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Efficient numerical methods for semilinear one dimensional parabolic singularly perturbed convection-diffusion systems

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Abstract

In this work we deal with the numerical solution of one dimensional semilinear parabolic singularly perturbed systems of convection-diffusion type. We assume that the coupling in the convection terms is weak and also that the coupling reaction terms are nonlinear. In the case of considering different small diffusion parameters at each equation with different orders of magnitude, the exact solution usually shows overlapping boundary layers on the outflow of the spatial domain. To approximate it, we construct a numerical scheme which combines the upwind finite difference scheme, defined on a piecewise uniform Shishkin mesh, to discretize in space and a linearized version of the fractional implicit Euler method together with an appropriate splitting by components to discretize in time. Then, the fully discrete method is uniformly convergent with respect to both diffusion parameters, having first order in time and almost first order in space. The choice of this time integrator provokes that only tridiagonal linear systems must be solved at each time step; in this way, the computational cost of the algorithm is considerably lower than this one associated to classical implicit methods. The numerical results obtained for different test problems corroborate in practice the good performance of the numerical algorithm.

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1 Introduction

In this work, a reliable and highly efficient numerical algorithm is proposed to solve one dimensional semilinear parabolic singularly perturbed coupled convection-diffusion

systems. Such initial and boundary value singularly perturbed problems are given by

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t}(x,t) + \mathcal{L}_{x,\boldsymbol{\varepsilon}}(t)\mathbf{u}(x,t) + \mathcal{A}(x,t,\mathbf{u}) = \mathbf{0}, \ (x,t) \in Q, \\ \mathbf{u}(0,t) = \mathbf{g}_0(t), \mathbf{u}(1,t) = \mathbf{g}_1(t), \ t \in [0,T], \ \mathbf{u}(x,0) = \boldsymbol{\varphi}(x), \ x \in (0,1), \end{cases} \tag{1}$$

where $Q \equiv (0,1) \times (0,T]$, the spatial differential operator $\mathcal{L}_{x,\boldsymbol{\varepsilon}}(t)$ is defined as

$$\mathcal{L}_{x,\boldsymbol{\varepsilon}}(t)\mathbf{u} \equiv -\mathcal{D}_{\boldsymbol{\varepsilon}} \frac{\partial^2 \mathbf{u}}{\partial x^2} + \mathcal{B}(x) \frac{\partial \mathbf{u}}{\partial x}, \tag{2}$$

with $\mathbf{u} = (u_1, u_2)^T$, the diffusion matrix is $\mathcal{D}_{\boldsymbol{\varepsilon}} = \mathrm{diag}(\varepsilon_1, \varepsilon_2)$, the convection matrix is $\mathcal{B} = \mathrm{diag}(b_{11}, b_{22})$, the nonlinear reaction term is $\mathcal{A} = (a_1, a_2)^T$, the boundary conditions are $\mathbf{g}_0 = (g_{1,0}, g_{2,0})^T$, $\mathbf{g}_1 = (g_{1,1}, g_{2,1})^T$ and the initial condition is $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)^T$.

We suppose that the diffusion parameters ε_k, $k = 1, 2$, can be very small and, in general, they can have different order of magnitude. We denote by $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)^T$; henceforth, we assume that the equations of (1) have been ordered in such a way that $0 < \varepsilon_1 \leq \varepsilon_2 \leq 1$. Moreover, we assume that the convection terms satisfy

$$b_{kk}(x) \geq \ \beta > 0, \ \forall x \in [0,1], \ k = 1, 2, \tag{3}$$

and the reaction term $\mathcal{A}$ satisfies

$$\begin{aligned} &L \geq \frac{\partial a_k}{\partial u_k}(x,t,\mathbf{v}) \geq 0, \ -L \leq \frac{\partial a_k}{\partial u_j}(x,t,\mathbf{v}) \leq 0, \ k \neq j, \ k,j = 1,2, \ \forall \ (x,t,\mathbf{v}) \in \overline{Q} \times \mathbf{R}^2 \ \text{ and} \\ &\min_{\mathbf{v} \in \mathbb{R}^2} \sum_{j=1}^{n} \frac{\partial a_k}{\partial u_j}(x,t,\mathbf{v}) \geq 0, \ k = 1,2, \ \forall \ (x,t) \in \overline{Q}. \end{aligned} \tag{4}$$

As well, we assume sufficient smoothness on all of the data of problem (1) and sufficient compatibility conditions among them (see [21]), in order to the exact solution holds

$$\mathbf{u} \in C^{4,2}(\overline{Q}). \tag{5}$$

Many physical phenomena can be modeled by singularly perturbed systems; some well known areas where this type of problems appear are (for instance) saturated flow in fractured porous media, convective heat transport with large Péclet number, reaction-diffusion enzyme model, turbulent interactions of waves and currents, tubular models in chemical reactor theory, combustion processes, diffusion processes in electro-analytic chemistry or neutron transport models (see [2,17,26,29]). In the case of convection-diffusion systems, when the diffusion parameters take sufficiently small values. in general, the exact solution has boundary layers on the outflow boundary of the spatial domain. So, to obtain good numerical approximations, uniformly convergent methods are mandatory, i.e., methods for which the numerical approximation is reliable for any value of the diffusion parameters.

In the context of singular perturbation problems, systems of this type are one of the most popular topics in the literature of the last years. There exist many works for this type

of problems for both reaction-diffusion and convection-diffusion systems. For instance, in [4,5,10,14–16,20,22–25,27,28], linear elliptic or parabolic, one or two dimensional in space, coupled systems were analyzed. In all of these works, different uniformly convergent numerical methods were proposed and analyzed. However, there are few works where semilinear coupled systems have been considered. We can mention [6,19], where a scalar one dimensional semilinear elliptic problem of reaction-diffusion type was studied. In [7], the case of a scalar three dimensional semilinear elliptic reaction-diffusion problem was analyzed. Again, in these works, uniformly convergent methods were constructed for the corresponding systems.

To our knowledge, semilinear singularly perturbed systems was firstly considered in [3], where a nonlinear finite difference scheme was defined and a monotone iterative method was used to obtain the numerical approximation; in that work, a first order in time and almost first order in space uniformly convergent method was defined. It is well known that the use of classical implicit time integrators requires to solve nonlinear systems to advance in time. Typically, such systems are tackled via Newton iterative processes, which enlarge the computational cost of the algorithm. This increasing becomes more and more relevant as long as the number of components in the system increases; in such cases, the linear systems involved in the iterative processes have longer bandwidths. To avoid this drawback, we propose a linearly implicit time integrator which, besides, requires to solve only tridiagonal systems, independently of the number of equations in the system, to advance in time. In this way we obtain a numerical algorithm which has a computational cost comparable to a classical explicit method which, besides, has the main advantages of robustness of the classical implicit methods (specially their unconditional convergence). From the point of view of the numerical analysis, this paper represents an extension of the techniques introduced in [11], where another linearly implicit method was introduced to solve efficiently semilinear reaction-diffusion systems.

The paper is structured as follows. In Section 2, we give the asymptotic behavior of the exact solution of the continuous problem and also we prove appropriate estimates for its partial derivatives. In section 3, we construct the spatial discretization of (1) by using the simple upwind finite difference scheme; if this scheme is defined on an adequate piecewise uniform Shishkin mesh then the resulting method is uniformly convergent of almost first order. In Section 4, we describe the time integration process, which combines a locally linearized fractional implicit Euler method joint to a splitting by components of the discrete convection-diffusion operator. We prove that this integration process is uniformly convergent of first order. Combining the results obtained for both discretizations, we deduce that the proposed numerical algorithm is uniformly convergent with respect to the diffusion parameters ε_1 and ε_2. In Section 5, we show the numerical results obtained for some test problems. Finally, in Section 6, some conclusions and remarks are included.

