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RESEARCH

Wrong hypotheses in the generalized RTBP

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Abstract
Generalized restricted three body problems consist of adding some extra hypotheses to the Restricted three body problem (RTBP) in order to have a new problem, not very different of the original RTBP. However, not any additional hypothesis is allowed; it must satisfy the laws of Physics. Among the several generalizations found in literature, we prove that at least there are two hypotheses that cannot be used, namely: 1) Perturbation in Coriolis and/or centrifugal forces, and 2) primaries are spheroids moving on elliptical orbits.

Keywords Generalized restricted three body problem · Coriolis force · Potential of two ellipsoids

1 Introduction

The restricted three body problem (RTBP) is with no doubt the most studied problem in Celestial Mechanics. Its applications span solar system dynamics, stellar dynamics, lunar theory, spacecraft dynamics, etc. Essentially, it consists of the motion of an *infinitesimal mass* that does not influence on the motion of other two point masses, called primaries, which move in Keplerian orbits around their mutual center of mass. However, some times this model is not sufficient to adequately represent actual cases: thus, since most of planets are not spherical, the oblateness must be considered; in close binary stars the tidal deformation should be taken into account, or even the radiation pressure force when one or both primaries are radiating forces. Hence, it is necessary to extend the problem to a *generalized restricted three body problem*.

Usually, the generalization is only applied to the Circular RTBP, which in principle is correct, see e.g. (Sharma and Subba Rao 1975; Bhatnagar and Chawla 1977; Schuerman 1980; Simmons et al. 1985; Elipe and Ferrer 1985, 1986; Elipe 1987, 1992; Bhatnagar et al. 1994; Elipe and Lara 1997; Ishwar and Elipe 2001), but several authors consider this problem to be like a *coat rack*, where it is possible to hang on it many items in order to have a "new problem" to which they apply the same techniques used in the RTBP; unfortunately, some of these new problems are wrong because their hypotheses are against physical laws. Nevertheless, these papers are published even in well reputed journals, likely due to the fact that neither authors nor reviewers paid attention on whether these hypotheses are possible or not.

One of these assumptions is that there are *perturbations in the Coriolis and/or in the centrifugal forces*. Szebehely (1967) opened this line by assuming perturbations in the Coriolis force, although he also mentioned that this was not a real case. This work was extended by Bhatnagar and Hallan (1978) who also included perturbations in the centrifugal forces, and once the gate was open, other authors continued in this line, as for example (Elshaboury 1989; Shu and Lu 2005; Raheem et al. 2006; Kaur et al. 2020; Singh and Amuda 2018; Ansari et al. 2019). But authors continued applying these hypotheses of perturbations in Coriolis and/or centrifugal forces to new models, like four, five, six, ...-bodies; see e.g. (Raheem et al. 2006; Singh and Vincent 2015; Abouelmagd and Guirao 2016; Suraj et al. 2017; Aggarwal et al. 2018; Singh and Amuda 2018; Suraj et al. 2019b; Ansari et al. 2019; Idrisi et al. 2021) among others, not realizing that this is not possible from the physical point of view (Elipe 2022) and that we prove in Sect. 2.

Another "new problem" is to consider that primaries are ellipsoids or radiating bodies that move not on a Keplerian circular orbit, but in a *Keplerian elliptic orbit*; see for instance (Kumar and Ishwar 2011; Narayan and Usha 2014; Singh and Tyokyaa 2016; Idrisi and Ullah 2020; Radwan and Moltep 2021; Singh and Isah 2021) and many others.

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Again, this hypothesis that seems to be a natural extension of the problem, is wrong as we prove in Sect. 3.

2 Perturbation in Coriolis and centrifugal forces

Let us denote by $\boldsymbol{R}$ the position vector of a particle with respect to an inertial reference frame $(OXYZ)$, and by $\boldsymbol{r}$ the same vector expressed in a moving reference frame $(Oxyz)$ with the same origin and that is moving with an angular velocity $\boldsymbol{\omega}(t)$ with respect to the inertial frame. As we can find in any undergraduate textbook on Mechanics (see e.g. (Goldstein 1980; Scheck 2005)), the absolute velocity and acceleration vectors are related with the relative ones by

$$\dot{\boldsymbol{R}} = \dot{\boldsymbol{r}} + \boldsymbol{\omega} \times \boldsymbol{r},$$

$$\ddot{\boldsymbol{R}} = \ddot{\boldsymbol{r}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + 2\boldsymbol{\omega} \times \dot{\boldsymbol{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}).$$

