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Summing Sneddon–Bessel series explicitly

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We sum in a closed form the Sneddon–Bessel series

$$\sum_{m=1}^{\infty} \frac{J_\alpha(x j_{m,\nu}) J_\beta(y j_{m,\nu})}{j_{m,\nu}^{2n+\alpha+\beta-2\nu+2} J_{\nu+1}(j_{m,\nu})^2},$$

where $0 < x$, $0 < y$, $x + y < 2$, n is an integer, $\alpha, \beta, \nu \in \mathbb{C}\setminus\{-1,-2,\dots\}$ with $2\operatorname{Re}\nu < 2n + 1 + \operatorname{Re}\alpha + \operatorname{Re}\beta$ and $\{j_{m,\nu}\}_{m\ge 0}$ are the zeros of the Bessel function J_ν of order ν. In most cases, the explicit expressions for these sums involve hypergeometric functions ${}_pF_q$. As an application, we prove some extensions of the Kneser–Sommerfeld expansion. For instance, we show that

$$\sum_{m=1}^{\infty} \frac{j_{m,\nu}^{\nu-\beta} J_\nu(x j_{m,\nu}) J_\beta(y j_{m,\nu})}{(j_{m,\nu}^2 - z^2) J_{\nu+1}(j_{m,\nu})^2} = \frac{\pi J_\beta(yz)}{4 z^{\beta-\nu} J_\nu(z)} \left(Y_\nu(z) J_\nu(xz) - J_\nu(z) Y_\nu(xz)\right),$$

if $\operatorname{Re}\nu < \operatorname{Re}\beta + 1$ and $0 < y \le x$, $x + y < 2$ (here, Y_ν denotes the Bessel function of the second kind), which becomes the Kneser–Sommerfeld expansion when $\beta = \nu$.

KEYWORDS

Bessel functions, Bessel series, hypergeometric functions, Kneser–Sommerfeld expansions, Sneddon–Bessel series, zeros

MSC CLASSIFICATION

33C10, 33C20

1 | INTRODUCTION

Sneddon considered in [1, § 2.2] the following Bessel series in two variables:

$$S_q^{\alpha,\beta;\nu}(x,y) = \sum_{m=1}^{\infty} \frac{J_\alpha(x j_{m,\nu}) J_\beta(y j_{m,\nu})}{j_{m,\nu}^{q} J_{\nu+1}(j_{m,\nu})^2}, \tag{1.1}$$

where $0 < x$, $0 < y$, $x + y < 2$, and $\{j_{m,\nu}\}_{m\ge 0}$ are the zeros of the Bessel function J_ν of order ν.

The purpose of this paper is to compute explicitly these Sneddon–Bessel series for

$$q = q_n = 2n + \alpha + \beta - 2\nu + 2, \quad n \in \mathbb{Z},$$

In celebration of the centenary of the first edition of Watson's "Treatise on the theory of Bessel functions."

under mild conditions on the parameters α, β, and ν.

The case $n < 0$ was computed by Sneddon [1, § 2.2] and more recently by Martin [2]. There are other particular cases that are already known. For instance,

(a) The case $x = 1$, $\alpha = \mu + \nu + 1$, $\beta = \nu$, and $n = -1$ is [3, p. 690, (9)].
(b) The case $\alpha = \beta = \nu$, $n \geq 0$, $0 \leq y \leq x \leq 1$, is packaged in the Kneser–Sommerfeld expansion (see [2, (2)])

$$\sum_{m=1}^{\infty} \frac{J_\nu(xj_{m,\nu})J_\nu(yj_{m,\nu})}{(j_{m,\nu}^2 - z^2)J_{\nu+1}(j_{m,\nu})^2} = \frac{\pi J_\nu(yz)}{4J_\nu(z)} (Y_\nu(z)J_\nu(xz) - J_\nu(z)Y_\nu(xz)) \tag{1.2}$$

(more precisely: $S_{q_n}^{\nu,\nu;\nu}(x, y)$, $n \geq 0$, are the Taylor coefficients at $z = 0$ of the analytic function of z on the right-hand side).

The problem of the explicit summation of Bessel series is a classical topic, but there is no doubt that it remains of interest today and active research is still being done ([4, § 6.8], [5]) (among other reasons, for its usefulness in applied mathematics, mathematical physics, and engineering, as explicitly explained in Sneddon's book [1] or, more recently, in Grebenkov's paper [5]).

The content of the paper is as follows. In Section 3, we use the calculus of residues to find a partial fraction expansion of functions of the form $\frac{z^{2\nu} f(z)}{J_\nu(z)^2}$, where f is an entire function satisfying a suitable bound in $\mathbb{C}$ (see Theorem 1 for details). We then particularize for $f(z) = \frac{J_\alpha(xz)J_\beta(yz)}{(xz)^\alpha(yz)^\beta}$. Although in this first step the series we manage to sum using residues for this particular f are not the Sneddon–Bessel series $S_{q_n}^{\alpha,\beta;\nu}(x, y)$, this approach allows us to avoid the problems that appear when using residues to compute $S_{q_n}^{\alpha,\beta;\nu}(x, y)$ when $n \geq 0$ (and which do not appear for $n < 0$; see [1, § 2.2] or [2]). Then, Theorem 1 leads us to a partial differential equation for the Sneddon–Bessel series $S_{q_n}^{\alpha,\beta;\nu}(x, y)$, $n \geq 0$, which we solve in Section 4. To do that, we have to use the integral transform

$$T_{\mu,\eta}(f)(x) = \frac{1}{2^{\mu-\eta-1}\Gamma(\mu - \eta)} \int_0^1 f(xs)s^{2\eta+1}(1 - s^2)^{\mu-\eta-1}\, ds$$

(which is introduced and studied in Section 2). Once we have found the sum of $S_{q_n}^{\alpha,\beta;\nu}(x, y)$, $n \geq 0$, Sneddon's (and Martin's) results for $n < 0$ can be easily deduced from the case $n = 0$ by differentiation (see Section 4.2).

In order to state our result in full detail, we need some notation. For $\nu \in \mathbb{C}\setminus\{-1, -2, \dots\}$, let us consider the entire function

$$\Phi_\nu(z) = 2^\nu\Gamma(\nu + 1)\frac{J_\nu(z)}{z^\nu} = \Gamma(\nu + 1) \sum_{n=0}^{\infty} \frac{(-1)^n(z/2)^{2n}}{n!\Gamma(n + \nu + 1)}. \tag{1.3}$$

Define now the polynomial $\delta_n^{\alpha,\beta;\nu}(x, y)$ (of degree $2n$), $n \geq 0$, by the generating function

$$\frac{\Phi_\alpha(xz)\Phi_\beta(yz)}{\Phi_\nu(z)^2} = \sum_{n=0}^{\infty} \delta_n^{\alpha,\beta;\nu}(x, y)z^{2n}.$$

This generating function allows the explicit computation of the polynomials $\delta_n^{\alpha,\beta;\nu}(x, y)$ recursively from the Taylor coefficients of the functions Φ_α, Φ_β, and Φ_ν.

We also define recursively the functions $\phi_n^{\alpha,\beta;\nu}(t)$, $n \geq 0$, by

$$\phi_0^{\alpha,\beta;\nu}(t) = \frac{1}{\nu} \binom{\alpha}{\nu} {}_2F_1 \left(\begin{matrix} \nu - \alpha, \nu \\ \beta + 1 \end{matrix} ; t^2 \right), \tag{1.4}$$

$$\phi_n^{\alpha,\beta;\nu}(t) = \frac{1}{2\nu - 2n} \left(\frac{1}{2(\alpha + 1)}\phi_{n-1}^{\alpha+1,\beta;\nu}(t) + \frac{t^2}{2(\beta + 1)}\phi_{n-1}^{\alpha,\beta+1;\nu}(t) \right), \tag{1.5}$$

where as usual ${}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; t \right)$ denotes the hypergeometric function

$$ {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix};t\right) = \sum_{m=0}^{\infty} \frac{\Gamma(a+m)\Gamma(b+m)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+m)} \cdot \frac{t^m}{m!}. $$

If we write

$$ \delta_n^{\alpha,\beta;\nu}(x,y) = \sum_{2j+2k\le 2n} A_{2j,2k,n}^{\alpha,\beta;\nu} x^{2j} y^{2k}, $$

then for $\nu \notin \{0,1,\dots,n\}$, $2\,\mathrm{Re}\,\nu < 2n+1+\mathrm{Re}\,\alpha+\mathrm{Re}\,\beta$, and $0<y\le x$, $x+y<2$ (and also for $x+y=2$ if $2\,\mathrm{Re}\,\nu < 2n+\mathrm{Re}\,\alpha+\mathrm{Re}\,\beta$), we show that

$$ S_{q_n}^{\alpha,\beta;\nu}(x,y) = \frac{\Gamma(\nu+1)^2 x^\alpha y^\beta}{2^{q_0}\Gamma(\alpha+1)\Gamma(\beta+1)} \left(x^{2n-2\nu}\phi_n^{\alpha,\beta;\nu}(y/x) - \sum_{2j+2k\le 2n} \frac{A_{2j,2k,n}^{\alpha,\beta;\nu}}{j+k+\nu-n} x^{2j} y^{2k} \right), \tag{1.6} $$

if $n \ge 0$, and

$$ S_{q_n}^{\alpha,\beta;\nu}(x,y) = \frac{x^{\alpha-2\nu+2n} y^\beta \Gamma(\nu-n)}{2^{q_n}\Gamma(\beta+1)\Gamma(n+\alpha-\nu+1)}\, {}_2F_1\left(\begin{matrix} \nu-n, \nu-\alpha-n \\ \beta+1 \end{matrix}; \frac{y^2}{x^2}\right), \tag{1.7} $$

if $n<0$. The case $\nu \in \{0,1,\dots,n\}$ can be computed by passing to the limit.