Henceforth, C denotes a generic positive constant independent of the diffusion parameters ε_k, $k = 1, 2$, and also of the discretization parameters N and M. As well, $\mathbf{v} \leq \mathbf{w}$ will mean that $v_k \leq w_k$, $k = 1, 2$, $|\mathbf{v}| = (|v_1|, |v_2|)^T$ and $\|\mathbf{f}\|_G = \max\{\|f_1\|_G, \|f_2\|_G\}$, where $\|f\|_G$ is the maximum norm of f on the closed set G; we will use $\mathbf{v} \leq \mathbf{C}$ meaning that

$v_k \leq C,\ k = 1, 2.$

2 Qualitative properties of the exact solution

In this section we study the asymptotic behavior of the solution of (1), and also we prove appropriate estimates for its first partial derivatives, which will be useful in the analysis of the uniform convergence of the numerical method introduced later on.

Following the ideas in [3,11], we use firstly the mean-value theorem to deduce

$$a_k(x,t,\mathbf{u}) = a_k(x,t,\mathbf{0}) + \sum_{j=1}^{2} \frac{\partial a_k}{\partial u_j}(x,t,\mathbf{v_k})u_j,\ k = 1, 2. \tag{6}$$

Using (6), it is immediate that $\mathbf{u}$ can be described as the solution of the following linear convection-diffusion system

$$\begin{cases} \dfrac{\partial u_1}{\partial t}(x,t) - \varepsilon_1 \dfrac{\partial^2 u_1}{\partial x^2}(x,t) + b_{11}\dfrac{\partial u_1}{\partial x}(x,t) + \dfrac{\partial a_1}{\partial u_1}(x,t,\mathbf{v_1})u_1 + \dfrac{\partial a_1}{\partial u_2}(x,t,\mathbf{v_1})u_2 = -a_1(x,t,\mathbf{0}), \\ \dfrac{\partial u_2}{\partial t}(x,t) - \varepsilon_2 \dfrac{\partial^2 u_2}{\partial x^2}(x,t) + b_{22}\dfrac{\partial u_2}{\partial x}(x,t)\dfrac{\partial a_2}{\partial u_1}(x,t,\mathbf{v_2})u_1 + \dfrac{\partial a_2}{\partial u_2}(x,t,\mathbf{v_2})u_2 = -a_2(x,t,\mathbf{0}), \\ \mathbf{u}(0,t) = \mathbf{g_0}(t),\ \mathbf{u}(1,t) = \mathbf{g_1}(t),\ \forall\ t \in [0,T],\ \mathbf{u}(x,0) = \boldsymbol{\varphi}(x),\ \forall\ x \in \overline{\Omega}. \end{cases} \tag{7}$$

Taking now into account the hypotheses (34) and (4), by following similar ideas to the used ones in [9,11] for diffusion reaction nonlinear systems, the next two results are deduced for the convection-diffusion case.

Lemma 1. *(Inverse positivity). If* $-\mathbf{A}(x,t,\mathbf{0}), \mathbf{g_0}(t), \mathbf{g_1}(t), \boldsymbol{\varphi}(x)$ *have non-negative components, then* $\mathbf{u}(x,t)$ *has non-negative components.*

Lemma 2. *(Uniform bounding). The exact solution of (1) satisfies*

$$\|\mathbf{u}\|_{\overline{Q}} \leq C_0, \tag{8}$$

being

$$C_0 \equiv \max\{\|\boldsymbol{\varphi}\|_{[0,1]}, \|\mathbf{g}_0\|_{[0,T]}, \|\mathbf{g}_1\|_{[0,T]}\} + \frac{\|\mathbf{A}(x,t,\mathbf{0})\|_{\overline{Q}}}{\beta}. \tag{9}$$

Analogously, under the same smoothness and compatibility requirements assumed to assure (5), the first two time derivatives of $\mathbf{u}$ can be described as solution of suitable linear parabolic systems of convection-diffusion type and, using again Lemma 2, we deduce

$$\left\| \frac{\partial^l \mathbf{u}}{\partial t^l} \right\|_{\overline{Q}} \leq C,\quad l = 1, 2. \tag{10}$$

To prove appropriate estimates for the remaining first order partial derivatives of the

exact solution, which will be required later on, we follow our ideas exposed in [12] for linear convection-diffusion systems and we decompose the solution of (7) as $\mathbf{u} = \mathbf{v} + \mathbf{w}$ where $\mathbf{v}$ is the regular component and $\mathbf{w}$ is the singular component (such components are defined in detail in [12]). Let us denote $\mathbf{v} = (v_1, v_2)^T$, $\mathbf{w} = (w_1, w_2)^T$ and

$$B_\gamma(x) = e^{-\beta(1-x)/\gamma}, \tag{11}$$

where γ is any positive constant and β is defined in (34). Then, from [12], it follows

$$\begin{array}{l} \left|\dfrac{\partial^r v_k}{\partial t^r}\right| \le C,\ r = 0,1,2,\ \left|\dfrac{\partial^l v_k}{\partial x^l}\right| \le C,\ l = 1,2,\ \left|\dfrac{\partial^2 v_k}{\partial t \partial x}\right| \le C,\ \left|\dfrac{\partial^3 v_k}{\partial t \partial x^2}\right| \le C,\ k = 1,2, \\ \left|\dfrac{\partial^3 v_1}{\partial x^3}\right| \le C\varepsilon_1^{-1},\ \left|\dfrac{\partial^3 v_2}{\partial x^3}\right| \le C\varepsilon_2^{-1}, \end{array} \tag{12}$$

and

$$\left|\frac{\partial^r w_k}{\partial t^r}\right| \le C B_{\varepsilon_2}(x),\ r = 0,1,2,\ k = 1,2, \tag{13}$$

$$\left|\frac{\partial^l w_1}{\partial x^l}\right| \le C\left(\varepsilon_1^{-l} B_{\varepsilon_1}(x) + \varepsilon_2^{-l+1} B_{\varepsilon_2}(x)\right),\ l = 1,2,3, \tag{14}$$

$$\left|\frac{\partial^l w_2}{\partial x^l}\right| \le C\varepsilon_2^{-l} B_{\varepsilon_2}(x),\ l = 1,2, \tag{15}$$

$$\left|\frac{\partial^3 w_2}{\partial x^3}\right| \le C\varepsilon_2^{-1}\left(\varepsilon_1^{-1} B_{\varepsilon_1}(x) + \varepsilon_2^{-2} B_{\varepsilon_2}(x)\right). \tag{16}$$

These bounds describe precisely the asymptotic behavior of the exact solution with respect to the diffusion parameters ε_1 and ε_2, showing the character of the overlapping boundary layer which appears near the outflow boundary $x = 1$.

