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Benoît Guerville-Ballé

Topological invariants of line arrangements

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UNIVERSIDAD DE ZARAGOZA

TOPOLOGICAL INVARIANTS OF LINE ARRANGEMENTS

BENOÎT GUERVILLE-BALLÉ

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"Je ne cherche pas à connaître les réponses,
je cherche à comprendre les questions."

Confucius

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Benoît

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Introduction

The first drawing of any child is a line. What is more natural than lines? What is the difference between two drawings of lines? This is roughly speaking the subject of this thesis. The support is slightly different of a sheet of paper, since we are focusing on line arrangements in the complex projective plane.

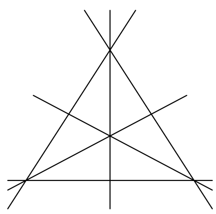

Figure 1: Ceva's arrangement (1678)

The first idea to describe a line arrangement is to define the set of lines, the set of singular points, and the incidence relations between them. This combinatorial information can be summarized in graphs, like Hasse's graph [57] or the incidence graph. But, as shown in [71, 60, 7, 6], this description is not sufficient to determine an arrangement.

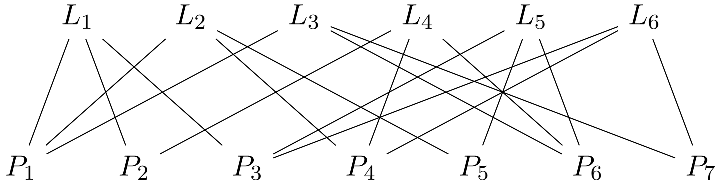

Figure 2: Incidence graph of the Ceva's arrangement

In mathematics, the study of line arrangements is a domain with a history reaching back only forty-five years. Of course the prehistory goes back much further in time with the Greeks like Pappus of Alexandria [55], or in the XVII$^{\text{th}}$ century with G. Ceva [16]. Its first materialization in modern mathematics is with E. Brieskorn [11, 12] where he shows that the braid group can be constructed as the fundamental group of the complement of a complex arrangement of hyperplanes. This relative young area takes thus its origins in the study of discriminants, combinatorics, configuration spaces, braids, etc.

Owing to its multiple origins, the study of arrangements admits different facets: combinatorial, geometrical, arithmetical and topological. The interaction between all these facets is its most attractive feature. Furthermore, arrangements provide a fertile source of examples and problems. Techniques developed for their studies have been successfully exported to other settings. One main interesting facet of an arrangement is determined by its ground field. We take an interest in line arrangements defined over the complex field $\mathbb{C}$ and their topologies. Our general aim is to investigate the relationship between their combinatorics and their topology.

The motivations

Definition. The *topological type* of an arrangement $\mathcal{A}$ in $\mathbb{CP}^2$ is the homeomorphism type of the pair $(\mathbb{CP}^2, \mathcal{A})$.

From the beginning of the study of arrangements, questions about their topologies were frequent. In 1969, V.I. Arnold gives a presentation of the cohomology of the complement as a graded algebra, see [2]. In 1973, P. Deligne [27] shows that the complement of any arrangement associated with a real reflection groups is aspherical. P. Orlik and L. Solomon [56] proved in 1980, that the cohomology ring of the complement only depends on the combinatorics of the arrangement.

Question. *What is the influence of the combinatorics on the topology of arrangements?*

Theoretically, a line arrangement is a specific case of algebraic plane curve. Fundamental results on curves are obtained by O. Zariski [71]. He shows the existence of a pair of curves admitting the same combinatorial data and different topological types. It is then natural to define a *Zariski pair*:

Definition ([4])**.** A *Zariski pair* of arrangement is a pair $(\mathcal{A}_1, \mathcal{A}_2)$ such that $\mathcal{A}_1$ and $\mathcal{A}_2$ admit the same combinatorics but different topological types.

For technical reasons, we consider also blow-ups of arrangements at their multiple points. A NC-*Zariski pair* is a pair of arrangements with the same combinatorics and blow-ups with different topological types, –NC stands for normal crossing–.

In [60], G. Rybnikov shows the existence of a Zariski pair in the case of line arrangements, and gives an explicit example. Studying the result of G. Rybnikov, the authors of [7, 6]: E. Artal, J. Carmona, J.I. Cogolludo and M.A. Marco, obtain a Zariski pair of complexified real arrangements. Examples of NC-Zariski pairs are obtained by A. Degtyarev [25] in the case of sextic curves.

The principal topological invariant of an arrangement $\mathcal{A}$ is it *complement* $E_{\mathcal{A}}$:

$$E_{\mathcal{A}} = \mathbb{CP}^2 \setminus \mathcal{A}.$$

From it, several invariants are constructed, the major of them is its fundamental group, see [46]. It is a strong invariant as shown in [60, 22, 21].

In [65], E.R. van Kampen, gives a method to compute a presentation of the fundamental group of an algebraic plane curve complement (known as the *Zariski-van Kampen method*). He uses the position of the curve relative to pencils of lines in $\mathbb{CP}^2$. This notion, the *braid monodromy*, receives its first modern approach in studies by D. Chéniot [17], O. Chisini [18] and B. Moishezon [51]. In [43], A. Libgober constructs a 2-dimensional CW-complex from the braid monodromy, admitting the same homotopy type as the complement of the curve. There are other contributions by M. Salvetti [63], D. Cohen and A. Suciu [21], V.S. Kulikov and M. Taĭkher [41] and J. Carmona [15] on the relationship between topology and braid monodromy.

The Zariski-van Kampen method is studied by R. Randell, in [58], in the case of complexified real arrangement. W. Arvola generalizes, in [9], the algorithm of R. Randell to the case of any complex line arrangement using the *braided wiring diagram* that encodes the braid monodromy of the arrangement.

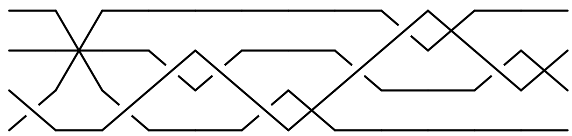

Figure 3: Example of wiring diagram

Since the problem of group isomorphism is undecidable, related topological invariants are studied. One of the main invariant we study is the set of characteristic varieties, which are subsets of the character torus $\mathbb{T}(\mathcal{A})$, the group of characters of the fundamental group in $\mathbb{C}^*$. They are first introduced by A. Libgober in the case of curves. They are cohomological invariants defined as the jumping loci of twisted cohomology of rank one local system on the variety, and they only depend on its fundamental group.

Definition. The characteristic varieties of $\mathcal{A}$ are:

$$\mathcal{V}_k(\mathcal{A}) = \left\{ \ \xi \in \mathbb{T}(\mathcal{A}) \ \mid \ \dim_{\mathbb{C}} \left(\mathrm{H}^1(E_{\mathcal{A}}; \mathcal{L}_\xi) \right) \geq k \ \right\},$$

where $\mathcal{L}_\xi$ is the local system of coefficient defined by ξ. The *depth* of a character ξ is $\dim_{\mathbb{C}} \left(\mathrm{H}^1(E_{\mathcal{A}}; \mathcal{L}_\xi) \right)$.

The first general structure theorem for these loci is discovered by D. Arapura, who described them in terms of fibrations over curves, see [1]; the structure was completed by several authors, E. Artal, J.I.Cogolludo and D. Matei [8], N. Budur [14], A. Dimca [29] and A. Libgober[47]. The characteristic varieties of a space can be seen as a generalization of the Alexander polynomial. This invariant has been extensively studied from different perspectives, in general case by S. Novikov [54], W. Dwyer and D. Fried [32] and A. Libgober [44]. In the case of arrangements, A. Libgober [45] gives an algorithm on how to find irreducible components of $\mathcal{V}_k(\mathcal{A})$, E. Hironaka [39] shows the equality of the characteristic varieties with a stratification of the character torus of $\pi_1(E_{\mathcal{A}})$ coming from the Fox derivatives, and D. Cohen and A. Suciu [23] prove that all the irreducible components of $\mathcal{V}_k(\mathcal{A})$ which pass through the identity element are combinatorially determined. They are related with the Green-Lazarsfeld set [36, 35].

Question. *Are the characteristic varieties of arrangements combinatorially determined?*

More recent works of A. Libgober and A. Dimca, D. Ibadula and D.A. Macinic show that for diagonal characters the projective depth is of combinatorial nature if the singular points of $\mathcal{A}$ have multiplicity less than 3 (see [48]), and in the case with multiplicity less than 5 if $\mathcal{A}$ contains less than 14 lines (see [30]).

The study of the *boundary manifold* is another approach of this question of the relationship between combinatorics and topology. It provides partial answers to this question. Indeed, this variety only depends on the combinatorics and can be constructed as a graph manifold over the incidence graph. E. Westlund describes, in his thesis [68], the graph manifold structure of the boundary manifold, and D. Cohen and A. Suciu in [24] obtain several results on its topology. The boundary manifold is intimately related to the complement of arrangements. In [40], E. Hironaka studies its inclusion in the complement in the case of complexified real arrangements, and shows that the complement can be constructed –up to homotopy– as a quotient of the boundary manifold.

Our advances and results

This thesis is at the intersection point of these two directions: the boundary manifold and the characteristic varieties. We have studied the inclusion map of the boundary manifold in the complement, see [34]. From it, attempts were made to find new topological invariants. This application was also used to study the non combinatorially part of the quasi-projective part of characteristic varieties. Indeed, E. Artal has developed an algorithm to compute the quasi-projective part of the depth of a character, using the image of some particular cycles of the boundary manifold in the complement, see [5].

We have then worked on the combinatoriality of the characteristic varieties using the combinatorial part of the complement given by the boundary manifold, and the relationship –without complete algorithm– with the characteristic varieties obtained by E. Artal. Even if we have not found the answer to this difficult question, this study allows to construct an explicit and complete method to compute the quasi-projective depth of a character, and a new topological invariant permitting to distinguish some NC-Zariski pairs of line arrangements.

In the first part of this work, we generalize the results obtained by E. Hironaka in [40] to the case of any complex line arrangement. To get around the problem due to the case of non complexified real arrangement we have studied the braided wiring diagram, and used it fully fledged as a topological object. So we have to develop a `Sage` program to compute it from the equation of the complex line arrangement. This diagram allows to give two explicit descriptions of the map:

$$i_* : \pi_1(B_{\mathcal{A}}) \longrightarrow \pi_1(E_{\mathcal{A}}).$$

From these descriptions, see Theorem 3.2.2 and Theorem 3.2.7, we obtain two new presentations of the fundamental group of the complement, see Corollary 3.2.5 and Corollary 3.2.8. The second corollary is, in fact, a generalization of Randell's Theorem [58].

In the next step of our work, we study the map induced on the first homology groups. We obtain simple descriptions of this application, see Theorem 4.1.1 and Theorem 4.1.3. Then inspired by ideas of J.I. Cogolludo, we give a canonical description of the homology of the boundary manifold as the product of the 1-homology with the 2-cohomology of the complement:

Theorem (4.3.2)**.** *Let $E_{\mathcal{A}}$ be the complement of $\mathcal{A}$ and let $B_{\mathcal{A}}$ be its boundary manifold, then we have a split exact sequence:*

$$0 \to \mathrm{H}^2(E_{\mathcal{A}}; \mathbb{C}) \longrightarrow \mathrm{H}_1(B_{\mathcal{A}}; \mathbb{C}) \xrightarrow{i_*} \mathrm{H}_1(E_{\mathcal{A}}; \mathbb{C}) \to 0.$$

Finally, we obtain an isomorphism between the 2-cohomology of the complement with the 1-homology of the incidence graph of the arrangement. This provides an intrinsic decomposition of $\mathrm{H}_1(B_{\mathcal{A}}; \mathbb{C})$ as $\mathrm{H}_1(E_{\mathcal{A}}; \mathbb{C}) \oplus \mathrm{H}_1(\Gamma_{\mathcal{A}}; \mathbb{C})$.

In the second part, we apply this work to the study of characters on the complement. We start from the results of E. Artal [5] on the computation of the depth of a character. This depth can be decomposed into a *projective term* and a *quasi-projective term*, vanishing for characters that ramify along all the lines. An algorithm to compute the projective part is given by A. Libgober [48]. E. Artal focuses on the quasi-projective part and gives a method to compute it from the image by the character of certain cycles of the complement. We use our results on the inclusion map of the boundary manifold to determine these cycles explicitly. Combined with the work of E. Artal we obtain an algorithm to compute the quasi-projective depth of any character.

Theorem 5.2.23 gives a combinatorial condition on characters to admit a quasi-projective depth potentially not determined by the combinatorics. With this property, we define the *inner-cyclic characters*. From their study, we observe a strong condition on the combinatorics of an arrangement to have only characters with null quasi-projective depth, see Proposition 6.1.1. Related to this, in order to reduce the number of computations, we introduce the notion of *prime combinatorics*. If a combinatorics is not prime, then the characteristics varieties of its realizations are completely determined by realization of a prime combinatorics with fewer lines.

In parallel, we observe that the composition of the map induced by the inclusion with specific characters provide topological invariants of the blow-up of arrangements, see Theorem 5.3.14. We show in Theorem 6.4.6 and Corollary 6.4.7 that the invariant captures more than combinatorial information. Thereby, we detect two new examples of NC-Zariski pairs. The first, is an oriented and ordered pair, see Theorem 6.3.2. The second, comes from the following combinatorics:

$$\begin{aligned}\mathscr{C} = [\ &[1,2],[1,3,5,7,12],[1,4,6,8],[1,9],[1,10,11],[2,3,6,9],\\ &[2,4,5,10],[2,7,11],[2,8],[2,12],[3,4],[3,8,11],[3,10],[4,7],\\ &[4,9,11],[4,12],[5,6],[5,8,9],[5,11],[6,7,10],[6,11],[6,12],\\ &[7,8],[7,9],[8,10],[8,12],[9,10],[9,12],[10,12],[11,12]\ ].\end{aligned}$$

It admits four realizations denoted by $\mathscr{A}^{\pm}$ and $\mathscr{B}^{\pm}$.

Theorem (6.4.10 and 6.4.13)**.** *The pairs $(\mathscr{A}^{\pm}, \mathscr{B}^{\pm})$ are NC-Zariski pairs, and the 4-tuplet $(\mathscr{A}^{+}, \mathscr{B}^{+}, \mathscr{A}^{-}, \mathscr{B}^{-})$ is an oriented NC-Zariski 4-tuplet.*

This provides the third example, known in the literature, of NC-Zariski pair. The two other being also Zariski pairs are obtained in [60, 6].

Thesis plan

This manuscript is composed of six chapters, following our work:

The first Chapter is essentially composed of basic tools. We define the notion of line arrangements in the complex projective plane. Then we explain the notions of combinatorics and their realizations. In the second section, we recall the desingularization of a space using the blow-up method, and apply it on line arrangements.

In Chapter 2, we study the braided wiring diagram, introduced by W. Arvola in [9] to compute the fundamental group of an arrangement. The first section contains the construction of this diagram. Then we show: how to use it to compute the fundamental group, how it is related to the braid monodromy, and how to find the braided wiring diagram of the conjugate arrangement. Then we explain the construction of a `Sage` program to compute the diagram. In the last section, we compute wiring diagrams of MacLane's arrangements and Fan's arrangements.

The generalization of the result of E. Hironaka in done in Chapter 3. First, we describe the boundary manifold as a graph manifold over the incidence graph of the arrangement, and we deduce a presentation of its fundamental groups. In the second section, we give an explicit description of the map induced by the inclusion of the boundary manifold in the complement on the fundamental groups. We obtain two new presentations of the fundamental group and a new proof of the Randell theorem. To conclude this Chapter, we give the detailed example of the positive MacLane arrangement.

Chapter 4 is composed of the homological version of the inclusion map described in the previous Chapter. Two examples are detailed in the second section: extended Ceva's arrangement and both MacLane's arrangements. The third, is composed of a canonical decomposition of the first homology group of the complement, and then a study of the link between the 2-cohomology of the complement and the 1-homology of the incidence graph.

The results of E. Artal form the principal part of Chapter 5. In the first section, we recall the definition and some properties of the characteristic varieties. The second section, is a detailed description of the result of E. Artal. In the last section, we define the notion of inner-cyclic combinatorics, and inner-cyclic character. Then we show that the image of particular cycles of the boundary manifold composed by the associated inner-cyclic character is a topological invariant of the topology of the blow-up of the arrangement.

The last Chapter, is the intersection point of the previous work. The first section contains the definition of a prime combinatorics, and why we study them. In the second section, we present two programs. One in `Sage` to detect the prime realizable combinatorics, and the second in `Maple` which gives the equation of the arrangement from its combinatorics. With these two programs, we have tested a lot of examples, and we present here the most interesting ones. The fourth section is the study of an example with eleven lines, and from it we construct NC-Zariski pairs.

Introduction (French)

Le premier dessin de tout enfant est un trait. Quoi de plus naturel que des traits, des droites ? Quelle est la différence entre deux dessins de droites ? Voilà approximativement le sujet de cette thèse. Le support est bien entendu légèrement différent d'une feuille de papier, puisque nous travaillons dans le plan projectif.

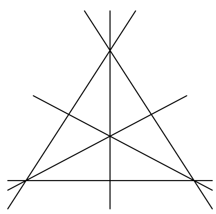

Figure 4 – Arrangement de Ceva (1678)

La première idée pour décrire un arrangement de droites est de donner l'ensemble des droites, l'ensemble des points d'intersections et les relations entre eux. Cette information combinatoire peut être résumée dans des graphes, tel le graphe de Hasse [57] ou le graphe d'incidence. Mais comme il fut montré dans [71, 60, 7, 6], cette description n'est pas suffisante pour caractériser un arrangement.

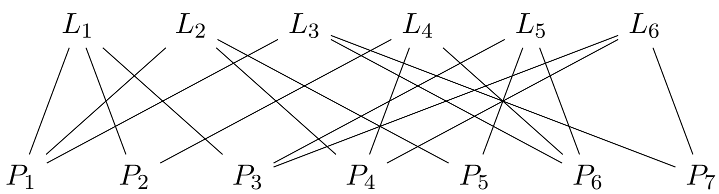

Figure 5 – Graphe d'incidence de l'arrangement de Ceva

En mathématiques, l'étude des arrangements de droites est un sujet ayant une histoire relativement courte, remontant à quarante-cinq années seulement. Bien entendu sa pré-histoire remonte beaucoup plus loin dans le temps –avec les grecs comme Pappus d'Alexandrie [55], ou au XVIIième siècle avec G. Ceva [16]–. La première approche moderne de l'étude des arrangements a été faite par E. Brieskorn [11, 12], où il a montré que le groupe de tresses pouvait être construit comme le groupe fondamental du complémentaire d'un arrangement d'hyperplans complexes. Ce nouveau domaine des mathématiques prend ses origines dans l'étude des discriminants, de la combinatoire, des configurations d'espace, des tresses, etc.

De par ses multiples origines, l'étude des arrangements admet plusieurs facettes : combinatoire, géométrique, arithmétique et topologique. L'interaction entre ces diverses facettes est le point le plus attractif de l'étude des arrangements. De plus, ils forment une source fertile d'exemples et de problèmes. Les techniques développées pour leur étude ont été exportées avec succès vers d'autres domaines des mathématiques. Le corps sur lequel est défini un arrangement détermine sa facette la plus intéressante. Ici, nous allons nous intéresser aux arrangements définis sur le corps des complexes $\mathbb{C}$ et à leurs topologies. Notre but est de saisir le lien entre la combinatoire et la topologie de tels arrangements.

Les motivations

Definition. Le *type topologique* d'un arrangement $\mathcal{A}$ dans $\mathbb{CP}^2$ est le type d'homéomorphisme de la paire $(\mathbb{CP}^2, \mathcal{A})$.

Depuis le début de l'étude des arrangements, les questions sur leurs topologies ont été fréquentes. En 1969, V.I. Arnauld donna une présentation de la cohomologie du complémentaire, vue comme une algèbre graduée, cf [2]. En 1973, P. Deligne [27] montra que le complémentaire de tout arrangement associé à un groupe de réflexion réel est asphérique. P. Orlik et L. Solomon [56] ont montré en 1980, que l'anneau de cohomologie du complémentaire ne dépend que de la combinatoire de l'arrangement.

