Semi-exponential operators connected to x^3. A probabilistic perspective

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Abstract

In this paper, we consider semi-exponential operators connected to $\boldsymbol{x^3}$. We obtain for them preservation properties, explicit estimates of the rates of convergence, and closed form expressions for their moments. This is done by using probabilistic representations for such operators in terms of expectations involving appropriate random variables.

Keywords: Semi-exponential operators, probabilistic representation, moment generating function, convergence.

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1 Introduction

The concept of semi-exponential operators was introduced in [11], where the class of exponential type operators (see [10]) was further generalized in a unified manner. An operator $L_n^\beta, \beta \geq 0$, defined on an interval I by

$$(L_n^\beta f)(x) := \int_I \rho_{n,x}^\beta(v) f(v) dv,$$

is of semi-exponential type and connected to $p(x)$, if its kernel satisfies

$$\frac{\partial}{\partial x}\rho_{n,x}^{\beta}(v) = \left[\frac{n(v-x)}{p(x)} - \beta\right]\rho_{n,x}^{\beta}(v),$$

where $p(x)$ is analytic in any finite subinterval of I. When $\beta = 0$, we get the exponential operator L_n^0 connected to $p(x)$. Also, the operators L_n^β fulfill the normalization condition $(L_n^\beta e_0) = 1$, where $e_i(t) = t^i, i = 0, 1, 2, ...$ In [11], semi-exponential extensions of the classical Szász-Mirakyan and Weierstrsss operators were captured from their exponential variant. The semi-exponential extension of the Post-Widder operators was obtained by Herzog [9]. The tabular form of exponential operators was provided in [8, (1.1.14)]. Other semi-exponential type operators were given in [1]. Using orthogonal polynomials, Gupta and Milovanović [7] obtained the semi-exponential operators connected to $1 + x^2$.

Let $\beta \geq 0, x > 0$ and $n = 1, 2, ...$ The semi-exponential operators connected to x^3 are defined (see [1]) by

$$\begin{aligned}\left(Q_n^\beta f\right)(x) &= \frac{n^{1/2}e^{n/x-\beta x}}{\sqrt{2\pi}}\int_0^\infty \Bigg(v^{-3/2}e^{-\frac{n}{2}\left(\frac{1}{v}+\frac{v}{x^2}\right)} \\ &+ \sum_{k=1}^{\infty}\frac{1}{k!\Gamma(k/2)}\left(\frac{\beta\sqrt{n}}{\sqrt{2}}\right)^k \frac{1}{e^{nv/(2x^2)}}\int_0^v \frac{(v-u)^{k/2-1}}{u^{3/2}e^{n/(2u)}}du\Bigg) f(v)\,dv, \quad (1)\end{aligned}$$

where $f : (0, \infty) \to \mathbb{R}$ is any measurable function for which formula (1) makes sense. For $\beta = 0$, we simply write

$$\begin{aligned}(Q_n f)(x) &= \int_0^\infty \rho_{n,x}(v) f(v) dv \quad (2)\\ &= \frac{n^{1/2}e^{n/x}}{\sqrt{2\pi}}\int_0^\infty v^{-3/2}\exp\left(-\frac{n}{2v} - \frac{nv}{2x^2}\right) f(v)\,dv,\end{aligned}$$

where

$$\rho_{n,x}(v) = \frac{n^{1/2}e^{n/x}}{\sqrt{2\pi}}v^{-3/2}\exp\left(-\frac{n}{2v} - \frac{nv}{2x^2}\right), \quad v > 0. \quad (3)$$

Approximation and moment properties concerning the operators Q_n were established in [6].

The aim of this paper is to investigate the more involved operators Q_n^β by using a probabilistic approach in the spirit of [3]. In fact, we give in the following section probabilistic representations of such operators in terms of expectations of appropriate random variables built up from a standard normal random variable, as well as Poisson and gamma processes (see Proposition 3). Such representations allow us to obtain preservation properties of Q_n^β in a simple way (see Corollary 4 and the comments following it).