3 The spatial finite difference scheme

To discretize in space problem (1), we use the classical upwind finite difference scheme. From previous section we know that the exact solution has overlapping regular boundary layers at $x = 1$. The first step to define an efficient numerical method is the construction of a special nonuniform mesh, which concentrates the grid points in the boundary layer region. Here, we use a piecewise uniform mesh of Shishkin type, $\overline{\Omega}^N \equiv \{0 = x_0 < x_1 < \ldots < x_N = 1\}$, which is defined as follows (see [23]). Let N be a positive integer multiple of 3; we define the transition parameters

$$\sigma_{\varepsilon_2} = \min\{2/3, \sigma_0 \varepsilon_2 \ln N\}, \quad \sigma_{\varepsilon_1} = \min\{\sigma_{\varepsilon_2}/2, \sigma_0 \varepsilon_1 \ln N\}, \tag{17}$$

which separate the coarse, the fine and the finest parts of the mesh, where σ_0 is a constant to be fixed later on. Then, the grid points are given by

$$x_i = \begin{cases} iH, & i = 0, \ldots, N/3, \\ x_{N/3} + (i - N/3)h_{\varepsilon_2}, & i = N/3+1, \ldots, 2N/3, \\ x_{2N/3} + (i - 2N/3)h_{\varepsilon_1}, & i = 2N/3+1, \ldots, N, \end{cases} \tag{18}$$

with $H = 3(1-\sigma_{\varepsilon_2})/N,\ h_{\varepsilon_2} = 3(\sigma_{\varepsilon_2} - \sigma_{\varepsilon_1})/N,\ h_{\varepsilon_1} = 3\sigma_{\varepsilon_1}/N.$

From now on, let us denote by $h_i = x_i - x_{i-1},\ i = 1, \ldots, N$, and $\overline{h}_i = (h_i + h_{i+1})/2,\ i = 1, \ldots, N-1$; let us denote by Ω^N the subgrid of $\overline{\Omega}^N$ composed only by the interior points of it, i.e., by $\overline{\Omega}^N \bigcap \Omega$, and let us denote by $[\mathbf{v}]_{\Omega^N}$ (analogously $[v]_{\Omega^N}$ for scalar functions) the restriction operators, applied to vector functions defined on Ω, to the mesh Ω^N.

For all $x_i \in \Omega^N$, we introduce the semidiscretization approach $\mathbf{U}^N(t) \equiv (\mathbf{U}_i^N(t)),\ i = 1, \ldots N-1$, with $\mathbf{U}_i^N(t) \equiv (U_{1,i}, U_{2,i})^T \approx \mathbf{u}(x_i, t)$, as the solution of the following Initial Value Problem

$$\begin{cases} \dfrac{d\mathbf{U}^N}{dt}(t) + \mathcal{L}_{\boldsymbol{\varepsilon}}^N \overline{\mathbf{U}}^N(t) + \mathcal{A}^N(t, \mathbf{U}^N(t)) = \mathbf{0}^N, \text{ in } \Omega^N \times [0,T], \\ \mathbf{U}_0^N(t) = \mathbf{g}_0(t), \text{in } [0,T], \\ \mathbf{U}_N^N(t) = \mathbf{g}_1(t), \text{in } [0,T], \\ \mathbf{U}^N(0) = [\boldsymbol{\varphi}(x)]_{\Omega^N}, \end{cases} \tag{19}$$

where $\overline{\mathbf{U}}^N(t)$ is the natural extension to $\overline{\Omega}^N \times [0,T]$ of the semidiscrete functions $\mathbf{U}^N(t)$, defined on $\Omega^N \times [0,T]$, by adding the corresponding boundary data, $\mathcal{L}_{\boldsymbol{\varepsilon}}^N = (\mathcal{L}_{N,\boldsymbol{\varepsilon},1}, \mathcal{L}_{N,\boldsymbol{\varepsilon},2})^T$ where

$$\mathcal{L}_{N,\boldsymbol{\varepsilon},k}(\mathbf{U}_N(t))_i \equiv -\varepsilon_k(\delta_{xx}U_{N,k})_i + b_{kk}D_x^-(U_{N,k})_i =$$

$$-\frac{\varepsilon_k}{\overline{h}_i}\left(\frac{U_{N,i+1,k} - U_{N,i,k}}{h_{i+1}} - \frac{U_{N,i,k} - U_{N,i-1,k}}{h_i}\right) + b_{kk}\frac{U_{N,i,k} - U_{N,i-1,k}}{h_i},\ i = 1, \ldots N-1,\ k = 1, 2,$$

is the discretization of the operator $\mathcal{L}_{x,\boldsymbol{\varepsilon}}(t)$ by using the simple upwind scheme. It is well known that this scheme can be rewritten as

$$(\mathcal{L}_{\boldsymbol{\varepsilon}}^N \overline{\mathbf{U}}^N)_{k,i} = r_{k,i}^- U_{k,i-1}^N + r_{k,i}^+ U_{k,i+1}^N + r_{k,i}^c U_{k,i}^N, \tag{20}$$

with

$$r_{k,i}^- = \frac{-\varepsilon_k}{h_i\overline{h}_i} - \frac{b_{kk}(x_i)}{h_i},\ r_{k,i}^+ = \frac{-\varepsilon_k}{h_{i+1}\overline{h}_i},\ r_{k,i}^c = -(r_{k,i}^- + r_{k,i}^+), \tag{21}$$

and

$$(\mathcal{A}^N(t, \mathbf{U}^N))_{k,i} = a_k(x_i, t, \mathbf{U}_i^N), \tag{22}$$

for $k = 1, 2$ and $i = 1, \ldots, N-1$.

To deduce some of the theoretical properties of this stiff problem, we reproduce similar reasonings to the used ones in previous section to study the continuous problem (1). Firstly, we rewrite the nonlinear reaction terms in the equivalent form given by (6); in this way, problem (19) can rewritten as a stiff linear system and we deduce for it the next result, which is the discrete analogue of Lemma 1 (see [9,11] for details).

Lemma 3. *Assuming that all of the data* $(-\mathbf{A}(x_i, t, \mathbf{0}), \mathbf{g_0}(t), \mathbf{g_1}(t), \boldsymbol{\varphi}(x_i)),\ i = 0, \ldots, N$, *of semidiscrete problem have non-negative values in their components then, the components of* $\mathbf{U}_N(t)$ *have only non-negative values.*

From this result, using adequate discrete barrier function techniques, it is straightforward to deduce that

$$\begin{array}{l} \|\mathbf{U}_N(t)\|_{\overline{\Omega}_N} \leq \\ \max\left\{\|\mathbf{g_0}(t)\|, \|\mathbf{g_1}(t)\|, \|[\boldsymbol{\varphi}(x)]_N\|_{\overline{\Omega}_N},\ \dfrac{\|[-\mathbf{A}(x,t,\mathbf{0})]_N\|_{\Omega_N\times[0,T]}}{\beta}\right\},\ \forall t \in [0,T]. \end{array} \tag{23}$$

This bound can be viewed as a uniform stability property of the spatial discretization process.

To study the uniform convergence of the spatial discretization, first we define the local truncation error at any time $t \in [0, T]$, at the grid point $x_i \in \overline{\Omega}_N,\ i = 1, \ldots, N-1$, which is given by

$$\upsilon_{N,i}(t) \equiv \mathcal{L}_{\boldsymbol{\varepsilon}}(\mathbf{u})(x_i, t) - (\mathcal{L}_{\boldsymbol{\varepsilon},N}([\mathbf{u}(x,t)]_N))_i. \tag{24}$$

Clearly the contributions of the time derivatives and the reaction terms in the local error are zero; therefore, it holds

$$\upsilon_{N,i}(t) = -\mathcal{D}_\varepsilon\left(\frac{\partial^2 \mathbf{u}}{\partial x^2}(x_i) - (\delta_{xx}[\mathbf{u}]_N)_i\right) + \mathcal{B}(x)\left(\frac{\partial \mathbf{u}}{\partial x} - (D_x^-[\mathbf{u}]_N)_i\right).$$

A detailed analysis of this local truncation error, depending on the value the transition parameters σ_{ε_1} and σ_{ε_2}, and the location of the grid points, can be seen in [12], where a linear version of problem (1) is studied. In that paper, the classical technique developed in the seminal paper [18] is used.

To complete the analysis of the uniform convergence of (19), we follow a similar technique as in [11], where a semilinear parabolic coupled system of reaction-diffusion was considered. The main idea consists of rewriting the global error for it, given by

$$\mathbf{e^{sp}}_N(t) \equiv [\mathbf{u}(x,t)]_N - \mathbf{U}_N(t),$$

as the solution of a linearized version of (19), where the boundary conditions are zero and the source term is the local truncation error. Then, using the same technique as in [12] we prove the main result for the spatial discretization process.