Using our identity (1.6) for the Sneddon–Bessel series, we find in Section 5 some extensions of the Kneser–Sommerfeld expansion (1.2), among which is the following:

$$ \sum_{m=1}^{\infty} \frac{j_{m,\nu}^{\nu-\beta} J_\nu(xj_{m,\nu}) J_\beta(yj_{m,\nu})}{(j_{m,\nu}^2 - z^2) J_{\nu+1}(j_{m,\nu})^2} = \frac{\pi J_\beta(yz)}{4z^{\beta-\nu} J_\nu(z)} \left(Y_\nu(z)J_\nu(xz) - J_\nu(z)Y_\nu(xz)\right), \tag{1.8} $$

if $\mathrm{Re}\,\nu < \mathrm{Re}\,\beta + 1$ and $0<y\le x$, $x+y<2$ (as usual Y_ν denotes the Bessel function of the second kind).

2 | PRELIMINARIES

The zeros of the function $\Phi_\nu(w)$ defined by (1.3), that is, the zeros of the even function $J_\nu(w)/w^\nu$, are simple and can be ordered as a double sequence $\{j_{m,\nu}\}_{m\in\mathbb{Z}\setminus\{0\}}$ with $j_{-m,\nu} = -j_{m,\nu}$ and $0 \le \mathrm{Re}\,j_{m,\nu} \le \mathrm{Re}\,j_{m+1,\nu}$ for $m \ge 1$ ([6, § 15.41, p. 497]). Although these zeros depend on ν, we will often omit this dependence to avoid unnecessary complications in the notation. The imaginary part of these zeros is bounded, and when m is a sufficiently large integer, there is exactly one zero in the strip $m\pi + \frac{\pi}{2}\mathrm{Re}\,\nu + \frac{\pi}{4} < \mathrm{Re}\,z < (m+1)\pi + \frac{\pi}{2}\mathrm{Re}\,\nu + \frac{\pi}{4}$ ([6, § 15.4, p. 497]), so that

$$ \lim_{m\to+\infty} \frac{|j_m|}{\pi m} = 1. $$

It follows from the estimate

$$ J_\nu(z) = 2^{1/2}(\pi z)^{-1/2}\left(\cos\left(z - \frac{\nu}{2}\pi - \frac{\pi}{4}\right) + o(1)\right), \quad z\to\infty $$

([7, (10.7.8)], see also [6, § 7.21(1), p. 199]) that

$$ J_\nu(z)^2 + J_{\nu+1}(z)^2 = \frac{2}{\pi z}\left(1 + e^{2|\mathrm{Im}\,z|} o(1)\right), \quad z\to\infty, $$

where the limit $z\to\infty$ is to be taken inside a sector $|\arg(z)| \le \pi - \delta$. Thus,

$$ 0 < c \le |J_{\nu+1}(j_m)^2 j_m| \le C, \tag{2.1} $$

for some constants c and C not depending on m. In terms of $\Phi_{\nu+1}$,

$$0 < c \le |\Phi_{\nu+1}(j_m)|^2 |j_m|^{2\,\mathrm{Re}\,\nu+3} \le C,$$

for some constants c and C not depending on m.

Bessel functions satisfy the bound

$$|J_\nu(z)| \le C \frac{e^{|\mathrm{Im}\,z|}}{|z|^{1/2}}, \tag{2.2}$$

for $|z|$ large enough, with a constant C depending only on ν. To be precise, for $|z| > \varepsilon > 0$ and ν on a compact set K, there is a constant C depending only on ε and K, as follows from [7, (10.4.4) and § 10.17(iv)].

For μ and η satisfying $\mathrm{Re}\,\mu > \mathrm{Re}\,\eta > -1$, consider the integral transform $T_{\mu,\eta}$ given by

$$T_{\mu,\eta}(f)(x) = \frac{1}{2^{\mu-\eta-1}\Gamma(\mu-\eta)} \int_0^1 f(xs) s^{2\eta+1} (1-s^2)^{\mu-\eta-1}\, ds \tag{2.3}$$

(with a small abuse of notation, we will often write $T_{\mu,\eta}(f(x))$ if it does not cause confusion).

Sonine's formula for the Bessel functions ([6, 12.11(1), p. 373]) can be written as

$$\frac{J_\mu(x)}{x^\mu} = \frac{2^{\eta+1-\mu}}{\Gamma(\mu-\eta)} \int_0^1 \frac{J_\eta(xs)}{(xs)^\eta} s^{2\eta+1}(1-s^2)^{\mu-\eta-1}\, ds = T_{\mu,\eta}\left(\frac{J_\eta(x)}{x^\eta}\right), \tag{2.4}$$

valid for $\mathrm{Re}\,\mu > \mathrm{Re}\,\eta > -1$. For $2\,\mathrm{Re}\,\eta + r + 2 > 0$, we also have

$$T_{\mu,\eta}(x^r) = \frac{\Gamma\left(\eta + \frac{r}{2} + 1\right)}{2^{\mu-\eta}\Gamma\left(\mu + \frac{r}{2} + 1\right)} x^r, \tag{2.5}$$

as follows from the identity

$$\int_0^1 s^a (1-s^2)^b\, ds = \frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma(b+1)}{2\Gamma\left(\frac{a+1}{2} + b + 1\right)}, \qquad \mathrm{Re}\ a,\ \mathrm{Re}\ b > -1.$$

Identities (2.4) and (2.5) can be extended for $\mathrm{Re}\,\eta < -1$ as follows. For complex numbers μ, η and a positive integer h satisfying $\mathrm{Re}\ \eta > -\frac{h}{2} - 1$, $\mathrm{Re}\,\mu > \mathrm{Re}\,\eta + h$, consider the integral transform $T_{\mu,\eta,h}$ given by

$$T_{\mu,\eta,h}(f)(x) = \frac{(-1)^h 2^{\eta+1-\mu}\Gamma(2\eta+2)}{\Gamma(\mu-\eta)\Gamma(2\eta+2+h)} \int_0^1 \frac{d^h}{ds^h}\left(f(xs)(1-s^2)^{\mu-\eta-1}\right) s^{2\eta+h+1}\, ds. \tag{2.6}$$

To be precise, η should not be half a negative integer (in case it is, we will manage somehow).

It is then easy to check that

$$\begin{aligned} T_{\mu,\eta,h}(x^r) &= \frac{\Gamma\left(\eta + \frac{r}{2} + 1\right)}{2^{\mu-\eta}\Gamma\left(\mu + \frac{r}{2} + 1\right)} x^r, \\ T_{\mu,\eta,h}\left(\frac{J_\eta(x)}{x^\eta}\right) &= \frac{J_\mu(x)}{x^\mu}. \end{aligned} \tag{2.7}$$

3 | PARTIAL FRACTION DECOMPOSITION OF BESSEL FUNCTIONS

In this section, we use the calculus of residues to find a partial fraction expansion of functions of the form $\frac{f(z)}{\Phi_\nu(z)^2}$, where f is an entire function with some growth control.

Theorem 1. *Let f be an entire function satisfying*

$$|f(z)| \le c(1+|z|)^N e^{\kappa|Im\, z|},\ z \in \mathbb{C},$$

for certain constants $c>0$, $N\in\mathbb{R}$, and $\kappa \le 2$. Let $\nu \in \mathbb{C}\backslash\{-1,-2,\dots\}$ and n be a nonnegative integer such that

$$N+1+2\ Re\ \nu < n,\ \ if\ \kappa = 2,$$

or

$$N+2\ Re\ \nu < n,\ \ if\ \kappa < 2.$$

Then,

$$\frac{1}{n!}\frac{d^n}{dt^n}\left(\frac{f(t)}{\Phi_\nu(t)^2}\right) = \sum_{m\in\mathbb{Z}\backslash\{0\}} 4(\nu+1)^2 \frac{((2\nu+1)t-(2\nu-n)j_m)\,f(j_m) - j_m(j_m-t)f'(j_m)}{(j_m-t)^{n+2} j_m^3 \Phi_{\nu+1}(j_m)^2},$$

where the convergence is uniform in bounded subsets of $\mathbb{C}\backslash\{j_m : m\in\mathbb{Z}\backslash\{0\}\}$.

