)\right]. \end{aligned} \quad (5)$$

Next auxiliary result is worthwhile for proving preservation of log concavity by Bernstein operator (see Theorem 1).

Lemma 1 *If f is log concave on a convex set I, then for $x, y, x+\delta, y+\delta \in I$ such that $x \le y$ and $\delta > 0$,*

$$f(y)f(x+\delta) - f(x)f(y+\delta) \ge 0.$$

Proof: Assuming assumptions in the Lemma we can write

$$\begin{aligned} y &= \frac{\delta}{y-x+\delta}x + \frac{y-x}{y-x+\delta}(y+\delta) \\ x+\delta &= \frac{y-x}{y-x+\delta}x + \frac{\delta}{y-x+\delta}(y+\delta). \end{aligned}$$

Since f is log concave on I, it follows that

$$\begin{aligned} f(y) &\ge f^{\frac{\delta}{y-x+\delta}}(x) f^{\frac{y-x}{y-x+\delta}}(y+\delta) \\ f(x+\delta) &\ge f^{\frac{y-x}{y-x+\delta}}(x) f^{\frac{\delta}{y-x+\delta}}(y+\delta) \end{aligned}$$

As f is non negative, multiplying both inequalities leads to the claim. ■

We are in conditions to prove the main theorem.

Theorem 1 *$B_n(f,x)$ is a log concave function for a log concave function f on $[0,1]$ and $n = 1, 2, \ldots$*

Proof: Result is obvious either for $n = 1$ or $f\left(\frac{k}{n}\right)$ constant for $k = 0, \ldots, n$ as $B_n(f,x)$ is a linear function. Therefore in order to prove the theorem we can consider that $n \ge 2$ and f is non-constant in $\frac{k}{n}$, $k = 0, \ldots, n$. In this case $\Delta^1_{\frac{1}{n}} f\left(\frac{k}{n}\right) \ne 0$ for some $k \in \{1, \ldots, n-1\}$ and $B'_n(f,x)$ is a polynomial of order between 1 and $n-1$ (see (5)), so that there exists a natural number r and $0 = s_0 < s_1 < \cdots < s_{r-1} < s_r = 1$ such that

$B_n'(f,s_j)=0$, $j=1,\dots,r-1$ and $B_n'(f,x)\neq 0$ in (s_{j-1},s_j), $j=1,\dots,r$. Let us show that $B_n(f,x)$ is log concave on (s_{j-1},s_j), $j=1,\dots,r$. Here, it is obvious that $B_n(f,x)>0$ on (s_{j-1},s_j). Therefore by (1), (2) and (5), it holds that

$$(\log B_n(f,x))' = \frac{B_n'(f,x)}{B_n(f,x)} = \frac{n}{x+\frac{E\left[f\left(\frac{B(n-1,x)}{n}\right)\right]}{E\left[\Delta^1_{\frac{1}{n}} f\left(\frac{B(n-1,x)}{n}\right)\right]}}, \quad x\in(s_{j-1},s_j), \tag{6}$$

$j=1,\dots,r$.

Now we show that for $0\le x\le y$

$$\begin{aligned} &E\left[f\left(\frac{B(n-1,y)}{n}\right)\right] E\left[\Delta^1_{\frac{1}{n}} f\left(\frac{B(n-1,x)}{n}\right)\right] \\ &-E\left[f\left(\frac{B(n-1,x)}{n}\right)\right] E\left[\Delta^1_{\frac{1}{n}} f\left(\frac{B(n-1,y)}{n}\right)\right]\ge 0. \end{aligned} \tag{7}$$

Observe that

$$B(n-1,x)\le_{\mathrm{lr}} B(n-1,y).$$

Applying bivariate characterization of likelihood ratio order, (7) holds if

$$\begin{aligned} &f\left(\frac{b}{n}\right)\Delta^1_{\frac{1}{n}} f\left(\frac{a}{n}\right) - f\left(\frac{a}{n}\right)\Delta^1_{\frac{1}{n}} f\left(\frac{b}{n}\right) \\ =\ &f\left(\frac{b}{n}\right) f\left(\frac{a}{n}+\frac{1}{n}\right) - f\left(\frac{a}{n}\right) f\left(\frac{b}{n}+\frac{1}{n}\right)\ge 0, \quad 0\le a\le b\le 1. \end{aligned}$$

The claim above is fulfilled by Lemma 1 with $x=\frac{a}{n}$, $y=\frac{b}{n}$, $\delta=\frac{1}{n}$ and $I=[0,1]$. Hence, (7) holds.