With this, the Second Newton Law reads

$$\boldsymbol{F} = m\ddot{\boldsymbol{R}} = m\Big(\ddot{\boldsymbol{r}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + 2\boldsymbol{\omega} \times \dot{\boldsymbol{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})\Big). \tag{1}$$

When the rotation of the moving frame is about the fixed axis OZ at constant rate $\|\boldsymbol{\omega}\| = \omega$, the derivative $\dot{\boldsymbol{\omega}}$ is null. Hence, we can put Eq. (1) as

$$m\,\ddot{\boldsymbol{r}} = \boldsymbol{F} - 2m\,\boldsymbol{\omega} \times \dot{\boldsymbol{r}} - m\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}). \tag{2}$$

Note that the three terms of the right member of this equation have the dimensions of a force, hence, usually the term $-2m\,\boldsymbol{\omega} \times \dot{\boldsymbol{r}}$ is called *Coriolis force* and the expression $-m\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$ *centrifugal force*, but as a matter of fact they are no physical forces; these terms are originated by the rotation of the reference frame.

Let us now move to the Circular RTBP, where the primaries move around its mutual center of mass in a pure Keplerian circular motion. It is usual to define in this problem an inertial frame $OXYZ$ with origin at the center of mass of the primaries, and plane OXY the one containing the motion of the primaries. Besides, it is also usual to define a synodic frame $Oxyz$, such that the primaries are always on the axis Ox, thus, the angular velocity of the rotation of the frames is $\boldsymbol{\omega} = (0, 0, \omega)$, where ω is the mean motion of the circular motion of the primaries and consequently ω *is constant!* and even more, ω is determined in univocal manner by the sum of the masses and the period, as the Third Newton Law clearly states.

Some authors among several others, like Szebehely (1967), Bhatnagar and Hallan (1978), Elshaboury (1989), Hallan and Rana (2001), Shu and Lu (2005), Raheem et al. (2006), Singh and Bello (2014), Singh and Vincent (2015), Abouelmagd and Guirao (2016), Kaur et al. (2020), Suraj et al. (2017), Aggarwal et al. (2018), Singh and Amuda (2018), Ansari et al. (2019), Suraj et al. (2019a), Idrisi et al. (2021), Kaur et al. (2022), Singh and Ahmad (2022) consider the Circular RTBP, and even more bodies, by assuming that there are perturbations in the Coriolis force by changing ω by $\omega(1+\alpha)$ in the Coriolis term, and perturbations in the centrifugal force by putting $\omega^2(1+\beta)$ instead of ω^2 and even assuming that $(1+\beta) \neq (1+\alpha)^2$. As shown before, *these assumptions go against the laws of Physics*.

3 Primaries are symmetrical ellipsoids moving on a Keplerian elliptic orbit

Another additional complexity to the RTBP consists of assuming that one or both primaries are symmetrical ellipsoids moving on a Keplerian elliptic orbit. Let us see whether this motion is possible or not.

3.1 Potential of two spheroids

Let us consider the motion of two rigid bodies S_1 and S_2 of masses m_1, and m_2 and O_1 and O_2 their centers of mass. We also consider two parallel reference frames with origins at each of the centers of mass of the bodies $(O_i XYZ)$. It is known that the mutual gravitational potential of two rigid bodies may be expanded in harmonic series, and its first terms are (see *e.g.* Leimanis (1965), Elipe and Ferrer (1985))

$$\begin{aligned} U = &-\mathcal{G}\frac{m_1 m_2}{r} - \mathcal{G}m_1\frac{A_2 + B_2 + C_2 - 3I_{21}}{2r^3} \\ &- \mathcal{G}m_2\frac{A_1 + B_1 + C_1 - 3I_{12}}{2r^3}, \end{aligned} \tag{3}$$

where $\mathcal{G}$ is the gravitational constant; I_{ij} is the moment of inertia of the body S_i with respect to the straight line $(\overrightarrow{O_i O_j})$ joining the mass centers O_i and O_j; $(A_i,\ B_i,\ C_i)$ are the principal moments of inertia of body S_i; and $r = \|\overrightarrow{O_1 O_2}\|$ is the distance between the centers of mass of the rigid bodies.