Question. *Quel est l'influence de la combinatoire sur la topologie d'un arrangement ?*

D'un point de vue théorique, un arrangement de droites est un cas particulier de courbe algébrique plane. Des résultats fondamentaux ont été obtenus par O. Zariski [71]. Il a prouvé l'existence de paires de courbes admettant les mêmes données combinatoires et des topologies différentes. Il est donc naturel de définir la notion de *paire de Zariski* :

Definition ([4])**.** Une *paire de Zariski* est un couple d'arrangements $(\mathcal{A}_1, \mathcal{A}_2)$ ayant la même combinatoire mais des topologies différentes.

Pour des raisons techniques, nous considérons également le blow-up des arrangements en leurs points multiples. Une NC-paire de Zariski est donc une paire d'arrangements ayant la même combinatoire et des blow-ups avec des topologies différentes, –NC signifie croisement normaux–.

Dans [60], G. Rybnikov montre l'existence d'une paire de Zariski dans les cas des arrangements de droites, et en donne un exemple explicite. En étudiant l'exemple de

G. Rybnikov, les auteurs de [7, 6] : E. Artal, J. Carmona, J.I. Cogolludo et M.A. Marco, ont obtenu une paire de Zariski composée d'arrangements réels complexifiés. Des exemples de NC-paires de Zariski ont déjà été obtenues par A. Degtyarev dans le cas de courbes sextiques.

L'invariant topologique principal d'un arrangement $\mathcal{A}$ est son *complémentaire* $E_{\mathcal{A}}$:

$$E_{\mathcal{A}} = \mathbb{CP}^2 \setminus \mathcal{A}.$$

De cet invariant, de nombreux autres sont construits, le principal étant son groupe fondamental [46]. C'est un invariant très fort comme le prouvent [60, 22, 21].

Dans [65], E.R. van-Kampen donne une méthode pour calculer une présentation du groupe fondamental du complémentaire d'une courbe algébrique plane (connue sous le nom de *méthode de Zariski/van-Kampen*). Il utilise la position relative de la courbe par rapport à des pinceaux de droites dans $\mathbb{CP}^2$. Cette notion, la monodromie de tresse, a reçu sa première définition moderne et ses premières études avec D. Chéniot [17], O. Chisini [18] et B. Moishezon [51]. Dans [43], A. Libgober a construit un CW-complexe de dimension 2 à partir de la monodromie de tresse d'une courbe ayant le même type d'homotopie que le complémentaire de la courbe. Il existe d'autres contributions sur la monodromie faites par M. Salvetti [63], D. Cohen et A. Suciu [21], V.S. Kulikov et M. Taĭkher [41] et J. Carmona [15] qui décrit le lien entre la monodromie et la topologie.

La méthode de Zariski/van-Kampen est étudiée par R. Randell, dans [58], dans le cas des arrangements réels complexifiés. W. Arvola généralisa, dans [9], la méthode de R. Randell au cas des arrangements complexes en utilisant le diagramme de câblage qui encode la monodromie.

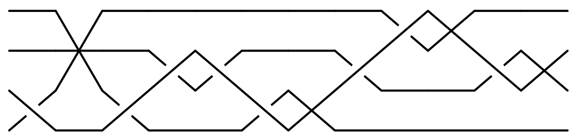

FIGURE 6 – Exemple de diagramme de câblage

Comme le problème des isomorphismes de groupes est indécidable, des invariants topologiques associés au groupe fondamental sont étudiés. L'un des principaux invariants que l'on étudiera est l'ensemble des variétés caractéristiques. Il forme un sous-ensemble du tore des caractères $\mathbb{T}(\mathcal{A})$: le groupe des caractères du groupe fondamental dans $\mathbb{C}^*$. Elles ont été introduites par A. Libgober dans le cas des courbes. Elles sont des invariants cohomologiques définies comme les lieux des sauts de la cohomologie tordue d'un système local de rang 1. Elles ne dépendent que du groupe fondamental.

Definition. Les variétés caractéristiques de $\mathcal{A}$ sont :

$$\mathcal{V}_k(\mathcal{A}) = \left\{ \xi \in \mathbb{T}(\mathcal{A}) \;\middle|\; \dim_{\mathbb{C}} \left(\mathrm{H}^1(E_{\mathcal{A}}; \mathcal{L}_\xi) \right) \geq k \right\},$$

où $\mathcal{L}_\xi$ est le système local de coefficients défini par ξ. La *profondeur* d'un caractère ξ est $\dim_{\mathbb{C}}\left(\mathrm{H}^1(E_{\mathcal{A}};\mathcal{L}_\xi)\right)$.

Le premier théorème général de structure pour les variétés caractéristiques a été découvert par D. Arapura. Il les décrit en termes de fibrations au dessus de courbes, cf [1] ; cette structure fut complétée par divers auteurs, E. Artal, J.I.Cogolludo et D. Matei [8], N. Budur [14], A. Dimca [29] et A. Libgober[47]. Les variétés caractéristiques d'un espace peuvent être vues comme une généralisation du polynôme d'Alexander. Ces invariants ont beaucoup été étudiés dans diverses perspectives, dans le cas général par S. Novik [54], W. Dwyer et D. Fried [32] et A. Libgober [44]. Dans le cas des arrangements, A. Libgober [45] donne un algorithme pour trouver les composantes irréductibles de $\mathcal{V}_k(\mathcal{A})$; E. Hironaka [39] montra l'égalité entre les variétés caratéristiques et une stratification du tore des caractères $\mathbb{T}(\mathcal{A})$ provenant du calcul des dérivées de Fox ; et D. Cohen et A. Suciu [23] ont prouvé que toutes les composantes irréductibles de $\mathcal{V}_k(\mathcal{A})$ passant par le caractère trivial sont déterminées par la combinatoire de l'arrangement. Elles sont reliées à l'ensemble de Green-Lazarsfeldt [36, 35].

Question. *Les variétés caractéristiques sont-elles déterminées par la combinatoire ?*

Des travaux plus récents de A. Libgober et, A. Dimca, D. Ibadula et D.A. Macinic ont montré que la partie projective de la profondeur des caractères diagonaux est de nature combinatoire si l'arrangement contient, au plus, des points de multiplicité 3 (cf [48]), et dans le cas des arrangements ayant des points de multiplicité 5, au plus, si l'arrangement contient moins de 14 droites (cf [30]).

L'étude de la *variété bord* est une autre approche de la question de la relation entre la combinatoire et la topologie. Elle donne une réponse partielle à cette question. En effet, cette variété ne dépend que de la combinatoire et peut être construite comme une variété graphée au-dessus du graphe d'incidence. E. Westlund décrit dans sa thèse [68] la structure de variété graphée de la variété bord ; D. Cohen et A. Suciu, dans [24], ont obtenu plusieurs résultats sur sa topologie. La variété bord est intimement liée au complémentaire d'un arrangement. Dans [40], E. Hironaka a étudié son inclusion dans le complémentaire pour les arrangements réels complexifiés, elle a ainsi montré que le complémentaire peut être construit –à homotopie près– comme un quotient de la variété bord.

Nos avancées et résultats

Cette thèse est le point d'intersection de ces deux directions : la variété bord et les variétés caractéristiques. Nous avons étudié l'application inclusion de la variété bord dans le complémentaire, cf [34]. De cette étude, nous avons tenté de définir de nouveaux invariants topologiques. Cette application a également servi à étudier la partie non combinatoire de la partie quasi-projective des variétés caractéristiques. En effet, E. Artal a développé un algorithme pour calculer la partie quasi-projective de la profondeur d'un caractère, en utilisant l'image de cycles particuliers de la variété bord dans le complémentaire, cf [5].

Nous avons ainsi travaillé sur la combinatorialité des variétés caractéristiques en utilisant la partie combinatoire du complémentaire donnée par la variété bord, et sur l'algorithme –non complet– d'E. Artal reliant la combinatoire et les variétés caractéristiques. Même si nous n'avons pas trouvé la réponse à cette question difficile qu'est le lien entre topologie et combinatoire, ces travaux ont permis de construire une méthode complète et explicite pour calculer la partie quasi-projective de la profondeur d'un caractère, ainsi que de découvrir un nouvel invariant permettant de distinguer des NC-paires de Zariski.

Dans la première partie de ce travail, nous avons généralisé le résultat obtenu par E. Hironaka dans [40] au cas des arrangements complexes. Pour contourner les problèmes dus au cas des arrangements non complexifiés, nous avons étudié le diagramme de câblage, et l'avons considéré comme un objet topologique à part entière. Afin de l'utiliser de façon systématique, nous avons développé un programme sous `Sage` permettant de le construire à partir des équations des droites de l'arrangement. Ce diagramme nous a permis de donner deux descriptions explicites de l'application :

$$i_* : B_{\mathcal{A}} \longrightarrow E_{\mathcal{A}}.$$

De ces deux descriptions, cf Théorème 3.2.2 et Théorème 3.2.7, nous avons obtenu deux nouvelles présentations du groupe fondamental du complémentaire, cf Corollaire 3.2.5 et Corollaire 3.2.8. Le second est une généralisation du Théorème de R. Randell [58].

Dans une deuxième étape, nous avons étudié l'application induite par l'inclusion sur les premiers groupes d'homologies. Nous avons donc obtenu deux descriptions simples de cette application, cf Théorème 4.1.1 et Théorème 4.1.3. Enfin, en s'inspirant des travaux de J.I. Cogolludo, nous avons trouvé une décomposition canonique du premier groupe d'homologie de la variété bord comme le produit de la 1-homologie et de la 2-cohomologie du complémentaire :

Theorem (4.3.2)**.** *Soient $E_{\mathcal{A}}$ le complémentaire de $\mathcal{A}$ et $B_{\mathcal{A}}$ sa variété bord, on a alors la suite exacte scindée :*

$$0 \to \mathrm{H}^2(E_{\mathcal{A}};\mathbb{C}) \longrightarrow \mathrm{H}_1(B_{\mathcal{A}};\mathbb{C}) \xrightarrow{i_*} \mathrm{H}_1(E_{\mathcal{A}};\mathbb{C}) \to 0.$$

Pour finir, nous avons obtenu un isomorphisme entre la 2-cohomologie du complémentaire et la 1-homologie du graphe d'incidence de l'arrangement. Ce qui fournit une décomposition intrinsèque de $\mathrm{H}_1(B_{\mathcal{A}};\mathbb{C})$ en $\mathrm{H}_1(E_{\mathcal{A}};\mathbb{C}) \oplus \mathrm{H}_1(\Gamma_{\mathcal{A}};\mathbb{C})$.

Dans la seconde partie de ce travail, nous avons appliqué les résultats précédents à l'étude des caractères sur le complémentaire. Notre point de départ fût les résultats d'E. Artal [5] sur le calcul de la profondeur d'un caractère. Cette profondeur se décompose en un terme projectif et un terme quasi-projectif. Un algorithme pour calculer la partie projective a été donné par A. Libgober [48]. E. Artal s'est concentré sur la partie quasi-projective, et a donné une méthode pour la calculer à partir de l'image de certains cycles du complémentaire. Nous avons donc utilisé le travail précédemment fait pour calculer explicitement ces cycles. En combinant ces deux travaux, nous avons obtenu un algorithme explicite pour calculer la profondeur quasi-projective d'un caractère.

Théorème 5.2.23 donne une condition combinatoire pour qu'un caractère admette une profondeur quasi-projective potentiellement non combinatoire. De cette propriété, nous avons défini la notion de *caractère inner-cyclic*. De leur étude, nous avons observé une condition forte sur la combinatoire pour n'avoir que des caractères de profondeur quasi-projective nulle, cf Proposition 6.1.1. Dans le but de réduire le nombre de calculs, nous avons introduit la notion de *combinatoire première*. Si une combinatoire n'est pas première, alors les variétés caractéristiques de ses réalisations sont complètement déterminées par une réalisation d'une combinatoire première avec moins de droites.

En parallèle, nous avons observé que la composition de l'application induite par l'inclusion avec un caractère spécifique fournissait des invariants topologiques du blow-up de l'arrangement, cf Théorème 5.3.14. Nous avons donc prouvé dans Théorème 6.4.6 et Corollaire 6.4.7, que cet invariant capture plus que la combinatoire. En effet, nous avons détecté deux nouveaux exemples de NC-paires de Zariksi. Le premier est une paire orientée et ordonnée, cf Théorème 6.3.2. Le second provient de la combinatoire suivante :

$$\begin{aligned}\mathscr{C} = [\ &[1,2],[1,3,5,7,12],[1,4,6,8],[1,9],[1,10,11],[2,3,6,9],\\ &[2,4,5,10],[2,7,11],[2,8],[2,12],[3,4],[3,8,11],[3,10],[4,7],\\ &[4,9,11],[4,12],[5,6],[5,8,9],[5,11],[6,7,10],[6,11],[6,12],\\ &[7,8],[7,9],[8,10],[8,12],[9,10],[9,12],[10,12],[11,12]\ ].\end{aligned}$$

Elle admet quatre réalisations notées $\mathscr{A}^{\pm}$ et $\mathscr{B}^{\pm}$.

Theorem (6.4.10 et 6.4.13)**.** *Les paires* $(\mathscr{A}^{\pm},\mathscr{B}^{\pm})$ *sont des* NC*-paires de Zariski, et le 4-tuplet* $(\mathscr{A}^{+},\mathscr{B}^{+},\mathscr{A}^{-},\mathscr{B}^{-})$ *est un* NC*-4-tuplet de Zariski orienté.*

Cela nous fournit le troisième exemple, connu dans la littérature, de NC-paire de Zariski. Les deux autres, étant aussi des paires de Zariski, sont obtenus dans [60, 6].

Plan de la thèse

Ce manuscrit est composé de six chapitres suivant le fil de notre travail :

Le premier chapitre est essentiellement composé des outils de base. Nous y définissons la notion d'arrangement de droites dans le plan projectif complexe. Ensuite, nous expliquons la notion de combinatoire et de réalisation. Dans la seconde section, nous rappelons la désingularisation d'un espace en utilisant la méthode du blow-up, et nous l'appliquons aux arrangements de droites.

Dans le Chapitre 2, nous étudions les diagrammes de câblage. La première section contient la construction théorique de ce diagramme. Ensuite, nous montrons comment l'utiliser pour calculer le groupe fondamental du complémentaire, comment en extraire la monodromie de tresse de l'arrangement, et comment construire le diagramme de câblage de l'arrangement conjugué. Dans la deuxième section, nous expliquons la construction d'un programme sous `Sage` afin de calculer ce diagramme. Dans la dernière section, nous calculons les diagrammes de câblage des arrangements de MacLane et de Fan.

La généralisation du résultat d'E. Hironaka est faite dans le Chapitre 3. Tout d'abord, nous décrivons la variété bord comme une variété graphée au-dessus du graphe d'incidence. De cette description, nous en déduisons une présentation de son groupe fondamental. Dans la deuxième section, nous donnons une description explicite de l'application induite par l'inclusion sur les groupes fondamentaux de la variété bord et du complémentaire. On en déduit deux nouvelles présentations du groupe fondamental du complémentaire, ainsi qu'une nouvelle preuve du Théorème de Randell. Pour conclure ce Chapitre, nous donnons le détail des calculs de cette application dans le cas de l'arrangement de MacLane positif.

Le Chapitre 4 est composé de la version homologique de l'application inclusion décrite dans le Chapitre précédent. Deux exemples sont détaillés dans la seconde section : Ceva-7 et MacLane (positif et négatif). La troisième section est composée d'une décomposition canonique du premier groupe d'homologie du complémentaire, ainsi que de l'étude du lien entre la 2-cohomologie du complémentaire et la 1-homologie du graphe d'incidence.

Les résultats d'E. Artal forment la partie principale du Chapitre 5. Dans la première section, nous rappelons la définition et des propriétés des variétés caractèristiques. La deuxième section est une description détaillée des résultats d'E. Artal. Dans la dernière section, nous définissons la notion de combinatoire inner-cyclic et de caractère inner-cyclic. Ensuite nous prouvons que l'image de cycle particulier de la variété bord par un caractère particulier est un invariant topologique du blow-up de l'arrangement.

Le dernier Chapitre est le point d'intersection des travaux précédents. La première section contient la définition d'une combinatoire première, et les raisons de leur étude. Dans la second section nous présentons deux programmes. L'un sous `Sage` pour détecter les combinatoires premières réalisables, et le second sous `Maple` qui donne les équations d'un arrangement à partir de sa combinatoire. Avec ces deux programmes, nous avons testé un grand nombre d'exemples, et nous présentons dans la troisième section les plus intéressants d'entre eux. La quatrième section est, quant à elle, composée de l'étude d'un exemple à onze droites, permettant la construction d'une NC-paire de Zariski.

Introducción (Spanish)

El primer dibujo que hace cualquier niño es una línea. ¿Qué es más natural que una recta?, ¿cuál es la diferencia entre dos dibujos de rectas?. Este es grosso modo el tema de estudio de la presente tesis. El soporte sobre el que vamos a trabajar es por supuesto ligeramente diferente a una hoja de papel, ya que trabajaremos sobre el plano proyectivo complejo.

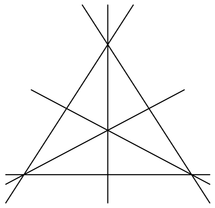

Figura 7: Configuración de Ceva (1678)

La primera idea para describir una configuración de rectas es definir el conjunto de rectas, el conjunto de puntos singulares y las relaciones de incidencia entre ambos conjuntos. Esta información combinatoria puede ser representada por grafos, como el grafo de Hasse [57] o el grafo de incidencia. Pero, como se puede ver en [71, 60, 7, 6], esta descripción no es suficiente para determinar una configuración.

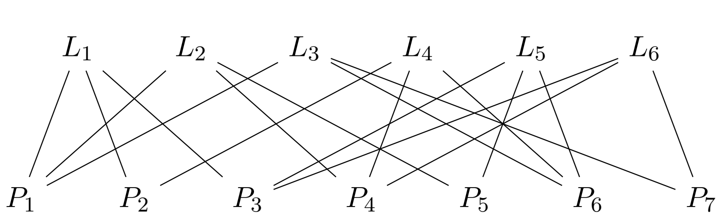

Figura 8: Grafo de incidencia de la configuración de Ceva

En matemáticas, el estudio de configuraciones de rectas es un dominio con sólo cuarenta y cinco años de historia. Por supuesto, la prehistoria de esta rama va mucho más lejos en el tiempo llegando al mundo griego, con Pappus de Alejandría [55], o en el siglo XVII con G. Ceva [16]. La primera materialización en las matemáticas modernas se debe a E. Brieskorn [11, 12], donde muestra que el grupo de trenzas puede ser construido como el grupo fundamental del complemento de una configuración de hiperplanos complejos. Este área relativamente joven de las matemáticas comienza en sus orígenes con el estudio de discriminantes, combinatoria, espacios de configuración, trenzas, etc.

Debido a sus múltiples orígenes, el estudio de configuraciones de hiperplanos admite diferentes aproximaciones: combinatorio, geométrico, aritmético y topológico. La interacción entre tales áreas constituye el rasgo más atractivo de este estudio. Más aun, las configuraciones de hiperplanos proporcionan una fuente fértil de ejemplos y problemas interesantes. Las técnicas desarrolladas para este estudio han sido satisfactoriamente exportadas a otras áreas. Las características de una configuración dependen del cuerpo sobre el que esta se define. Tenemos especial interés en el estudio de configuraciones de rectas definidas sobre el cuerpo complejo $\mathbb{C}$ y sus topologías. Nuestro propósito general es el de observar la relación entre sus combinatorias y sus topologías.

Motivaciones

Definition. El *tipo topológico* de una configuración de rectas $\mathcal{A}$ en $\mathbb{CP}^2$ es el tipo de homeomorfismo del par $(\mathbb{CP}^2, \mathcal{A})$.