Proof. Let us fix $t\in\mathbb{C}\backslash\{j_m : m\in\mathbb{Z}\backslash\{0\}\}$ and consider the holomorphic function $\frac{f(w)}{(w-t)^{n+1}\Phi_\nu(w)^2}$. It has a pole at t of order $n+1$, and a double pole at each j_m, $m\in\mathbb{Z}\backslash\{0\}$. The residue at t is, therefore,

$$\frac{1}{n!}\frac{d^n}{dw^n}\left(\frac{f(w)}{\Phi_\nu(w)^2}\right)_{w=t},$$

while the residue at each j_m is

$$\lim_{w\to j_m}\frac{d}{dw}\left(\frac{(w-j_m)^2 f(w)}{(w-t)^{n+1}\Phi_\nu(w)^2}\right) = \frac{d}{dw}\left(\frac{f(w)}{(w-t)^{n+1}}\right)_{w=j_m}\left(\lim_{w\to j_m}\frac{w-j_m}{\Phi_\nu(w)}\right)^2 + \frac{f(j_m)}{(j_m-t)^{n+1}}\frac{d}{dw}\left(\left(\frac{w-j_m}{\Phi_\nu(w)}\right)^2\right)_{w=j_m}. \tag{3.1}$$

Let us consider separately the last term:

$$\begin{aligned}\frac{d}{dw}\left(\left(\frac{w-j_m}{\Phi_\nu(w)}\right)^2\right)_{w=j_m} &= 2\frac{1}{\Phi'_\nu(j_m)}\lim_{w\to j_m}\frac{\Phi_\nu(w)-(w-j_m)\Phi'_\nu(w)}{\Phi_\nu(w)^2}\\ &= \frac{2}{\Phi'_\nu(j_m)}\lim_{w\to j_m}\frac{-(w-j_m)\Phi''_\nu(w)}{2\Phi_\nu(w)\Phi'_\nu(w)} = \frac{2}{\Phi'_\nu(j_m)}\cdot\frac{-\Phi''_\nu(j_m)}{2\Phi'_\nu(j_m)\Phi'_\nu(j_m)} = -\frac{\Phi''_\nu(j_m)}{\Phi'_\nu(j_m)^3}.\end{aligned}$$

Now, the identities

$$\begin{aligned}\Phi'_\nu(z) &= -\frac{z}{2(\nu+1)}\Phi_{\nu+1}(z),\\ \Phi''_\nu(z) &= -\Phi_\nu(z)+\frac{2\nu+1}{2(\nu+1)}\Phi_{\nu+1}(z)\end{aligned} \tag{3.2}$$

(see [6, § 3.2, p. 45]) prove that

$$-\frac{\Phi''_\nu(j_m)}{\Phi'_\nu(j_m)^3} = \frac{4(\nu+1)^2(2\nu+1)}{j_m^3\Phi_{\nu+1}(j_m)^2},$$

so that, going back to (3.1) and using (3.2) again, the residue at j_m is

$$\begin{aligned}
&\frac{(j_m-t)f'(j_m)-(n+1)f(j_m)}{(j_m-t)^{n+2}}\left(\frac{1}{\Phi_\nu'(j_m)}\right)^2+\frac{f(j_m)}{(j_m-t)^{n+1}}\cdot\frac{4(\nu+1)^2(2\nu+1)}{j_m^3\Phi_{\nu+1}(j_m)^2}\\
&\quad=\frac{(j_m-t)f'(j_m)-(n+1)f(j_m)}{(j_m-t)^{n+2}}\cdot\frac{4(\nu+1)^2}{j_m^2\Phi_{\nu+1}(j_m)^2}+\frac{f(j_m)}{(j_m-t)^{n+1}}\cdot\frac{4(\nu+1)^2(2\nu+1)}{j_m^3\Phi_{\nu+1}(j_m)^2}\\
&\quad=4(\nu+1)^2\frac{j_m(j_m-t)f'(j_m)+((2\nu-n)j_m-(2\nu+1)t)\,f(j_m)}{(j_m-t)^{n+2}j_m^3\Phi_{\nu+1}(j_m)^2}.
\end{aligned}$$

Thus, if $D=\{z\in\mathbb{C}:|z|=A\}$ is a large circle of radius $A>|t|$ with the only condition, at the moment, that none of the points j_m lie in D, the calculus of residues gives

$$\begin{aligned}
\frac{1}{2\pi i}\int_D\frac{f(w)}{(w-t)^{n+1}\Phi_\nu(w)^2}dw&=\frac{1}{n!}\frac{d^n}{dw^n}\left(\frac{f(w)}{\Phi_\nu(w)^2}\right)_{w=t}\\
&\quad+\sum_{|j_m|<A}4(\nu+1)^2\frac{j_m(j_m-t)f'(j_m)+((2\nu-n)j_m-(2\nu+1)t)\,f(j_m)}{(j_m-t)^{n+2}j_m^3\Phi_{\nu+1}(j_m)^2}.
\end{aligned}\tag{3.3}$$

Now, the value of A can be chosen arbitrarily large and such that there exists some constant $c>0$, independent of A, satisfying

$$c\frac{e^{|\mathrm{Im}\,w|}}{|w|^{1/2}}\le|J_\nu(w)|,$$

for $w\in D$ (see [8, formula (2.4)]). Thus,

$$\left|\frac{f(w)}{(w-t)^{n+1}\Phi_\nu(w)^2}\right|=C\frac{|f(w)||w^\nu|^2}{|w-t|^{n+1}|J_\nu(w)|^2}\le C\frac{A^{N+1+2\,\mathrm{Re}\,\nu}e^{(\kappa-2)|\mathrm{Im}\,w|}}{(A-|t|)^{n+1}},$$

for $w\in D$, where C is a constant, independent of A, but possibly different at each occurrence. The natural parametrization of D then gives

$$\begin{aligned}
\left|\frac{1}{2\pi i}\int_D\frac{f(w)}{(w-t)^{n+1}\Phi_\nu(w)^2}dw\right|&\le\frac{C}{2\pi}\int_{-\pi}^{\pi}\frac{A^{N+2+2\,\mathrm{Re}\,\nu}e^{(\kappa-2)A|\sin s|}}{(A-|t|)^{n+1}}ds\\
&=\frac{2C}{\pi}\frac{A^{N+2+2\,\mathrm{Re}\,\nu}}{(A-|t|)^{n+1}}\int_0^{\pi/2}e^{(\kappa-2)A|\sin s|}ds\le\frac{2C}{\pi}\frac{A^{N+2+2\,\mathrm{Re}\,\nu}}{(A-|t|)^{n+1}}\int_0^{\pi/2}e^{(\kappa-2)A2s/\pi}ds.
\end{aligned}$$

Now, the last integral is obviously a constant if $\kappa=2$, while it is $O(A^{-1})$ as $A\to\infty$ if $\kappa<2$. Taking this bound into (3.3) and letting A be arbitrarily large proves the theorem. □

Evaluating at $t=0$, the identity of Theorem 1 gives

$$\frac{1}{n!}\frac{d^n}{dt^n}\left(\frac{f(t)}{\Phi_\nu(t)^2}\right)_{t=0}=-4(\nu+1)^2\sum_{m\in\mathbb{Z}\setminus\{0\}}\frac{(2\nu-n)f(j_m)+j_mf'(j_m)}{j_m^{n+4}\Phi_{\nu+1}(j_m)^2},\tag{3.4}$$

under the assumption that

$$\begin{aligned}
N+1+2\,\mathrm{Re}\,\nu<n,&\quad\text{if }\kappa=2,\\
N+2\,\mathrm{Re}\,\nu<n,&\quad\text{if }\kappa<2.
\end{aligned}$$

We then define the double Bessel numbers $\delta_n^{f,\nu}$ by

$$\delta_n^{f,\nu} = \frac{1}{n!}\frac{d^n}{dt^n}\left(\frac{f(t)}{\Phi_\nu(t)^2}\right)_{t=0}.$$

These are the Taylor (or Maclaurin) coefficients of $\frac{f(t)}{\Phi_\nu(t)^2}$ at $t=0$, in other words,

$$\frac{f(t)}{\Phi_\nu(t)^2} = \sum_{n=0}^{\infty}\delta_n^{f,\nu}t^n,$$

in a neighborhood of 0.

4 | SUMMING SNEDDON–BESSEL SERIES EXPLICITLY

Our goal is to sum the Sneddon–Bessel series

$$\sum_{m\geq 1}\frac{J_\alpha(xj_m)J_\beta(yj_m)}{j_m^{2n+\alpha+\beta-2\nu+2}J_{\nu+1}(j_m)^2},$$

where $\alpha,\beta,\nu\in\mathbb{C}\backslash\{-1,-2,\dots\}$, $0<x$, $0<y$, $x+y\leq 2$, and n is an integer. Our method extends the known results, which mainly refer to some particular cases as we mentioned in Section 1, to a rather general setting. In particular, as far as we know, the explicit expressions for $S_{q_0}^{\alpha,\beta;\nu}(x,y)$ and $S_{q_1}^{\alpha,\beta;\nu}(x,y)$ in Sections 4.1.1 and 4.1.2 and $S_n^{\alpha;\nu}(x)$ in Section 4.3 have not been previously stated in the literature.