The moments of inertia I_{ij} with respect to the line $\overrightarrow{O_i O_j}$ are

$$\begin{aligned} I_{12} &= A_1\alpha_{12}^2 + B_1\beta_{12}^2 + C_1\gamma_{12}^2, \\ I_{21} &= A_2\alpha_{21}^2 + B_2\beta_{21}^2 + C_2\gamma_{21}^2, \end{aligned} \tag{4}$$

where α_{ij}, β_{ij}, and γ_{ij} are the cosines of the angles made by $\overrightarrow{O_i O_j}$ with the principal inertia axes $(O_i; \xi_i, \eta_i, \zeta_i)$ of the body S_i (Leimanis 1965) that satisfy

$$\alpha_{12}^2 + \beta_{12}^2 + \gamma_{12}^2 = 1, \qquad \alpha_{21}^2 + \beta_{21}^2 + \gamma_{21}^2 = 1.$$

Besides, it is well known from Vectorial Calculus (Marsden and Tromba 1988) that for a triaxial ellipsoid of semi-axes a, b and c, and mass m, its principal moments of inertia

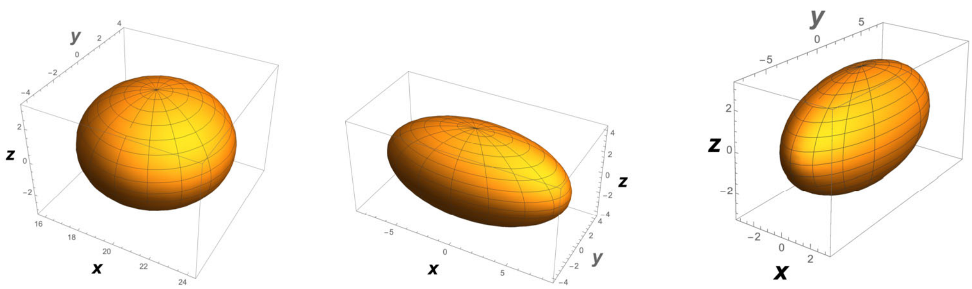

Fig. 1 The three possible cases: a) float; b) spoke; and c) arrow

are

$$A = \frac{m}{5}\left(b^2 + c^2\right), \qquad B = \frac{m}{5}\left(a^2 + c^2\right),$$
$$C = \frac{m}{5}\left(a^2 + b^2\right).$$

Let us obtain the expression of the moments of inertia with respect to a straight line (4) depending on the different chosen ellipsoids.

In what follows we shall consider three types of ellipsoids, the ones that as proven by Duboshin (1982) have "regular motions" in the orbital-rotational motion of two rigid bodies (see Fig. 1). Note that in the three cases, the system of principal axes of inertia coincides with the Cartesian frame; also we assume that the centers of mass O_i are continuously lying on the Ox-axis, thus, $\alpha_i^2 = 1$, $\beta_i^2 = \gamma_i^2 = 0$, and therefore, $A_i + B_i + C_i - 3I_{ij} = -2A_i + B_i + C_i$.

1. *Float case*:

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1, \tag{5}$$

hence,

$$A_i + B_i + C_i - 3I_{ij} = C_i - A_i = \frac{m}{5}(a^2 - c^2). \tag{6}$$

2. *Spoke case*:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1, \tag{7}$$

thus,

$$A_i + B_i + C_i - 3I_{ij} = 2(B_i - A_i) = \frac{2m}{5}(a^2 - b^2). \tag{8}$$

3. *Arrow case*:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1, \tag{9}$$

now,

$$A_i + B_i + C_i - 3I_{ij} = (B_i - A_i) = \frac{m}{5}(a^2 - b^2). \tag{10}$$

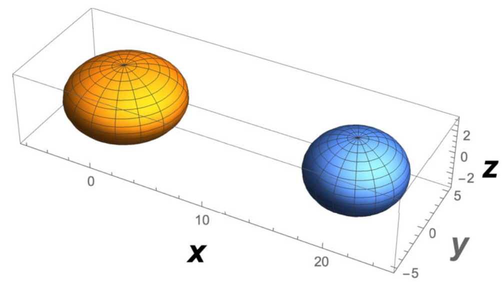

Fig. 2 The *float-float case*

Therefore, it is possible to take whatever combination of two ellipsoids belonging to the aforementioned axisymmetric ellipsoids. In particular we consider below only three configurations, already described by Duboshin (1982), although it is possible to take any combination of them.