Based on equation (6) for $x,y\in(s_{j-1},s_j)$ such that $x\le y$

$$\frac{B_n'(f,x)}{B_n(f,x)} \ge \frac{n}{y+\frac{E\left[f\left(\frac{S_{n-1,x}}{n}\right)\right]}{E\left[\Delta^1_{\frac{1}{n}} f\left(\frac{S_{n-1,x}}{n}\right)\right]}} \ge \frac{n}{y+\frac{E\left[f\left(\frac{S_{n-1,y}}{n}\right)\right]}{E\left[\Delta^1_{\frac{1}{n}} f\left(\frac{S_{n-1,y}}{n}\right)\right]}} = \frac{B_n'(f,y)}{B_n(f,y)},$$

where the second inequality holds by (7). Therefore, $(\ln B_n(f,x))'$ is non-increasing in x on (s_{j-1},s_j). Hence, $\ln B_n(f,x)$ is concave on (s_{j-1},s_j), and (4) holds true into these intervals. As Bernstein polynomials and their derivatives are continuous we conclude that $B_n(f,x)$ is a log concave function on

$[0,1]$, as (4) can be extended to the whole interval by using a limiting argument. ■

For the applications in Section 2, we will need a result concerning preservation of log-concavity for Bernstein operators when we have functions defined on the set $\{0,1,2,\ldots,n\}$ being log-concave. Note that Theorem 1 will also apply to this case, as shown in the next proposition.

Proposition 1 *Let $f:\{0,1,2,\ldots,n\}\to[0,\infty)$ being log-concave, that is, having no internal zeroes and such that*

$$f(i+1)^2 \geq f(i)f(i+2), \quad i=0,\ldots,n-2.$$

Then, $Ef\left(B(n,x)\right)$ is a log-concave function on x

Proof: Let us define the function $f_n:\{0,1/n,2/n,\ldots,1\}\to[0,\infty)$ such that $f_n(i/n)=f(i)$. Clearly we have

$$f_n\left(\frac{i+1}{n}\right)^2 \geq f_n\left(\frac{i}{n}\right) f_n\left(\frac{i+2}{n}\right), \quad i=0,\ldots,n-2.$$

Now consider $\hat{f}_n:[0,1]\to[0,\infty)$ such that $\hat{f}_n(i/n)=f_n(i/n)$, and for $i/n<x<(i+1)/n$ with $f(i)f(i+1)>0$, $\log\hat{f}_n(x)$ is obtained by linear interpolation between consecutive values of $\{log(\hat{f}_n(i/n))\}$. Note that $\log(\hat{f}_n(x))$ is a concave function in its domain of definition. We extend $\hat{f}_n$ to the interval $[0,1]$, if $f(0)f(n)=0$, as $\hat{f}_n(x)=0$ in the extreme lower and/or upper intervals. We have, by construction, that $\hat{f}_n$ is a log-concave function. Thus, by Theorem 1, $B_n(\hat{f}_n,x)$ is log-concave. The conclusion follows as

$$Ef\left(B(n,x)\right)=B_n(\hat{f}_n,x).$$

■

2 Application: Conditions for IFR and DRFR property of a coherent system with n i.i.d. components

Coherent systems are basic systems in reliability theory. A system is coherent if each component is relevant and its structure function increases in each component (the structure function indicates the state of a system (working or not), in terms of the state of each component). The lifetime of a system is determined by its components and its structure. See, for instance, [13], for a recent reference concerning coherent systems. Samaniego [17] introduced the concept of "signature" of a system, which depends on the structure of the system, and proved that the lifetime distribution of a coherent system, whose components have continuous, independent and identically distributed (i.i.d.) lifetimes, can be obtained as a linear combination of distributions of order statistics obtained from the lifetimes of the components. The signature $\mathbf{s} = (s_1, \ldots, s_n)$ of a coherent system with n i.i.d. lifetimes of the components is the n-dimensional probability vector whose ith element is $s_i = P(S_{\mathbf{X}}(\mathbf{s}) = X_{i:n})$, where $S_{\mathbf{X}}(\mathbf{s})$ denotes the lifetime of the coherent system and $X_{1:n}, \ldots, X_{n:n}$ denotes the order statistics of n i.i.d. component lifetimes $\mathbf{X} = (X_1, \ldots, X_n)$ with a common continuous distribution function. Let us define

$$R_j = \sum_{i=j}^{n} s_i, \text{ and } L_j = \sum_{i=1}^{j} s_i, \quad j = 1, \ldots, n, \tag{8}$$

and $R_{n+1} = L_0 = 0$. The reliability function of $S_{\mathbf{X}}(\mathbf{s})$ for i.i.d. components with common distribution function F and reliability function $\overline{F}$ can be expressed as (see equation (1) in [21]).