To this end, let us take

$$\xi_{n,\alpha,\beta,\nu}(x,y) = \sum_{m\geq 1}\frac{\Phi_\alpha(xj_m)\Phi_\beta(yj_m)}{j_m^{2n+4}\Phi_{\nu+1}(j_m)^2}, \tag{4.1}$$

with the condition that

$$2\ \mathrm{Re}\ \nu < 2n+1+\mathrm{Re}\ \alpha+\mathrm{Re}\ \beta. \tag{4.2}$$

According to (2.1) and (2.2), this guarantees that the series converges absolutely. These series are related to the Sneddon–Bessel series (1.1) by

$$S_{q_n}^{\alpha,\beta;\nu}(x,y) = \frac{\Gamma(\nu+2)^2x^\alpha y^\beta}{2^{\alpha+\beta-2\nu-2}\Gamma(\alpha+1)\Gamma(\beta+1)}\xi_{n,\alpha,\beta,\nu}(x,y). \tag{4.3}$$

Under the stronger condition

$$2\ \mathrm{Re}\,\nu < 2n+\mathrm{Re}\,\alpha+\mathrm{Re}\,\beta, \tag{4.4}$$

termwise differentiation in (4.1) is allowed. In particular, we obtain

$$\frac{\partial}{\partial x}\xi_{n,\alpha,\beta,\nu}(x,y) = -\frac{x}{2(\alpha+1)}\xi_{n-1,\alpha+1,\beta,\nu}(x,y) \tag{4.5}$$

(and the same for the other partial derivative).

4.1 | The case $n\geq 0$

Let us assume firstly that n is a nonnegative integer (later on, we will address the case when n is negative).

The function $f(z)=\Phi_\alpha(xz)\Phi_\beta(yz)$ meets the conditions of Theorem 1 with $N=-\mathrm{Re}\ \alpha-\mathrm{Re}\ \beta-1$ and $\kappa=x+y$, and the condition $N+2\ \mathrm{Re}\ \nu<2n$ of Theorem 1 is therefore (4.2). Thus, (3.4) becomes

$$\frac{\delta_n^{\alpha,\beta;\nu}(x,y)}{-8(\nu+1)^2} = (2\nu-2n)\xi_{n,\alpha,\beta,\nu}(x,y) + x\frac{\partial\xi_{n,\alpha,\beta,\nu}(x,y)}{\partial x} + y\frac{\partial\xi_{n,\alpha,\beta,\nu}(x,y)}{\partial y}, \tag{4.6}$$

where the function

$$\delta_n^{\alpha,\beta;\nu}(x,y) = \frac{1}{(2n)!}\frac{d^{2n}}{dz^{2n}}\left(\frac{\Phi_\alpha(xz)\Phi_\beta(yz)}{\Phi_\nu(z)^2}\right)_{z=0} \tag{4.7}$$

is a polynomial in x^2 and y^2 (i.e., even powers of x and y) which could be computed recursively from the Taylor coefficients of the functions Φ_α, Φ_β, and Φ_ν involved. Notice that

$$\frac{\Phi_\alpha(xz)\Phi_\beta(yz)}{\Phi_\nu(z)^2} = \sum_{n=0}^{\infty}\delta_n^{\alpha,\beta;\nu}(x,y)z^{2n}. \tag{4.8}$$

Let us write

$$\delta_n^{\alpha,\beta;\nu}(x,y) = \sum_{2j+2k\le 2n} A_{2j,2k,n}^{\alpha,\beta;\nu}x^{2j}y^{2k}, \tag{4.9}$$

and assume also, for simplicity, that $\nu \neq 0, 1, \ldots, n$. Then, it is easy to see that the solution to (4.6) is

$$\xi_{n,\alpha,\beta,\nu}(x,y) = \frac{1}{16(\nu+1)^2}\left(-\sum_{2j+2k\le 2n}\frac{A_{2j,2k,n}^{\alpha,\beta;\nu}}{j+k+\nu-n}x^{2j}y^{2k} + x^{2n-2\nu}\phi_{n,\alpha,\beta,\nu}(y/x)\right), \tag{4.10}$$

if (4.4) holds, where $\phi_{n,\alpha,\beta,\nu}$ is a one-variable function to be determined. In case $\nu \in \{0,1,\ldots,n\}$, some logarithmic terms appear also.

Before going on, let us focus on the dependence of these functions and constants on the parameter α and β. It is apparent from (4.7) and (4.9) that each $A_{2j,2k,n}^{\alpha,\beta;\nu}$ is a rational function of α, β, and ν. If n, ν, x, y, and β (respectively, α) are fixed, then the function $\Phi_\alpha(xj_m)$ is holomorphic on $\alpha \in \mathbb{C}\backslash\{-1,-2,\ldots\}$, and so is $\xi_{n,\alpha,\beta,\nu}(x,y)$ (resp., β) under the condition (4.2) (the series involved converge uniformly on α-compacts, as follows from 1.3 and 2.2). The same applies therefore to $\phi_{n,\alpha,\beta,\nu}(y/x)$. This analytic dependence on α (resp., β) will eventually allow us to extend some identities by analytic continuation.

Thus, formula (4.10), which in principle requires (4.4) to hold, extends to the whole range (4.2) in this way: Firstly, (4.5) can be written as

$$\xi_{n,\alpha,\beta,\nu}(x,y) = -\frac{2\alpha}{x}\frac{\partial}{\partial x}\xi_{n+1,\alpha-1,\beta,\nu}(x,y),$$

on the whole range (4.2); using now (4.10) on the right-hand side gives an expression for $\xi_{n,\alpha,\beta,\nu}(x,y)$ with holomorphic coefficients, which by analytic continuation must equal the coefficients in (4.10).

In view of (4.10), it is enough to find the function $\phi_{n,\alpha,\beta,\nu}$ to explicitly determine the function $\xi_{n,\alpha,\beta,\nu}$. So let us now find a recursion for the functions $\phi_{n,\alpha,\beta,\nu}$. Given $0 < t < 1$, let us write

$$\varphi_{n,\alpha,\beta,\nu}(s) = \xi_{n,\alpha,\beta,\nu}(s,ts),$$

for s small enough. Then, (4.5) yields

$$\begin{aligned}\varphi'_{n,\alpha,\beta,\nu}(s) &= \frac{\partial}{\partial x}\xi_{n,\alpha,\beta,\nu}(s,ts) + t\frac{\partial}{\partial y}\xi_{n,\alpha,\beta,\nu}(s,ts)\\ &= -\frac{s}{2(\alpha+1)}\xi_{n-1,\alpha+1,\beta,\nu}(s,ts) - \frac{st^2}{2(\beta+1)}\xi_{n-1,\alpha,\beta+1,\nu}(s,ts).\end{aligned}$$

The coefficient of $s^{2n-1-2\nu}$ on the right-hand side, as follows from (4.10), is

$$\frac{1}{16(\nu+1)^2}\left(-\frac{1}{2(\alpha+1)}\phi_{n-1,\alpha+1,\beta,\nu}(t) - \frac{t^2}{2(\beta+1)}\phi_{n-1,\alpha,\beta+1,\nu}(t)\right).$$

On the other hand, (4.10) translates into

$$\varphi_{n,\alpha,\beta,\nu}(s) = \frac{1}{16(\nu+1)^2}\left(-\sum_{2j+2k\le 2n}\frac{A_{2j,2k,n}^{\alpha,\beta;\nu}}{j+k+\nu-n}s^{2j+2k}t^{2k} + s^{2n-2\nu}\phi_{n,\alpha,\beta,\nu}(t)\right),$$

so that the coefficient of $s^{2n-1-2\nu}$ in $\varphi'_{n,\alpha,\beta,\nu}(s)$ is

$$\frac{2n-2\nu}{16(\nu+1)^2}\phi_{n,\alpha,\beta,\nu}(t).$$

Equating both formulas for the coefficient of $s^{2n-1-2\nu}$ results in

$$\phi_{n,\alpha,\beta,\nu}(t) = \frac{1}{2\nu-2n}\left(\frac{1}{2(\alpha+1)}\phi_{n-1,\alpha+1,\beta,\nu}(t) + \frac{t^2}{2(\beta+1)}\phi_{n-1,\alpha,\beta+1,\nu}(t)\right). \tag{4.11}$$

This recursion reduces the problem of finding $\xi_{n,\alpha,\beta,\nu}(x,y)$ to the case $n=0$, so let us concentrate on this. We first consider the case $\alpha=\beta=\nu$, then address the general case. Observe that condition (4.2) holds for $n=0$, $\alpha=\beta=\nu$. Now, (4.7) gives $\delta_0^{\alpha,\beta;\nu}(x,y)=1$, so that (4.10) is

$$\xi_{0,\nu,\nu,\nu}(x,y) = \frac{1}{16(\nu+1)^2}\left(-\frac{1}{\nu} + x^{-2\nu}\phi_{0,\nu,\nu,\nu}(y/x)\right).$$

Since $\Phi_\nu(j_m)=0$, the definition (4.1) trivially gives

$$\xi_{0,\nu,\nu,\nu}(1,t)=0,\ \ 0<t<1.$$

Therefore, $\phi_{0,\nu,\nu,\nu}(t)=\frac{1}{\nu}$ for $0<t<1$ and (by symmetry)

$$\xi_{0,\nu,\nu,\nu}(x,y) = \begin{cases} \frac{1}{16\nu(\nu+1)^2}\left(-1+x^{-2\nu}\right), & \text{if } y\le x,\\ \frac{1}{16\nu(\nu+1)^2}\left(-1+y^{-2\nu}\right), & \text{if } x<y. \end{cases} \tag{4.12}$$