Theorem 1. *The global error* $\mathbf{e^{sp}}_N(t) \equiv [\mathbf{u}(x,t)]_N - \mathbf{U}_N(t)$*, associated to the spatial discretization (20)-(22) on the Shishkin mesh given in (18), satisfies*

$$\|\mathbf{e^{sp}}_N(t)\|_{\overline{\Omega}_N} \leq CN^{-1}\ln N, \quad \forall\ t \in [0,T]. \tag{25}$$

Therefore, the solutions of the spatial semidiscrete problems (19) converge uniformly to the solution of (1) with almost first order.

4 The fully discrete scheme: uniform convergence

In this section we describe the numerical algorithm proposed to solve efficiently the continuous problem (1). Such method is going to be obtained by applying an appropriate one step time integrator, which consists of three fractional steps, to the semidiscrete problems (19).

Let us consider, for simplicity, a constant time step $\tau = T/M$; let us denote by $\mathbf{U}^{N,m} \equiv (\mathbf{U}_i^{N,m}), i = 1,\dots N-1$, where $\mathbf{U}_i^{N,m} \equiv (U_{1,i}^{N,m}, U_{2,i}^{N,m})^T$ the numerical approaches of $\mathbf{u}(x_i,t_m)$ to be computed with our method, for $i = 1,\dots,N-1$, being $t_m = m\tau$, for $m = 0,1,\dots,M$, and let us denote $\overline{\mathbf{U}}^{N,m} \equiv (\mathbf{U}_i^{N,m}), i = 0,\dots N$. Then, our fully discrete

scheme is given by

$$
\begin{array}{l}
\text{Initialize} \\
\overline{\mathbf{U}}^{N,0} = [\boldsymbol{\varphi}]_{\overline{\Omega}^N}, \\
\left\{ \begin{array}{l}
\text{For } m = 0, 1, \ldots, M-1, \\
\text{First fractional step} \\
\mathbf{U}_0^{N,m+1/3} = \mathbf{U}_0^{N,m} \\
\dfrac{\mathbf{U}^{N,m+1/3} - \mathbf{U}^{N,m}}{\tau} + \mathcal{A}^N(t_m, \mathbf{U}^{N,m}) = \mathbf{0}^N, \quad \text{in } \Omega^N \\
\mathbf{U}_N^{N,m+1/3} = \mathbf{U}_N^{N,m} \\
\text{Second fractional step} \\
U_{2,i}^{N,m+2/3} = U_{2,i}^{N,m+1/3}, \ i = 0, \ldots, N, \\
\left\{ \begin{array}{l}
U_{1,0}^{N,m+2/3} = g_{1,0}(t_{m+1}), \\
\dfrac{U_{1,i}^{N,m+2/3} - U_{1,i}^{N,m+1/3}}{\tau} + (\mathcal{L}_{\boldsymbol{\varepsilon}}^N \overline{\mathbf{U}}^{N,m+2/3})_{1,i} = 0, i = 1, \ldots, N-1, \\
U_{1,N}^{N,m+2/3} = g_{1,1}(t_{m+1}),
\end{array} \right. \\
\text{Third fractional step} \\
U_{1,i}^{N,m+1} = U_{1,i}^{N,m+2/3}, \ i = 0, \ldots, N, \\
\left\{ \begin{array}{l}
U_{2,0}^{N,m+1} = g_{2,0}(t_{m+1}), \\
\dfrac{U_{2,i}^{N,m+1} - U_{2,i}^{N,m+2/3}}{\tau} + (\mathcal{L}_{\boldsymbol{\varepsilon}}^N \overline{\mathbf{U}}^{N,m+1})_{2,i} = 0, i = 1, \ldots, N-1, \\
U_{2,n}^{N,m+1} = g_{2,1}(t_{m+1}).
\end{array} \right.
\end{array} \right.
\end{array}
\tag{26}
$$

Notice that the first fractional step involves only explicit calculations via evaluations of the reaction term; to compute the other two fractional steps of (26), only tridiagonal linear systems must be solved. A consequence of this fact, is that this algorithm provides a remarkable reduction of the computational cost when we compare it with classical implicit methods. Such systems can be rewritten in the form

$$
(I + \tau A_k)\bar{U}_k^{N,m+(k+1)/3} = b_{k,m}, k = 1, 2, \tag{27}
$$

being $a_{k,0,0} = a_{k,N,N} = 0$, $a_{k,i,i} = r_{k,i}^c$, $a_{k,i,i+1} = r_{k,i}^+$, $a_{k,i,i-1} = r_{k,i}^-, i = 1, \ldots N-1$, $\bar{U}_k = (U_{k,i})i = 0, \ldots N$ and $b_{k,m,0} = g_{k,0}(t_{m+1}), b_{k,m,N} = g_{k,1}(t_{m+1}), b_{k,m,i} = U_k^{N,m+k/3}, i = 1, \ldots N-1$.

In the next four results, we give the main theoretical qualities of our time integrator. Finally, we end this section with the uniform convergence result which comprehends the main results obtained for the space semidiscretization and the time integration processes.

Lemma 4. *The matrices $I + \tau A_k$, involved in the linear systems (27), are tridiagonal, inverse positive and they hold*

$$\|(I + \tau A_k)^{-1}\|_\infty \leq 1, k = 1, 2. \tag{28}$$

This result is a direct consequence of the well-known inverse positivity of the simple upwind schemes; jointing it with the Lipschitz condition on the reaction term, which we have assumed in (4), it is easy to prove the following uniform stability result.

Theorem 2. *Let $\overline{\mathbf{U}}^{N,m}$ and $\hat{\overline{\mathbf{U}}}^{N,m}$ be two solutions obtained with the scheme (26), starting from two different initial conditions $[\boldsymbol{\varphi}]_{\overline{\Omega}^N}$ and $[\hat{\boldsymbol{\varphi}}]_{\overline{\Omega}^N}$, respectively. Then, it holds*

$$\|\overline{\mathbf{U}}^{N,m} - \hat{\overline{\mathbf{U}}}^{N,m}\|_{\overline{\Omega}^N} \leq (1 + \tau L)\|\overline{\mathbf{U}}^{N,m-1} - \hat{\overline{\mathbf{U}}}^{N,m-1}\|_{\overline{\Omega}^N}, m = 1, 2 \dots M.$$

To study the uniform consistency of (26), we introduce the concept of the local error, at $t = t_m$ for this time integration process, as the difference $\overline{\mathbf{U}}^N(t_m) - \tilde{\overline{\mathbf{U}}}^{N,m}$, where $\overline{\mathbf{U}}^N(t)$ is the solution of (19) and $\tilde{\overline{\mathbf{U}}}^{N,m}$ is the result of applying step $m-1$ of (26) when we substitute $\overline{\mathbf{U}}^{N,m-1}$ by $\overline{\mathbf{U}}^N(t_{m-1})$. Then, the following uniform consistency result can be proven, to estimate the contribution of one step of the numerical method (26) to the global error.

Theorem 3. *Under the smoothness assumptions (5) made on the solution of the problem (1), the local error associated to (26) satisfies*

$$\|\overline{\mathbf{U}}^N(t_m) - \tilde{\overline{\mathbf{U}}}^{N,m}\|_{\overline{\Omega}^N} \leq CM^{-2}, m = 1, 2, \dots M.$$

Proof. In order to prove that our proposal does not suffer the drawback of order reduction, we follow a non standard technique to that our method is consistent of first order. The order reduction phenomenon is a well known drawback which is present very often when one step methods are used to integrate initial-boundary value problems, (see for instance [8]) specially when time dependent boundary conditions are considered (for more details, see [1,8] and references therein).