On the other hand, we give in Section 4 estimates for $(Q_n^\beta f)(x) - f(x)$ in terms of the usual first and second modulus of continuity of f with explicit constants, as well as similar results for the variant operator $\widetilde{Q}_n^\beta$, which exhibits better rates of convergence (see Theorems 7 and 8, respectively). To unify the proofs, we have included Section 3 dealing with general centered operators defined on $[0,\infty)$, which may have an independent interest. The final section is devoted to obtain closed form expressions for the moments of Q_n^β, by taking advantage of their probabilistic representations.

2 Probabilistic representations

In this section, we provide a probabilistic representation of formula (1) in terms of expectations of appropriate random variables. The approximation properties of the sequence of operators $(Q_n^\beta)_{n\geq 1}$ will be derived from such representations.

To this end, let Z be a random variable having the standard normal density

$$\rho(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad z \in \mathbb{R}. \tag{4}$$

Mathematical expectations are always denoted by E. We start with the case $\beta = 0$.
Proposition 1. *Let $x > 0$ and $n = 1, 2, \ldots$ The function $\rho_{n,x}(v)$ defined in (3) is the probability density of the random variable*

$$V_n(x) = \frac{1}{4}\left(\frac{xZ}{\sqrt{n}} + \sqrt{\frac{(xZ)^2}{n} + 4x}\right)^2. \tag{5}$$

As a consequence,

$$(Q_n f)(x) = Ef(V_n(x)). \tag{6}$$

Proof. A crucial remark to simplify computations is the identity

$$\int_0^\infty \frac{1}{u^2} e^{-\frac{n}{2}\left(\frac{u}{x}-\frac{1}{u}\right)^2} du = \frac{1}{x}\int_0^\infty e^{-\frac{n}{2}\left(\frac{u}{x}-\frac{1}{u}\right)^2} du, \tag{7}$$

which can be seen by making the change $u = x/t$. Setting $v = u^2$, we have from (2), (3), and (7)

$$\begin{aligned}(Q_n f)(x) &= \frac{n^{1/2}}{\sqrt{2\pi}} 2\int_0^\infty \frac{1}{u^2} e^{-\frac{n}{2}\left(\frac{u}{x}-\frac{1}{u}\right)^2} f(u^2)du \\ &= \frac{n^{1/2}}{\sqrt{2\pi}} \int_0^\infty \left(\frac{1}{x} + \frac{1}{u^2}\right) e^{-\frac{n}{2}\left(\frac{u}{x}-\frac{1}{u}\right)^2} f(u^2)du. \end{aligned}\tag{8}$$

Making the change $\theta = u/x - 1/u$, or equivalently

$$u = \frac{\theta x + \sqrt{(\theta x)^2 + 4x}}{2},$$

we have from (8)

$$(Q_n f)(x) = \frac{n^{1/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-n\theta^2/2} f\left(\frac{(\theta x + \sqrt{(\theta x)^2 + 4x})^2}{4}\right) d\theta.$$

Recalling (4), the result follows by setting $\theta\sqrt{n} = z$. □

Remark 1. *In view of (5), we can define* $V_n(0) = 0$. *Thus, the operator* Q_n *acts on real measurable functions* f *defined on* $[0,\infty)$ *for which the integral in (2) makes sense, with* $(Q_n f)(0) = f(0)$. *In other words,* Q_n *interpolates* f *at the origin. On the other hand, representation (5) makes clear that* $V_n(x) \to x$, $x \geq 0$, *as* $n \to \infty$. *This immediately implies that*

$$(Q_n f)(x) \to f(x), \quad x \geq 0, \quad \textit{as} \quad n \to \infty,$$

whenever we can apply dominated convergence.

Some preservation properties of Q_n can be easily derived from (5) and (6). We mention here the following.

Corollary 2. *Suppose that* f *is increasing. For any* $n = 1, 2, ...,$ *we have* $(Q_n f)(x) \leq (Q_n f)(y), 0 \leq x \leq y$, *and* $(Q_{n+1} f)(x) \leq (Q_n f)(x), 0 \leq x$.