Let us now find $\xi_{0,\alpha,\beta,\nu}(x,y)$. By symmetry, we can assume that $y\le x$ without loss of generality. Sonine's formula (2.4) easily gives

$$\xi_{0,\alpha,\beta,\nu}(x,y) = \frac{4\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\nu+1)^2\Gamma(\alpha-\nu)\Gamma(\beta-\nu)}\iint\limits_{[0,1]\times[0,1]} \xi_{0,\nu,\nu,\nu}(xr,ys)r^{2\nu+1}s^{2\nu+1}(1-r^2)^{\alpha-\nu-1}(1-s^2)^{\beta-\nu-1}\,dr\,ds,$$

with the additional restrictions Re $\alpha>$ Re $\nu>-1$, Re $\beta>$ Re $\nu>-1$. To evaluate this integral, let us separate the square $[0,1]\times[0,1]$ into the sets

$$A_{x,y} = \{(r,s) : xr\ge ys\},$$
$$B_{x,y} = \{(r,s) : xr< ys\},$$

so that

$$\xi_{0,\nu,\nu,\nu}(xr,ys) = \begin{cases} \frac{1}{16\nu(1+\nu)^2}\left(-1+x^{-2\nu}r^{-2\nu}\right), & (r,s)\in A_{x,y},\\ \frac{1}{16\nu(1+\nu)^2}\left(-1+y^{-2\nu}s^{-2\nu}\right), & (r,s)\in B_{x,y}. \end{cases}$$

The above integral is therefore

$$\frac{1}{16\nu(\nu+1)^2}\left(-\int_0^1\int_0^1 r^{2\nu+1}s^{2\nu+1}(1-r^2)^{\alpha-\nu-1}(1-s^2)^{\beta-\nu-1}\,dr\,ds\right.$$
$$+\iint_{A_{x,y}} x^{-2\nu}r^{-2\nu}r^{2\nu+1}s^{2\nu+1}(1-r^2)^{\alpha-\nu-1}(1-s^2)^{\beta-\nu-1}\,dr\,ds$$
$$\left.+\iint_{B_{x,y}} y^{-2\nu}s^{-2\nu}r^{2\nu+1}s^{2\nu+1}(1-r^2)^{\alpha-\nu-1}(1-s^2)^{\beta-\nu-1}\,dr\,ds\right).$$

The first of these three integrals is immediate:

$$\int_0^1\int_0^1 r^{2\nu+1}s^{2\nu+1}(1-r^2)^{\alpha-\nu-1}(1-s^2)^{\beta-\nu-1}\,dr\,ds=\frac{\Gamma(\alpha-\nu)\Gamma(\beta-\nu)\Gamma(\nu+1)^2}{4\Gamma(\alpha+1)\Gamma(\beta+1)}.$$

Taking into account that $y\le x$, the second integral is

$$\iint_{A_{x,y}} x^{-2\nu}r^{-2\nu}r^{2\nu+1}s^{2\nu+1}(1-r^2)^{\alpha-\nu-1}(1-s^2)^{\beta-\nu-1}\,dr\,ds$$
$$=x^{-2\nu}\int_0^1\left(\int_{y^2s^2/x^2}^1 r(1-r^2)^{\alpha-\nu-1}\,dr\right)s^{2\nu+1}(1-s^2)^{\beta-\nu-1}\,ds$$
$$=\frac{x^{-2\nu}}{2(\alpha-\nu)}\int_0^1\left(1-\frac{y^2}{x^2}s^2\right)^{\alpha-\nu}s^{2\nu+1}(1-s^2)^{\beta-\nu-1}\,ds$$
$$=\frac{x^{-2\nu}\Gamma(\nu+1)\Gamma(\beta-\nu)}{4(\alpha-\nu)\Gamma(\beta+1)}\,{}_2F_1\left(\begin{matrix}\nu-\alpha,\nu+1\\ \beta+1\end{matrix};\frac{y^2}{x^2}\right),$$

where the integral representation of the hypergeometric function ${}_2F_1$ is used [7, (15.6.1)].

The third integral is

$$\iint_{B_{x,y}} y^{-2\nu}s^{-2\nu}r^{2\nu+1}s^{2\nu+1}(1-r^2)^{\alpha-\nu-1}(1-s^2)^{\beta-\nu-1}\,dr\,ds$$
$$=y^{-2\nu}\int_0^{y^2/x^2}\left(\int_{x^2r^22/y^2}^1 s(1-s^2)^{\beta-\nu-1}\,ds\right)r^{2\nu+1}(1-r^2)^{\alpha-\nu-1}\,dr$$
$$=\frac{y^{-2\nu}}{2(\beta-\nu)}\int_0^{y^2/x^2}\left(1-\frac{x^2}{y^2}r^2\right)^{\beta-\nu}r^{2\nu+1}(1-r^2)^{\alpha-\nu-1}\,dr$$
$$=\frac{y^2x^{-2\nu-2}}{2(\beta-\nu)}\int_0^1(1-t^2)^{\beta-\nu}t^{2\nu+1}\left(1-\frac{y^2}{x^2}t^2\right)^{\alpha-\nu-1}\,dt$$
$$=\frac{y^2x^{-2\nu-2}\Gamma(\nu+1)\Gamma(\beta-\nu+1)}{4(\beta-\nu)\Gamma(\beta+2)}\,{}_2F_1\left(\begin{matrix}\nu-\alpha+1,\nu+1\\ \beta+2\end{matrix};\frac{y^2}{x^2}\right).$$

Putting together all the pieces,

$$\xi_{0,\alpha,\beta,\nu}(x,y)=\frac{1}{16\nu(\nu+1)^2}\left(-1+\frac{x^{-2\nu}\Gamma(\alpha+1)}{\Gamma(\nu+1)\Gamma(\alpha-\nu+1)}\right.$$
$$\left.\times\left({}_2F_1\left(\begin{matrix}\nu-\alpha,\nu+1\\ \beta+1\end{matrix};\frac{y^2}{x^2}\right)+\frac{y^2(\alpha-\nu)}{x^2(\beta+1)}\,{}_2F_1\left(\begin{matrix}\nu-\alpha+1,\nu+1\\ \beta+2\end{matrix};\frac{y^2}{x^2}\right)\right)\right).$$

Finally, the elementary relation

$${}_2F_1\left(\begin{matrix}a,b+1\\ c;\end{matrix}t\right)-t\frac{a}{c}\,{}_2F_1\left(\begin{matrix}a+1,b+1\\ c+1\end{matrix};t\right)={}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix};t\right)$$

gives

$$\xi_{0,\alpha,\beta,\nu}(x,y)=\frac{1}{16\nu(\nu+1)^2}\left(-1+x^{-2\nu}\binom{\alpha}{\nu}\,{}_2F_1\left(\begin{matrix}\nu-\alpha,\nu\\ \beta+1\end{matrix};\frac{y^2}{x^2}\right)\right),\tag{4.13}$$

valid for $\mathrm{Re}\ \alpha,\mathrm{Re}\ \beta>\mathrm{Re}\ \nu>-1$, and $0<y\le x, x+y<2$.

Assuming that $\mathrm{Re}\ \nu>-1$, identity (4.13) extends to the whole range given by (4.2), that is, $2\ \mathrm{Re}\ \nu<1+\mathrm{Re}\ \alpha+\mathrm{Re}\ \beta$, by an argument of analyticity (and also for $x+y=2$ if $2\ \mathrm{Re}\ \nu<\mathrm{Re}\ \alpha+\mathrm{Re}\ \beta$).

Let us consider now the case $\mathrm{Re}\ \nu<-1$. Take a positive integer h such that $\mathrm{Re}\ \nu>-h/2-1$ and α,β satisfying $\mathrm{Re}\alpha,\mathrm{Re}\ \beta>\mathrm{Re}\ \nu+h$. Let us assume for the moment that ν is not half a negative integer; using the integral transform $T_{\alpha,\nu,h}$ defined by (2.6) acting on x and $T_{\beta,\nu,h}$ acting on y, we get from (2.7):

$$\xi_{0,\alpha,\beta,\nu}(x,y)=\frac{4\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\nu+1)^2\Gamma(\alpha-\nu)\Gamma(\beta-\nu)}\cdot\frac{\Gamma(2\nu+2)^2}{\Gamma(2\nu+2+h)^2}$$
$$\times\iint_{[0,1]\times[0,1]}\frac{\partial^{2h}}{\partial r^h\,\partial s^h}\left(\xi_{0,\nu,\nu,\nu}(xr,ys)(1-r^2)^{\alpha-\nu-1}(1-s^2)^{\beta-\nu-1}\right)(rs)^{2\nu+h+1}\,dr\,ds.$$

The function $\xi_{0,\nu,\nu,\nu}(x,y)$ is given by (4.12). Therefore, $\xi_{0,\nu,\nu,\nu}$ is analytic in ν and so is the function on the right-hand side of the above identity (on the region $\mathrm{Re}\ \nu>-h/2-1$). For $\nu>-1$, integrating by parts, we deduce that this function is equal to

$$\frac{4\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\nu+1)^2\Gamma(\alpha-\nu)\Gamma(\beta-\nu)}\iint_{[0,1]\times[0,1]}\xi_{0,\nu,\nu,\nu}(xr,ys)(1-r^2)^{\alpha-\nu-1}(1-s^2)^{\beta-\nu-1}(rs)^{2\nu+1}\,dr\,ds.\tag{4.14}$$

Proceeding as before, we deduce that for $\nu>-1$ and $0<y\le x, x+y<2$, the function (4.14) is equal to the function on the right-hand side of (4.13), which is also analytic in ν. This proves identity (4.13) also for $\mathrm{Re}\ \nu>-h/2-1$ and $\mathrm{Re}\alpha,\mathrm{Re}\ \beta>\mathrm{Re}\ \nu+h$. Using again an argument of analyticity on the variables α and β, we prove that (4.13) holds indeed for $2\ \mathrm{Re}\ \nu<1+\mathrm{Re}\ \alpha+\mathrm{Re}\ \beta$ (and also for $x+y=2$ if $2\ \mathrm{Re}\ \nu<\mathrm{Re}\ \alpha+\mathrm{Re}\ \beta$). The requirement that ν is not half a negative integer can be suppressed by continuity.