Let us denote by $[\ \ ]_r$ the restriction of a discrete function defined on $\overline{\Omega}^N$ to Ω^N, let us denote by $[\ \ ]_e$ the extension, by adding zeros, of a function defined on Ω^N to $\overline{\Omega}^N$ and let us consider the following Taylor expansion for the semidiscrete solution

$$\mathbf{U}^N(t_{m-1}) = \mathbf{U}^N(t_m) - \tau\frac{d\mathbf{U}^N}{dt}(t_m) + \mathcal{O}(\tau^2) = \mathbf{U}^N(t_m) + \tau\left(\mathcal{L}_{\boldsymbol{\varepsilon}}^N\overline{\mathbf{U}}^N(t_m) + \mathcal{A}^N(t_m, \mathbf{U}^N(t_m))\right) + \mathcal{O}(\tau^2).$$

Then, taking into account the smoothness of the reaction terms, it is straightforward that it holds

$$\mathbf{U}^N(t_{m-1}) = \left[(I + \tau\mathcal{L}_{\boldsymbol{\varepsilon}}^{N,1})(I + \tau\mathcal{L}_{\boldsymbol{\varepsilon}}^{N,2})\overline{\mathbf{U}}^N(t_m)\right]_r + \mathcal{A}^N(t_{m-1}, \mathbf{U}^N(t_{m-1})) + \mathcal{O}(\tau^2),$$

where the linear discrete operators operators $\mathcal{L}_{\boldsymbol{\varepsilon}}^{N,1}$ and $\mathcal{L}_{\boldsymbol{\varepsilon}}^{N,2}$ concentrate the coefficients of the convection diffusion operator associated to the first and second components respectively; in other words, reordering the components of the discrete solution in the form $(U_{1,0}, U_{1,N}, U_{2.0} \dots U_{2,N})$, these matrices are given by

$$\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 0 & A_2 \end{pmatrix}.$$

On the other hand, it is clear that $\tilde{\mathbf{U}}^{N,m}$ satisfies

$$\left[(I + \tau\mathcal{L}_{\boldsymbol{\varepsilon}}^{N,1})(I + \tau\mathcal{L}_{\boldsymbol{\varepsilon}}^{N,2})\tilde{\overline{\mathbf{U}}}^{N,m}\right]_r = \mathbf{U}^N(t_{m-1}) - \tau\mathcal{A}^N(t_{m-1}, \mathbf{U}^N(t_{m-1})).$$

Subtracting now the last two equations, it is deduced that

$$\left[(I + \tau\mathcal{L}_{\boldsymbol{\varepsilon}}^{N,1})(I + \tau\mathcal{L}_{\boldsymbol{\varepsilon}}^{N,2})(\overline{\mathbf{U}}^N(t_m) - \tilde{\overline{\mathbf{U}}}^{N,m})\right]_r = \mathcal{O}(\tau^2),$$

and if we incorporate the boundary values of $\overline{\mathbf{U}}^N(t_m)$ and $\tilde{\overline{\mathbf{U}}}^{N,m}$, which are equal, it holds that

$$(I + \tau\mathcal{L}_{\boldsymbol{\varepsilon}}^{N,1})(I + \tau\mathcal{L}_{\boldsymbol{\varepsilon}}^{N,2})(\overline{\mathbf{U}}^N(t_m) - \tilde{\overline{\mathbf{U}}}^{N,m}) = \left[\mathcal{O}(\tau^2)\right]_e.$$

From this result, using now the stability bound (28), the required result follows. □

Now, a classical reasoning "uniform stability + uniform consistency ⇒ uniform convergence", permits us to obtain the following theorem, which proves the uniform convergence of the time discretization.

Theorem 4. *(Uniform convergence of the time integrator). Assuming that* $\mathbf{u} \in C^{4,2}(\overline{Q})$*, it holds that*

$$\|\mathbf{U}^{N,m} - \mathbf{U}^N(t_m)\|_{\overline{\Omega}^N} \leq CM^{-1}, \quad m = 1, \dots M,$$

being C *independent of* $\boldsymbol{\varepsilon}$*,* N *and* M*.*

Finally, jointing the results in Theorems 1 and 4, we deduce the main uniform convergence result of the work.

Theorem 5. *(Uniform convergence). Assuming that (34)-(5) hold, the global error associated to the numerical method defined by (26) on the Shishkin mesh given in (18), satisfies*

$$\max_{0 \leq m \leq M} \|\mathbf{U}^{N,m} - [\mathbf{u}(x, t_m)]_{\overline{\Omega}^N}\|_{\overline{\Omega}^N} \leq C\left(N^{-1} \ln N + M^{-1}\right). \tag{29}$$

Then, the numerical scheme (26) is uniformly convergent of first order in time and almost first order in space.

5 Numerical results

In this section, the numerical results obtained with our numerical algorithm for some test problems of type (1) are shown. All results have been obtained in a PC with an Intel(R) Core(TM) i7-10700 running at @ 2.90 GH processor with four cores. The computations have been performed in only one core using GNU Fortran with optimisation -O2. The tridiagonal linear systems involved in our method are solved by using our own implementation of the Thomas algorithm.

The data for the first test problem are given by

$$\begin{aligned} &T = 1,\ b_{11}(x) = 1 + \sin(\pi x),\ b_{22}(x) = 2 - e^{-x}, \\ &a_1(x,t,\mathbf{u}) = 3u_1 + u_1^3 - (1+u_1^2)u_2\ a_2(x,t,\mathbf{u}) = -(1+u_2^2)u_1 + 3u_2 + u_2^3, \\ &\boldsymbol{\varphi}(x) = (0,0)^T, \\ &\mathbf{g_0}(t) = (0,0)^T,\ \mathbf{g_1}(t) = (t^2(1-e^{-t}), \frac{1-\cos(\pi t)}{2})^T. \end{aligned} \tag{30}$$

Figure 1 displays the numerical approximation for both components, showing clearly the overlapping boundary layers at $x = 1$.

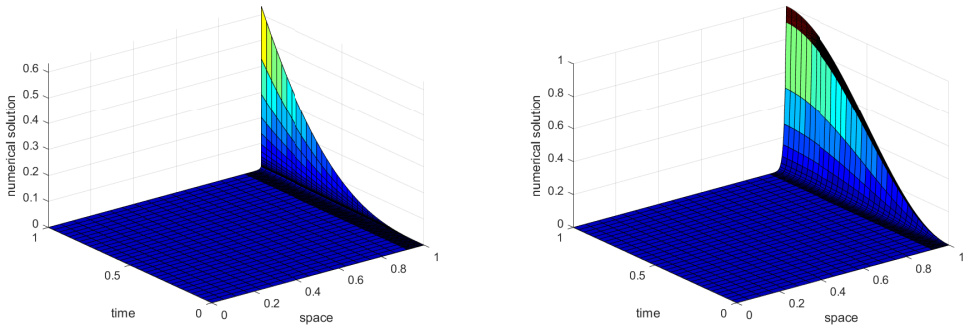

Fig. 1. Components 1 (left) and 2 (right) of problem (30) for $\varepsilon_1 = 10^{-4}$, $\varepsilon_2 = 10^{-2}$ with $N = 48, M = 32$