Proof. It suffices to observe that $V_n(x) \leq V_n(y), 0 \leq x \leq y$ and $V_{n+1}(x) \leq V_n(x), 0 \leq x$. □

To describe the operators Q_n^β, we consider the following two stochastic processes having independent stationary increments and right-continuous non-decreasing paths (see [5] for more details). In the first place, the gamma process $(S(t))_{t\geq 0}$, i.e., $S(0) = 0$ and for each $t > 0$, the random variable $S(t)$ has the gamma density

$$\rho_t(\theta) = \frac{1}{\Gamma(t)} \theta^{t-1} e^{-\theta}, \quad \theta > 0. \tag{9}$$

In the second place, the standard Poisson process $(N_t)_{t\geq 0}$, i.e., $N_0 = 0$ and for each $t > 0$, N_t has the Poisson distribution with mean t, that is,

$$P(N_t = k) = \frac{t^k}{k!} e^{-t}, \quad k = 0, 1, 2, ... \tag{10}$$

We assume that both processes are independent and consider the subordinated stochastic process $(S(aN_t))_{t\geq 0}$, where $a > 0$ is fixed. For any measurable function $g : [0,\infty) \to \mathbb{R}$ for which the expectations below make sense, we have

$$
\begin{aligned}
Eg(S(aN_t)) &= \sum_{k=0}^{\infty} Eg(S(ak))P(N_t = k) \\
&= g(0)e^{-t} + \sum_{k=1}^{\infty} \frac{t^k e^{-t}}{k!} \frac{1}{\Gamma(ak)} \int_0^{\infty} \theta^{ak-1} e^{-\theta} g(\theta) d\theta. \qquad (11)
\end{aligned}
$$

As follows from (9), the moment generating function of $S(t)$ is given by

$$
Ee^{\lambda S(t)} = \frac{1}{\Gamma(t)} \int_0^{\infty} \theta^{t-1} e^{-\theta} e^{\lambda\theta} d\theta = \frac{1}{(1-\lambda)^t}, \quad \lambda < 1,
$$

thus implying, by virtue of (11), that

$$
\begin{aligned}
G(\lambda) :&= Ee^{\lambda S(aN_t)} = \sum_{k=0}^{\infty} Ee^{\lambda S(ak)} P(N_t = k) \\
&= \sum_{k=0}^{\infty} \frac{1}{(1-\lambda)^{ak}} \frac{t^k}{k!} e^{-t} = e^{t((1-\lambda)^{-a}-1)}, \quad \lambda < 1. \qquad (12)
\end{aligned}
$$

□

Note that

$$
ES(aN_t) = G'(0) = at. \qquad (13)
$$

From now on, we assume that $Z, (S(t))_{t\geq 0}$ and $(N_t)_{t\geq 0}$ are mutually independent. We are in a position to give a probabilistic representation for Q_n^{β}.

Proposition 3. *Let $x > 0$ and $n = 1, 2, \ldots$. For any $\beta \geq 0$, we have*

$$
(Q_n^{\beta} f)(x) = Ef(W_n(\beta, x)), \qquad (14)
$$

where

$$
W_n(\beta, x) = V_n(x) + \frac{2x^2}{n} S(2^{-1} N_{\beta x}), \qquad (15)
$$

and $V_n(x)$ is defined in (5).

Proof. Let $k = 1, 2, 3, \ldots$ Using Fubini's theorem and making the change $v = u + 2x^2\theta/n$, we obtain

$$
\begin{aligned}
&\int_0^{\infty} e^{-nv/(2x^2)} f(v) dv \int_0^{v} \frac{(v-u)^{k/2-1}}{u^{3/2}} e^{-n/(2u)} du \\
&= \int_0^{\infty} u^{-3/2} e^{-n/(2u)} du \int_u^{\infty} (v-u)^{k/2-1} e^{-nv/(2x^2)} f(v) dv
\end{aligned}
$$

$$= \left(\frac{2x^2}{n}\right)^{k/2} \int_0^\infty u^{-3/2} e^{-\frac{n}{2}\left(1/u+u/x^2\right)} du \int_0^\infty \theta^{k/2-1} e^{-\theta} f\left(u + \frac{2x^2}{n}\theta\right) d\theta.$$