1.- *Float-float configuration*:

Because of the similitude with planets, most authors studying the generalization of the circular RTBP consider that primaries are spheroids, that is, ellipsoids of revolution defined by Eq. (5) in such a way that the equatorial planes of the spheroids coincide with the plane of the motion of the centers of mass. This case corresponds to the case denoted by Duboshin (1982) as *float-float case* (see Fig. 2). The most usual case in literature is for oblate spheroids ($a > c$), but also may be used for prolate spheroids ($a < c$) or even a combination of both; when spheres, $a = c$.

2.- *Spoke-spoke case*:

In this case both ellipsoids are of type spoke (7), and they rotate synchronously about their z-axes with the same angular velocity as the orbital angular velocity of the motion of O_2 around O_1 (see Fig. 3).

This configuration has been recently used by Elipe et al. (2024) to model tidal deformations in close binary stars, fact that has already been confirmed by astronomical observations (Baron et al. 2012).

3.- *Float-arrow case*:

Now, one ellipsoid is of the type float (5) whereas the other is of the type arrow (9); the configuration can be seen in Fig. 4. To the knowledge of the author none application of this model to any real case is done.

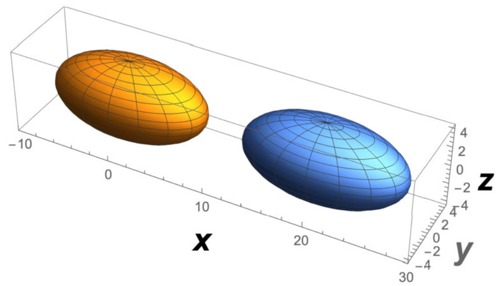

Fig. 3 The *spoke-spoke case*

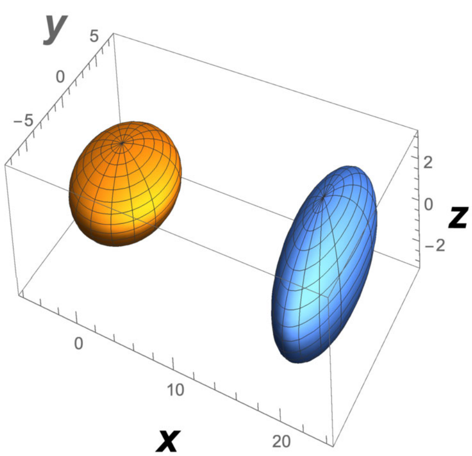

Fig. 4 The *float-arrow case*

What is most remarkable is that whatever the configuration we may choose, the second and third terms of the expression of the potential given in Eq. (3) are

$$\mathcal{G}m_1\frac{A_2+B_2+C_2-3I_{21}}{2r^3}=\frac{\mathcal{G}m_1m_2}{r^3}\mathcal{F}_2(a_2^2,b_2^2,c_2^2),$$
$$\mathcal{G}m_2\frac{A_1+B_1+C_1-3I_{12}}{2r^3}=\frac{\mathcal{G}m_1m_2}{r^3}\mathcal{F}_1(a_1^2,b_1^2,c_1^2),$$

where $\mathcal{F}_i(a_i^2,b_i^2,c_i^2)$ stands for a linear combination of the semi-axes of the ellipsoid S_i given by Equations (6), (8), and (10). In sum, the gravitational potential of the mutual attraction of two spheroids given by Eq. (3) reduces to

$$U(r)=-\mathcal{G}\frac{m_1m_2}{r} - \frac{\mathcal{G}m_1m_2}{r^3}\left[\mathcal{F}_2(a_2^2,b_2^2,c_2^2)+\mathcal{F}_1(a_1^2,b_1^2,c_1^2)\right]. \tag{11}$$

At this point it is worth noting first, that the motion is a central force motion; and second, that the semi-axes of the ellipsoids are much smaller than the radial distance r, hence the quantity $\mathcal{F}=\mathcal{F}_1+\mathcal{F}_2$ may be considered a small parameter.

3.2 Motion of two spheroids

Proceeding as in the two-body problem (see e.g. Danby (1988)), and by virtue of the six integrals of the center of masses of the bodies, the relative motion of O_2 with respect to O_1 is the solution of the differential equation

$$\ddot{\boldsymbol{r}}=-\nabla_{\boldsymbol{r}}U(r)=-\frac{\mu}{r^3}\boldsymbol{r}-\frac{3\,\mu\,\mathcal{F}}{r^5}\boldsymbol{r}, \tag{12}$$

where $\mu=\mathcal{G}(m_1+m_2)$, and $\mathcal{F}\ll 1$.