$$\overline{F}_{S_{\mathbf{X}(\mathbf{s})}}(t) = \sum_{j=1}^{n} s_j \overline{F}_{j:n}(t) = \sum_{j=0}^{n-1} (\sum_{i=j+1}^{n} s_i) \binom{n}{j} F(t)^j \overline{F}(t)^{n-j} = E\left[R_{B(n,F(t))+1}\right].$$

As an immediate consequence of Proposition 1 together with the fact that R_j is non increasing in j we have the following (see [10, p.689]).

Corollary 1 *If R_j is log concave, and F is convex into its support, then $S_{\mathbf{X}}(\mathbf{s})$ has reliability function being log concave, that is, $S_{\mathbf{X}}(\mathbf{s})$ is increasing failure rate. (IFR)*

Moreover, the cumulative distribution function of $S_{\mathbf{X}}(\mathbf{s})$ for i.i.d. components is given by

$$F_{S_{\mathbf{X}(\mathbf{s})}}(t) = \sum_{j=1}^{n} s_j \sum_{i=j}^{n} \binom{n}{i} F(t)^i \overline{F}(t)^{n-i}$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{i} s_j \binom{n}{i} F(t)^i \overline{F}(t)^{n-i} = E\left[L_{B(n,F(t))}\right].$$

As an immediate consequence of Proposition 1 together with the fact that L_j is non decreasing in j we have the following (see [10, p.689]).

Corollary 2 *If L_j is log concave, and F is concave into its support, then $S_{\mathbf{X}}(\mathbf{s})$ has log concave cumulative distribution function, that is, $S_{\mathbf{X}}(\mathbf{s})$ is decreasing reversed failure rate (DRFR).*

Remark 1 *Conditions given in Corollary 1 (Corollary 2) are different to those in Tavangar [20], and those given in Navarro et al. [12]. The conditions for IFR and DRFR in Tavangar [20] which are denoted by C1 and C2, respectively, are given as follows:*

- *(C1) $\overline{F}$ is log concave and $(n-i)\frac{s_{i+1}}{R_{i+1}}$ is non decreasing on i whenever the expression has sense;*
- *(C2) F is log concave and $i\frac{s_i}{L_i}$ is non increasing on i whenever the expression has sense.*

Arnold et al. [3] showed that Tavangar conditions for the monotone failure rate hold for coherent systems with identically distributed components, but not necessarily independent.

In Navarro et al. [12], ageing properties for a system $S_{\mathbf{X}}(\mathbf{s})$ with exchangeable components are derived on the basis of functional properties of the domination function denoted by H. In i.i.d. case, the domination function is a polynomial which is defined as

$$H(p) = \sum_{j=0}^{n} R_{n+1-j} \binom{n}{j} p^j (1-p)^{n-j} = E[R_{n+1-B(n.p)}], \quad 0 \le p \le 1. \tag{9}$$

It is shown in [12] that if $\overline{F}$ (F) is log concave and

$$p\frac{H'(p)}{H(p)} \left(p\frac{H'(1-p)}{1-H(1-p)} \right), \quad 0 \le p \le 1 \tag{10}$$

is non increasing on p, the IFR (DRFR) property for the system $S_{\mathbf{X}}(\mathbf{s})$ *holds.*

Next, using straightforward algebra and the bivariate characterization of the lr stochastic order, we show that under Tavangar condition C1 (C2), $p\frac{H'(p)}{H(p)}$ $(p\frac{H'(1-p)}{1-H(1-p)})$ *is non increasing on p. Therefore, Tavangar results are a particular case of results in [12].*