Las cuestiones en torno a la topología de las configuraciones son frecuentes desde el inicio de su estudio. En 1969, V.I. Arnold da una presentación de la cohomología del complementario como un álgebra graduada [2]. En 1973, P. Deligne [27] muestra que el complementario de una configuración de rectas asociada a un grupo de reflexión real es un espacio de Eilenberg-McLane. P. Orlik y L. Solomon [56] probaron en 1980 que el anillo de cohomología del complementario depende únicamente de la combinatoria de la configuración.

Question. *¿Cuál es la influencia de la combinatoria en la topología de las configuraciones?*

Teóricamente, una configuración de rectas es un caso particular de curvas algebraicas, cuyos principales resultados fueron obtenidos por O. Zariski [71]. Zariski mostró la existencia de pares de curvas admitiendo la misma combinatoria pero diferentes tipos topológicos, que son definidos como *pares de Zariski*:

Definition ([4])**.** Un *par de Zariski* de una configuración de rectas es un par $(\mathcal{A}_1, \mathcal{A}_2)$ tal que $\mathcal{A}_1$ y $\mathcal{A}_2$ admiten la misma estructura combinatoria pero poseen diferentes tipos topológicos.

Por razones técnicas, vamos a considerar también *blow-ups* o *explosiones* de una configuración de rectas en sus puntos múltiples. Un NC-*par de Zariski* es un par de configuraciones de rectas con la misma estructura combinatoria pero cuyos blow-ups poseen diferentes tipos topológicos, – donde NC proviene de los cruces normales (*normal crossing*)–.

En [60], G. Rybnikov muestra la existencia de un par de Zariski en el caso de configuraciones de rectas dando un ejemplo explícito. Estudiando el resultado de G. Rybnikov, los autores de [7, 6]: E. Artal, J. Carmona, J.I. Cogolludo y M.A. Marco, obtienen un par de Zariski de configuraciones reales complexificadas. Ejemplos de NC-pares de Zariski fueron obtenidos por A. Degtyarev [25] en el caso de curvas séxticas.

El principal invariante topológico de una configuración $\mathcal{A}$ es su *complementario* $E_{\mathcal{A}}$:

$$E_{\mathcal{A}} = \mathbb{CP}^2 \setminus \mathcal{A}.$$

Numerosos invariantes están construidos a partir del complementario, siendo el más importante el grupo fundamental de la configuración [46]. Como se puede ver en [60, 22, 21] el grupo fundamental es un invariante fuerte de la configuración.

En [65], E.R. van Kampen, da un método para calcular la presentación del grupo fundamental del complementario de una curva algebraica plana (conocido como el *método de Zariski-van Kampen*). En este método, utiliza la posición relativa de haces de rectas en $\mathbb{CP}^2$. Esta noción, llamada *monodromía de trenzas*, tiene su primer estudio moderno de la mano de D. Chéniot [17], O. Chisini [18] y B. Moishezon [51]. En [43], A. Libgober construye un CW-complejo 2-dimensional a partir de la monodromía de trenzas admitiendo el mismo tipo de homotopía que el complementario de una cierta curva. Existen numerosas contribuciones dadas por M. Salvetti [63], D. Cohen y A. Suciu [21], V.S. Kulikov y M. Taĭkher [41] y J. Carmona [15] sobre la relación entre la topología y la monodromía de trenzas.

El método de Zariski-van Kampen para el caso de configuraciones de rectas reales complexificadas es estudiado por R. Randell en [58]. W. Arvola generaliza, en [9], el algoritmo de R. Randell para el caso de configuraciones de rectas complejas utilizando el *diagrama de cableado de trenzas* que codifica la monodromía de trenzas de la configuración.

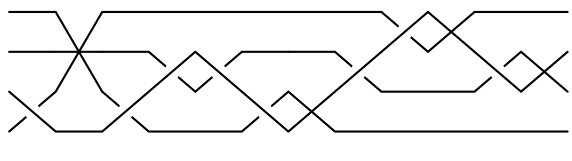

Figura 9: Ejemplo de diagrama de cableado

Ya que el problema de saber si dos grupos son isomorfos a partir de dos presentaciones es indecidible, se opta por estudiar los invariantes topológicos asociados. Uno de los principales invariantes que estudiamos es el conjunto de variedades características, que son subconjuntos del toro de caracteres $\mathbb{T}(A)$, el grupo de caracteres en $\mathbb{C}^*$ del grupo fundamental de la configuración. Éstos fueron introducidos por primera vez por A. Libgober en el caso de curvas. Son invariantes cohomológicos definidos como los lugares de saltos de la cohomología torcida de sistemas locales de rango uno en la variedad, quienes sólo dependen del grupo fundamental.

Definition. Las variedades características de $\mathcal{A}$ son:

$$\mathcal{V}_k(\mathcal{A}) = \left\{ \ \xi \in \mathbb{T}(\mathcal{A}) \ \mid \ \dim_{\mathbb{C}} \left(\mathrm{H}^1(E_{\mathcal{A}}; \mathcal{L}_\xi) \right) \geq k \ \right\},$$

donde $\mathcal{L}_\xi$ es el sistema local de coeficientes definido por ξ. La *profundidad* de un carácter ξ es $\dim_{\mathbb{C}} \left(\mathrm{H}^1(E_{\mathcal{A}}; \mathcal{L}_\xi) \right)$.

El primer teorema general de estructura para estos lugares de salto fue descubierto por D. Arapura, quien los describió en términos de fibraciones sobre curvas [1]; esta estructura fue completamente descrita por varios autores, E. Artal, J.I.Cogolludo y D. Matei [8], N. Budur [14], A. Dimca [29] y A. Libgober[47]. Las variedades características de un espacio pueden verse como una generalización del polinomio de Alexander. Este invariante ha sido ampliamente estudiado desde diversos puntos de vista y en el caso general por S. Novik [54], W. Dwyer y D. Fried [32] y A. Libgober [44]. En el caso de configuraciones, A. Libgober [45] da un algoritmo para encontrar las componentes irreducible de $\mathcal{V}_k(\mathcal{A})$, E. Hironaka [39] demuestra la igualdad entre variedades características con una estratificación del toro de caracteres de $\pi_1(E_{\mathcal{A}})$ a partir de las derivadas de Fox, y D. Cohen y A. Suciu [23] prueba que todas las componentes irreducibles de $\mathcal{V}_k(\mathcal{A})$ conteniendo el elemento trivial están combinatoriamente determinadas. Las variedades características están relacionadas con el conjunto de Green–Lazarsfeld [36, 35].

Question. *¿Las variedades características de las configuraciones de rectas están determinadas combinatoriamente?*

Trabajos más recientes de A. Libgober y A. Dimca, D. Ibadula y D.A. Macinic muestran que para el carácter diagonal la profundidad proyectiva es de naturaleza combinatoria si el lugar singular de $\mathcal{A}$ tiene multiplicidad inferior a tres (ver [48]), y si tiene multiplicidad inferior a 5 y $\mathcal{A}$ contiene menos de 14 rectas (ver [30]).

El estudio de la *variedad borde* es otro enfoque de esta cuestión sobre la relación entre la combinatoria y la topología, pero nos da una respuesta parcial a esta cuestión. En efecto, esta variedad depende únicamente de la combinatoria y puede ser construida como una variedad grafo sobre el grafo de incidencia. E. Westlund describió, en su tesis [68], la estructura de variedad grafo de la variedad borde, así como D. Cohen y A. Suciu in [24] obtuvieron diversos resultados sobre su topología. La variedad borde está íntimamente relacionada con el complementario de una configuración. En [40], E. Hironaka estudia su inclusión en el complementario de configuraciones reales complexificados, también demuestra que el complementario puede ser construido –salvo homotopía– como cociente de la variedad borde.

Nuestros avances y resultados

Esta tesis estudia problemas que se encuentran en la intersección de dos objetos de estudio: la variedad borde y las variedades características. Hemos estudiado la aplicación inclusión de la variedad borde en el complementario, ver [34]. El objetivo de este estudio era el de encontrar y definir nuevos invariantes topológicos. Esta aplicación nos ha ayudado también a estudiar la parte no combinatoria de la parte cuasi-proyectiva de las variedades características. De hecho, E. Artal ha desarrollado un algoritmo para calcular la parte cuasi-proyectiva de la profundidad de un carácter utilizando la imagen de ciclos particulares de la variedad borde en el complementario, ver [5].

Hemos trabajado también sobre la combinatorialidad de las variedades características utilizando la parte combinatoria del complementario dado por la variedad borde, y sobre el algoritmo –no completo– de E. Artal relacionando la combinatoria y las variedades características. Aunque no hayamos encontrado una respuesta a esta difícil cuestión que es la relación entre la topología y la combinatoria, estos estudios nos han permitido construir un método completo y explícito para calcular la parte cuasi-projectiva de la profundidad de un carácter, así como descubrir un nuevo invariante permitiendo distinguir algunos NC-pares de Zariski de configuraciones de rectas.

En la primera parte de este trabajo, hemos generalizado el resultado obtenido por E. Hironaka en [40] para el caso de configuraciones de rectas complejas. Para resolver los problemas provenientes de configuraciones no complexificadas, hemos estudiado el diagrama de cableado considerándolo enteramente como un objeto topológico. Con el fin de utilizarlo de forma sistemática, hemos desarrollado un programa en `Sage` para construirlo a partir de las ecuaciones de las rectas de la configuración. Este diagrama nos ha permitido dar dos expresiones explícitas de la aplicación:

$$i_* : \pi_1(B_{\mathcal{A}}) \longrightarrow \pi_1(E_{\mathcal{A}}).$$

A partir de estas descripciones, ver Teorema 3.2.2 y Teorema 3.2.7, obtenemos dos nuevas presentaciones del grupo fundamental del complementario, ver Corolario 3.2.5 y Corolario 3.2.8. El segundo corolario es, de hecho, una generalización del Teorema de Randell [58].

En una segunda parte de este trabajo, hemos estudiado la aplicación inducida por la inclusión sobre los primeros grupos de homología, obteniendo descripciones simples de esta aplicación, ver Teorema 4.1.1 y Teorema 4.1.3. Luego, inspirados en los trabajos de J.I. Cogolludo, hemos encontrado una descomposición canónica del primer grupo de homología de la variedad borde como el producto de la 1-homología y de la 2-cohomología del complementario:

Theorem (4.3.2)**.** *Sea $E_{\mathcal{A}}$ el complementario de $\mathcal{A}$ y $B_{\mathcal{A}}$ su variedad borde, tenemos entonces una sucesión exacta escindida:*

$$0 \to \mathrm{H}^2(E_{\mathcal{A}}; \mathbb{C}) \longrightarrow \mathrm{H}_1(B_{\mathcal{A}}; \mathbb{C}) \xrightarrow{i_*} \mathrm{H}_1(E_{\mathcal{A}}; \mathbb{C}) \to 0.$$

Finalmente, hemos obtenido un isomorfismo entre la 2-cohomología del complementario con la 1-homología del grafo de incidencia de la configuración. Esto nos da una descomposición intrínseca de $\mathrm{H}_1(B_{\mathcal{A}};\mathbb{C})$ as $\mathrm{H}_1(E_{\mathcal{A}};\mathbb{C}) \oplus \mathrm{H}_1(\Gamma_{\mathcal{A}};\mathbb{C})$.

En una segunda parte de este trabajo, hemos aplicado los resultados precedentes en el estudio de los caracteres sobre el complementario. Nuestro punto de partida viene de resultados de E. Artal [5] en el cálculo de la profundidad de un carácter. Esta profundidad puede descomponerse en una *parte proyectiva* y una *parte cuasi-proyectiva*; esta última es nula en los caracteres que ramifican a lo largo de todas las rectas. Para calcular la parte proyectiva, A. Libgober da un algoritmo en [48]. E. Artal se centra en la parte cuasi-proyectiva y da un método para calcularla a partir de la imagen de ciertos ciclos del complementario. Usando estos resultados sobre la aplicación inclusión de la variedad borde, podemos determinar estos ciclos explícitamente. Combinando estos dos estudios, hemos obtenido un algoritmo explícito para calcular la profundidad cuasi-proyectiva de un carácter.

El Teorema 5.2.23 nos da una condición combinatoria para que un carácter admita una profundidad cuasi-proyectiva potencialmente no combinatoria. A partir esta propiedad, hemos definido la noción de *carácter inner-cyclic*. A partir de su estudio, hemos observado una condición fuerte sobre la combinatoria para que una configuración pueda tener caracteres en los que la profundidad cuasi-proyectiva es no nula, ver Proposición 6.1.1. En relación a tal condición y con la intención de reducir el número de cálculos, hemos introducido la noción de *combinatoria prima*. Si una combinatoria no es prima, entonces las variedades características de sus realizaciones están definidas por las de una subconfiguración con un número menor de rectas.

Paralelamente a este estudio, hemos observado que la composición de la aplicación inducida por la inclusión sobre el primer grupo de homología con un carácter proporciona un invariante topológico de configuraciones de rectas desingularizadas, ver Teorema 5.3.14. También mostramos en el Teorema 6.4.6 y el Corolario 6.4.7 que este invariante contiene más información que la combinatoria. De hecho, hemos podido detectar dos nuevos ejemplos de NC-pares de Zariski. El primero es un par orientado y ordenado, ver Teorema 6.3.2. El segundo proviene de la siguiente combinatoria:

$$\begin{aligned}\mathscr{C} = [\ & [1,2],[1,3,5,7,12],[1,4,6,8],[1,9],[1,10,11],[2,3,6,9],\\ & [2,4,5,10],[2,7,11],[2,8],[2,12],[3,4],[3,8,11],[3,10],[4,7],\\ & [4,9,11],[4,12],[5,6],[5,8,9],[5,11],[6,7,10],[6,11],[6,12],\\ & [7,8],[7,9],[8,10],[8,12],[9,10],[9,12],[10,12],[11,12]\ ].\end{aligned}$$

Esta combinatoria admite cuatro realizaciones denotadas por $\mathscr{A}^\pm$ y $\mathscr{B}^\pm$.

Theorem (6.4.10 y 6.4.13)**.** *Los pares $(\mathscr{A}^\pm,\mathscr{B}^\pm)$ son* NC*-pares de Zariski, y la 4-tupla $(\mathscr{A}^+,\mathscr{B}^+,\mathscr{A}^-,\mathscr{B}^-)$ es un* NC*-4-tupla de Zariski orientado.*

Esto nos da el tercer ejemplo conocido en la literatura de NC-par de Zariski. Los otros dos –que son también pares de Zariski– fueron obtenidos en [60, 6].

Plan de tesis

Este manuscrito se compone de seis capítulos, siguiendo nuestro trabajo:

El primer capítulo está esencialmente compuesto con los conceptos y herramientas de base. Definimos la noción de configuraciones de rectas en el plano complejo proyectivo. Luego explicamos la noción de combinatoria y sus realizaciones. En la segunda sección, recordamos la desingularización de un espacio mediante blow-ups o explosiones, que aplicaremos a las configuraciones de rectas.

En el Capítulo 2, estudiamos los diagramas de cableado, introducidos por W. Arvola en [9] para calcular el grupo fundamental de una configuración. La primera sección contiene la construcción de este diagrama. Luego mostramos cómo utilizarlo para calcular el grupo fundamental del complementario, cómo obtener la monodromía de trenzas de la configuración, y cómo construir el diagrama de cableado de la configuración conjugada. En la segunda sección, explicaremos la construcción del programa en `Sage` para calcular este diagrama. En la última sección, calculamos los diagramas de cableado de las configuraciones de MacLane y Fan.

Presentamos la generalización del resultado de E. Hironaka en el Capítulo 3. Primeramente, describimos la variedad borde como la variedad grafo sobre el grafo de incidencia de la configuración y deducimos una presentación de su grupo fundamental. En la segunda sección, damos una descripción explícita de la aplicación inducida por la inclusión sobre los grupos fundamentales de la variedad borde y del complementario. A partir de aquí, deducimos dos nuevas presentaciones del grupo fundamental del complementario, así como una nueva demostración del Teorema de Randell. Para concluir este capítulo, damos los detalles de los cálculos de esta aplicación en el caso de la configuración de MacLane positiva.

El Capítulo 4 está compuesto por la versión homológica de la aplicación inclusión descrita en el capítulo precedente. Daremos los detalles de dos ejemplos en la segunda sección: Ceva 7 y MacLane (positivo y negativo), La tercera sección está compuesta por una descomposición canónica del primer grupo de homología del complementario, así como de el estudio de la relación entre la 2-cohomología del complementario y la 1-homología del grafo de incidencia.

Los resultados de E. Artal conforman la parte principal del Capítulo 5. En la primera sección, recordaremos la definición y algunas propiedades de las variedades características. La segunda parte es una descripción detallada de los resultados de E. Artal. En la última sección, definiremos la noción de combinatoria *inner-cyclic* y de carácter *inner-cyclic*. A partir de esto, probaremos que la imagen de un ciclo particular de la variedad borde por un carácter particular es un invariante topológico del blow-up de la configuración.

El último Capítulo es el punto de intersección de los trabajos precedentes. La primera sección contiene la definición de combinatoria prima y las razones principales de su estudio. En la segunda sección presentaremos dos programas: uno sobre `Sage` para detectar las combinatorias primas realizables, y otro sobre `Maple` que da las ecuaciones de una configuración a partir de su combinatoria. Con estos dos programas hemos podido analizar un gran número de ejemplos, presentando en la tercera sección los más interesantes de entre ellos. La cuarta sección es principalmente el estudio de un ejemplo de configuración de once rectas, a partir del cual construimos un NC-par de Zariski.

Chapter 1

Preliminaries

In this chapter, we introduce the basic tools used in the study of topological and combinatorial aspects of complex line arrangements. Along the way, we preview some of the important problems already solved connecting these two facets. This chapter has been inspired by the books [57, 20].

We start with an elementary presentation of the notion of complex line arrangements and its topological type (see [6]), then we define the combinatorics of an arrangement and the realization of a combinatorics. Lastly, we construct a graph containing all the combinatorial information of an arrangement.

In Section 1.2, we introduce the blow-up of a complex surface at a singular point (see [13]), and we apply this construction to line arrangements. Then, we construct a space X with a normal crossing divisor $\hat{\mathcal{A}}$ such that its complement in X is homeomorphic to the complement of the line arrangement $\mathcal{A}$ in $\mathbb{CP}^2$. From this construction we define the dual graph of the arrangement.

1.1 Definitions

1.1.1 Line arrangements

A *hyperplane arrangement* $\mathcal{A}$ in a vector space V (or in a projective space) is a finite set of hyperplanes $\{L_0, \cdots, L_n\}$, with $L_i \neq L_j$ for $i \neq j$. In this thesis, we consider the case of lines in $\mathbb{CP}^2$, and $\mathcal{A}$ is a *complex line arrangement*. There exist n homogenous polynomials $F_i(x, y, z)$ of degree 1 such that:

$$L_i = \{F_i(x, y, z) = 0\}.$$

Definition 1.1.1. The product

$$P_{\mathcal{A}}(x,y,z)=\prod_{i=0}^{n}F_i(x,y,z),$$

is a *defining polynomial* of $\mathcal{A}$.

Definition 1.1.2. A *real line arrangement* is a finite set of lines in $\mathbb{RP}^2$. It can be *complexified* to form a complex line arrangement in $\mathbb{CP}^2\simeq\mathbb{RP}^2\otimes\mathbb{C}$.

Remark 1.1.3. If $\mathcal{A}$ is a complexified real arrangement, then the L_i's can be defined by polynomial with real coefficients. In this case $P_{\mathcal{A}}$ is a real polynomial. Nevertheless, the converse is wrong.

Consider the local chart $\{[x:y:z]\in\mathbb{CP}^2 \mid z\neq 0\}\simeq\mathbb{C}^2$ (i.e. the line $\{z=0\}$ is the line at infinity). The image of $\mathcal{A}$ by this chart is the *affine part* $\mathcal{A}^{\text{aff}}$ of the arrangement. The affine part of each L_i (also denoted by L_i) is defined by $\{f_i(x,y)=F_i(x,y,1)=0\}$.

Remark 1.1.4. With a linear changing of coordinates, we may assume that $L_0=\{z=0\}$. In this case, the affine part $\mathcal{A}^{\text{aff}}$ of $\mathcal{A}$ is composed of the affine part of the lines $L_1,\cdots,L_n$.