It is easy to check that the angular momentum $\boldsymbol{G}=\boldsymbol{r}\times\dot{\boldsymbol{r}}$ is constant and whence its norm $G=\|\boldsymbol{G}\|=$ constant, and the motion is planar. Besides, because the independent variable t does not appear explicitly in Eq. (12), the energy h is constant along the motion

$$h=\frac{1}{2}\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}}+U(r)=\frac{1}{2}\dot{r}^2+\frac{G^2}{2r^2}-\frac{\mu}{r}-\frac{\mu\mathcal{F}}{r^3}, \tag{13}$$

and the effective potential becomes

$$U_{\text{eff}}=U_{\text{eff}}(r)=\frac{G^2}{2r^2}-\frac{\mu}{r}-\frac{\mu\mathcal{F}}{r^3}=U_{\text{Keff}}-\frac{\mu\mathcal{F}}{r^3}, \tag{14}$$

where the term U_{Keff} is the effective potential of Kepler's problem. Thus, by Eq. (13), at a certain level of energy h, the motion is only possible when $h\geq U_{\text{eff}}$.

This problem has been analyzed with detail by Elipe et al. (2024) and the reader is addressed to it. We put in Table 1 the main differences between both potentials. Note that $r_{\max}<r_{\min}$. Observe too that those critical points correspond to two

Table 1 Main differences between the two potentials U_{Keff} and U_{eff}

	U_{Keff}	U_{eff}
$r\to 0$	$\to +\infty$	$\to -\infty$
$r\to +\infty$	$\to 0$	$\to 0$
maximum	–	$r_{\max}=(G^2+\sqrt{G^2-12\mu\varepsilon})/(2\mu)$
minimun	G^2/μ	$r_{\min}=(G^2+\sqrt{G^2+12\mu\varepsilon})/(2\mu)$

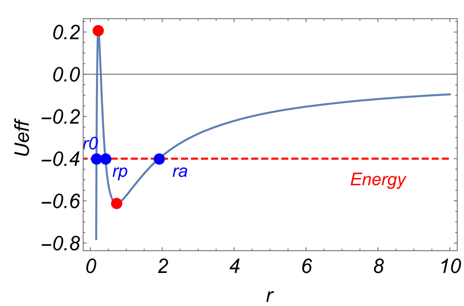

Fig. 5 Effective potential $U_{\text{eff}}=U_{\text{eff}}(r)$. Points in red are the relative extrema, and points in blue ($r_0<r_p\leq r_a$), the intersection of the potential with the constant energy level h. When $r_p\leq r\leq r_a$ the motion is bounded within an annulus; When $r_p=r=r_a=r_{\min}$ the motion is circular

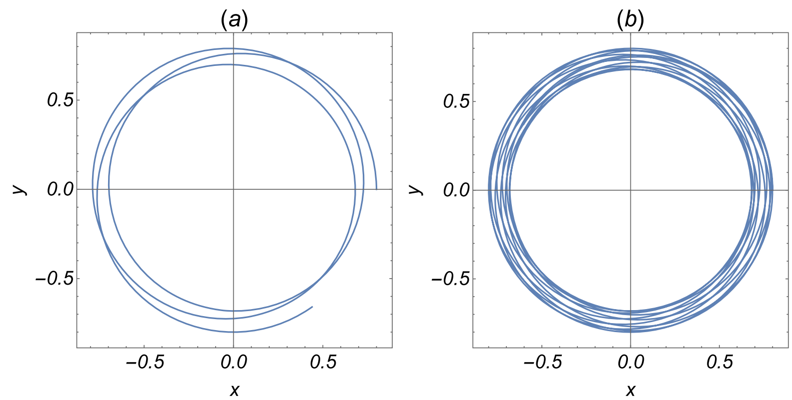

Fig. 6 Orbit in case 2, that is, oscillatory motion with lower and upper bounds, $r_p \le r \le r_a$; the motion takes place inside an annulus, and it is not elliptical. Both plots belong to same orbit but computed for different spans of time; (a) for $0 \le t \le 10$, and (b) for $0 \le t \le 40$. The initial conditions for the orbit are: $\mu = 1$, $\varepsilon = 0.055$, $a = 1$, $e = 0.2$, position vector $\boldsymbol{x}(t=0) = (0.8,\ 0)$, and velocity vector $\dot{\boldsymbol{x}}(t=0) = (0,\ 1.2247448714)$

circular orbits; one unstable ($r = r_{\max}$) and the other stable ($r = r_{\min}$), whereas in the Keplerian case, there is only one minimum that corresponds to a stable circular orbit.