As the exact solution of this problem is unknown, to approximate the maximum errors for each component $u_k, k = 1, 2$, given by

$$\max_{0\le m\le M}\max_{0\le i\le N} |U^m_{N,i,k} - u_k(x_i, t_m)|, k = 1, 2,$$

we use the double-mesh principle (see [13] for instance), which calculates

$$d^{N,M}_{\boldsymbol{\varepsilon},k} = \max_{0\le m\le M}\max_{0\le i\le N} |U^m_{N,i,k} - \widehat{U}^{2m}_{2N,2i,k}|,\ d^{N,M}_k = \max_{\varepsilon} d^{N,M}_{\boldsymbol{\varepsilon},k}, k = 1, 2, \tag{31}$$

to approximate the errors, where $\{\widehat{\mathbf{U}}^m_{2N,j}\}$ is the numerical solution on a finer mesh $\{(\hat{x}_j, \hat{t}_m)\}$ which contains the mesh points of the coarse mesh and their midpoints both in space and time. From the double-mesh approximated errors we obtain the corresponding approximated orders of convergence by

$$p_k = \log(d^{N,M}_{\boldsymbol{\varepsilon},k}/d^{2N,2M}_{\boldsymbol{\varepsilon},k})/\log 2,\ p^{uni}_k = \log(d^{N,M}_k/d^{2N,2M}_k)/\log 2, k = 1, 2. \tag{32}$$

Table 1 shows maximum errors and their corresponding orders of convergence for some values of diffusion parameter ε_2, when ε_1 belong to the set $R = \{\varepsilon_1; \varepsilon_1 = \varepsilon_2, 2^{-2}\varepsilon_2, \ldots, 2^{-32}\}$ and for different values of the discretization parameters N taking $M = N/3$ and $\sigma_0 = 1$ in (17). For each value of ε_2 the first two rows show, for the first component u_1, the estimated errors and the numerical orders of convergence respectively; the following two rows show the same data for the second component u_2. From them, the almost first order of uniform convergence clearly arises in accordance with the theoretical results.

Table 1
Maximum errors and orders of convergence for the two components of the numerical solution of (30)

ε_2	N=24	N=48	N=96	N=192	N=384	N=768
	3.0256E.2	2.1333E.2	1.4043E.2	8.6613E.3	5.1229E.3	2.9346E-3
2^{-6}	0.5041	0.6032	0.6972	0.7576	0.8038	
	6.8773E.2	4.9271E.2	3.3570E.2	2.1563E.2	1.3009E.2	7.7038E-3
	0.4811	0.5536	0.6386	0.7290	0.7559	
	3.1385E.2	2.1430E.2	1.4098E.2	8.6522E.3	5.1130E.3	2.9824E-3
2^{-8}	0.5504	0.6041	0.7044	0.7589	0.7777	
	6.8301E.2	4.8387E.2	3.2984E.2	2.1264E.2	1.2820E.2	7.6109E-3
	0.4973	0.5528	0.6333	0.7300	0.7523	
	3.1879E.2	2.1549E.2	1.4114E.2	8.6506E.3	5.1109E.3	2.9231E-3
2^{-10}	0.5650	0.6105	0.7062	0.7592	0.8061	
	6.8158E.2	4.8165E.2	3.2848E.2	2.1195E.2	1.2784E.2	7.4223E-3
	0.5009	0.5522	0.6321	0.7294	0.7844	
	3.2018E.2	2.1614E.2	1.4138E.2	8.6823E.3	5.1104E.3	2.9528E-3
2^{-12}	0.5669	0.6124	0.7035	0.7646	0.7913	
	6.8122E.2	4.8109E.2	3.2800E.2	2.1230E.2	1.2765E.2	7.4565E-3
	0.5018	0.5526	0.6276	0.7339	0.7756	
...	...	...	...	...	...	...
...	...	...	...	...	...	...
	3.2056E.2	2.1637E.2	1.4153E.2	8.6657E.3	5.1188E.3	3.0566E-3
2^{-24}	0.5671	0.6124	0.7078	0.7595	0.7439	
	6.7245E.2	4.7160E.2	3.2445E.2	2.0924E.2	1.2546E.2	7.2725E-3
	0.5119	0.5396	0.6328	0.7379	0.7867	
$d^{N,M}_1$	3.2063E.2	2.1638E.2	1.4154E.2	8.6911E.3	5.1229E.3	3.0566E-3
p^{uni}_1	0.5673	0.6124	0.7036	0.7626	0.7450	
$d^{N,M}_2$	6.8773E.2	4.9271E.2	3.3570E.2	2.1563E.2	1.3009E.2	7.7038E-3
p^{uni}_2	0.4811	0.5536	0.6386	0.7290	0.7559	

An important improvement which our algorithm provides is related to its low computational cost; to observe this fact, we compare the CPU times derived of solving (30) with a classical method and our algorithm for some values of $N, M = N/3$ and fixed values of the diffusion parameters. The classical method combines the implicit Euler method to discretize in time, which is a well known option, and the simple upwind finite difference scheme defined on the same piecewise uniform Shishkin mesh as before. In this case, to obtain the numerical approximation of both components, a nonlinear system must be solved at each time step; for that, we use the Newton method to obtain a short sequence of approximations $\mathbf{U}_N^{m,k}$ to the numerical approach $\mathbf{U}_N^m$; this iterative process is ended when this following stopping criterion holds

$$\|\mathbf{U}_N^{m,k+1} - \mathbf{U}_N^{m,k}\| \leq 10^{-1} \min\{M^{-2}, M^{-1}N^{-1} \ln N\}, \tag{33}$$

In all cases, the initial iteration of the Newton's method is taken as $\mathbf{U}_N^{m,0} = \mathbf{U}_N^{m-1}$.

Table 2 shows the required CPU time in seconds using our algorithm and the classical method described previously. From it, we clearly see that our algorithm is considerably faster than the classical one as it was expected.

Table 2
CPU times for problem (30) with $\varepsilon_2 = 2^{-16}$ and $\varepsilon_1 = 2^{-20}$ using splitting and classical methods

	N=384	N=768	N=1536	N=3072	N=6144
splitting	0.04600	0.12500	0.46900	1.78100	7.19000
classical	0.67100	2.53100	10.01500	39.49900	157.28100

In this second test problem we have considered a more general problem; the convection matrix $\mathcal{B}$ depends also on the time variable, but it preserves the positivity of their convection coefficients, i.e.

$$b_{kk}(x,t) \geq \ \beta > 0, \ \forall (x,t) \in [0,1] \times (0,T], \ k = 1,2. \tag{34}$$

As well, in this example we have taken a reaction term which does not satisfy some of the bounds of (4) and we have weakened the compatibility conditions among data in order to $u \notin C^{4,2}(\overline{Q})$.

Concretely, the data of the problem are given by

$$\begin{aligned}
&T = 1, \ b_{11}(x) = 2 + \sin(x+t) - xt, \ b_{22}(x) = 3 - (x^2 + t^2) \\
&a_1(x,t,\mathbf{u}) = t^2(\cos(u_2) - u_2) + u_1^3 + 2u_1, \\
&a_2(x,t,\mathbf{u}) = te^{-u_1-u_2} + 2(e^{-u_1} + 1)u_2, \\
&\boldsymbol{\varphi}(x) = (0,0)^T, \\
&\mathbf{g_0}(t) = (t^2, 1 - \cos(\pi t))^T, \ \mathbf{g_1}(t) = (1 - \cos(\pi t), \sin(\pi t))^T;
\end{aligned} \tag{35}$$

and the exact solution of this test problem is unknown too.

Figure 2 displays the numerical approximation for both components, showing clearly the overlapping boundary layers at $x = 1$.