We therefore have from (1) and (3)

$$\begin{aligned}(Q_n^\beta f)(x) &= \int_0^\infty \rho_{n,x}(u) du \Bigg(e^{-\beta x} f(u) + \sum_{k=1}^\infty \frac{(\beta x)^k}{k!} e^{-\beta x} \\ &\quad \frac{1}{\Gamma(k/2)} \int_0^\infty \theta^{k/2-1} e^{-\theta} f\left(u + \frac{2x^2}{n}\theta\right) d\theta \Bigg) \\ &= \int_0^\infty \rho_{n,x}(u) E f\left(u + \frac{2x^2}{n} S(2^{-1} N_{\beta x})\right) du, \qquad (16)\end{aligned}$$

where we have used formula (11) with $a = 1/2$, $t = \beta x$, and $g(\theta) = f(u + 2x^2\theta/n)$, $\theta \geq 0$. Thus, the result follows from Proposition 1, (16), and the independence of the random variables involved. □

Remark 2. *As in Remark 1, the operator Q_n^β acts on real measurable functions f defined on $[0, \infty)$, with $(Q_n^\beta f)(0) = f(0)$, since $W(\beta, 0) = 0$, as follows from (15). Again by (15), we have*

$$(Q_n^\beta f)(x) \to f(x), \quad x \geq 0, \quad \beta \geq 0, \quad \text{as} \quad n \to \infty,$$

whenever we can apply dominated convergence.

The following result extends Corollary 2.

Corollary 4. *Suppose that f is increasing. For any $n = 1, 2, \ldots$ and $\beta \geq 0$, we have*

$$\begin{aligned}(Q_n^\beta f)(x) &\leq (Q_n^\beta f)(y), \quad 0 \leq x \leq y \\ (Q_{n+1}^\beta f)(x) &\leq (Q_n^\beta f)(x), \quad 0 \leq x. \qquad (17)\end{aligned}$$

In addition,

$$(Q_n^{\beta_1} f)(x) \leq (Q_n^{\beta_2} f)(x), \quad 0 \leq \beta_1 \leq \beta_2, \quad 0 \leq x. \qquad (18)$$

Proof. The proof is similar to that of Corollary 2 by taking into account that the stochastic processes $(S(t))_{t \geq 0}$ and $(N_t)_{t \geq 0}$ have non decreasing paths. This property implies that $W_n(\beta_1, x) \leq W_n(\beta_2, x)$, whenever $0 \leq \beta_1 \leq \beta_2$. □

By Corollary 4, the operator Q_n^β preserves monotonicity. This implies that this operator diminish the ϕ-variation (see [3] for more details).

It has been shown in [6, Lemma 3] that the mean and the variance of $V_n(x)$ are respectively given by

$$EV_n(x) = x, \quad \sigma_n^2(x) := E(V_n(x) - x)^2 = \frac{x^3}{n}. \qquad (19)$$

This entails, by virtue of (13) and (15), that

$$EW_n(\beta, x) \;=\; x + \frac{\beta x^3}{n}. \tag{20}$$

Rates of convergence for the approximation of $f(x)$ by $(Q_n^\beta f)(x)$ are obtained in Section 4. Such rates are given in terms of different moduli of smoothness according to $\beta = 0$ or $\beta > 0$. This is due to the fact that $EW_n(\beta, x) \neq x$ for $\beta > 0$.

3 Rates of convergence for centered operators

Let $x \geq 0$ and $n = 1, 2, ...$ Let $Y_n(x)$ be a random variable taking values in $[0, \infty)$. We consider the positive linear operator

$$(L_n f)(x) = Ef(Y_n(x))$$

acting on real measurable functions $f : [0, \infty) \to \mathbb{R}$ for which the preceding expectation is well defined. The operation L_n is called centered if

$$EY_n(x) \;=\; x, \tag{21}$$

or equivalently, if $(L_n e_1)(x) = x$. We assume that

$$E(Y_n(x) - x)^2 \;=\; \frac{\varphi^2(x)}{n} < \infty. \tag{22}$$

We will estimate $(L_n f)(x) - f(x)$ in terms of the usual second modulus of smoothness of f defined as

$$\omega_2(f; \delta) = \sup\{|f(y - h) - 2f(y) + f(y + h)| : y \geq h, \quad 0 \leq h \leq \delta\}, \quad \delta \geq 0.$$

To this end, let U_1 and U_2 be two independent identically distributed random variables having the uniform distribution on $[-1, 1]$. The random variable $U = (U_1 + U_2)/2$ has probability density

$$\tau(u) = (1 + u)1_{[-1,0]} + (1 - u)1_{(0,1]},$$

where 1_A stands for the indicator function of the set A. For $m = 0, 1...$, denote by $f_{(m)}$ an antiderivative of f of order m, that is, $f_{(m)}^{(m)} = f$.