As we proved in Elipe et al. (2024), there are only three possible types of motions depending on where r is with respect to the three possible roots $r_0 < r_p \le r_a$ of the equation $U_{\rm eff}(r) - h = 0$ (see Fig. 5), namely

1. Bounded motion when $0 < r \le r_0$;
2. Oscillatory motion with lower and upper bounds, that is, $r_p \le r \le r_a$;
3. Unbounded motion for $h \ge 0$.

In what follows, we restrict our analysis to motion of type 2.

The Laplace vector (also know by physicists as Runge-Lenz vector) is defined (Goldstein 1980) as

$$\boldsymbol{A} = \dot{\boldsymbol{r}} \times \boldsymbol{G} - \frac{\mu}{r}\boldsymbol{r}. \tag{15}$$

In the Keplerian problem, it is a constant vector always pointing to the pericenter and is related with the eccentricity ($e = \|\boldsymbol{e}\|$) in such a way that $\boldsymbol{e} = \boldsymbol{A}/\mu$. In our case, by using Eq. (12), there results that

$$\frac{d\boldsymbol{A}}{dt} = -\frac{3\,\mu\,\mathcal{F}}{r^5}\,(\boldsymbol{r} \times \boldsymbol{G}) \neq 0. \tag{16}$$

From Eq. (16), Laplace's vector suffers a tangential push and its norm is

$$\|\dot{\boldsymbol{A}}\| = 3\,\mu\,\mathcal{F}\,\frac{G}{r^4}, \tag{17}$$

but since the radial distance r oscillates in the interval $[r_p,\ r_a]$ (see Fig. 5), $\|\dot{\boldsymbol{A}}\|$ has an oscillatory behavior, and hence e also does.

In consequence, the Laplace vector is not constant, and then neither the eccentricity e nor the pericenter angle ω are constant, and therefore, *the motion is not elliptical*. We easily can check this fact by integrating the equations of motion (12) and plotting the planar orbit (see Fig. 6). The orbit is bounded and takes place inside a circular annulus ($r_p \le r \le r_a$); the orbit is precessing and has a rosetta-like motion.

Let us note that the equation of motion (12) is essentially the same that appear in the so-called Cid's radial intermediary (Cid and Lahulla 1969) in Artificial Satellite Theory, and its analytical integration has been done from different points of view; in terms of elliptic functions (Belen'kii 1981; Ferrándiz 1986; Lara and Gurfil 2012; Elipe et al. 2024) of by using a regularizig function and then applying (Abad et al. 2021) the Krylov and Bogoliubov (1947) averaging method.

Although the motion is not elliptical, circular motion is possible, although not Keplerian. This corresponds to the case $h = U_{\rm eff}(r_{\min})$, that is, when the initial conditions are such that the constant energy coincides with the relative minimum of the effective potential (see Fig. 5).

Let us consider the circular stable orbit that corresponds to the value $r = r_{\min}$ of the effective potential. From the radial acceleration ($\ddot{r} - r\dot{\theta}^2$), we have that

$$\ddot{r} - r\dot{\theta}^2 = -\frac{dU}{dr},$$

and because the radial distance in a circle is constant, $r = r_{\min} =$ constant, there results that

$$\dot{\theta}^2 = \frac{\mu}{r_{\min}^3} + \frac{3\,\mu\,\mathcal{F}}{r_{\min}^5} = n_K^2 + \frac{3\,\mu\,\mathcal{F}}{r_{\min}^5},$$

where n_K is the mean motion of the Kepler circular orbit; hence, the mean motion of the circular orbit here considered and the Keplerian one are related through the expression

$$n^2 = n_K^2\left(1 + \frac{3\,\mathcal{F}}{r_{\min}^2}\right).$$

4 Conclusions

We prove that some assumptions used in generalizations of the restricted three body problem are not correct, because they are contravening Physics Laws. In particular, we focus our attention on two cases: 1) Perturbation in Coriolis and/or centrifugal forces, and 2) Primaries are spheroids moving on elliptical orbits. Both cases are not possible.

Author contributions I am the only author.

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Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

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