The set of singular points of $\mathcal{A}$ (i.e. the intersection of the lines) is denoted by $\mathcal{Q}=\{P_1,\cdots,P_l\}$, with $P_i=[x_i:y_i:z_i]$. For $P\in\mathcal{Q}$, let us define:

$$\mathcal{A}_P=\{L\in\mathcal{A}\mid P\in L\}.$$

For $P_i\in\mathcal{Q}$, let $\mathcal{A}_{P_i}$ be the set $\{L_{i_1},\cdots,L_{i_m}\}$, if necessary the point P_i is denoted by $P_{i_1,\cdots,i_m}$. The number $m=\#\mathcal{A}_{P_i}$ is the *multiplicity* of P_i.

The set of singular points of $\mathcal{A}^{\text{aff}}$ is –up to a reordering of $\mathcal{Q}$– $\mathcal{P}=\{P_1,\cdots,P_k\}$, with $k\leq l$. They are contained in the chart $\{[x:y:z]\in\mathbb{CP}^2\mid z\neq 0\}$, assume that $z_i=1$ for $i\leq k$, then we have affine coordinates (x_i,y_i) for P_i.

Definition 1.1.5. The *topological type* of an arrangement is the homeomorphism type of the pair $(\mathbb{CP}^2,\bigcup L_i)$.

Notation. By abuse of notation, the union $\bigcup\limits_{L_i\in\mathcal{A}} L_i$ is also denoted by $\mathcal{A}$.

Two arrangements $\mathcal{A}$ and $\mathcal{A}'$ are *topologically equivalent* if there exists a homeomorphism $\Phi:\mathbb{CP}^2\to\mathbb{CP}^2$ such that $\Phi(\mathcal{A})=\mathcal{A}'$.

An *ordered line arrangement* is a line arrangement where the lines are ordered. Let Φ be a topological equivalence between $\mathcal{A}$ and $\mathcal{A}'$. Assume that they are ordered line arrangements. Then Φ is a *topological equivalence of the ordered line arrangement* $\mathcal{A}$ and $\mathcal{A}'$ if it respects their orders. An *oriented topology* of a line arrangement is the oriented homeomorphism type of the pair $(\mathbb{CP}^2,\bigcup L_i)$, keeping the orientations of both the plane and the lines.

Definition 1.1.6. The *complement* of $\mathcal{A}$ is denoted by $E_{\mathcal{A}}$, and defined by:

$$E_{\mathcal{A}}=\mathbb{CP}^2\setminus\mathcal{A}.$$

Remark 1.1.7. If L_0 is the line at infinity, then the complement of $\mathcal{A}^{\text{aff}}$ in $\mathbb{C}^2$ coincides with $E_{\mathcal{A}}$.

Theorem 1.1.8. *Let $\mathcal{A}$ and $\mathcal{A}'$ be two topologically equivalent arrangements, then $E_{\mathcal{A}}$ and $E_{\mathcal{A}'}$ have isomorphic fundamental groups.*

Proof. By definition $\mathcal{A}$ and $\mathcal{A}'$ have homeomorphic complements, then they have isomorphic fundamental groups. □

However, there are some arrangements with isomorphic fundamental groups but non homeomorphic complements. A classical example of such arrangements is given by M. Falk in [33], and it is pictured in Figure 1.1.

∞ ∞

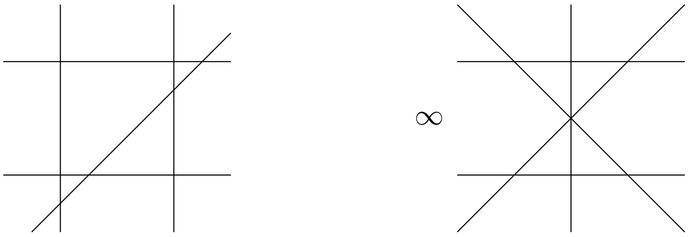

Figure 1.1: Falk's example

Notation. In the whole manuscript, only the affine part of the arrangement are pictured. The symbol ∞ means that the infinity line is in the arrangement.

Proposition 1.1.9 ([64, 46])**.** *If two arrangements lie in a connected family of equisingular curves, then they have isomorphic fundamental groups.*

In [24], D. Cohen and A. Suciu construct a regular neighborhood U of $\mathcal{A}$ as follows. Choose homogenous coordinates $\boldsymbol{x} = [x_0 : x_1 : x_2]$ on $\mathbb{CP}^2$. A closed, regular neighborhood of $\mathcal{A}$ may be constructed as follows. Define $\psi : \mathbb{CP}^2 \to \mathbb{R}$ by $\psi(\boldsymbol{x}) = |P_{\mathcal{A}}(\boldsymbol{x})|^2/||\boldsymbol{x}||^{2(n+1)}$, and let $U = \psi^{-1}([0, \delta])$, for $\delta > 0$ sufficiently small. An alternative construction using triangulation is done by A.H. Durfee in [31].

Definition 1.1.10. Let U be a compact regular neighborhood of $\mathcal{A}$. The *exterior* of $\mathcal{A}$ is the closure of $\mathbb{CP}^2 \setminus U$, and it is denoted by $Ext_{\mathcal{A}}$.

Remark 1.1.11. The exterior of $\mathcal{A}$ is a manifold with boundary, whose boundary is a graph manifold, see [66, 53] for definition and construction of graph manifolds.

Proposition 1.1.12. *The spaces $E_{\mathcal{A}}$ and $Ext_{\mathcal{A}}$ have the same homotopy type.*

Proof. By construction $Ext_{\mathcal{A}}$ is a deformation retract of $E_{\mathcal{A}}$. □

1.1.2 Combinatorics

The combinatorics of a line arrangement is studied in [57]. The formal definition of a *line combinatorics* is the following, see [57] for definitions and properties of combinatorics of line and hyperplane arrangements.

Definition 1.1.13. A *line combinatorics* is a triple $(\mathcal{L}, \mathcal{P}, \in)$ where $\mathcal{L}$ and $\mathcal{P}$ are finite subsets, and $\in$ a relation between $\mathcal{P}$ and $\mathcal{L}$ satisfying:

- $\forall \ell, \ell' \in \mathcal{L}, \ell \neq \ell', \exists! p \in \mathcal{P}$ such that $p \in \ell$ and $p \in \ell'$ (noted $p = \ell \cap \ell'$).
- $\forall p \in \mathcal{P}, \exists \ell, \ell' \in \mathcal{L}, \ell \neq \ell'$ such that $p = \ell \cap \ell'$.

Analogously we define an *ordered line combinatorics*, where we order the elements of $\mathcal{L} = \{\ell_0, \cdots, \ell_n\}$. For convenient we denote a combinatorics by a list of m-tuples. Each m-tuple corresponds to an element p of $\mathcal{P}$, which contains the index of lines containing p.

Definition 1.1.14. Let $C_1 = (\mathcal{L}_1, \mathcal{P}_1, \in_1)$ and $C_2 = (\mathcal{L}_2, \mathcal{P}_2, \in_2)$ be two line combinatorics. An *isomorphism* between C_1 and C_2 is composed of two one-to-one applications $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ and $\psi : \mathcal{P}_1 \to \mathcal{P}_2$, such that:

$$P \in_1 L \text{ and } P \in_1 L' \iff \psi(P) \in_2 \phi(L) \text{ and } \psi(P) \in_2 \phi(L').$$

We then write, $C_1 \cong C_2$. The set of the *automorphisms* of a combinatorics C is the *automorphism group* of C, its is denoted by Aut_C.

Example 1.1.15.

(1) The automorphism group of the combinatorics $[[1, \cdots, n]]$ (i.e. a pencil of n lines) is the symmetric group on n elements $\mathfrak{S}_n$.

(2) The automorphism group of the combinatorics $[[1, 2, 3], [1, 4], [2, 4], [3, 4]]$ is the symmetric group on $\{1, 2, 3\}$ elements $\mathfrak{S}_3$.

Up to automorphism, each line arrangement $\mathcal{A}$ has a unique line combinatorics $C_{\mathcal{A}}$. It is composed of the set of the lines, the set of the singular points $\mathcal{P}$, and the incidence relations.

Example 1.1.16. The combinatorics of the arrangement pictured in Figure 1.2 is:

[[0,1],[0,2,3],[0,4,5],[0,6,7],[1,2,4],[1,3,6],[1,5,7],[2,5],[2,6],[2,7],[3,4],[3,5],[3,7],[4,6],[4,7],[5,6]].

Proposition 1.1.17. *Let $\mathcal{A}$ and $\mathcal{A}'$ be two topologically equivalent arrangements, then: $C_{\mathcal{A}} \cong C_{\mathcal{A}'}$.*

However, the converse is wrong. G. Rybnikov shows in [60] the existence of two line arrangements with the same combinatorics but with non-isomorphic fundamental groups, then with different topological types. Another example of such pair is given in [6].

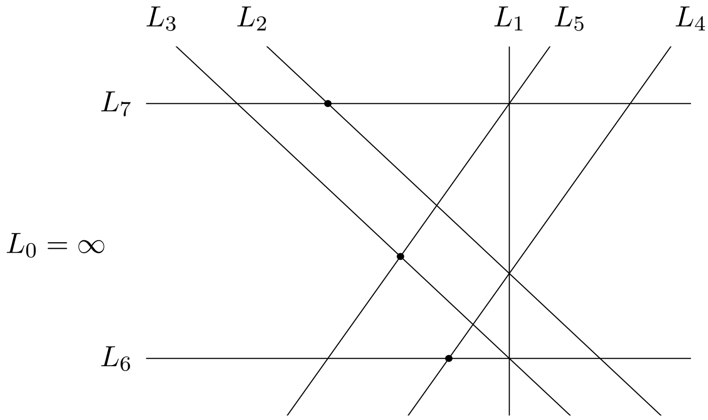

Figure 1.2: Pappus arrangement

Remark 1.1.18. These are the only two examples currently known.

Definition 1.1.19 ([28])**.** A *Zariski pair* is a pair of arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ with the same combinatorics, but with different topological type.

Definition 1.1.20. A *realization* of a line combinatorics $C = (\mathcal{L}, \mathcal{P}, \in)$, is a line arrangement $\mathcal{A}$ with combinatorics C. An *ordered realization* of an ordered line combinatorics is defined accordingly.

Notation. The space of all realizations of a line combinatorics C is denoted by $\Sigma(C)$. If C is ordered, we denote by $\Sigma^{ord}(C)$ the space of all ordered realizations of C.

There is a natural action of $\mathrm{PGL}_3(\mathbb{C})$ on $\Sigma(C)$ and $\Sigma^{ord}(C)$. This justifies the following definition.

Definition 1.1.21. The *moduli space* of a combinatorics C is the quotient:

$$\mathcal{M}(C) = \Sigma(C)/\,\mathrm{PGL}_3(\mathbb{C}).$$

The *ordered moduli space* $\mathcal{M}^{ord}(C)$ of an ordered combinatorics C is defined accordingly.

For more details about moduli space, see [52].

Remark 1.1.22. Let us consider an ordered line combinatorics C, $\mathrm{Aut}(C)$ acts on $\mathcal{M}^{ord}(C)$.

Proposition 1.1.23. *All realizable combinatorics admit at least a realization in a number field.*

Proof. Let $L_i = \big\{[x : y : z] \in \mathbb{CP}^2 \mid a_i x + b_i y + c_i z = 0\big\}$ for $i = 1, 2, 3$ be lines of $\mathbb{CP}^2$. They are concurrent in $\mathbb{CP}^2$ if and only if:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

Then the equations satisfied by the coefficients of the lines of $\mathcal{A}$ are polynomials of degree 3. Thus there is at least one solution in a number filed. $\square$

Remark 1.1.24. There are combinatorics that do not admit realization.

Consider the combinatorics C of the Pappus arrangement shown in Example 1.1.16. A simple computation shows that Pappus' arrangement is –up to deformation– the only one realization of C. Add a line L_8 to C, and assume that it passes through $P_{2,7}$ and $P_{3,5}$, and that it generically intersects the other lines. We then obtain:

$$[\ [0,1],[0,2,3],[0,4,5],[0,6,7],[0,8],[1,2,4],[1,3,6],[1,5,7],\\ [1,8],[2,5],[2,6],[2,7,8],[3,4],[3,5,8],[3,7],[4,6],[4,7],[4,8],[5,6],[7,8]\ ].$$

Pappus' Theorem implies that the three points marked with a $\bullet$ in Figure 1.2 (i.e. $P_{2,7}$, $P_{3,5}$ and $P_{4,6}$) are aligned. But the line L_8 added to C passes through by $P_{2,7}$ and $P_{3,5}$, but not by $P_{4,6}$. Then this combinatorics is not realizable.

1.1.3 Incidence graph

The incidence graph is a tool to encode the combinatorial information of an arrangement, see [57] for details. It is equivalent to the combinatorics. For $P \in \mathcal{Q}$, let us recall that $\mathcal{A}_P = \{L \in \mathcal{A} \mid P \in L\}$.

Definition 1.1.25. The *incidence graph* $\Gamma_\mathcal{A}$ of $\mathcal{A}$ is a non-oriented bipartite graph where the set of vertices $V(\mathcal{A})$ decomposes as $V_P(\mathcal{A}) \amalg V_L(\mathcal{A})$, with

$$V_P(\mathcal{A}) = \{v_P \mid P \in \mathcal{Q}\}, \quad \text{and} \quad V_L(\mathcal{A}) = \{v_L \mid L \in \mathcal{A}\}.$$

The vertices of $V_P(\mathcal{A})$ are called *point-vertices* and those of $V_L(\mathcal{A})$ are called *line-vertices*. The edges of $\Gamma_\mathcal{A}$ join v_L to v_P if and only if $L \in \mathcal{A}_P$. They are denoted by $e(L,P)$.

Since two lines always intersect, then it is not necessary to conserve this information in the incidence graph if it is not in a singular point of multiplicity greater than 3. Then we can define the *reduced incidence graph*:

Definition 1.1.26. The *reduced incidence graph* $\Gamma'_\mathcal{A}$ of $\mathcal{A}$ is a non-oriented graph where the set of vertices $V'(\mathcal{A})$ decomposes as $V'_P(\mathcal{A}) \amalg V'_L(\mathcal{A})$, with

$$V'_P(\mathcal{A}) = \{v_P \mid P \in \mathcal{Q}, m_P \geq 3\} \subset V_P, \quad V'_L(\mathcal{A}) = \{v_L \mid L \in \mathcal{A}\} = V_L.$$

The edges of $\Gamma'_\mathcal{A}$ are of two types:

- It joins v_L to v_P if and only if $L \in \mathcal{A}_P$ and $m_P \geq 3$.
- It joins v_{L_i} to v_{L_j}, with $i \neq j$, if and only if the point $P = L_i \cap L_j$ has multiplicity 2.

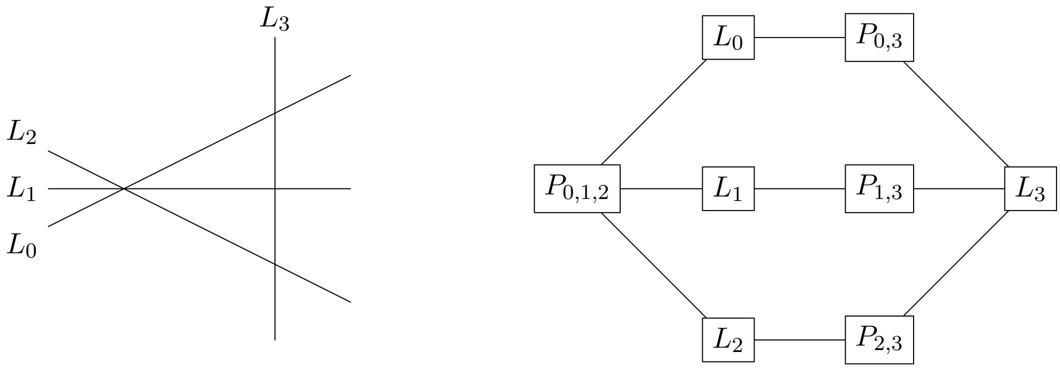

Figure 1.3: Example of an incidence graph

Example 1.1.27. The incidence graph of the arrangement composed of three concurrent lines L_0, L_1, L_2 and a generic line L_3 is pictured in Figure 1.3.

An isomorphism of incidence graphs sends elements of $V_P(\mathcal{A})$ (resp. $V_L(\mathcal{A})$) to elements of $V_P(\mathcal{A})$ (resp. $V_L(\mathcal{A})$), and preserve the incidence relations. The automorphisms of an incidence graph induce automorphisms of the corresponding line combinatorics.

Proposition 1.1.28. *Let $\mathcal{A}$ and $\mathcal{A}'$ be two topologically equivalent line arrangements, then $\mathcal{A}$ and $\mathcal{A}'$ have the same combinatorics, and isomorphic incidence graphs.*

Proof. By the definition of topologically equivalent arrangement, there exists a homeomorphism Φ sending $\mathcal{A}$ on $\mathcal{A}'$. This implies that they have the same combinatorics. Therefore, they have isomorphic incidence graphs. □

1.2 Blow-up

1.2.1 Description

The blow-up is a geometric transformation of the projective plane in a projective surface which helps to simplify the singularities of the curves in the surface. Initially, we define the blow-up of $\mathbb{C}^2$ at the point $(0,0)$. Since, the blow-up is a local modification of $\mathbb{C}^2$, using local charts on a complex surface, we then define the blow-up of a surface M in a point $P \in M$.

Consider the quasi-projective space $\mathbb{C}^2 \times \mathbb{CP}^1$, where (x_1, x_2) are the complex affine coordinates of $\mathbb{C}^2$, and $[z_1 : z_2]$ the homogeneous coordinates of $\mathbb{CP}^1$.

Definition 1.2.1. The *blow-up* of $\mathbb{C}^2$ at the point $(0,0)$ is the subset X of $\mathbb{C}^2 \times \mathbb{CP}^1$ defined by:

$$X = \{((x_1, x_2), [z_1 : z_2]) \mid x_i z_j = x_j z_i,\ i, j \in \{1, 2\}\}.$$

Remark 1.2.2. Since the projective line is the set of lines in $\mathbb{C}^2$ passing through the origin, the blow-up of $\mathbb{C}^2$ at the point $(0,0)$, can be defined as the set:

$$X = \left\{(P, L) \mid P \in \mathbb{C}^2, P \in L, (0,0) \in L\right\}.$$

Remark 1.2.3. To blow-up $\mathbb{C}^2$ in a point Q different of $P = (0,0)$, we make a linear change of coordinates such that the new coordinates of Q are $(0,0)$.

We have a natural projection $\sigma : X \to \mathbb{C}^2 \times \mathbb{CP}^1$ defined by the restriction of the usual projection $\mathbb{C}^2 \times \mathbb{CP}^1 \to \mathbb{C}^2$. It can be pictured by the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] & \mathbb{C}^2 \times \mathbb{CP}^1 \arrow[d] \\ & \mathbb{C}^2 \end{tikzcd}$$

Property 1.2.4. *The blow-up map $\sigma : X \to \mathbb{C}^2 \times \mathbb{CP}^1$, is a birational map, which is an isomorphism outside the origin.*

Definition 1.2.5. If U is an open set of $\mathbb{C}^2$ containing $P = (0,0)$. The blow-up $\hat{U}$ of U at the point P is defined by: $\hat{U} = \sigma^{-1}(U)$, where $\sigma : X \to \mathbb{C}^2$ is the blow-up of $\mathbb{C}^2$ at the point P.

Let M be a complex surface and P a point of M. Consider a local chart

$$c : V \to U \subset \mathbb{C}^2,$$

where V is an open neighborhood of P and U an open set of $\mathbb{C}^2$ containing $(0,0)$. Assume that the local chart c is centered at P (i.e. $c(P) = (0,0)$). Then, the blow-up $\hat{M}$ of M at the point P is:

$$\hat{M} = (M \setminus \{P\}) \bigsqcup \hat{U},$$

where $\hat{U}$ is the blow-up of U over $(0,0)$ and the points of $V \setminus \{P\}$ are identified with the points of $\hat{U} \setminus \sigma^{-1}((0,0))$.