Let $h > 0$. The Steklov means of f can be defined in various equivalent ways as follows

$$\begin{aligned} P_h f(y) \;&=\; Ef(y + hU) = \int_{-1}^{1} f(y + hu)\tau(u)du \\ &= \int_{-1}^{1}\int_{-1}^{1} f\left(y + \frac{h}{2}(u_1 + u_2)\right) du_1 du_2 \end{aligned}$$

$$= \frac{1}{h^2}\left(f_{(2)}(y-h) - 2f_{(2)}(y) + f_{(2)}(y+h)\right), \quad y \geq h > 0. \tag{23}$$

It is well known (see, for instance, [4]) that

$$\begin{aligned} |P_h f(y) - f(y)| &\leq \frac{1}{2}\omega_2(f;h), \\ |(P_h f)''(y)| &\leq \frac{1}{h^2}\omega_2(f;h), \quad y \geq h > 0. \end{aligned} \tag{24}$$

Following the ideas in [4], we define a smooth approximant of f in the whole interval $[0,\infty)$ as

$$T_h f(y) = \begin{cases} P_h f(y), & y \geq h \\ 2P_h f(h) - P_h f(2h-y), & 0 \leq y < h \end{cases} \tag{25}$$

Note that $T_h f$ is twice differentiable except at the point h, where $T_h f$ has only one sided second derivatives. Despite this fact, $(T_h f)'$ is absolutely continuous and therefore satisfies the Taylor's formula

$$\begin{aligned} T_h f(y) &= T_h f(x) + (T_h f)'(y-x) \\ &\quad + \frac{1}{2}(y-x)^2 E(T_h f)''(x+(y-x)\beta), \quad x,y \in [0,\infty), \end{aligned} \tag{26}$$

where β is a random variable having the beta density $\nu(\theta) = 2(1-\theta), 0 \leq \theta \leq 1$, and it is understood that $(T_h f)''(y)$ is the right second derivative of $T_h f$.

Lemma 5. *Let $h > 0$ and $y \geq 0$. Then,*

$$|T_h f(y) - f(y)| \leq \frac{5}{2}\omega_2(f;h) \tag{27}$$

and

$$|(T_h f)''(y)| \leq \frac{1}{h^2}\omega_2(f;h). \tag{28}$$

Proof. If $y \geq h$, inequality (27) follows from (24) and (25), whereas if $0 \leq y < h$, we have from (25)

$$\begin{aligned} T_h f(y) - f(y) &= 2(P_h f(h) - f(h)) - (P_h f(2h-y) - f(2h-y)) \\ &\quad - (f(y) - 2f(h) + f(2h-y)). \end{aligned} \tag{29}$$

Thus, (27) follows from the first inequality in (24). Inequality (28) readily follows from (24) and (25). □

We are in a position to state the following result concerning the centered operator L_n.

Theorem 6. *For any $x \geq 0$ and $n = 1, 2, \ldots$, we have*

$$|(L_n f)(x) - f(x)| \leq \frac{11}{2}\omega_2\left(f; \frac{|\varphi(x)|}{\sqrt{n}}\right).$$

Proof. Without loss of generality, we can assume that the random variables $Y_n(x)$ and β, as defined in (26), are independent. For any $h > 0$, we have

$$\begin{aligned} |(L_n f)(x) - f(x)| &\leq |(L_n f)(x) - (L_n T_h f)(x)| + |T_h f(x) - f(x)| \\ &\quad + |(L_n T_h f)(x) - T_h f(x)| \\ &\leq 5\omega_2(f; h) + |(L_n T_h f)(x) - T_h f(x)|, \end{aligned} \tag{30}$$

as follows from (27). On the other hand, replace y by the random variable $Y_n(x)$ in (26) and then take expectations. Using (21), (22), (28), and the independence of the random variables involved, we get

$$\begin{aligned} |(L_n T_h f)(x) - T_h f(x)| &= |E T_h f(Y_n(x)) - T_h f(x)| \\ &\leq \frac{1}{2} E(Y_n(x) - x)^2 |(T_h f)''(x + (Y_n(x) - x)\beta)| \\ &\leq \frac{\omega_2(f; h)}{2h^2} E(Y_n(x) - x)^2 \\ &= \frac{\varphi^2(x)}{2nh^2}\omega_2(f; h). \end{aligned} \tag{31}$$