Under Tavangar condition C1, let us define the bivariate function g as follows

$$g(i,j) = R_{i+1}(n-j)s_{j+1}, \quad i,j = 1,\dots,n-1.$$

Note that, if $R_{i+1}R_{j+1} > 0$*, we have, as* $R_{i+1} \geq R_{j+1}$ *and C1, that*

$$g(i,j) - g(j,i) = R_{i+1}R_{j+1}\left(\frac{(n-j)s_{j+1}}{R_{j+1}} - \frac{(n-i)s_{i+1}}{R_{i+1}}\right) \geq 0, \quad i \leq j,$$

and obviously $R_{i+1} \geq R_{j+1}$ *implies that* $g(i,j) - g(j,i) \geq 0$ *for* $i \leq j$ *in case* $R_{i+1}R_{j+1} = 0$*. Hence, by bivariate characterization of lr stochastic order and the fact that* $B(n,p_1) \leq_{lr} B(n,p_2)$ $(p_1 \leq p_2)$*, we have*

$$E[R_{B(n,p_1)+1}]E[(n-B(n,p_2))s_{B(n,p_2)+1}] \geq E[R_{B(n,p_2)+1}]E[(n-B(n,p_1))s_{B(n,p_1)+1}],$$

or equivalently as $B(n,p) \stackrel{\mathrm{d}}{=} n - B(n,1-p)$

$$\begin{aligned} &E[R_{n+1-B(n,1-p_1)}]E[B(n,1-p_2)s_{n+1-B(n,1-p_2)}] \\ &\geq E[R_{n+1-B(n,1-p_2)}]E[B(n,1-p_1)s_{n+1-B(n,1-p_1)}]. \end{aligned} \tag{11}$$

We now show that $p\frac{H'(p)}{H(p)}$ *is non increasing iff (11) holds. Indeed, the derivative of (9) by (5) is given by*

$$H'(p) = n\sum_{j=0}^{n-1}(R_{n+1-j-1}-R_{n+1-j})\binom{n-1}{j}p^j(1-p)^{n-1-j} = n\sum_{j=0}^{n-1}s_{n-j}\binom{n-1}{j}p^j(1-p)^{n-1-j}.$$

Thus, for $Y_i(p)$*,* $i = 1,\dots,n$ *independent Bernoulli random variables with parameter p,*

$$\begin{aligned} pH'(p) &= nE\left[[Y_n(p)s_{n-\sum_{i=1}^{n-1}Y_i(p)}\right] = nE\left[Y_n(p)s_{n+1-\sum_{i=1}^{n}Y_i(p)}\right] \\ &= nE\left[\frac{\sum_{i=1}^{n}Y_i(p)}{n}s_{n+1-\sum_{i=1}^{n}Y_i(p)}\right] = \sum_{j=0}^{n} js_{n+1-j}\binom{n}{j}p^j(1-p)^{n-j} \\ &= E[B(n,p)s_{n+1-B(n,p)}], \end{aligned}$$

where the second equality holds because the terms inside the expectation in both sides give the same value either if $Y_n(p) = 0$ or $Y_n(p) = 1$, and the third equality holds by identical distribution. Finally, the sought monotonicity result follows by (11) as

$$p\frac{H'(p)}{H(p)} = \frac{E[B(n,p)s_{n+1-B(n,p)}]}{E[R_{n+1-B(n.p)}]}$$

Next, under Tavangar condition C2, let us define the bivariate function g as follows

$$g(i,j) = L_j i s_i, \quad i,j = 1,\ldots,n.$$

Using the same arguments as above, we can show that $g(i,j) - g(j,i) \geq 0$ for $i \leq j$ due to C2 and the fact that $L_i \leq L_j$. Hence, by bivariate characterization of lr stochastic order and the fact that $B(n,p_1) \leq_{lr} B(n,p_2)$ ($p_1 \leq p_2$), we have

$$E[L_{B(n,p_2)}]E[(B(n,p_1)s_{B(n,p_1)}] \geq E[L_{B(n,p_1)}]E[(B(n,p_2)s_{B(n,p_2)}]. \tag{12}$$

Simple algebra analogous to the IFR case leads to

$$1 - H(1-p) = \sum_{j=1}^{n} L_j \binom{n}{j} p^j (1-p)^{n-j} = E[L_{B(n,p)}], \tag{13}$$

and

$$H'(1-p) = n\sum_{j=0}^{n-1} s_{j+1}\binom{n-1}{j} p^j(1-p)^{n-1-j}.$$

Similarly to the IFR case, we have

$$\begin{aligned} pH'(1-p) &= nE\left[Y_n(p)s_{1+\sum_{i=1}^{n-1}Y_i(p)}\right] = nE\left[Y_n(p)s_{\sum_{i=1}^{n}Y_i(p)}\right] \\ &= nE\left[\frac{\sum_{i=1}^{n}Y_i(p)}{n}s_{\sum_{i=1}^{n}Y_i(p)}\right] = E[B(n,p)s_{B(n,p)}]. \end{aligned} \tag{14}$$

Hence, the DRFR case under Tavangar conditions C2 holds by (12) taking into account (13) and (14).