Remark 1.2.6. To blow-up a surface at a point P, we replace its neighborhood V by the blow-up at the point $(0,0)$ of the local chart.

Consider the blow-up $\hat{M}$ at the point P. Let $Q \in M \setminus \{P\}$ and let U be any neighborhood of Q which did not contain P, then $\sigma^{-1}(U) \simeq U$. In fact, we have $M \setminus \{P\} \simeq \sigma^{-1}(M \setminus \{P\}))$, and $\sigma^{-1}(P) \simeq \mathbb{CP}^1$.

Remark 1.2.7. The points of $\sigma^{-1}(P)$ are in one-to-one correspondence with the set of all the complex directions passing through P.

Since the blow-up of M at the point P make only a local modification around P, it is possible to blow-up any other points of X.

1.2.2 Case of arrangements

Let us recall the notion of divisor (also called Weil's divisor). If X is an algebraic variety, a divisor on X is a finite formal sum with integer coefficients of closed and irreducible sub-varieties of codimension 1. It is an *effective divisor* if all the coefficients are positive. It is a *normal crossing divisor* if the irreducible sub-varieties are smooth, intersect pairwise in a transversal way, and all points of X are at most in two irreducible sub-varieties.

Proposition 1.2.8. *If we blow-up one time each singular point of multiplicity greater than 3 of an arrangement $\mathcal{A}$, then the preimage $\hat{\mathcal{A}}$ of $\mathcal{A}$ is an effective divisor with normal crossings.*

Proof. The case of a singular point of multiplicity 3 is detailed in Example 1.2.11. With this equation it is clear that $\hat{\mathcal{A}}$ is a divisor with normal crossings. The case of points with greater multiplicity are similar. □

Definition 1.2.9. Let $\mathcal{A}$ be a line arrangement. The blow-up of $\mathcal{A}$ is the pair $(X, \hat{\mathcal{A}})$, where X is the blow-up of $\mathbb{CP}^2$ over all the singular points of $\mathcal{A}$ with multiplicity greater than 3, and $\hat{\mathcal{A}}$ is preimage of $\mathcal{A}$ by the blow-up.

Definition 1.2.10. Let $\sigma : X \to \mathbb{CP}^2$ be the blow-up of $\mathbb{CP}^2$ over the singular points of multiplicity greater than 3. The *strict transformation* of $L \in \mathcal{A}$ is the inverse image of the smooth part of L by σ. The *exceptional component* associated with a singular point $P \in \mathcal{Q}$ of multiplicity greater than 3, is $\sigma^{-1}(P)$.

Example 1.2.11. The blow-up of a singular point of multiplicity 3 is pictured in Figure 1.4 and the computation is detailed below.

Consider a pencil of three lines $\mathcal{A} = \{L_0, L_1, L_2\}$ defined by the polynomial:

$$P_{\mathcal{A}}(x_1, x_2) = (x_1 + x_2)x_2(x_1 - x_2).$$

The blow-up X of $\mathbb{C}^2$ is given by the equation $x_1 z_2 = x_2 z_1$. It is homeomorphic to $\mathbb{C}^2$ except that the point $(0, 0)$ has been replaced by $\mathbb{CP}^1$.

To obtain the total inverse image of $\mathcal{A}$ in X, we consider the equations in $\mathbb{C}^2 \times \mathbb{CP}^1$: $(x_1+x_2)x_2(x_1-x_2) = 0$ and $x_1 z_2 = x_2 z_1$. Since the coordinates of $\mathbb{CP}^1$ are homogenous, either z_1 or z_2 is non-zero. Suppose $z_1 \neq 0$, then we may assume that $z_1 = 1$ and we have the equations $(x_1 + x_2)x_2(x_1 - x_2) = 0$ and $x_1 z_2 = x_2$. Substituting, we obtain:

$$x_1^3 z_2(1 + z_2)(1 - z_2) = 0.$$

Hence the total inverse image $\hat{\mathcal{A}}$ of $\mathcal{A}$ is composed of the four irreducible components:

$\sigma^{-1}(P)$	: $x_1 = 0$, $x_2 = 0$, z_2 arbitrary, corresponds to the inverse image of P.
$\hat{L}_0$	: $z_2 = -1$, x_2 arbitrary, $x_1 = -x_2$, corresponds to the transformation of L_0.
$\hat{L}_1$	: $z_2 = 0$, $x_2 = 0$, x_2 arbitrary, corresponds to the transformation of L_1.
$\hat{L}_2$	: $z_2 = 1$, x_2 arbitrary, $x_1 = x_2$, corresponds to the transformation of L_2.

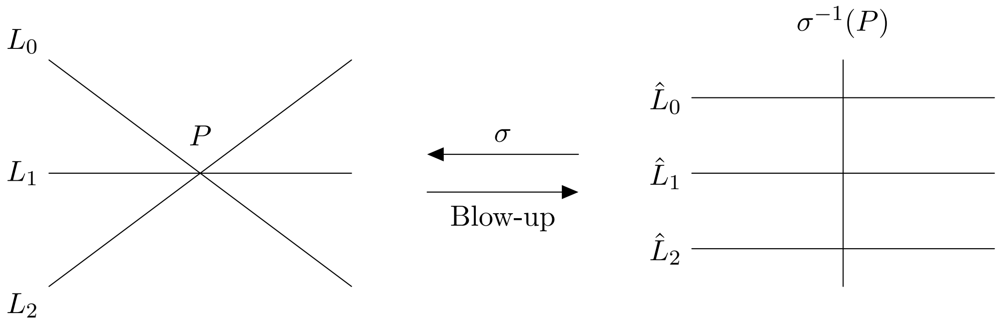

Figure 1.4: Local blow-up of a singular point

Definition 1.2.12. A NC-*Zariski pair* is a pair of arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ with the same combinatorics and such that there is no homeomorphism between the couples $(X_1, \hat{\mathcal{A}}_1)$ and $(X_2, \hat{\mathcal{A}}_2)$, preserving strict transform of lines and exceptional components.

Remark 1.2.13. The initials NC mean normal crossing.

In both definitions of Zariski pair (usual and NC) we can consider oriented (resp. ordered) pair, in this case, we are only interested in homeomorphism preserving the orientation (resp. order).

Proposition 1.2.14. *If a pair $\mathcal{A}_1$ and $\mathcal{A}_2$ of arrangement is a Zariski pair, then it is an* NC*-Zariski pair.*

Proof. We proceed by contradiction. Assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ is not a NC-Zariski pair, then there is a homeomorphism ϕ from $(X_1, \hat{\mathcal{A}}_1)$ to $(X_2, \hat{\mathcal{A}}_2)$. Since the map $\sigma_i : X_i \to \mathbb{CP}^2$ preserve strict transformations and exceptional components and are continuous, then ϕ induce a homeomorphism from $(\mathbb{CP}^2, \mathcal{A}_1)$ to $(\mathbb{CP}^2, \mathcal{A}_2)$. □

Question. *Let $\mathcal{A}_1$, $\mathcal{A}_2$ be a* NC*-Zariski pair, is it a Zariski pair ?*

1.2.3 Dual Graph

Definition 1.2.15. Let $\hat{\mathcal{A}}$ be the blow-up of an arrangement $\mathcal{A}$. The dual graph of $\hat{\mathcal{A}}$ is defined as follows:

- The set of vertices is in a one-to-one correspondence with the irreductible components of $\hat{\mathcal{A}}$.
- An edge joins two vertices if and only if the corresponding components intersect.

The dual graph is denoted by $\hat{\Gamma}_{\mathcal{A}}$.

Proposition 1.2.16. *The graphs $\Gamma'_{\mathcal{A}}$ and $\hat{\Gamma}_{\mathcal{A}}$ are isomorphic.*

Proof. An irreducible component of $\hat{\mathcal{A}}$ can be of two types:

- It corresponds to a line L_i of $\mathcal{A}$, and it is denoted $\hat{L}_i$.
- It corresponds to a singular point $P_j \in \mathcal{Q}$ of $\mathcal{A}$, and it is denoted by E_j.

The isomorphism sends each $\hat{L}_i$ to v_{L_i} and each E_j to v_{P_j}. Thus the edges of $\hat{\Gamma}_{\mathcal{A}}$ are sent on the edges of $\Gamma'_{\mathcal{A}}$. □

Remark 1.2.17. If we consider the construction of $\hat{\mathcal{A}}$ in which we blow-up all the singular points, then the dual graph is isomorphic to the incidence graph $\Gamma_{\mathcal{A}}$.

Chapter 2

Wiring diagram

One of the difficult points in the study of objects in the complex plane is the comprehension of a 4-dimensional space. To overcome this difficulty, we consider smaller dimensional objects reflecting their properties.

The braid monodromy is one of the more powerful of such objects. It has been first initiated by O. Zariski in [71] and E.R. van Kampen in [65], where he gives a method to compute the fundamental group of the complement of an algebraic curve (known as the *Zariski-van Kampen method*) using the positions of the curves along pencils of lines. Refinements of van Kampen's algorithm were given by O. Chisini [18]. In the early 80's, B. Moishezon [51] introduced the modern notion of braid monodromy, which he used to recover van Kampen's presentation. In [43], A. Libgober introduces a 2-dimensional CW-complex constructed from the braid monodromy admitting the same homotopy type than the complement.

In [9], W. Arvola generalizes the method of R. Randell in [58] to compute a Zariski-van Kampen–like presentation of the fundamental group of a complex line arrangement. For that, he introduces the notion of braided wiring diagram, which is a singular braid derived from the braid monodromy of the arrangement.

First, we give the definition and explain how to construct this diagram. Then we describe Arvola's algorithm to compute the presentation of the fundamental group of $E_{\mathcal{A}}$, and we give some other properties of the braided wiring diagram.

In Section 2.2, we describe the construction of a `Sage` program to compute the braided wiring diagram from the equations of the lines of $\mathcal{A}^{\text{aff}}$.

Finally, in Section 2.3, we picture the braided wiring diagram of two particular examples: MacLane's arrangements, and Fan's arrangements –transmit by K.M. Fan to E. Artal–.

2.1 Braided Wiring Diagram

2.1.1 Braided wiring diagram

Up to a change of coordinates, let us assume that $L_0 = \{z = 0\}$, and consider the chart

$$\Big\{[x : y : z] \in \mathbb{CP}^2 \mid z \neq 0\Big\} \simeq \mathbb{C}^2.$$

The affine part $\mathcal{A}^{\text{aff}} \subset \mathbb{C}^2$ of $\mathcal{A}$ can be viewed as the set of the affine part of the lines $L_1, \cdots, L_n$. Let $\pi : \mathbb{C}^2 \to \mathbb{C}$, be a *generic* linear projection, in the sense that:

- For all $i \in \{1, \cdots, n\}$, the restriction of $\pi_{|L_i}$ is a homeomorphism (i.e. there is no vertical line).
- Each multiple point lies in a different fiber of π.

We suppose also that the points $x_i = \pi(P_i)$ have distinct real parts, then we can order the points of $\pi(\mathcal{P})$ by increasing real parts, so that $\Re(x_1) < \Re(x_2) \cdots < \Re(x_k)$. A path $\gamma : [0,1] \to \mathbb{C}$ emanating from x_0 with $\Re(x_0) < \Re(x_1)$, passing through $x_1, \cdots, x_k$ in order, with no self-intersection, and horizontal (i.e. with constant imaginary part) in a neighborhood of each x_i is said to be *admissible*.

Definition 2.1.1. The *wiring diagram* associated to an admissible path γ is defined by:

$$W_{\mathcal{A}} = \{(x, y) \in \mathcal{A} \mid \ \exists t \in [0, 1],\ \pi(x, y) = \gamma(t)\ \} = \pi^{-1}(\gamma([0, 1])) \cap \mathcal{A}.$$

The trace $\omega_i = W_{\mathcal{A}} \cap L_i$ is the *wire* associated to the line L_i.

Note that if $\mathcal{A}$ is a real complexified arrangement, then we can take $\gamma = [x_0 - \eta, x_k + \eta] \subset \mathbb{R}$; and $W_{\mathcal{A}} \simeq \mathcal{A} \cap \mathbb{R}^2$.

Remark 2.1.2. We can define also γ on $\mathbb{R}$ instead of $[0, 1]$. In this case, $W_{\mathcal{A}}$ is the generalization of the trace $\mathcal{A}^{\text{aff}} \cap \mathbb{R}^2$ for a complexified real arrangement.

Remark 2.1.3.
i) The wiring diagram depends on the path γ, and on the projection π.
ii) The set of singular points $\mathcal{P}$ is contained in $W_{\mathcal{A}}$.

We re-index the lines $L_1, \cdots, L_n$ such that:

$$I_i > I_j \iff i < j,$$

where $I_i = Im(L_i \cap \pi^{-1}(x_0))$. On the representation of the braided wiring diagram described below, this re-indexation implies that the lines are ordered, at the left of the diagram, from top to bottom.

Since the x coordinates of the points of $W_{\mathcal{A}}$ are parametrized by γ, the wiring diagram can be seen as a one dimensional object inside $\mathbb{R}^3 \simeq [0, 1] \times \mathbb{C}$. It is then called *braided wiring diagram*. Consider its image by a generic projection $\gamma([0, 1]) \times \mathbb{C} \to \mathbb{R}^2$. If we take a plane projection of this diagram (assume, for example, that it is in the direction of the vector $(0,0,1)$ -that is, in the direction of the imaginary axis of the fibre-), we obtain a planar graph. Observe that there are nodes corresponding to the image of actual nodes in the wiring diagram in $\mathbb{R}^3$ (that is, to a singular point of the arrangement). Other nodes appear from the projection of underpassing and overpassing wires of the wiring diagram in $\mathbb{R}^3$. The two types of nodes are called by Arvola *actual and virtual crossing*.

If we represent the virtual crossings in the same way that they are represented in the case of braid diagrams, we obtain a schematic representation of the wiring diagram as in Figure 2.1. From now on, we will refer to this representation as the wiring diagram itself. By genericity, we assume that two crossings (actual or virtual) do not lie on the same vertical line.

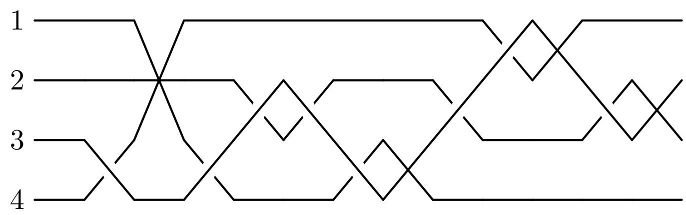

Figure 2.1: Example of braided wiring diagram

It is worth noticing that from the braided wiring diagram, one may extract the braid monodromy of $\mathcal{A}$ (see Subsection 2.1.3), related to the generic projection π. The local equation of a multiple point is of the form $y^m - x^m$, where m is the multiplicity, and the corresponding local monodromy is a full twist in the braid group with m strands.

Another description of the wiring diagram is explained below, it was introduced by D. Cohen and A. Suciu in [21].

The construction of γ implies that the $x_i \in [x_i - \eta, x_i + \eta]$; so we can modify the construction of γ and taken smaller η such that, in the pre-image of $[x_i - \eta, x_i + \eta]$ by π, the wires form a multiple point (without virtual crossings). We denote $f_{i,l}$ (resp. $f_{i,r}$) the fiber over $x_i - \eta$ (resp. $x_i + \eta$). Up to isotopy of $\mathcal{A}$, we may assume that the global position of the wires in all the $f_{i,l}$'s and $f_{i,r}$'s are identical. They are respectively called $\Pi_{i,l}$ and $\Pi_{i,r}$. Furthermore, between two consecutive fibers $f_{i,r}$, $f_{i+1,l}$, the wires form a braid. We call this braid β_i. Finally a braided wiring diagram can be viewed as a sequence of positions, vertices, and braids:

$$\Pi_{0,r} \xrightarrow{\beta_0} \Pi_{1,l} \xrightarrow{V_1} \Pi_{1,r} \xrightarrow{\beta_1} \Pi_{2,l} \to \cdots \to \Pi_{k-1,l} \xrightarrow{V_{k-1}} \Pi_{k-1,r} \xrightarrow{\beta_{k-1}} \Pi_{k,l} \xrightarrow{V_k} \Pi_{k,r},$$

where the vertex V_i corresponds to the cross of the wires associated to the lines incident on P_i.

2.1.2 Complement of the arrangement: Arvola's presentation

We recall briefly the method due to Arvola [9] to obtain a presentation of the fundamental group of the complement from a braided wiring diagram $W_{\mathcal{A}}$. The algorithm goes as follows: start from the left of the diagram, assigning a generator α_i to each strand. Then follow the diagram from the left to the right, assigning a new word to the strands each time going through a crossing. The rules for this new assignation are given in Figure 2.2, where the a_i's are words in the α_i's.

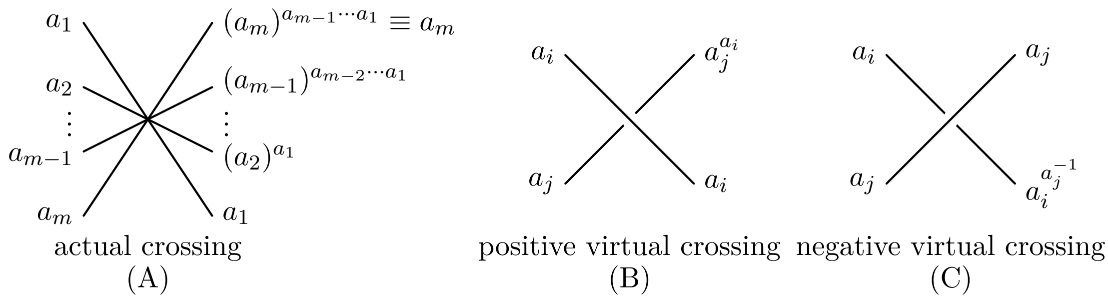

Figure 2.2: Computation of Arvola's words

The notation of Figure 2.2 is $a_i^{a_j} = a_j^{-1}a_ia_j$. For each actual crossing P –that corresponds to a singular point of $\mathcal{A}^{\text{aff}}$–, suppose that the strands are labeled with the words $a_1, \ldots, a_m$ with respect to their order in the diagram at this point P, from top to bottom, where $m = m_P$ is the multiplicity of P. Then the following relations are added to the presentation

$$R_P = [a_m, \cdots, a_1] = \{a_m \cdots a_1 = a_1 a_m \cdots a_2 = \cdots = a_{m-1} \cdots a_1 a_m\}.$$

Remark 2.1.4. They correspond to the action of a half-twist on the free group, whereas the action of a virtual crossing is given by the corresponding braid.

For $i = 1, \cdots, n$, let α_i be the meridian of the lines L_i near L_0.

Theorem 2.1.5 (Arvola [9]). *The fundamental group of the complement of $\mathcal{A}^{\mathit{aff}}$ admits the following presentation*

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{A}^{\mathit{aff}}) = \langle \alpha_1, \cdots, \alpha_n \mid \bigcup_P R_P \rangle,$$

where P ranges over all the actual crossings of the wiring diagram $W_{\mathcal{A}}$.

Remark 2.1.6. Since $E_{\mathcal{A}} \simeq \mathbb{C}^2 \setminus \mathcal{A}^{\text{aff}}$ then the presentation given in Theorem 2.1.5 is also a presentation of $\pi_1(E_{\mathcal{A}})$.

Example 2.1.7. The fundamental group associated with the wiring diagram pictured in Figure 2.1 is:

$$\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_4^{\alpha_3}, \alpha_2, \alpha_1],\ [\alpha_3, \alpha_1],\ [\alpha_4, \alpha_3],\ [\alpha_3, \alpha_2^{\alpha_1}] \rangle.$$

2.1.3 Braid monodromy

The braid monodromy of a complex curve was introduced by O. Chisini in [18], and B. Moishezon in [51]. In [43], A. Libgober constructs a presentation of the fundamental group of $E_{\mathcal{A}}$, using the braid monodromy, such that the CW-complex built on this presentation has the same homotopy type than $E_{\mathcal{A}}$. We explain how the braid monodromy of an arrangement can be found from its wiring diagram.