By the way, this means that

$$\phi_{0,\alpha,\beta,\nu}(t)=\frac{1}{\nu}\binom{\alpha}{\nu}\,{}_2F_1\left(\begin{matrix}\nu-\alpha,\nu\\ \beta+1\end{matrix};t^2\right),\quad 0<t<1,\tag{4.15}$$

which, together with (4.11), allows to find the functions $\phi_{n,\alpha,\beta,\nu}$ and, therefore, $\xi_{n,\alpha,\beta,\nu}$ for every positive integer n.

For the sake of completeness, we display in full extension the cases $n=0,1$.

4.1.1 | The case $n=0$

Identities (4.3) and (4.13) give

$$S_{q_0}^{\alpha,\beta;\nu}(x,y)=\frac{\Gamma(\nu+1)^2x^\alpha y^\beta}{2^{q_0}\nu\Gamma(\alpha+1)\Gamma(\beta+1)}\left(-1+x^{-2\nu}\binom{\alpha}{\nu}\,{}_2F_1\left(\begin{matrix}\nu-\alpha,\nu\\ \beta+1\end{matrix};\frac{y^2}{x^2}\right)\right),\tag{4.16}$$

valid for $2\ \mathrm{Re}\nu<1+\mathrm{Re}\alpha+\mathrm{Re}\beta$, $\nu\neq0$, and $0<y\le x, x+y<2$ (and also for $x+y=2$ if $2\ \mathrm{Re}\ \nu<\mathrm{Re}\ \alpha+\mathrm{Re}\ \beta$).

Now, $S_{q_0}^{\alpha,\beta;0}(x,y)$ can be obtained taking $\nu \to 0$ in the above formula: On one hand, after writing the hypergeometric function as a power series and looking for a hypergeometric representation of the resulting limit, it turns out that

$$\lim_{\nu\to 0}\frac{1}{\nu}\frac{x^{-2\nu}\Gamma(\alpha+1)}{\Gamma(\nu+1)\Gamma(\alpha-\nu+1)}\left(-1+{}_2F_1\left(\begin{matrix}\nu-\alpha,\nu\\ \beta+1\end{matrix};\frac{y^2}{x^2}\right)\right)=-\frac{\alpha}{\beta+1}\frac{y^2}{x^2}\,{}_3F_2\left(\begin{matrix}1,1,1-\alpha\\ 2,\beta+2\end{matrix};\frac{y^2}{x^2}\right).$$

On the other hand, L'Hôpital's rule gives

$$\lim_{\nu\to 0}\frac{1}{\nu}\left(-1+\frac{x^{-2\nu}\Gamma(\alpha+1)}{\Gamma(\nu+1)\Gamma(\alpha-\nu+1)}\right)=-2\log x+H_\alpha,$$

where $H_\alpha=\gamma+\frac{\Gamma'(\alpha+1)}{\Gamma(\alpha+1)}$ is the harmonic number of order α (as usual γ denotes the Euler constant). Then, we conclude that

$$S_{q_0}^{\alpha,\beta;0}(x,y)=\frac{x^\alpha y^\beta}{2^{q_0}\Gamma(\alpha+1)\Gamma(\beta+1)}\left(-2\log x+H_\alpha-\frac{\alpha}{\beta+1}\frac{y^2}{x^2}\,{}_3F_2\left(\begin{matrix}1,1,1-\alpha\\ 2,\beta+2\end{matrix};\frac{y^2}{x^2}\right)\right).$$

4.1.2 | The case $n=1$

Identities (4.3), (4.10), (4.11), and (4.15) lead to

$$\begin{aligned}S_{q_1}^{\alpha,\beta;\nu}(x,y)=\frac{\Gamma(\nu+1)^2x^\alpha y^\beta}{2^{q_1-1}\Gamma(\alpha+1)\Gamma(\beta+1)}\Bigg(&\frac{x^2}{2\nu(\alpha+1)}+\frac{y^2}{2\nu(\beta+1)}-\frac{1}{\nu^2-1}\\&+\frac{x^{2-2\nu}\binom{\alpha}{\nu}}{2\nu(\nu-1)}\left(\frac{{}_2F_1\left(\begin{matrix}\nu-\alpha-1,\nu\\ \beta+1\end{matrix};\frac{y^2}{x^2}\right)}{\alpha+1-\nu}+\frac{\frac{y^2}{x^2}\,{}_2F_1\left(\begin{matrix}\nu-\alpha,\nu\\ \beta+2\end{matrix};\frac{y^2}{x^2}\right)}{\beta+1}\right)\Bigg),\end{aligned}$$

valid for $2\,\mathrm{Re}\,\nu<3+\mathrm{Re}\,\alpha+\mathrm{Re}\,\beta$, $\nu\neq 0,1$, and $0<y\le x$, $x+y<2$ (also for $x+y=2$ if $2\,\mathrm{Re}\,\nu<2+\mathrm{Re}\,\alpha+\mathrm{Re}\,\beta$).

Now, $S_{q_1}^{\alpha,\beta;0}(x,y)$ and $S_{q_1}^{\alpha,\beta;1}(x,y)$ follow taking limits as $\nu\to 0$ and $\nu\to 1$ in the above formula with the same kind of manipulations of the case $n=0$. We thus obtain that

$$\begin{aligned}S_{q_1}^{\alpha,\beta;0}(x,y)=\frac{x^\alpha y^\beta}{2^{q_1-2}\Gamma(\alpha+1)\Gamma(\beta+1)}\Bigg(&2-\frac{x^2}{(\alpha+1)^2}-\frac{((\alpha+1)y^2+(\beta+1)x^2)(1+H_\alpha-2\log x)}{(\alpha+1)(\beta+1)}\\&+\frac{y^2}{\beta+1}\,{}_3F_2\left(\begin{matrix}1,1,-\alpha\\ 2,\beta+2\end{matrix};\frac{y^2}{x^2}\right)+\frac{\alpha y^4}{(\beta+1)(\beta+2)x^2}\,{}_3F_2\left(\begin{matrix}1,1,1-\alpha\\ 2,\beta+3\end{matrix};\frac{y^2}{x^2}\right)\Bigg),\end{aligned}$$

and

$$S_{q_1}^{\alpha,\beta;1}(x,y)=\frac{x^\alpha y^\beta}{2^{q_1}\Gamma(\alpha+1)\Gamma(\beta+1)}\left(\frac{x^2}{\alpha+1}+\frac{y^2}{\beta+1}-\frac{-3+2H_\alpha-4\log x}{2}+\frac{\alpha}{\beta+1}\frac{y^2}{x^2}\,{}_3F_2\left(\begin{matrix}1,1,1-\alpha\\ 2,\beta+2\end{matrix};\frac{y^2}{x^2}\right)\right).$$

4.2 | The case $n<0$

Once we have determined the case $n\ge 0$, let us consider now the case when n is a negative integer.

From (4.5), we get

$$\xi_{-1,\alpha,\beta,\nu}(x,y)=-\frac{2\alpha}{x}\frac{\partial}{\partial x}\xi_{0,\alpha-1,\beta,\nu}(x,y).$$

An easy computation using (4.13) gives

$$\xi_{-1,\alpha,\beta,\nu}(x,y)=\frac{\Gamma(\alpha+1)}{4(\nu+1)^2\Gamma(\nu+1)\Gamma(\alpha-\nu)}x^{-2\nu-2}\,{}_2F_1\left(\begin{matrix}\nu-\alpha+1,\nu+1\\ \beta+1\end{matrix};\frac{y^2}{x^2}\right),$$

from where identity (1.7) for $n = -1$ follows easily using (4.3). Identity (1.7) for an integer $n < -1$ can be proved similarly. The case $\nu \in \{0, 1, \dots, n\}$ follows by passing to the limit.

4.3 | The one-variable case

We will need later the following Sneddon–Bessel series in one variable:

$$\mathcal{S}_n^{\alpha;\nu}(x) = \sum_{m=1}^{\infty} \frac{J_\alpha(x j_{m,\nu})}{j_{m,\nu}^{2n+\alpha-2\nu+2} J_{\nu+1}(j_{m,\nu})^2}, \tag{4.17}$$

where $0 \le x \le 2$.