The conclusion follows from (30) and (31) by choosing $h = |\varphi(x)|/\sqrt{n}$. □

4 Rates of convergence for Q_n^β

As seen in (19) and (20), the operator Q_n is centered, whereas Q_n^β, for $\beta > 0$, is not. This means that in this last case we cannot give rates of convergence only in terms of the second modulus. For this reason, we consider the usual first modules of continuity of f defined as

$$\omega_1(f; \delta) = \sup\{|f(x) - f(y)| : x, y \geq 0, \quad |x - y| \leq \delta\}, \quad \delta \geq 0.$$

Theorem 7. *Let $x \geq 0$, $\beta \geq 0$ and $n = 1, 2, \ldots$ Then,*

$$|(Q_n^\beta f)(x) - f(x)| \leq \frac{11}{2}\omega_2\left(f; \frac{x^{3/2}}{\sqrt{n}}\right) + 2\omega_1\left(f; \frac{\beta x^3}{n}\right).$$

Proof. Since Q_n is centered, we have from (19) and Theorem 6

$$\begin{aligned} |(Q_n^\beta f)(x) - f(x)| &\leq |(Q_n f)(x) - f(x)| + |(Q_n^\beta f)(x) - (Q_n f)(x)| \\ &\leq \frac{11}{2}\omega_2\left(f; \frac{x^{3/2}}{\sqrt{n}}\right) + |(Q_n^\beta f)(x) - (Q_n f)(x)|. \end{aligned} \tag{32}$$

We simply denote by

$$Y = \frac{2x^2}{n} S(2^{-1} N_{\beta x}). \tag{33}$$

Using the subadditivity of $\omega_1(f;.)$ and Propositions 1 and 3, we obtain for any $\delta > 0$

$$\begin{aligned} |(Q_n^\beta f)(x) - (Q_n f)(x)| &= |Ef(V_n(x) + Y) - Ef(V_n(x))| \\ &\le E\omega_1(f;Y) \le \left(1 + \frac{EY}{\delta}\right)\omega_1(f;\delta). \end{aligned} \tag{34}$$

As follows from (20), $EY = \beta x^3/n$. The conclusion follows from (32) and (34) by choosing $\delta = \beta x^3/n$. □

Let $m = 1, 2, ...$ be fixed. We consider the operator $\widetilde{Q}_n^\beta$ defined as

$$(\widetilde{Q}_n^\beta f)(x) = (Q_{mn}^\beta f(nt))\left(\frac{x}{n}\right).$$

By Propositions 1 and 3, this operator can be represented in probabilistic terms as

$$(\widetilde{Q}_n^\beta f)(x) = Ef\left(nV_{mn}\left(\frac{x}{n}\right) + \frac{2x^2}{mn^2} S(2^{-1} N_{\beta x/n})\right). \tag{35}$$

We simply denote by

$$\widetilde{Y}_n(x) = nV_{mn}\left(\frac{x}{n}\right), \quad \widetilde{Y} = \frac{2x^2}{mn^2} S(2^{-1} N_{\beta x/n}). \tag{36}$$

Using (13) and the first equality in (19), we see that

$$E\widetilde{Y}_n(x) = x, \quad E\widetilde{Y} = \frac{\beta x^3}{mn^3}. \tag{37}$$

Observe that $E\widetilde{Y}$ is much less than EY, as defined in (33). By the second equality in (19), we have

$$\widetilde{\sigma}_n^2(x) := E(\widetilde{Y}_n(x) - x)^2 = n^2 E\left(V_{mn}\left(\frac{x}{n}\right) - \frac{x}{n}\right)^2 = \frac{x^3}{mn^2}. \tag{38}$$

Again this variance is much less than that in (19). As we will see in the following result, these two facts imply that the rate of convergence for the operator $\widetilde{Q}_n^\beta$ is much more faster than that for Q_n^β.