Let $W_{\mathcal{A}}$ be the wiring diagram of $\mathcal{A}$ associated with an admissible path γ. Consider γ' a small perturbation of γ such that near each x_i, the path γ' turns around x_i in the positive direction, along a small half circle of radius η, as pictured in Figure 2.3. Let $t_k \in (0,1)$ be such that $\gamma(t_k) = x_k - \eta$. We define γ_k as the restriction of γ' to $[0, t_k]$. We denote c_k the oriented circle of radius η centered in x_k. The fibers over $\mu_k = \gamma_k^{-1} c_k \gamma_k$ are all generic (i.e. their intersection with $\mathcal{A}$ is a set of n distinct points).

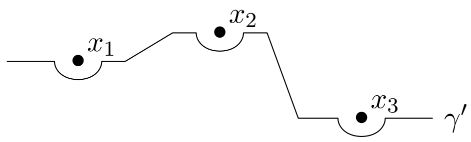

Figure 2.3: Construction of γ'

Definition 2.1.8. The local monodromy b_k around $P_k \in \mathcal{P}$ is:

$$b_k = \pi^{-1}(c_k) \cap \mathcal{A} \in \mathbb{B}_n.$$

The braid monodromy λ_k associated with $P_k \in \mathcal{P}$ is:

$$\lambda_k = \pi^{-1}(\mu_k) \cap \mathcal{A} \in \mathbb{B}_n.$$

The local monodromy around P_k is the full-twist on the local index of P_k (it is the product of two half-twist h_k also on the local index of P_k). Let $I_k = \{j, \cdots, j+m-1\}$ be the local index of P_k, then:

$$h_k = (\sigma_j \cdots \sigma_{j+m-2})(\sigma_j \cdots \sigma_{j+m-3}) \cdots (\sigma_j \sigma_{j+1})\sigma_j,$$

and $b_k = h_k^2$. To extract the braid monodromy from the braided wiring diagram of $\mathcal{A}$, we consider the notation used in the Cohen-Suciu description of $W_{\mathcal{A}}$.

Proposition 2.1.9. *The braid monodromy associated with $P_k \in \mathcal{P}$ is:*

$$\lambda_k = (\beta_{k-1} h_{k-1} \cdots \beta_1 h_1 \beta_0)^{-1}\, b_k\, (\beta_{k-1} h_{k-1} \cdots \beta_1 h_1 \beta_0)$$

2.1.4 Properties

a) Relation between incidence graph and wiring diagram

The braided wiring diagram encodes all the combinatorics information contained in $\Gamma_{\mathcal{A}}$, but it also contains topological information. Indeed, it contains the homotopical type of the arrangement since the 2 dimensional CW-complex constructed on Arvola's presentation have the same homotopical type than $E_{\mathcal{A}}$, see [43]. However, we do not know if it contains the topological type of the arrangement.

Proposition 2.1.10. *The graph $\Gamma_{\mathcal{A}}$ is a deformation retract of $W_{\mathcal{A}}$.*

Proof. First, send the line-vertex of $\Gamma_{\mathcal{A}}$ on the left most point of the corresponding wire. Then, send the point-vertex on the corresponding point in $W_{\mathcal{A}}$. And the edges are sent on the part of the wire joining the image of its boundary. Finally, retract the right part of the wires on the last singular points. □

b) Complex conjugate

Consider the involution of $\mathbb{CP}^2$ defined by the complex conjugation. The conjugate arrangement $\overline{\mathcal{A}}$ is the arrangement defined by the action of the complex conjugation on $\mathcal{A}$. Let $W_{\mathcal{A}}$ be the wiring diagram of $\mathcal{A}$ associated to a projection π and an admissible path γ. The set of singular points $\overline{\mathcal{P}}$ of $\overline{\mathcal{A}}^{\text{aff}}$ is composed of the conjugates of the points of $\mathcal{P}$. Since π is generic for $\mathcal{A}$, then it is also for $\overline{\mathcal{A}}$, and $\overline{\gamma}$ the conjugate path of γ is also admissible for $\overline{\mathcal{A}}$. So we consider $W_{\overline{\mathcal{A}}}$ the braided wiring diagram of $\overline{\mathcal{A}}$ associated with π and $\overline{\gamma}$.

Proposition 2.1.11. *The braided wiring diagram $W_{\overline{\mathcal{A}}}$ of $\overline{\mathcal{A}}$ is obtained by reversing the virtual crossing of $W_{\mathcal{A}}$.*

Proof. Since complex conjugation transforms the imaginary part of a complex number in its opposite, then at a virtual crossing the over arc becomes an under arc and conversely. Then virtual crossings are reversed to pass from $W_{\mathcal{A}}$ to $W_{\overline{\mathcal{A}}}$. □

2.2 Computation

Computing the braided wiring diagram of an arrangement is not a trivial question. Some algorithms designed to compute it were developed by J. Carmona in [15] and D. Bessis in [10] for the more general case of braid monodromy of plane curves.

2.2.1 Algorithm

We have developed a program with Sage to compute the braided wiring diagram. The complete code is given in Appendix A.1.

The entry data is a list of the affine equations defining the lines of $\mathcal{A}^{\text{aff}}$ on the form $f_i(x, y) = 0$. The algorithm works as follow:

- (function *sing*) Computes the set $\mathcal{P}$ of the singular points of $\mathcal{A}^{\text{aff}}$.
  Solves the system $\begin{cases} f_i(x,y) &= 0 \\ f_j(x,y) &= 0 \end{cases}$, for all $i \neq j$, and add the result to the set if it is not already in it.

- (functions *Xprojection*, *sort* and *order*) Computes the image of $\mathcal{P}$ by the projection π, and orders the set by increasing real part.
- (functions *cloud* and *linear_interp*) Defines the admissible path γ.
  Creates a cloud of points in $\mathbb{C}$ by translating the image of $\mathcal{P}$ in the direction $-\eta$ and η (with η small enough). And then the piecewise linear path joining all the points of this cloud is an admissible path passing horizontally thought the x_i's.
- (function *lift*) Computes the different lifts $\gamma_i \subset \omega_i$ of γ in $\mathcal{A}$.
  On each linear part of γ and for all the lines, solves the equation $f_i(\gamma(t), y) = 0$ in function of y to obtain $\gamma_i = \pi^{-1}(\gamma) \cap \omega_i$.
- (function *found_cross*) Detects the different types of crossing of the wiring diagram.
  The first step is to solve $\Re(\gamma_i(t)) = \Re(\gamma_j(t))$ to detect the crossings (without distinction between virtual and actual) and conserve the crossing locus t. The second is to compare the imaginary parts to fix the crossing type.
- (function *Wiring* and *Wiring_diagram*) Returns the ordering set of all the different crossings pairwise, with the corresponding value of t.
  It is the composition of all the previous functions.

A function *display* is added to obtain a better presentation of the results.

Remark 2.2.1. The result only presents pairwise crossing. So to draw the braided wiring diagram, we need to use the information given by the crossing locus t returned with all crossings to regroup the pairs of crossing forming the actual crossings with multiplicity greater than 2.

Remark 2.2.2. The speed performance of this program is not optimized.

On Sage it is possible to make formal computation on number fields. Since each arrangement admits at least one realization on a number field (see Proposition 1.1.23), it is possible to make a formal computation. For the sake of simplicity and speed, we implement the program on $\mathbb{C}$. But on $\mathbb{C}$, Sage cannot make formal computation. Then we have added an error when we solve the equations in *found_cross*. Indeed after successive approximations actual cross may appear as virtual if a possible error is not added. For the moment it is bounded by 10^{-6}. Note however that in the experiment the risk of error is very small, but a program with formal computation is in project for the future.

2.2.2 Output

a) Genericity of the projection

One of the most important hypothesis to define the braided wiring diagram is the genericity of the projection π. To obtain such a projection, we usually make a random change of variables. It appears that with this approach, the result is very often unusable. Indeed, for an arrangement with ten lines and a random projection, the braided wiring diagram contains more than one hundred crossings.

To simplify the computation and then the picture of the arrangement, we consider a projection with a maximum number of vertical lines. Then, for the program, they do not appear in the wiring diagram. And using the information given by the crossing locus t of each crossing we can add the vertical lines in the arrangement. The computation of the example given in Section 2.3 was done with this method. This explains why there is no function to check the genericity of the projection.

Better, from such a non-generic projection it is simple to come back to a generic projection applying a smaller perturbation to the arrangement (and therefore to the picture), see [3].

Remark 2.2.3. With the lack of function to check the genericity, some special cases may appear. For example, consider the case pictured in Figure 2.4. Let $L_1 = \{x = 0\}$, $L_2 = \{x + y - i = 0\}$ and $L_3 = \{x + y + i = 0\}$. The singular points are $(0, i)$ and $(0, -i)$. If $\gamma(t) = 2t - 1$, then the function *wiring_diagram* returns two intersections at $t = 0.5$: one between L_1 and L_2 and the other between L_1 and L_3. But it is not a point of multiplicity 3, because L_2 and L_3 do not intersect in $\mathbb{C}^2$. This situation is not drawable in the sense of the description of Subsection 2.1.1.

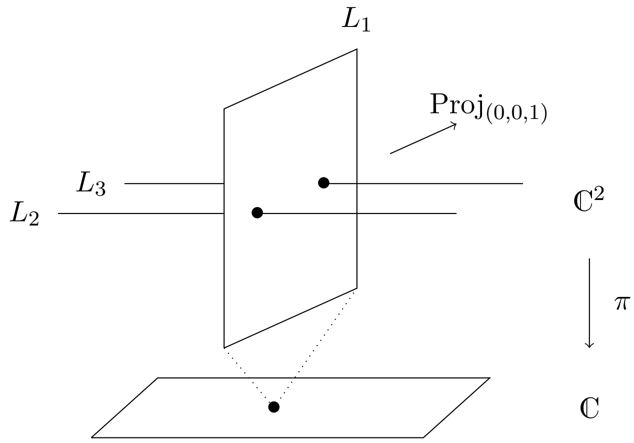

Figure 2.4: Example of problem with a non-generic projection

b) Possible ameliorations

As previously mentioned this program is not optimized. Indeed, since the admissible path γ is piecewise linear, the positions $\Pi_{i,l}$ and $\Pi_{i,r}$ are sufficient to compute the braids β_i. With this method the time of computation may be reduced.

Some other modifications or extensions are possible:

- Regroup the pairs of crossings forming the multiple points.

- Add a function to extract the braid monodromy of the braided wiring diagram. This function needs that the previous one already exists.
- Draw the wiring diagram (when it is possible, see Remark 2.2.3).
- Obtain the LATEXcode for the picture using the package Tikz.

2.3 Examples

2.3.1 MacLane arrangements

The MacLane arrangements are two conjugated arrangements coming from MacLane's matroid [50]. It is the arrangement with a minimal number of lines such that the combinatorics admits a realization over $\mathbb{C}$ but not over $\mathbb{R}$ (see, [69]). These arrangements are constructed as follows. Let us consider the 2-dimensional vector space on the field $\mathbb{F}_3$ of three elements. Such a plane contains 9 points and 12 lines, 4 of them pass through the origin. Consider $\mathcal{L} = \mathbb{F}_3^2 \setminus \{(0,0)\}$ and $\mathcal{P}$, the set of lines in $\mathbb{F}_3^2$ (i.e. the set of lines in $\mathbb{F}_3^2$ not passing through the origin). This provides a line combinatorics $(\mathcal{L}, \mathcal{P}, \Subset)$, where for all $\ell \in \mathcal{L}, P \in \mathcal{P}$, we have $P \Subset \ell \Leftrightarrow (\ell \in P,\ \text{in } \mathbb{F}_3^2)$. Figure 2.5 represents the ordered MacLane's combinatorics viewed in $\mathbb{F}_3^2$. As ordered combinatorics, it admits two ordered realizations, denoted by $\mathcal{Q}^+$ and $\mathcal{Q}^-$. They are defined by the following equations:

$$
\begin{array}{ccc}
L_0 \ :\ z = 0 & L_1 \ :\ z - x = 0 & L_2 \ :\ x = 0 \\
L_3 \ :\ y = 0 & L_4 \ :\ z + \omega^2 x + \omega y = 0 & L_5 \ :\ y - x = 0 \\
L_6 \ :\ z - x - \omega^2 y = 0 & L_7 \ :\ z + \omega y = 0 &
\end{array}
$$

where $\omega = \exp(\frac{2i\pi}{3})$ for Q^+, and $\omega = \exp(\frac{-2i\pi}{3})$ for Q^-.

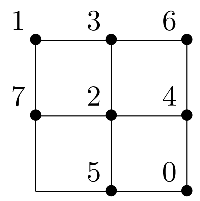

Figure 2.5: Ordered MacLane combinatorics, viewed in $\mathbb{F}_3^2$

The braided wiring diagram of Q^+, is pictured in Figure 2.6. It is computed using the Sage program described in Subsection 2.2.1, with the lines L_1 and L_2 taken vertically (i.e. they are fibers of the projection). The diagram is then pictured with a small perturbation of the projection, such that L_1 and L_2 are generics relative to the new projection. Using Subsection 2.1.4, we have computed the braided wiring diagram of Q^-, pictured in Figure 2.7.

As they are minimal (in terms of number of lines such that the combinatorics admits a realization over $\mathbb{C}$ but not over $\mathbb{R}$), these arrangements are good examples of complex

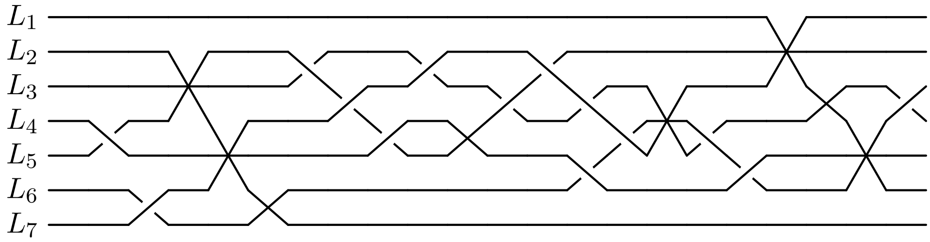

Figure 2.6: Braided wiring diagram of MacLane positive

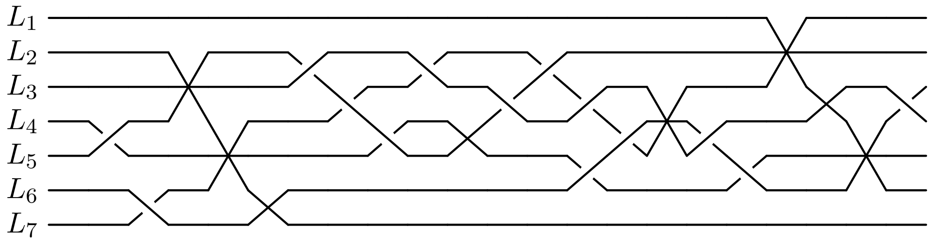

Figure 2.7: Braided wiring diagram of MacLane negative

arrangements. Then Q^+ is used as example in the next chapter, to illustrate the method developed.

2.3.2 Fan's arrangements

K.M. Fan transmitted to E. Artal a combinatorics admitting three different realizations (one real and two complex). The question about them is:

Question. *Is the real arrangement topologically different from the complex realizations ?*

Their equations are:

$$L_0 \ : \ z = 0 \qquad L_1 \ : \ x = 0 \qquad L_2 \ : \ x - z = 0$$
$$L_3 \ : \ y = 0 \qquad L_4 \ : \ y - z = 0 \qquad L_5 \ : \ x - y = 0$$
$$L_6 \ : \ x - y - z = 0 \qquad L_7 \ : \ x - \alpha z = 0 \qquad L_8 \ : \ \alpha y + (2-\alpha)x - \alpha z = 0$$
$$L_9 \ : \ (3\alpha - 4)y + (2-\alpha)x - \alpha z = 0 \qquad L_{10} \ : \ (3\alpha-4)y + (2-\alpha)x - 2\alpha(\alpha-1)z = 0$$

where α is a solution of the equation $x^3 - \frac{7}{2}x^2 + 6x - 4 = 0$. The realization associated to the real solution is denoted by F^0, and F^+ (resp. F^-) is the realization associated to the complex solution with a positive (resp. negative) imaginary part.

The braided wiring diagram of F^0, is its trace in $\mathbb{R}^2$. It is pictured in Figure 2.8.

The braided wiring diagram of the positive Fan's arrangement F^+ is pictured in Figure 2.9. It is computed with the Sage program with a non generic projection. And the negative case is obtained using Subsection 2.1.4, and it is pictured in Figure 2.10.

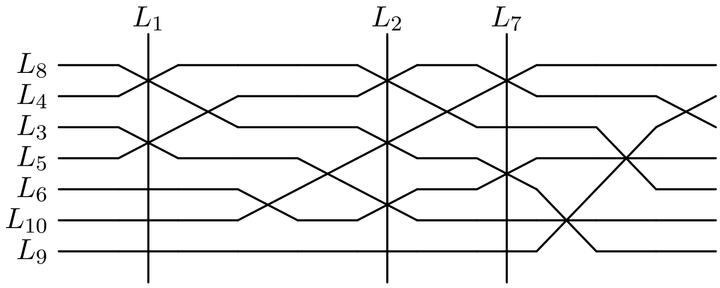

Figure 2.8: Braided wiring diagram of Fan real

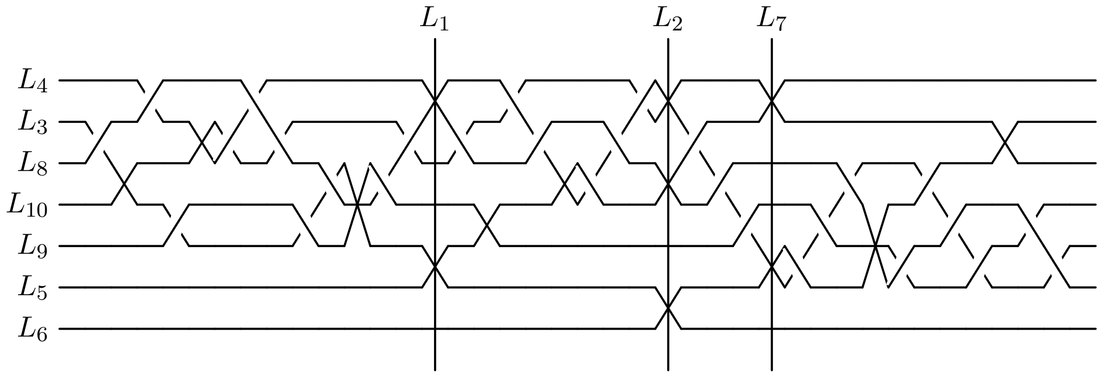

Figure 2.9: Braided wiring diagram of Fan Positive

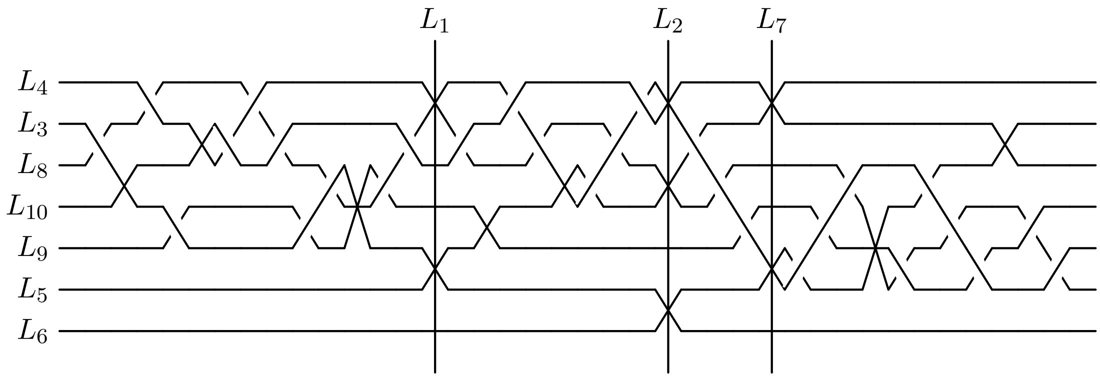

Figure 2.10: Braided wiring diagram of Fan negative

As F^+ and F^- are conjugated arrangement, then their fundamental groups are isomorphic. These wiring diagrams allow to compute some topological invariants. But they give no hint about topological equivalence.