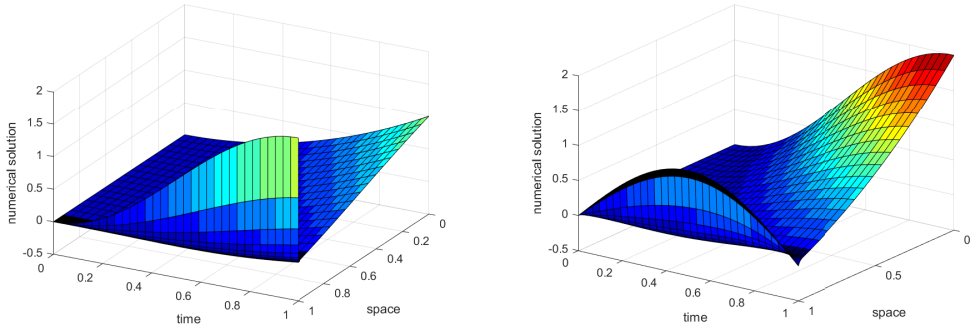

Fig. 2. Components 1 (left) and 2 (right) of problem (35) for $\varepsilon_1 = 10^{-4}$, $\varepsilon_2 = 10^{-2}$ with $N = 48, M = 32$

Table 3 shows the maximum errors and their corresponding orders of convergence for the same values that in example (30). From them, again we clearly observe the almost first order of uniform convergence. So, we can conclude that, in a more general case where $\mathcal{B}$ depends also on the time variable and with less restricted reaction terms, the numerical method gives similar good results. As well, we have observed that weakening the compatibility conditions among data in such way that $\mathbf{u} \in C^{2,1}(\overline{Q}) \cap C^{4,2}(Q)$, we obtain good numerical results too.

In the next third test problem, we have chosen a system with three equations, given by

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \varepsilon_1 \dfrac{\partial^2 u_1}{\partial x^2} + (1+x)\dfrac{\partial u_1}{\partial x} + u_1 + t(e^{3u_1 - 2u_2 - u_3} - 2\sin(\pi x) - 1) = 0, \\ \dfrac{\partial u_2}{\partial t} - \varepsilon_2 \dfrac{\partial^2 u_2}{\partial x^2} + (1+\cos(x))\dfrac{\partial u_2}{\partial x} + 2u_2 + \sin(-u_1 + 2u_2 - u_3) = 0, \\ \dfrac{\partial u_3}{\partial t} - \varepsilon_3 \dfrac{\partial^2 u_3}{\partial x^2} + (1+e^{-x})\dfrac{\partial u_3}{\partial x} + 3u_3 - u_2 - u_1 + e^{-u_3^2} - 1 = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x,t) \in (0,1) \times (0,1], \\ \mathbf{u}(0,t) = \mathbf{u}(1,t) = (0,0,0)^T, \ t \in (0,1], \\ \mathbf{u}(x,0) = (0,0,0)^T, \ x \in [0,1], \end{cases} \tag{36}$$

whose exact solution is again unknown.

Figure 3 displays the numerical approximation for the three components, showing again the overlapping boundary layers at $x = 1$.

According to the character of its boundary layer, in this case, the Shishkin mesh $\overline{\Omega}^N$ is

Table 3
Maximum errors and orders of convergence for the two components of the solution of (35)

ε_2	N=24	N=48	N=96	N=192	N=384	N=768
	4.3007E-1	2.6866E-1	1.5620E-1	9.0131E-2	5.1715E-2	2.9287E-2
2^{-6}	0.6788	0.7824	0.7933	0.8014	0.8203	
	1.2228E+0	9.1953E-1	5.5604E-1	3.0543E-1	1.6253E-1	8.5433E-2
	0.4112	0.7257	0.8643	0.9101	0.9279	
	4.3726E-1	2.6465E-1	1.5547E-1	9.1877E-2	5.3244E-2	3.0260E-2
2^{-8}	0.7244	0.7675	0.7588	0.7871	0.8152	
	1.3108E+0	1.0205E+00	6.2328E-1	3.3861E-1	1.7790E-1	9.2558E-2
	0.3611	0.7114	0.8803	0.9286	0.9426	
	4.3899E-1	2.6351E-1	1.5618E-1	9.3138E-2	5.4077E-2	3.0437E-2
2^{-10}	0.7363	0.7547	0.7458	0.7843	0.8292	
	1.3360E+0	1.0538E+00	6.4647E-1	3.4988E-1	1.8280E-1	9.4705E-2
	0.3423	0.7049	0.8857	0.9366	0.9488	
	4.3942E-1	2.6322E-1	1.5643E-1	9.3599E-2	5.4304E-2	3.0618E-2
2^{-12}	0.7393	0.7508	0.7410	0.7854	0.8267	
	1.3420E+0	1.0626E+00	6.5289E-1	3.5300E-1	1.8409E-1	9.5255E-2
	0.3368	0.7026	0.8872	0.9393	0.9506	
...	...	...	...	...	...	...
...	...	...	...	...	...	...
	4.3954E-1	2.6313E-1	1.5653E-1	9.3684E-2	5.4399E-2	3.0902E-2
2^{-24}	0.7403	0.7493	0.7406	0.7842	0.8159	
	1.3436E+0	1.0651E+00	6.5473E-1	3.5377E-1	1.8434E-1	9.5310E-2
	0.3350	0.7021	0.8881	0.9405	0.9516	
$d_1^{N,M}$	4.3956E-1	2.6866E-1	1.5653E-1	9.3735E-2	5.4403E-2	3.0902E-2
p_1^{uni}	0.7103	0.7794	0.7398	0.7849	0.8160	
$d_2^{N,M}$	1.3439E+0	1.0654E+00	6.5502E-1	3.5403E-1	1.8451E-1	9.5430E-2
p_2^{uni}	0.3350	0.7018	0.8877	0.9402	0.9512	

constructed as follows. Let N be multiple of 4; then, the grid points of $\bar{I}_x^N$ are given by

$$x_i = \begin{cases} iH, & i = 0, \ldots, N/4, \\ x_{N/4} + (i - N/4)h_1, & i = N/4 + 1, \ldots, N/2, \\ x_{N/2} + (i - N/2)h_2, & i = N/2 + 1, \ldots, 3N/4, \\ x_{3N/4} + (i - 3N/4)h_3, & i = 3N/4 + 1, \ldots, N/, \end{cases} \tag{37}$$

where $H = 4(1 - \sigma_3)/N$, $h_1 = 4(\sigma_3 - \sigma_2)/N$, $h_2 = 4(\sigma_2 - \sigma_1)/N$, $h_3 = 4\sigma_1/N$, and the transition parameters $\sigma_1, \sigma_2, \sigma_3$ are defined by

$$\sigma_3 = \min\left\{3/4, \sigma_0\varepsilon_3 \ln N\right\}, \ \sigma_2 = \min\left\{2\sigma_3/3, \sigma_0\varepsilon_2 \ln N\right\}, \ \sigma_1 = \min\left\{\sigma_2/2, \sigma_0\varepsilon_1 \ln N\right\}. \tag{38}$$

As the exact solution is again unknown, we use the same double mesh principle to approximate the maximum errors for each value of the diffusion parameter. Table 4 shows these