Theorem 8. *Let $x \geq 0$, $\beta \geq 0$ and $m, n = 0, 1, 2, \ldots$ Then,*

$$|(\widetilde{Q}_n^\beta f)(x) - f(x)| \leq \frac{11}{2}\omega_2\left(f; \frac{x^{3/2}}{n\sqrt{m}}\right) + 2\omega_1\left(f; \frac{\beta x^3}{mn^3}\right).$$

Proof. By (35) and (36), we can write

$$(\widetilde{Q}_n^\beta f)(x) - f(x) = Ef(\widetilde{Y}_n(x)) - f(x) + Ef(\widetilde{Y}_n(x) + \widetilde{Y}) - Ef(\widetilde{Y}_n(x)). \tag{39}$$

Recalling (22) and (38), we can apply Theorem 6 with $\varphi(x) = (x^3/(mn))^{1/2}$ to obtain

$$|Ef(\widetilde{Y}_n(x)) - f(x)| \leq \frac{11}{2}\omega_2\left(f; \frac{x^{3/2}}{n\sqrt{m}}\right). \tag{40}$$

As in the proof of Theorem 7, we have

$$|Ef(\widetilde{Y}_n(x) + \widetilde{Y}) - Ef(\widetilde{Y}_n(x)| \leq E\omega_1(f; \widetilde{Y}) \leq 2\omega_1(f; E\widetilde{Y}) = 2\omega_1\left(f; \frac{\beta x^3}{mn^3}\right),$$

where the last equality follows from (37). This, together with (39) and (40), completes the proof. □

5 Moments of Q_n^β

Let $x > 0$. The moment generating function of the random variable $V_n(x) - x$ is given by

$$H_n(\lambda) := Ee^{\lambda(V_n(x)-x)} = \exp\left(\frac{n}{x}\left(1 - \sqrt{\frac{n - 2x^2\lambda}{n}}\right) - \lambda x\right), \lambda < \frac{n}{2x^2}. \tag{41}$$

Formula (41) was shown in [6, Lemma 1]. In fact, this formula follows from (5) and (6) by choosing

$$f(y) = e^{\lambda(y-x)}, \quad \lambda < \frac{n}{2x^2}.$$

On the other hand, we get from (12)

$$J_\beta(\lambda) := Ee^{\lambda S(2^{-1}N_{\beta x})} = \exp\left(\beta x\left(\frac{1}{\sqrt{1-\lambda}} - 1\right)\right), \quad \beta \geq 0, \quad \lambda < 1. \tag{42}$$

Finally, denote by

$$f_m(y) = (y - x)^m, \quad y \geq 0, \quad m = 0, 1, 2, \ldots$$

With these ingredients, the moments of the operator Q_n^β are computed in the following result.

Theorem 9. *Let $x > 0, \beta \geq 0$ and $n = 1, 2, \ldots$ For any $m = 0, 1, 2, \ldots$, we have*

$$(Q_n f_m)(x) = \sum_{k=0}^{m} \binom{m}{k} \left(\frac{2x^2}{n} \right)^k J_\beta^{(k)}(0) H_n^{(m-k)}(0).$$

Proof. Using Proposition 3 and the independence of the random variables involved, we have

$$\begin{aligned}(Q_n f_m)(x) &= E(W_n(\beta, x) - x)^m = E\left(V_n(x) - x + \frac{2x^2}{n} S(2^{-1} N_{\beta x}) \right)^m \\ &= \sum_{k=0}^{m} \binom{m}{k} \left(\frac{2x^2}{n} \right)^k E(S(2^{-1} N_{\beta x}))^k E(V_n(x) - x)^{m-k} \\ &= \sum_{k=0}^{m} \binom{m}{k} \left(\frac{2x^2}{n} \right)^k J_\beta^{(k)}(0) H_n^{(m-k)}(0),\end{aligned}$$

where the last equality follows from (41) and (42). The proof is complete. □

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