Chapter 3

Boundary manifold

The boundary manifold $B_{\mathcal{A}}$ is a compact graph manifold, in the sense of F. Waldhausen [66]. It is determined by the combinatorics and modeled on the incidence graph of the arrangement. In [40], E. Hironaka describes the embedding of $B_{\mathcal{A}}$ in $E_{\mathcal{A}}$ in the case of the complexified real arrangement. In this Chapter, we generalize this result to the case of complex line arrangements using the braided wiring diagram. This work gives rise to a submitted paper [34].

In Section 3.1, we describe the procedure to construct the boundary manifold from the incidence graph. Then we use this description to compute its fundamental group.

In Section 3.2, we generalize the results of E. Hironaka [40] and give the explicit algorithm to compute the map induced by the inclusion on the fundamental groups. From this map, we deduce a new minimal presentation of $\pi_1(E_{\mathcal{A}})$ such that the CW-complex built on it has the same homotopical type that $E_{\mathcal{A}}$. After being given a simpler version of the previous result, we recover the presentation of the fundamental group of $E_{\mathcal{A}}$ obtained by R. Randell in [58].

To conclude this Chapter, in Section 3.3, we explicitly compute the fundamental group of the positive MacLane's arrangement, hence also of the negative one.

3.1 Boundary manifold

3.1.1 Definition & construction

Let U be a compact regular neighborhood of $\mathcal{A}$ as constructed in Subsection 1.1.1. We define the boundary manifold $B_{\mathcal{A}}$ as ∂U. Remark that –up to orientation– $B_{\mathcal{A}}$ is also the boundary of $Ext_{\mathcal{A}}$. This manifold is combinatorially determined and can be computed from the incidence graph $\Gamma_{\mathcal{A}}$ as follows:

For every singular point $P \in \mathcal{Q}$ of $\mathcal{A}$, consider a 4-ball $\mathbb{B}_P$ of radius η, centered in

P. Let $\mathcal{S}_P = \partial(\mathbb{B}_P)\backslash\mathbf{T}$, where $\mathbf{T}$ is an open regular neighborhood of the link $L_P = (\partial\mathbb{B}_P \cap \mathcal{A})$. The boundary of $\mathcal{S}_P$ is a union of disjoint tori T indexed by the lines L_i passing through P, and $T_L = (L \cap \partial\mathbb{B}_P) \times S^1$.

Definition 3.1.1. Let $P \in \mathcal{Q}$ and $L \in \mathcal{A}$ be such that $P \in L$. The *meridian* m_L and the *longitude* l_L of the torus T_L are the pair of oriented simple closed curves in $T_L = \partial\overline{\mathbf{T}}$ which are determined up to isotopy by the homology and linking relations:

$$m_L \sim 0, \quad l_L \sim (L \cap \overline{\mathbf{T}}) \quad \text{ in } \mathrm{H}_1(\overline{\mathbf{T}});$$
$$\ell(m_L, L \cap \overline{\mathbf{T}}) = 1, \quad \ell(l_L, L \cap \overline{\mathbf{T}}) = 0,$$

where $\ell(\cdot,\cdot)$ denotes the linking number in $\partial\mathbb{B}_P \simeq S^3$.

Consider the surface:

$$F = \mathcal{A} \setminus \coprod_{P \in \mathcal{Q}} \left(\mathcal{A} \cap \mathring{\mathbb{B}}_P \right),$$

it is obtained by removing of $\mathcal{A}$ open discs from the $\mathbb{B}_P$'s. One sees that F is a union $\coprod_{i=0}^{n} F_i$ where each F_i corresponds to the line L_i of $\mathcal{A}$. Let $\mathcal{N}_i = F_i \times S^1$ whose boundary is a union of disjoint tori T indexed by the points $P \in \mathcal{P} \cap L_i$.

Let D be a generic line (i.e. for all $P \in \mathcal{Q}$, $P \notin D$), and consider D as the line at infinity. We decompose $\mathcal{N}_i$ in the solid torus $\mathcal{N}_i \cap T(D) = \mathbf{T}_i^\infty$, where $T(D)$ is regular neighborhood of D, and the affine part $\mathcal{N}_i^{\mathrm{aff}}$ defined as the closure of $\mathcal{N}_i \setminus \mathbf{T}_i^\infty$. Viewed as the affine part, $\mathcal{N}_i^{\mathrm{aff}}$ admits a natural trivialization in the affine space, and we choose a section s of $\mathcal{N}_i^{\mathrm{aff}}$.

Definition 3.1.2. Let $L_i \in \mathcal{A}$ and $P \in \mathcal{P}$ be such that $P \in L_i$. The *longitude* l_P of the torus T_P is the intersection of the section s with T_P. The *meridian* m_P of T_P is the class in $\mathrm{H}_1(T_P)$ of $\{*\} \times S^1$.

Remark 3.1.3. To reconstruct $\mathcal{N}_i$ from $\mathcal{N}_i^{\mathrm{aff}}$ and $\mathbf{T}_i^\infty$, we glue the boundary component of $\mathcal{N}_i^{\mathrm{aff}}$ different of the T_P's with $\partial\mathbf{T}_i^\infty$. The gluing is done identifying the intersection of the section in this component with the sum of a longitude of $\partial\mathbf{T}_i^\infty$ (i.e. a curve in $\partial\mathbf{T}_i^\infty$ homologically equivalent to $L_i \cap \mathbf{T}_i^\infty$ in $\mathbf{T}_i^\infty$) and a meridian; and the meridian with a fiber of the S^1-fibration of T_i^∞.

For each edge $e(L_i, P)$ of $\Gamma_\mathcal{A}$, glue $\mathcal{S}_P$ with $\mathcal{N}_i$ along T_{L_i} and T_P identifying meridian with meridian, and longitude with longitude. The manifold then obtained is the boundary manifold of $\mathcal{A}$. From this construction of $B_\mathcal{A}$, we deduce its structure of graph manifold.

Proposition 3.1.4 ([68, 40])**.** *Let $\mathcal{A}$ be a complex line arrangement. The boundary manifold $B_\mathcal{A}$ is a graph manifold over the incidence graph $\Gamma_\mathcal{A}$.*

Remark 3.1.5. The construction previously done is different of the classic one using plumbing graph as defined by W.D. Neumann in [53]. With elementary computations, one may show that two constructions coincide, though gluings are described differently. Indeed, to obtain this construction from the one present here, we need to blow-up the singular points, but this transformation does not change the $\mathcal{S}_P$'s.

Corollary 3.1.6. *The boundary manifold of a complex line arrangement depends only on the combinatorics of the arrangement.*

Proof. The plumbing used to compute $B_{\mathcal{A}}$ as a graph manifold over $\Gamma_{\mathcal{A}}$ is combinatorial. Then the equivalence of $\Gamma_{\mathcal{A}}$ and the combinatorics of $\mathcal{A}$ induce the result. □

A similar construction is done by [68] and [24] but using the blow-up $\hat{\mathcal{A}}$ of $\mathcal{A}$, and its dual graph $\hat{\Gamma}_{\mathcal{A}}$. Since the incidence graph of $\mathcal{A}$ and the dual graph of $\hat{\mathcal{A}}$ are equivalent, using a result of Waldhausen [66], they obtain homeomorphic manifold. However these two constructions give different presentations of the fundamental group of the boundary manifold.

3.1.2 Combinatorial groups

In [65], E.R. van Kampen explains how compute the fundamental group of the gluing of topological spaces (known as Seifert-Van Kampen method).

Definition 3.1.7. Let G and H be two groups admitting the following presentations $G = \langle g_1, \cdots, g_m \mid r_1, \cdots r_n \rangle$ and $H = \langle h_1, \cdots, h_k \mid s_1, \cdots, s_l \rangle$. Suppose that $A \subset G$ and $B \subset H$ are subgroups with an isomorphism $\Phi : A \to B$. Then the free product of G and H with an amalgamation of subgroups A and B by Φ is the group with presentation:

$$\langle g_1, \cdots, g_n, h_1, \cdots, h_k \mid r_1, \cdots, r_n, s_1, \cdots, s_l, a = \Phi(a), \forall a \in A \rangle,$$

abbreviated as $\langle G * H \mid A = B, \Phi \rangle$.

Definition 3.1.8. Let A and B be subgroups of G, with an isomorphism $\Phi : A \to B$. The Higman-Neumann-Neumann (HNN) extension of G, with stable letter t and associated subgroups A and B is the group:

$$\langle G, t \mid t^{-1} \cdot a \cdot t = \Phi(a), \forall a \in A \rangle.$$

That is, the presentation contains all the generators and relators of G, with an additional generator t, and a set of new relations of the form $t^{-1} \cdot a \cdot t = \Phi(a)$.

a) Product with amalgamation and fundamental group of a union

Suppose that X and Y are path-connected spaces having open path-connected subspaces U and V, respectively. Suppose $\Phi : U \to V$ is a homeomorphism. Let $u \in U$ be a base point for the fundamental groups of U and X, and $\Phi(u) \in V$ be a base point for the fundamental groups of V and Y. The homeomorphism Φ induces an isomorphism $\Phi_* : \pi_1(U, u) \to \pi_1(V, \Phi(u))$. Let Z be the space obtained by identifying U and V via Φ.

Proposition 3.1.9 (Seifert-Van Kampen)**.** *With the previous notations, we have:*

$$\pi_1(Z) = \langle \pi_1(X) * \pi_1(Y) \mid \pi_1(U) = \pi_1(V), \Phi_* \rangle.$$

If $U = V = X \cap Y$, then this gives $\pi_1(X \cup Y)$.

b) HNN extension and fundamental group of a space with a handle

Now suppose U and V are disjoint homeomorphic path-connected subspaces in a path-connected space X. Let I be the unit interval $[0,1]$. We attach a handle $U \times I$ to X by identifying $U \times \{0\}$ with U and $U \times \{1\}$ with V, using the homeomorphism Φ (see Figure 3.1). We call Z this new space.

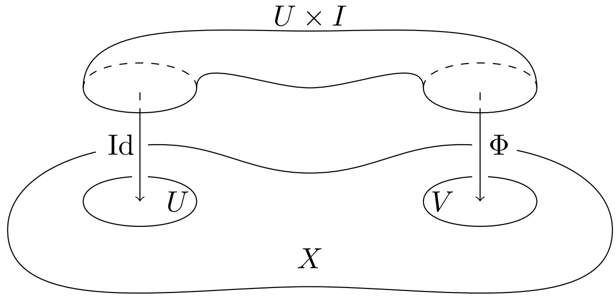

Figure 3.1: Handle gluing

Proposition 3.1.10 (Seifert-Van Kampen). *With the previous notations, the fundamental group of Z is the HNN extension of $\pi_1(X)$ with stable letter t and associated subgroups $\pi_1(U)$ and $\pi_1(V)$. In other terms:*

$$\pi_1(Z) = \langle \pi_1(X), t \mid t^{-1} \cdot \pi_1(U) \cdot t = \pi_1(V), \Phi_* \rangle.$$

Remark 3.1.11. The stable letter t obtained is a cycle arising from the gluing of the handle.

3.1.3 Fundamental group of $B_{\mathcal{A}}$

The fundamental group of $B_{\mathcal{A}}$ is the group associated to the incidence graph, see [68, 24]. Two types of generators naturally appear: the *meridians* of the lines and the *cycles* related to the graph.

Definition 3.1.12. Let L be a line in $\mathbb{CP}^2$, and $b \in \mathbb{CP}^2 \setminus \mathcal{A}$. A homotopy class $\alpha \in \pi_1(\mathbb{CP}^2 \setminus \mathcal{A}, b)$ is a *meridian of* L if α has a representative δ constructed as follows:

- there is a smooth complex analytic disc $\Delta \subset \mathbb{CP}^2$ transverse to L at a smooth point of $\mathcal{A}$ and such that $\Delta \cap L = \{b'\} \subset L$, and pick out a point $b'' \in \partial\Delta$.
- there is a path a in $\mathbb{CP}^2 \setminus \mathcal{A}$ from b to $b'' \in \partial\Delta$;
- $\delta = a^{-1} \cdot \beta \cdot a$, where β is the closed path based in b' given by $\partial\Delta$ (in the positive direction).

Choose arbitrarily a line L_0 of the arrangement. Note that a meridian of L_0 is the product of the inverse of some meridians of the lines $L_1, \cdots, L_n$, in $E_{\mathcal{A}}$. Let $\mathcal{P}$ be the set of singular points of the affine arrangement $\mathcal{A}^{\text{aff}} = \mathcal{A} \setminus L_0$. We fix a braided wiring diagram $W_{\mathcal{A}}$ and assume that $\mathcal{A}$ is ordered as described in Subsection 2.1.1.

Definition 3.1.13. A *cycle* of the incidence graph $\Gamma_{\mathcal{A}}$ is an element of $\pi_1(\Gamma_{\mathcal{A}}, v_{L_0})$.

Remark 3.1.14. The group $\pi_1(\Gamma_{\mathcal{A}}, v_{L_0})$ is a free group on $b_1(\Gamma_{\mathcal{A}})$ generators.

We construct a generating system $\mathscr{E}$ of cycles of $\Gamma_{\mathcal{A}}$ as follows. Let $\mathcal{T}$ be the maximal tree of $\Gamma_{\mathcal{A}}$ containing the following edges:

- $e(L, P)$ for all $P \in L_0$, and $L \in \mathcal{A}$;
- $e(L_{\nu(P)}, P)$ for all $P \in \mathcal{P}$ and $\nu(P) = \min\{\ j \ \mid\ L_i \in \mathcal{A}_P\ \}$.

Remark 3.1.15. Up to the choice of a projection and an admissible path for the wiring diagram $W_{\mathcal{A}}$, and then of an order on $\mathcal{A}$, this maximal tree is uniquely determined.

An edge in $\Gamma_{\mathcal{A}} \setminus \mathcal{T}$ is of the form $e(L_j, P)$, with $P \in \mathcal{P}$ and $L_j \in \mathcal{A} \setminus L_0$. By definition of a maximal tree, there exists a unique path $\lambda_{P,j}$ in $\mathcal{T}$ joining v_P and v_{L_j}. The unique cycle of $\Gamma_{\mathcal{A}}$ containing the three line-vertices v_{L_0}, $v_{L_{\nu(P)}}$ and v_{L_j}, and no other line-vertex, is denoted by:

$$\xi_{\nu(P),j} = \lambda_{P,j} \cup e(L_j, P).$$

Let $\mathscr{E}$ be the set of cycles of $\Gamma_{\mathcal{A}}$ of the form $\xi_{s,t}$. To each $\xi_{s,t}$ in $\mathscr{E}$ will correspond an element (a stable letter) of $\pi_1(B_{\mathcal{A}}, X_0)$ (where $X_0 \in \mathcal{N}_0$), that we denote $\mathfrak{e}_{s,t}$.

Notation. As in Chapter 2, we denote $[a_1, \cdots, a_m]$ the equality of all the cyclic permutations

$$a_1 \cdots a_m = a_2 \cdots a_m a_1 = \cdots = a_m a_1 \cdots a_{m-1}.$$

For $i = 0, \cdots, n$, let α_i be a meridian of L_i contained in the boundary of a regular neighborhood of L_0, and for $\xi_{s,t} \in \mathscr{E}$, let $\mathfrak{e}_{s,t}$ be a non trivial cycle contained in the boundary of a regular neighborhood of $L_0 \cup L_s \cup L_t$ (it comes from the gluing over the edge $e(L_t, P)$ where P is $L_s \cap L_t$). By Definition 3.1.12, α_i is $a_i \cdot \beta_i \cdot a_i^{-1}$. We assume that $\mathfrak{e}_{s,t}$ can be decomposed in $a_t \cdot^{-1} \tau \cdot a_s$, where τ is a path in the boundary of a regular neighborhood of $L_s \cup L_t$. We also supposed that for $(s,t) \neq (s', t')$, $\mathfrak{e}_{s,t} \cap \mathfrak{e}_{s',t'} = X_0$.

Proposition 3.1.16. *Let α_i and $\mathfrak{e}_{s,t}$ be as previously defined. For any singular point $P = P_{i_1,\cdots,i_m}$ with multiplicity m and $i_1 = \nu(P)$, let*

$$\mathcal{R}_P = [\alpha_{i_m}^{c_{i_m}}, \cdots, \alpha_{i_2}^{c_{i_2}}, \alpha_{i_1}], \text{ where } c_{i_j} = \mathfrak{e}_{i_1,i_j} \text{ for all } j = 2, \cdots, m.$$

The fundamental group of the boundary manifold $B_{\mathcal{A}}$ admits the following presentation:

$$\pi_1(B_{\mathcal{A}}, X_0) = \langle \alpha_0, \alpha_1, \cdots, \alpha_n, \mathfrak{e}_{s_1,t_1}, \cdots, \mathfrak{e}_{s_l,t_l} \mid \bigcup_{P \in \mathcal{P}} \mathcal{R}_P \rangle.$$

It is worth noticing that the $\mathfrak{e}_{s,t}$ are not uniquely defined (see details in the proof).

Proof. Consider $P \in \mathcal{Q}$. Assume that $P = P_{i_1,\cdots,i_m}$. Let $y_{P,i_1}, \ldots, y_{P,i_m}$ be the 'local' meridians of the line L_i in $\partial\mathbb{B}_P$. We have the following presentation of $\pi_1(\mathcal{S}_P)$:

$$\pi_1(\mathcal{S}_P) = \langle y_{P,i_1}, \cdots, y_{P,i_m} \mid [y_{P,i_m}, \cdots, y_{P,i_1}]\rangle.$$

Remark that, according with Definition 3.1.1, y_{P,i_j} is a meridian of T_{i_j} and a longitude is the product of the other y_{P,i_k}.

Consider $k \in \{0, \cdots, n\}$. Let $\mathcal{Q} \cap L_k = \{P_{k_1}, \cdots, P_{k_l}\}$. Let g_{k,k_i} be the image of a meridian in F_k around P_{k_i}, viewed in $F_k \times \{1\} \subset \mathcal{N}_k$, and $\alpha_k \in \pi_1(\mathcal{N}_k)$ a meridian of L_k contained in a regular neighborhood of L_0. We have the following presentation of $\pi_1(\mathcal{N}_k)$:

$$\pi_1(\mathcal{N}_k) = \langle g_{k,k_1}, \cdots, g_{k,k_l}, \alpha_k \mid \forall i \in \{1, \cdots, l\}\,,\ \alpha_k^{-1} \cdot g_{k,k_i} \cdot \alpha_k = g_{k,k_i}\rangle.$$

Remark that according with Definition 3.1.2, g_{k,k_i} is a longitude and α_k is a meridian of T_{P_i}.

In a first step, we only glue the $\mathcal{N}_k$'s and the $\mathcal{S}_P$'s over the edges of $\mathcal{T}$. To do this, we use Proposition 3.1.9, and we consider a contractible set Θ homeomorphic to $\mathcal{T}$ and joining the base points of the $\mathcal{N}_k$'s and the $\mathcal{S}_P$'s. The fundamental group of $B_\mathcal{A}$ is compute relatively to Θ.

In a second step, we glue over the edges of $\Gamma_\mathcal{A} - \mathcal{T}$ (or equivalently the elements of $\mathscr{E}$). Then we use Proposition 3.1.10, and we denote $\mathfrak{e}_{s,t}$ the stable letters coming from the glue due to the edge $e(L_t, P)$, with $P = L_s \cap L_t$.