If we assume $2\,\mathrm{Re}\,\nu < 2n + 1/2 + \mathrm{Re}\,\alpha$, then the uniform convergence of the Sneddon–Bessel series (1.1) holds also for $y = 0$; hence, the Sneddon–Bessel series (4.17) arises after dividing the Sneddon–Bessel series (1.1) by y^β and taking $y \to 0$. This can be done in identities (1.6) and (1.7). To this end, we have to compute the sequence

$$d_n^{\alpha;\nu} = \phi_n^{\alpha,\beta;\nu}(0), \ \ n \ge 0.$$

After some easy computations, using (1.4) and (1.5), we arrive at

$$d_n^{\alpha;\nu} = \frac{\Gamma(\alpha+1)\Gamma(\nu-n)}{2^{2n}\Gamma(\nu+1)^2\Gamma(n+1+\alpha-\nu)}, \ \ n \ge 0.$$

Let us define the polynomial $\delta_n^{\alpha;\nu}(x)$ (of degree $2n$), $n \ge 0$, by the generating function

$$\frac{\Phi_\alpha(xz)}{\Phi_\nu(z)^2} = \sum_{n=0}^{\infty} \delta_n^{\alpha;\nu}(x) z^{2n}. \tag{4.18}$$

This generating function allows the explicit computation of the polynomials $\delta_n^{\alpha;\nu}(x)$ recursively from the Taylor coefficients of the functions Φ_α and Φ_ν.

If we write

$$\delta_n^{\alpha;\nu}(x) = \sum_{j=0}^{n} A_{j,n}^{\alpha;\nu} x^{2j}, \tag{4.19}$$

then for $\nu \notin \{0, 1, \dots, n\}$, $2\ \mathrm{Re}\ \nu < 2n + 1/2 + \mathrm{Re}\ \alpha$, and $0 < x < 2$ (and also for $x = 2$ if $2\ \mathrm{Re}\ \nu < 2n - 1/2 + \mathrm{Re}\ \alpha$), we have

$$\mathcal{S}_n^{\alpha;\nu}(x) = \frac{\Gamma(\nu+1)^2 x^\alpha}{2^{\alpha-2\nu+2}\Gamma(\alpha+1)} \left(\frac{\binom{\alpha}{\nu}\Gamma(\alpha-\nu+1)\Gamma(\nu-n)}{2^{2n}\Gamma(\nu+1)\Gamma(n+1+\alpha-\nu)} x^{2n-2\nu} - \sum_{j=0}^{n} \frac{A_{j,n}^{\alpha;\nu}}{j+\nu-n} x^{2j} \right), \tag{4.20}$$

if $n \ge 0$, and

$$\mathcal{S}_n^{\alpha;\nu}(x) = \frac{\Gamma(\nu-n)}{2^{2n+\alpha-2\nu+2}\Gamma(n+\alpha-\nu+1)} x^{\alpha-2\nu+2n},$$

if $n < 0$.

For instance, for $n = 0$, we get

$$\mathcal{S}_0^{\alpha;\nu}(x) = \frac{\Gamma(\nu+1)^2 x^\alpha}{2^{2+\alpha-2\nu}\nu\Gamma(\alpha+1)} \left(-1 + \binom{\alpha}{\nu} x^{-2\nu} \right),$$

assuming $0 < x < 2$, $2\ \mathrm{Re}\ \nu < \frac{1}{2} + \mathrm{Re}\ \alpha$, and $\nu \ne 0$ (also for $x = 2$ if $2\ \mathrm{Re}\ \nu < \mathrm{Re}\ \alpha$). And for $\nu = 0$ and $-\frac{1}{2} < \mathrm{Re}\ \alpha$,

$$\mathcal{S}_0^{\alpha;0}(x) = \frac{x^\alpha}{2^{\alpha+1}\Gamma(\alpha+1)} \left(-\log x + \frac{1}{2} H_\alpha \right),$$

which was previously computed using a different method in [9, (4)].

And for $n = 1$, the corresponding Sneddon–Bessel series is

$$\mathcal{S}_1^{\alpha;\nu}(x) = \frac{\Gamma(\nu+1)\Gamma(\nu-1)x^{\alpha}}{2^{3+\alpha-2\nu}\Gamma(\alpha+1)}\left(\frac{(\nu-1)x^2}{2(\alpha+1)} - \frac{\nu}{\nu+1} + \frac{\binom{\alpha}{\nu}x^{2-2\nu}}{2(\alpha+1-\nu)}\right),$$

assuming $0 < x < 2$, $2\ \mathrm{Re}\ \nu < \frac{5}{2} + \mathrm{Re}\ \alpha$, and $\nu \neq 0, 1$.

The cases $\nu = 0, 1$ can be deduced taking limits as $\nu \to 0$ and $\nu \to 1$, respectively. As a result,

$$\mathcal{S}_1^{\alpha;0}(x) = \frac{x^{\alpha}}{2^{\alpha+3}\Gamma(\alpha+2)}\left(x^2\log x - \frac{1+H_{\alpha+1}}{2}x^2 + \alpha + 1\right),$$

for $-\frac{5}{2} < \mathrm{Re}\ \alpha$, and

$$\mathcal{S}_1^{\alpha;1}(x) = \frac{x^{\alpha}}{2^{\alpha+1}\Gamma(\alpha+1)}\left(-\log x + \frac{-3+2H_{\alpha}}{4} + \frac{1}{2(\alpha+1)}x^2\right),$$

whenever $-\frac{1}{2} < \mathrm{Re}\ \alpha$.

5 | EXTENDING THE KNESER–SOMMERFELD EXPANSION

In this section, we use identity (4.10) to prove some extensions of the Kneser–Sommerfeld expansion (1.2). These new extensions, which provide the sum of some series in a closed form, are (1.8) and (5.5).

For the sake of completeness, we first prove the Kneser–Sommerfeld expansion (1.2). In terms of the functions Φ_ν defined by (1.3), the identity to be proved is

$$\sum_{m=1}^{\infty} \frac{\Phi_\nu(xj_{m,\nu})\Phi_\nu(yj_{m,\nu})}{j_{m,\nu}^2(j_{m,\nu}^2 - z^2)\Phi_{\nu+1}(j_{m,\nu})^2} = \frac{\pi J_\nu(yz)\left(Y_\nu(z)J_\nu(xz) - J_\nu(z)Y_\nu(xz)\right)}{16(\nu+1)^2(xy)^\nu J_\nu(z)}. \tag{5.1}$$

Write $\varphi(x, y, z)$ and $\psi(x, y, z)$ for the left- and right-hand sides of (5.1), respectively. Using the geometric series, we can write

$$\varphi(x, y, z) = \sum_{m=1}^{\infty} \frac{\Phi_\nu(xj_{m,\nu})\Phi_\nu(yj_{m,\nu})}{j_{m,\nu}^2(j_{m,\nu}^2 - z^2)\Phi_{\nu+1}(j_{m,\nu})^2} = \sum_{n=0}^{\infty} \xi_{n,\nu,\nu,\nu}(x, y)z^{2n},$$

where the function $\xi_{n,\nu,\nu,\nu}(x, y)$ is defined by (4.1).

Consider now the partial differential equation

$$-\frac{\Phi_\nu(xz)\Phi_\nu(yj_{m,\nu})}{8(\nu+1)^2\Phi_\nu(z)^2} = 2\nu U(x, y, z) - z\frac{\partial U}{\partial z}(x, y, z) + x\frac{\partial U}{\partial x}(x, y, z) + y\frac{\partial U}{\partial y}(x, y, z). \tag{5.2}$$

On the one hand, using the partial differential equation (4.6) for $\xi_n(x, y)$ and (4.8), we get that $\varphi(x, y, z)$ satisfies the partial differential equation (5.2). On the other hand, it is a matter of computation to check that the function $\psi(x, y, z)$ satisfies the partial differential equation (5.2) as well. Hence, we deduce that

$$\varphi(x, y, z) - \psi(x, y, z) = x^{-2\nu}\rho(y/x, z/x),$$

for certain two-variable function ρ. But the definition of φ and ψ as both sides of (5.1) shows that

$$\varphi(1, y, z) = \psi(1, y, z) = 0,$$

so that $\rho = 0$ and identity (5.1) holds.

For Re β > Re ν > −1, identity (1.8) follows by applying the integral transform $T_{\beta,\nu}$ defined by (2.3) acting in the variable y to both sides of the Kneser–Sommerfeld expansion (1.2) and using Sonine's formula (2.4). With a standard argument of analyticity, identity (1.8) extends to $-1 < \text{Re } \nu < \text{Re } \beta + 1$.

If Re $\nu < -1$, we can take a positive integer h satisfying Re $\nu > -h/2 - 1$. When $\text{Re}\,\beta > \text{Re}\,\nu + h$, using the integral transform $T_{\beta,\nu,h}$ defined by (2.6), we prove identity (1.8) for Re β > Re $\nu + h$, and using an argument of analyticity for $\text{Re}\,\nu < \text{Re } \beta + 1$.

The Kneser–Sommerfeld expansion has the following one-variable version:

$$\sum_{m=1}^{\infty} \frac{j_{m,\nu}^{\nu} J_\nu(x j_{m,\nu})}{(j_{m,\nu}^2 - z^2) J_{\nu+1}(j_{m,\nu})^2} = \frac{\pi z^\nu}{4 J_\nu(z)} \left(Y_\nu(z) J_\nu(xz) - J_\nu(z) Y_\nu(xz) \right), \tag{5.3}$$

valid for Re $\nu < 1/2$ (it follows easily dividing identity 1.2 by y^ν and then taking limit as $y \to 0$).