On the other hand, if we compare conditions obtained in Corollaries 1 and 2 and the ones in [12], we see that F concave implies that F is log-concave, and F convex implies $\overline{F}$ concave (and therefore, log-concave). Thus, *conditions for F in both corollaries are stronger than the corresponding ones in [12]. However, with respect to Corollary 2, taking into account Proposition 1 and (13), we have that $1-H(1-p)$ is log concave, and therefore*

$$\frac{H'(1-p)}{1-H(1-p)} \quad \textit{is non-increasing on } p.$$

But if we multiply it by the increasing factor p, the monotonicity condition can be lost, thus condition in Corollary 2 does not necessarily imply condition in [12]. For instance, let us consider a coherent system having 4 components with vector signature $\mathbf{s} = (1/4, 7/12, 1/6, 0)$ (see [16, p.25, system 3]). The vector of values for L_j is $(1/4, 10/12, 1, 1)$ which is log-concave. As a consequence of Corollary 2 the lifetime of the system is DRHR when F is concave. However,

$$p\frac{H'(1-p)}{1-H(1-p)} = \frac{p(1-p)^3 + 7p^2(1-p)^2 + 2p^3(1-p)}{p(1-p)^3 + 5p^2(1-p)^2 + 4p^3(1-p) + 2p^4} \tag{15}$$

does not satisfy the monotonicity condition in [12] (see Figure 1).

With respect to Corollary 1, we can consider the dual model of system 3 in [16, p.25]), which is system 16, and has signature $\mathbf{s}^\star = (0, 1/6, 7/12, 1/4)$, and call $H_\star(p)$ the function defined in (9) for the dual system. It is clear that $R_i^\star = 1 - L_i$, so that

$$\begin{aligned} H_\star(p) &= 1 - \sum_{j=0}^{n} L_{n+1-j}\binom{n}{j} p^j (1-p)^{n-j} = 1 - \sum_{k=0}^{n} L_k \binom{n}{k} (1-p)^k p^{n-k} \\ &= 1 - H(1-p), \qquad\qquad 0 \le p \le 1 \end{aligned}$$

and the desired monotonicity condition for $H_\star(p)$ is then not satisfied, due to (15).

Remark 2 *Under the assumptions in Corollary 1 (Corollary 2) if R_j (L_j) is a log concave sequence, $\overline{F}_{S_{\mathbf{X}(\mathbf{s})}}$ ($F_{S_{\mathbf{X}(\mathbf{s})}}$) being log concave, does not entail that the common distribution is IFR (DRFR). Indeed, let consider the system $\max(\min(X_1, X_2), X_3, X_4, X_5)$ for $X_1, \ldots, X_5$ i.i.d with common distribution function F with support on $(0,1)$ defined as $F(t) = t^{\frac{1}{2}}$, $0 < t < 1$. The signature of this system is $(0, 0, 0, \frac{2}{5}, \frac{3}{5})$. the signature itself is log-concave, R_j is a log concave sequence. It is straightforward to show that F is not IFR*