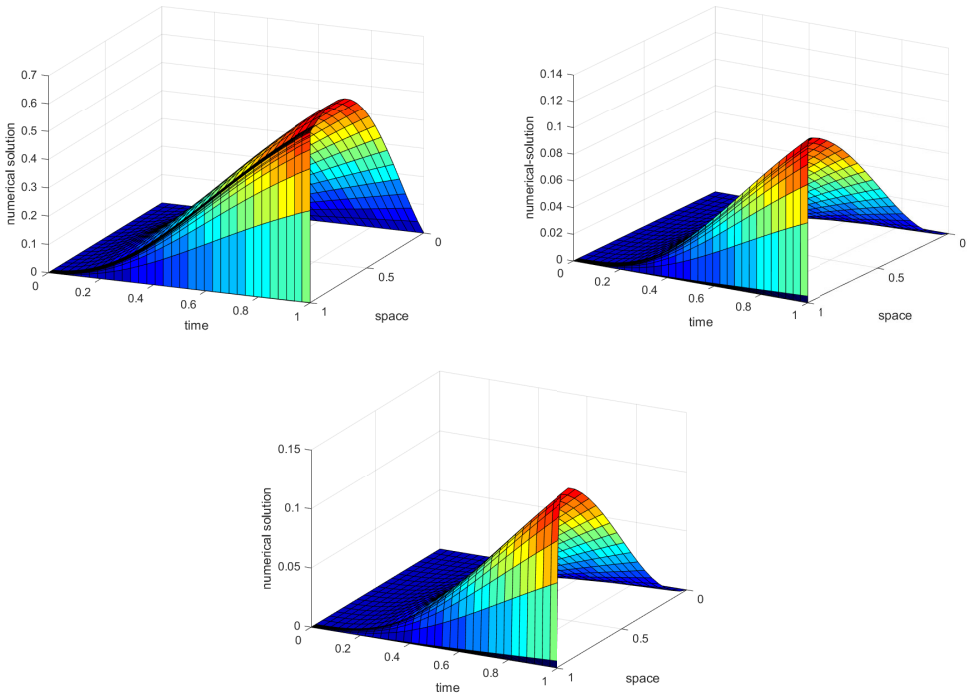

Fig. 3. Components of problem (36) for $\varepsilon_1 = 10^{-6}\, \varepsilon_2 = 10^{-4}$, $\varepsilon_3 = 10^{-2}$, with $N = 36, M = 32$ (left up u_1, right up u_2, bottom u_3)

maximum errors and their corresponding numerical orders of convergence for the three components, choosing some values of ε_3, of the discretization parameters N and choosing $M = N$ and $\sigma_0 = 1$. Besides ε_2 covers the set $R_2 = \{\varepsilon_2; \varepsilon_2 = \varepsilon_3, 2^{-2}\varepsilon_3, \ldots, 2^{-22}\}$ and ε_1 the set $R_1 = \{\varepsilon_1; \varepsilon_1 = \varepsilon_2, 2^{-2}\varepsilon_2, \ldots, 2^{-26}\}$. For each value of ε_3, the first and second rows correspond to errors and orders for the first component, the third and the fourth ones to the second component and the fifth and sixth ones to the third component. From it, we clearly observe that the numerical algorithm shows uniform convergence of almost first order, like in the previous example.

We compare again the CPU times of our method and the same classical method as in the first example, when solving the problem (36) for some values of $N(M = N)$ and fixed values for the diffusion parameters $\varepsilon_i,\ i = 1, 2, 3$. We have used the same stopping criterion for the Newton's method, which we have used in the previous experiment. Table 5 shows the required CPU time in seconds using our algorithm and the classical method. From it, we see a speed up for our algorithm respect to the classical one, due to the increase of the number of components in the system,. As well, the ratios between the CPU times are considerably larger than the ratios in the previous test problem because of the increase in the components of the system, as it was expected.

Table 4
Maximum errors and orders of convergence for the three components of the solution of (36)

ε_3	N=36	N=72	N=144	N=288	N=576	N=1152
	4.3242E-2	2.9828E-2	1.9423E-2	1.1978E-2	7.0899E-3	4.0529E-3
	0.5358	0.6189	0.6974	0.7566	0.8068	
	6.8286E-3	4.7355E-3	3.0004E-3	1.8401E-3	1.0781E-3	6.1394E-4
2^{-6}	0.5281	0.6584	0.7054	0.7713	0.8122	
	6.3320E-3	4.6459E-3	2.9993E-3	1.8474E-3	1.0849E-3	6.1917E-4
	0.4467	0.6313	0.6991	0.7679	0.8092	
	4.3881E-2	3.0282E-2	1.9674E-2	1.2117E-2	7.1615E-3	4.0904E-3
	0.5351	0.6222	0.6993	0.7587	0.8080	
	6.9697E-3	4.7773E-3	3.0181E-3	1.8480E-3	1.0815E-3	6.1528E-4
2^{-8}	0.5449	0.6625	0.7077	0.7729	0.8137	
	7.1176E-3	5.0304E-3	3.2420E-3	1.9612E-3	1.1472E-3	6.5187E-4
	0.5007	0.6338	0.7252	0.7736	0.8155	
	4.4069E-2	3.0422E-2	1.9754E-2	1.2163E-2	7.1867E-3	4.1043E-3
	0.5347	0.6230	0.6996	0.7591	0.8082	
	7.0026E-3	4.7893E-3	3.0223E-3	1.8508E-3	1.0833E-3	6.1636E-4
2^{-10}	0.5481	0.6642	0.7075	0.7727	0.8136	
	7.3322E-3	5.1303E-3	3.3101E-3	1.9965E-3	1.1669E-3	6.6260E-4
	0.5152	0.6322	0.7294	0.7748	0.8165	
	4.4116E-2	3.0457E-2	1.9775E-2	1.2175E-2	7.1934E-3	4.1080E-3
	0.5345	0.6231	0.6997	0.7592	0.8082	
	6.9677E-3	4.7772E-3	3.0196E-3	1.8505E-3	1.0834E-3	6.1666E-4
2^{-12}	0.5445	0.6618	0.7064	0.7723	0.8131	
	7.3687E-3	5.1477E-3	3.3197E-3	2.0014E-3	1.1701E-3	6.6464E-4
	0.5175	0.6329	0.7300	0.7744	0.8160	
...	...	...	...	...	...	...
	...	...	...	...	...	
	4.4099E-2	3.0462E-2	1.9781E-2	1.2179E-2	7.1956E-3	4.1092E-3
	0.5337	0.6229	0.6997	0.7592	0.8082	
	6.4552E-3	4.5781E-3	2.9308E-3	1.7906E-3	1.0510E-3	5.9767E-4
2^{-18}	0.4957	0.6434	0.7109	0.7686	0.8144	
	6.4920E-3	4.5551E-3	2.8514E-3	1.6910E-3	9.6938E-4	5.3840E-4
	0.5112	0.6758	0.7538	0.8028	0.8484	
$d_1^{N,M}$	4.4121E-2	3.0465E-2	1.9781E-2	1.2179E-2	7.1956E-3	4.1092E-3
p_1^{uni}	0.5343	0.6230	0.6997	0.7592	0.8082	
$d_2^{N,M}$	7.0026E-3	4.7893E-3	3.0223E-3	1.8508E-3	1.0834E-3	6.1666E-4
p_2^{uni}	0.5481	0.6642	0.7075	0.7725	0.8131	
$d_3^{N,M}$	7.3687E-3	5.1477E-3	3.3197E-3	2.0014E-3	1.1701E-3	6.6464E-4
p_3^{uni}	0.5175	0.6329	0.7300	0.7744	0.8160	

6 Conclusions

In this work, we have constructed and analyzed a new numerical algorithm to solve a type of semilinear parabolic singularly perturbed systems of convection–diffusion type.

Table 5
CPU times for problem (36) with $\varepsilon_3 = 2^{-12}, \varepsilon_2 = 2^{-16}$ and $\varepsilon_1 = 2^{-20}$ and $M = N$ when solving with splitting and classical methods

	N=384	N=768	N=1536	N=3072	N=6144
splitting	0.12500	0.46800	1.87500	7.39000	29.46800
classical	1.76500	7.09300	28.20300	113.14000	446.35900

The numerical method combines the classical upwind finite difference scheme, which is defined on a appropriate piecewise uniform Shishkin mesh, to discretize in space and a a linearized version of the fractional implicit Euler method together with an appropriate splitting by components to discretize in time on a uniform mesh. The resulting fully discrete scheme is uniformly and unconditionally convergent, of first order in time and of almost first order in space. Moreover, the chosen time integrator conduces to obtain the numerical approximation at each time level by solving only small linear tridiagonal systems must be solved. The numerical results obtained for two test problems, corroborate in practice the uniform convergence of the method, according with the theoretical results, and also show the reduction in the computational cost when it is compared with classical methods which need iterate to calculate the numerical solution of the semilinear problem. Such differences respect to the computational cost become more remarkable when the number of components of the coupled system increases.

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