Now, the well-known properties of the Bessel function of the second kind Y_ν allow us to rewrite (5.3) as

$$\sum_{m=1}^{\infty} \frac{j_{m,\nu}^{\nu} J_\nu(x j_{m,\nu})}{(j_{m,\nu}^2 - z^2) J_{\nu+1}(j_{m,\nu})^2} = \frac{\pi z^\nu}{4 \sin(\pi\nu) J_\nu(z)} \left(J_\nu(z) J_{-\nu}(xz) - J_{-\nu}(z) J_\nu(xz) \right) \tag{5.4}$$

(as usual, if $\nu = n$ is a nonnegative integer, the function on the right can be understood as the limit as $\nu \to n$).

We finish this paper proving the following extension of identity (5.4):

$$\sum_{m=1}^{\infty} \frac{j_{m,\nu}^{2\nu-\alpha} J_\alpha(x j_{m,\nu})}{(j_{m,\nu}^2 - z^2) J_{\nu+1}(j_{m,\nu})^2} = \frac{\pi}{4 \sin(\pi\nu) J_\nu(z)} \left(\frac{x^{\alpha-2\nu} J_\nu(z)\, {}_1F_2\left(\begin{matrix} 1 \\ -\nu+1, \alpha-\nu+1 \end{matrix} ; -\frac{(xz)^2}{4} \right)}{2^{\alpha-2\nu} \Gamma(-\nu+1) \Gamma(\alpha-\nu+1)} - z^{2\nu-\alpha} J_{-\nu}(z) J_\alpha(xz) \right), \tag{5.5}$$

valid for 2 Re ν < Re $\alpha + 1/2$, $\alpha - \nu + 1 \neq 0, -1, -2, \dots$, $\nu \notin \mathbb{Z}$ and $0 < x \leq 2$ (for $\nu = n \in \mathbb{Z}$, we can extend it passing to the limit $\nu \to n$).

For Re $\nu < 1/2$ and 2 Reν < Re$\alpha + 1/2$, the proof is similar to that of identity (1.8). For Reα > Reν and $-1 <$ Re $\nu < 1/2$, identity (5.5) follows by applying the integral transform $T_{\alpha,\nu}$ defined by (2.3) to both sides of the one-variable Kneser–Sommerfeld expansion (5.4): In the left-hand side, we use Sonine's formula (2.4) and in the right-hand side identity (2.5) applied to the power expansion of $J_{-\nu}(xz)$. Using a standard argument of analyticity, identity (5.5) extends to $-2 < 2$ Re ν < Re $\alpha + 1/2$. If Re $\nu < -1$, we can take a positive integer h satisfying Re $\nu > -h/2 - 1$ and use the integral transform $T_{\alpha,\nu,h}$ defined by (2.6).

In order to extend identity (5.4) to 2 Re ν < Re $\alpha + 1/2$, $\alpha - \nu + 1 \neq 0, -1, -2, \dots$, $\nu \notin \mathbb{Z}$, and $0 < x \leq 2$, we proceed as follows. First of all, since both sides of identity (5.4) are analytic functions of z, it would be enough to prove (5.4) for $|z|$ small enough. To this end, let us find suitable bounds for the Sneddon–Bessel series (4.17). Looking at (4.20), for each ν-compact set $K \subset \mathbb{C}\setminus\mathbb{Z}$, we have

$$\left| \frac{\Gamma(\nu+1) \binom{\alpha}{\nu} \Gamma(\alpha-\nu+1) \Gamma(\nu-n) x^{\alpha+2n-2\nu}}{2^{\alpha-2\nu+2+2n} \Gamma(\alpha+1) \Gamma(n+1+\alpha-\nu)} \right| \leq c_{\alpha,K}, \tag{5.6}$$

with a positive constant $c_{\alpha,K}$ depending only on α and the ν-compact K; this follows from the fact that $0 < x \leq 2$ and

$$\Gamma(\nu-n) = \frac{\Gamma(\nu+1)}{\nu(\nu-1)\cdots(\nu-n)},$$
$$\Gamma(n+1+\alpha-\nu) = (n+\alpha-\nu)\cdots(\alpha-\nu+2)(\alpha-\nu+1)\Gamma(\alpha-\nu+1).$$

Now, let us consider the analytic function

$$F_{j,\nu}(z) = \frac{z^{2j}}{\Phi_\nu(z)^2},$$

for each $j \geq 0$. There exists some constant C_K depending on the ν-compact K such that

$$|F_{j,\nu}(z)| \leq C_K,$$

on the circle $|z| = 1/2$. Then, Cauchy's integral formula gives

$$\left| \frac{F_{j,\nu}^{(2n)}(0)}{(2n)!} \right| \leq C_K 2^{2n}.$$

Using again Cauchy's integral formula, together with (4.19) and (4.18), it follows that

$$|A_{j,n}^{\alpha;\nu}| = \left| \frac{\Phi_\alpha^{(2j)}(0) F_{j,\nu}^{(2n)}(0)}{(2j)!(2n)!} \right| \leq d_\alpha C_K 2^{4n}. \tag{5.7}$$

Inserting estimates (5.6) and (5.7) in (4.20) proves that

$$|S_n^{\alpha;\nu}(x)| \leq e_{\alpha,K} 2^{6n}, \tag{5.8}$$

where $e_{\alpha,K}$ is a constant depending only on α and the ν-compact K. Now, the power series expansion of $(j_{m,\nu}^2 - z^2)^{-1}$ and the definition (4.17) lead to

$$\sum_{m=1}^{\infty} \frac{j_{m,\nu}^{2\nu-\alpha} J_\alpha(x j_{m,\nu})}{(j_{m,\nu}^2 - z^2) J_{\nu+1}(j_{m,\nu})^2} = \sum_{n=0}^{\infty} S_n^{\alpha;\nu}(x) z^{2n}. \tag{5.9}$$

To be precise, estimate (5.8) shows that if $2\,\mathrm{Re}\;\nu < \mathrm{Re}\;\alpha + 1/2$, $\alpha - \nu + 1 \neq 0, -1, -2, \dots$, $\nu \notin \mathbb{Z}$, $0 < x \leq 2$, and $|z| < 1/2^3$, identity (5.9) holds and the right-hand side is an analytic function of ν. Since we have already proved that identity (5.5) for $-1 < \mathrm{Re}\;\nu < 1/2$ and its right-hand side is also an analytic function of ν, we deduce that identity (5.5) holds indeed for $2\,\mathrm{Re}\;\nu < \mathrm{Re}\;\alpha + 1/2$, $\alpha - \nu + 1 \neq 0, -1, -2, \dots$, $\nu \notin \mathbb{Z}$, $0 < x \leq 2$.

6 | CONCLUSION

We have summed in a closed form the Sneddon–Bessel series

$$\sum_{m=1}^{\infty} \frac{J_\alpha(x j_{m,\nu}) J_\beta(y j_{m,\nu})}{j_{m,\nu}^{2n+\alpha+\beta-2\nu+2} J_{\nu+1}(j_{m,\nu})^2},$$

where $0 < x$, $0 < y$, $x + y < 2$, n is an integer, $\alpha, \beta, \nu \in \mathbb{C} \setminus \{-1, -2, \dots\}$ with $2\,\mathrm{Re}\;\nu < 2n + 1 + \mathrm{Re}\;\alpha + \mathrm{Re}\;\beta$.

We have also given closed expressions, for some range of the parameters and the variables, for series like

$$\sum_{m=1}^{\infty} \frac{j_{m,\nu}^{\nu-\beta} J_\nu(x j_{m,\nu}) J_\beta(y j_{m,\nu})}{(j_{m,\nu}^2 - z^2) J_{\nu+1}(j_{m,\nu})^2}$$

and also, as an application, for the series

$$\sum_{m=1}^{\infty} \frac{j_{m,\nu}^{2\nu-\alpha} J_\alpha(x j_{m,\nu})}{(j_{m,\nu}^2 - z^2) J_{\nu+1}(j_{m,\nu})^2}$$

which constitutes an extension of the Kneser–Sommerfeld expansion.

There are some interesting problems for a future work in this field. For instance, would it be possible to extend the Sneddon–Bessel series to more than two variables? For the simple case $n = 0$, the problem consists in finding the explicit sum of the series

$$\sum_{m=1}^{\infty} \frac{\prod_{i=1}^{k} J_{\alpha_i}(x_i j_{m,\nu})}{j_{m,\nu}^{-2\nu+2+\sum_{i=1}^{k} \alpha_i} J_{\nu+1}(j_{m,\nu})^2}$$

(see (4.16) for the case $k = 2$).

Taking into account our extension (1.8) of the Kneser–Sommerfeld expansion (1.2), we would like to mention also that it remains as a challenge to find explicitly the sum of the series

$$\sum_{m=1}^{\infty} \frac{j_{m,\nu}^{2\nu-\alpha-\beta} J_\alpha(x j_{m,\nu}) J_\beta(y j_{m,\nu})}{(j_{m,\nu}^2 - z^2) J_{\nu+1}(j_{m,\nu})^2}.$$

CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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