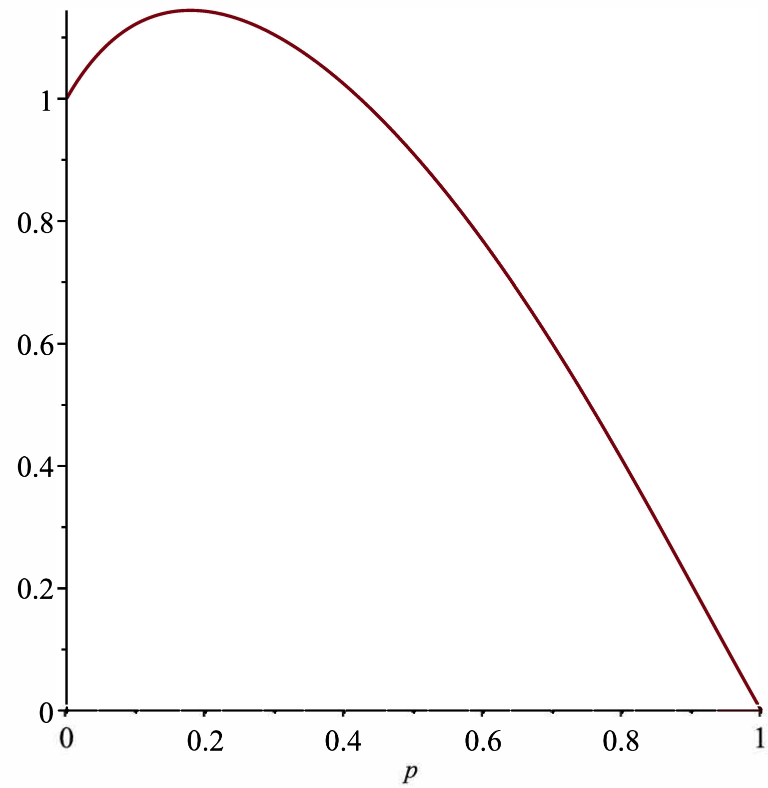

Figure 1: Plot of the function $\frac{p(1-p)^3+7p^2(1-p)^2+2p^3(1-p)}{p(1-p)^3+5p^2(1-p)^2+4p^3(1-p)+2p^4}$

and in this case

$$\begin{aligned}
\overline{F}_{S_{\mathbf{X}(\mathbf{s})}}(t) &= \sum_{j=0}^{5} R_{j+1}\binom{5}{j}F(t)^j\overline{F}(t)^{5-j} \\
&= 1-\frac{2}{5}\binom{5}{4}F^4(t)\overline{F}(t)-F^5(t) \\
&= 1-2t^2(1-t^{\frac{1}{2}})-t^{\frac{5}{2}} = 1-2t^2+t^{\frac{5}{2}}, \quad 0<t<1
\end{aligned}$$

Obviously, $\overline{F}_{S_{\mathbf{X}(\mathbf{s})}}$ is a concave function on its support, therefore $\overline{F}_{S_{\mathbf{X}(\mathbf{s})}}$ is log concave. For Corollary 2 we consider the dual system of the previous one whose signature is $(\frac{3}{5}, \frac{2}{5}, 0, 0, 0)$ and for which L_j is a log concave sequence. In this case the common distribution function of the components of the system *is F with support $(0,1)$, defined as $F(t) = 1-(1-t)^{\frac{1}{2}}$, $0<t<1$. Obviously F is not DRFR and simple algebra leads in this case to*

$$\begin{aligned} F_{S_{\mathbf{X}(\mathbf{s})}}(t) &= \sum_{j=0}^{5} L_j \binom{5}{j} F(t)^j \overline{F}(t)^{5-j} \\ &= 1 - \frac{2}{5}\binom{5}{1} F(t)\overline{F}^4(t) - \overline{F}^5(t) \\ &= 1 - 2(1-t)^2(1-(1-t)^{\frac{1}{2}}) - (1-t)^{\frac{5}{2}} \\ &= 1 - 2(1-t)^2 + (1-t)^{\frac{5}{2}} \end{aligned}$$

It is derived easily that $F_{S_{\mathbf{X}(\mathbf{s})}}$ is a concave function on its support, therefore $F_{S_{\mathbf{X}(\mathbf{s})}}$ is log concave.

On the other hand, for a given signature satisfying assumptions in Corollary 1, $\overline{F}$ the common survival function for the components being log-concave, does not imply that $\overline{F}_{S_{\mathbf{X}(\mathbf{s})}}$ is log-concave. For instance, it is shown in Samaniego et al. [17, p. 71] that $\overline{F}_{S_{\mathbf{X}(\mathbf{s})}}$ is not log-concave for the system with log-concave signature $(0, 2/3, 1/3)$ and lifetimes of the components being exponential. However, the DRFR property holds, as a consequence of Corollary 2, because the exponential distribution has concave distribution function. To find a a signature in the settings of Corollary 2, and a log-concave distribution function F which does not imply $F_{S_{\mathbf{X}(\mathbf{s})}}$ log-concave, we can take the dual signature as before, and a random variable with support in $(-\infty, 0]$ such that $F(t) = e^t$, $t \le 0$. that is $(1/3, 2/3, 0)$. It is straightforward to see that

$$\begin{aligned} F_{S_{\mathbf{X}(\mathbf{s})}}(t) &= \sum_{j=0}^{3} L_j \binom{3}{j} F(t)^j \overline{F}(t)^{3-j} = 1 - \frac{2}{3}\binom{3}{1} F(t)\overline{F}(t)^2 - \overline{F}(t)^3 \\ &= 1 - 2F(t)\overline{F}(t)^2 - \overline{F}(t)^3 = 1 - 2e^t(1-e^t)^2 - (1-e^t)^3, \quad t \le 0, \end{aligned}$$

which is not a log-concave function.

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