Note that if $P_i \in L_j$, then the meridian α_j is identified with $y_{P_i,j}$ and $g_{j,i}$ with the product of generators of $\pi_1(\mathcal{S}_{P_i})$ different of $y_{P_i,j}$. Then after a first simplification, we obtain the following presentation of the fundamental group of the boundary manifold:

$$\pi_1(B_\mathcal{A}) = \langle \alpha_0, \alpha_1, \cdots, \alpha_n, \mathfrak{e}_{s_1,t_1}, \cdots, \mathfrak{e}_{s_l,t_l} \mid \bigcup_{P\in\mathcal{Q}} \mathcal{R}_P\rangle.$$

Remark that the presentation of $\pi_1(\mathcal{N}_0)$ implies that α_0 commutes with the $g_{0,i}$ and by identification with α_i, for $i \in \{1, \cdots, n\}$. By construction of $\mathcal{T}$, the relators $\mathcal{R}_P$, for $P \in L_0$, are of the form:

$$[\alpha_{0_m}, \alpha_{0_{m-1}}, \cdots, \alpha_{0_2}, \alpha_0].$$

Then they are all trivial. □

3.2 Inclusion map

Let $W_\mathcal{A}$ be the wiring diagram associated to the choice of a generic projection π and an admissible path γ. We start by choosing a generating system $\mathscr{E} = \{\xi_{s,t}\}$ of cycles of the incidence graph $\Gamma_\mathcal{A}$. These cycles can be directly seen in $W_\mathcal{A}$, since it contains all the singular points and the vertices of $\Gamma_\mathcal{A}$ can be identified with their corresponding wires between two singular points. Then, the first step is to "push" each cycle $\xi_{s,t}$ from

$W_{\mathcal{A}}$ to the boundary manifold $B_{\mathcal{A}}$. The procedure is described in Section 3.2.1, and gives an explicit family $\{\varepsilon_{s,t}\}$ of $\pi_1(B_{\mathcal{A}})$, indexed by $\mathscr{E}$. This family, with the set of meridians of the lines, generates $\pi_1(B_{\mathcal{A}})$. The second step is to compute the images of these generators $\varepsilon_{s,t}$ by the inclusion map. We use an ad hoc Arvola's algorithm to make the computations directly from $W_{\mathcal{A}}$, see Section 3.2.2. Then the map is described in Theorem 3.2.2.

In Subsection 3.2.3, we examine the kernel of the map; this provides an exact sequence involving $\pi_1(B_{\mathcal{A}})$ and $\pi_1(E_{\mathcal{A}})$, see Theorem 3.2.4. We deduce in Subsection 3.2.4 a presentation of $\pi_1(E_{\mathcal{A}})$ where the generators of $\pi_1(B_{\mathcal{A}})$ appear explicitly. This presentation provides a complex whose homotopy type is the same that $E_{\mathcal{A}}$, see Proposition 3.2.6. Finally, in Subsection 3.2.5 we give a simpler description of the map i_*, and obtain a generalization of the Randell Theorem.

3.2.1 Cycles of the boundary manifold

We suppose that the admissible path γ emanates from x_0 and goes through $x_1, \dots, x_k$, the images of the singular points of $\mathcal{A}$ by π, ordered by their real parts. Let $\mathscr{E} = \{\xi_{s,t}\}$ be the generating set defined in Section 3.1 of cycles of $\Gamma_{\mathcal{A}}$.

Each cycle $\xi_{s,t} \in \mathscr{E}$ is sent to $B_{\mathcal{A}}$ via $W_{\mathcal{A}}$, as follows. Let $X_0 = (x_0, y_0)$ be a point of $\mathcal{N}_0$ such that $x_0 = \gamma(0)$. The vertices of $\xi_{s,t}$ of the form v_{L_0} or v_P with $P \in L_0$ and the edges of the form $e(L_0, P)$, with $P \in L_0$, are all sent to the point X_0. The edges $e(L_i, P)$, with $i \neq 0$ and $P \in L_0$, are sent to segments from X_0 to the points $L_i \cap \pi^{-1}(x_0)$. Then the remaining vertices of the form $v_L \in V_L(\mathcal{A})$ are sent to $L \cap \pi^{-1}(x_0)$. Let $\xi_{s,t}$ denote now the cycle of $W_{\mathcal{A}}$, relative to the left endpoints, where the vertices $v_P \in V_P(\mathcal{A})$ are identified with the singular points P, and the edges with their corresponding wire of $W_{\mathcal{A}}$.

A *framed cycle* in $B_{\mathcal{A}}$ is obtained as a perturbation of a cycle $\xi_{s,t}$. This cycle $\xi_{s,t}$ consists of two arcs, the parts in L_s and L_t. Each arc goes through several actual crossings of $W_{\mathcal{A}}$ and can be viewed as a union of small arcs. Each of them is projected to $B_{\mathcal{A}}$ in the direction $(0, i)$ and their images are glued together as follows. For each actual crossing P, modify γ slightly so that it makes a half circle of (small) radius η_P around $x = \pi(P)$ in the positive sense. Choose η_P so that the preimage of this half circle lies in $B_{\mathcal{A}}$. Note that we avoid the intersection point $P : (x_P, y_P)$ of L_s and L_t as follows. Consider $\mathcal{N}_P$ as a polydisc, and join the two end points with the union of the two segments joining these end points with the point of $\pi^{-1}(x_P - \eta_P) \cap \mathcal{N}_P$ having the smallest real part, see Figure 3.3. The class of the obtained cycle in $\pi_1(B_{\mathcal{A}}, X_0)$ denoted by $\varepsilon_{s,t}$ is a stable letter in the computation of $\pi_1(B_{\mathcal{A}})$ done in Proposition 3.1.16 (i.e. we take $\mathfrak{e}_{s,t} = \varepsilon_{s,t}$). The following map σ is a morphism:

$$\begin{array}{rcl} \sigma : \pi_1(\Gamma_{\mathcal{A}}, \mathcal{T}) & \longrightarrow & \pi_1(B_{\mathcal{A}}, X_0), \\ \xi_{s,t} & \longmapsto & \varepsilon_{s,t}. \end{array}$$

In order to compute the image of the framed cycles in the complement $E_{\mathcal{A}}$, we introduce *geometric cycles* $\mathcal{E}_{s,t}$, defined as parallel copies of the $\xi_{s,t}$'s. Indeed, let $\mathcal{E}_{s,t}$ in

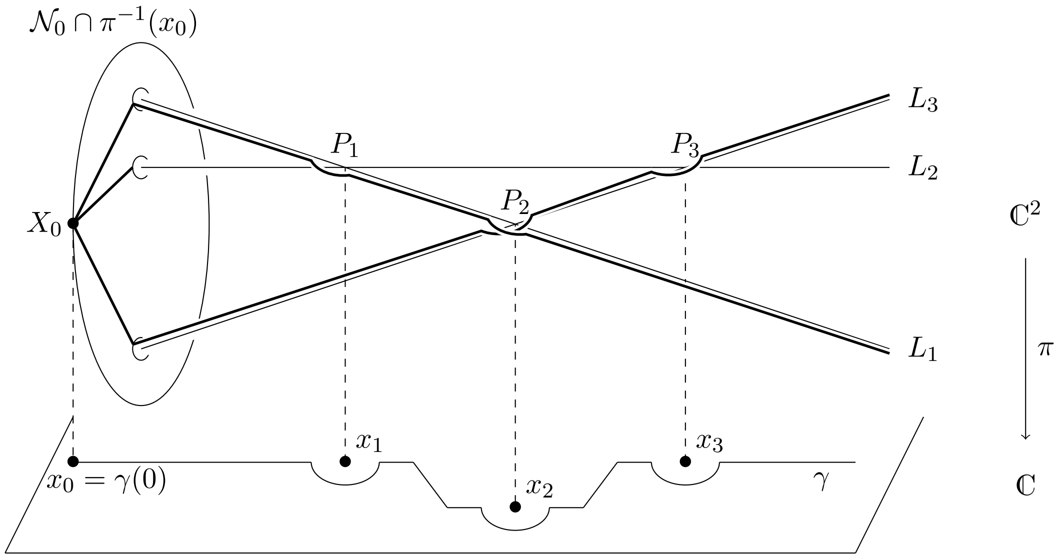

Figure 3.2: Construction of $\delta(\varepsilon)$

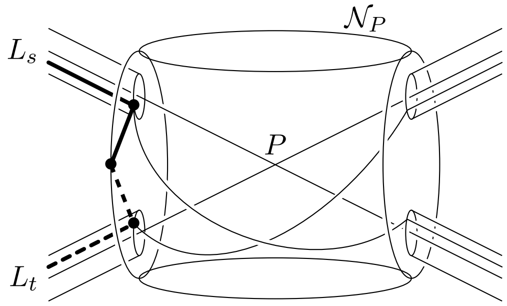

Figure 3.3: Construction of $\delta(\varepsilon)$ near a singular point

$\pi_1(B_{\mathcal{A}})$ be the image of $\xi_{s,t}$ by the projection in the direction $[0 : i : 0]$. We define an *unknotting map*:

$$\begin{array}{rcl} \delta : \pi_1(B_{\mathcal{A}}, X_0) & \longrightarrow & \pi_1(B_{\mathcal{A}}, X_0), \\ \varepsilon_{s,t} & \longmapsto & \delta^l_{s,t}\ \varepsilon_{s,t}\ \delta^r_{s,t}, \end{array}$$

where $\delta^l_{s,t}$ (resp. $\delta^r_{s,t}$) corresponds to the arc in L_s (resp. L_t) of $\varepsilon_{s,t}$. They are defined as the products over all actual crossings P of the arc L_s (resp. L_t) of $\xi_{s,t}$, different from $L_s \cap L_t$, of the following words. Suppose that $P = L_{i_1} \cap \cdots \cap L_{i_m}$ where the order of the lines corresponds to Figure 3.4.

- If $P \in L_s$, let $h \in \{1, \cdots, m\}$ be such that $i_h = s$, then P contributes to $\delta^l_{s,t}$ by:

$$\varepsilon^{-1}_{i_1,i_h} \left(\alpha^{-1}_{i_1} \left(\varepsilon_{i_1,i_2} \alpha^{-1}_{i_2} \varepsilon^{-1}_{i_1,i_2} \right) \cdots \left(\varepsilon_{i_1,i_{h-1}} \alpha^{-1}_{i_{h-1}} \varepsilon^{-1}_{i_1,i_{h-1}} \right) \right) \varepsilon_{i_1,i_h},$$

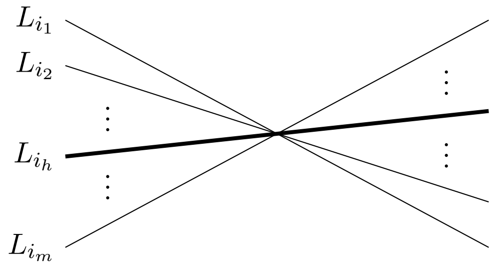

Figure 3.4: Indexation of a crossing

- If $P \in L_t$, let $h \in \{1, \cdots, m\}$ be such that $i_h = t$, then P contributes to $\delta^r_{s,t}$ by:

$$\varepsilon^{-1}_{i_1,i_h}\left(\left(\varepsilon_{i_1,i_{h-1}}\alpha_{i_{h-1}}\varepsilon^{-1}_{i_1,i_{h-1}}\right)\cdots\left(\varepsilon_{i_1,i_2}\alpha_{i_2}\varepsilon^{-1}_{i_1,i_2}\right)\alpha_{i_1}\right)\varepsilon_{i_1,i_h}.$$

Proposition 3.2.1. *The image of a framed cycle by the unknotting map δ is a geometric cycle. In particular, we have*

$$\delta(\varepsilon_{s,t}) = \delta^l_{s,t}\ \varepsilon_{s,t}\ \delta^r_{s,t} = \mathcal{E}_{s,t}.$$

Proof. The contribution of P is induced by the action of a half-twist, given by the pre-image by γ of the half circle around each $x = \pi(P)$, in the positive sense. We obtain the description of $\delta^l_{s,t}$ and $\delta^r_{s,t}$ above, and then $\varepsilon_{s,t} = \left(\delta^l_{s,t}\right)^{-1}\mathcal{E}_{s,t}\left(\delta^r_{s,t}\right)^{-1}$. □

3.2.2 Inclusion map

Geometric cycles were constructed by taking parallel copies of cycles of $\Gamma_{\mathcal{A}}$, via $W_{\mathcal{A}}$, to the boundary manifold $B_{\mathcal{A}}$. Their image in $E_{\mathcal{A}}$ can then be computed directly from $W_{\mathcal{A}}$.

Let $\xi_{s,t}$ be a cycle of $W_{\mathcal{A}}$, relative to the left endpoints. An *over arc* ς is an arc of $W_{\mathcal{A}}$ that goes over $\xi_{s,t}$ through a virtual crossing. Denote $\mathrm{sgn}(\varsigma) \in \{\pm 1\}$ the sign of the crossing. It is positive if the orientations of ς and $\xi_{s,t}$ (in this order) at the crossing form a positive base, and is negative otherwise.

Let $S_{\xi_{s,t}}$ be the set of over arcs of $\xi_{s,t}$, oriented from left to right. The element $\mu_{s,t}$ is defined by :

$$\mu_{s,t} = \prod_{\varsigma \in S_{\xi_{s,t}}} a_\varsigma^{\mathrm{sgn}(\varsigma)},$$

where a_ς is the word associate to the arc ς by the Arvola's algorithm (see Subsection 2.1.2), and the order in the product respects the order of the virtual crossings in the cycle $\xi_{s,t}$. Note that $\mu_{s,t}$ is a product of conjugates of meridians.

Theorem 3.2.2. *For $i = 0, \ldots, n$, let α_i be the meridians of the lines and let $\{\varepsilon_{s,t}\}$ be a set of cycles indexed by a generating system $\mathscr{E}$ of cycles of the incidence graph $\Gamma_{\mathcal{A}}$.*

Then the fundamental group of $B_{\mathcal{A}}$ is generated by $\{\alpha_0, \dots, \alpha_n, \varepsilon_{s_1,t_1}, \dots, \varepsilon_{s_l,t_l}\}$, and the map $i_* : \pi_1(B_{\mathcal{A}}) \to \pi_1(E_{\mathcal{A}})$ induced by the inclusion is described as follows :

$$i_* : \begin{cases} \alpha_i & \longmapsto \quad \alpha_i, \\ \varepsilon_{s,t} & \longmapsto \quad \left(\delta^l_{s,t}\right)^{-1} \mu_{s,t} \left(\delta^r_{s,t}\right)^{-1}, \end{cases}$$

It is worth noticing that by a recursive argument on the set of $\varepsilon_{s,t}$, the words $\left(\delta^l_{s,t}\right)^{-1} \mu_{s,t} \left(\delta^r_{s,t}\right)^{-1}$ are products of conjugates of the meridians $\alpha_1, \dots, \alpha_n$.

Proof. Since $B_{\mathcal{A}} \subset E_{\mathcal{A}}$, then a class in $\pi_1(B_{\mathcal{A}})$ can be viewed as a class in $\pi_1(E_{\mathcal{A}})$, and the both are denoted in the same way.

By Proposition 3.2.1, each class $\delta^l_{s,t}\varepsilon_{s,t}\delta^r_{s,t}$ in $\pi_1(B_{\mathcal{A}})$ can be represented by a geometric cycle $\mathcal{E}_{s,t}$, obtained as a parallel copy of $\xi_{s,t}$ from $W_{\mathcal{A}}$ to $B_{\mathcal{A}}$. Consider a 2-cell homotopic to a disc with card$(S_{\xi_{s,t}})$ holes. Then glue the boundary of the disc to $\mathcal{E}_{s,t}$ and the other boundary components to the meridians of over arcs $\varsigma \in S_{\xi_{s,t}}$. As the 2-cell is in $E_{\mathcal{A}}$, then, in the exterior, $\mathcal{E}_{s,t}$ can be retracted to the product $\mu_{s,t}$ of the a_ς, with $\varsigma \in S_{\xi_{s,t}}$. It follows that in $\pi_1(E_{\mathcal{A}})$, $\delta^l_{s,t}\varepsilon_{s,t}\delta^r_{s,t} = \mu_{s,t}$. $\square$

Remark 3.2.3. The advantage of Theorem 3.2.2 is that the cycles $\varepsilon_{s,t}$ are well defined. Furthermore they are nearby the lines L_0, L_s and L_t. This will be very useful in Chapter 5.

3.2.3 Exact sequence

Theorem 3.2.4. *The following sequence is exact*

$$0 \longrightarrow K \xrightarrow{\phi} \pi_1(B_{\mathcal{A}}) \xrightarrow{i_*} \pi_1(E_{\mathcal{A}}) \longrightarrow 0,$$

where K is the normal subgroup of $\pi_1(B_{\mathcal{A}})$ generated by the elements of the form $\delta^l_{s,t}\varepsilon_{s,t}\delta^r_{s,t}\mu^{-1}_{s,t}$, and the product $\alpha_0 \cdots \alpha_n$.

Proof. By Theorem 3.2.2, the map i_* is onto and K is included in $\ker(i_*)$. It remains to show that the relations induced by the images $i_*(\varepsilon_{s,t})$ are enough to determine a presentation of $\pi_1(E_{\mathcal{A}})$. We compare these relations to those coming from braid monodromy and Zariski-Van Kampen's method, see [43] for example.

Let $P = L_{i_1} \cap \cdots \cap L_{i_m}$ (as in Figure 3.4), be a singular point of $\mathcal{A}$, with $i_1 = \nu(P)$. Consider a small ball in $\mathbb{P}^2$ with center P and a local base point b in its boundary sphere. Let λ be a path from X_0 to b, and let y_j be the (local) meridian of L_j with base b, for $j = 1, \dots, m$. The path λ can be chosen in such a way that the Zariski-Van Kampen relation associated to P are :

$$[y^\lambda_{i,i_m}, \cdots, y^\lambda_{i,i_1}].$$

We can assume that b is a point of $\varepsilon_{i_1,j}$, for all $j = i_2, \cdots, i_m$. Then $\varepsilon_{i_1,j} = \beta_j^{-1}\beta_{i_1}$ where β_{i_1} goes from X_0 to b, and β_j^{-1} from b to X_0. We get

$$\begin{aligned}
[\alpha_{i_m}^{\varepsilon_{i_1,i_m}}, \cdots, \alpha_{i_2}^{\varepsilon_{i_1,i_2}}, \alpha_{i_1}] &\Leftrightarrow [\alpha_{i_m}^{\beta_{i_m}^{-1}\beta_{i_1}}, \cdots, \alpha_{i_2}^{\beta_{i_2}^{-1}\beta_{i_1}}, \alpha_{i_1}^{\beta_{i_1}^{-1}\beta_{i_1}}],\\
&\Leftrightarrow [\alpha_{i_m}^{\beta_{i_m}^{-1}}, \cdots, \alpha_{i_2}^{\beta_{i_2}^{-1}}, \alpha_{i_1}^{\beta_{i_1}^{-1}}]^{\beta_{i_1}},\\
&\Leftrightarrow [\alpha_{i_m}^{\beta_{i_m}^{-1}}, \cdots, \alpha_{i_2}^{\beta_{i_2}^{-1}}, \alpha_{i_1}^{\beta_{i_1}^{-1}}],\\
&\Leftrightarrow [\alpha_{i_m}^{\beta_{i_m}^{-1}}, \cdots, \alpha_{i_2}^{\beta_{i_2}^{-1}}, \alpha_{i_1}^{\beta_{i_1}^{-1}}]^{\lambda}.
\end{aligned}$$

Note that during this computation, the base point may have changed, but the first and the last relations are based in X_0. Since $\alpha_j^{\beta_j^{-1}} = y_{i,j}$, for all $j = i_1, \cdots, i_m$, then:

$$\begin{aligned}
[\alpha_{i_m}^{\varepsilon_{i_1,i_m}}, \cdots, \alpha_{i_2}^{\varepsilon_{i_1,i_2}}, \alpha_{i_1}] &\Leftrightarrow [y_{i_m}, \cdots, y_{i_2}, y_{i_1}]^{\lambda},\\
&\Leftrightarrow [y_{i_m}^{\lambda}, \cdots, y_{i_2}^{\lambda}, y_{i_1}^{\lambda}].
\end{aligned}$$

□

3.2.4 Homotopy type of the complement

From Theorem 3.2.3, we obtain a presentation of the fundamental group of $\pi_1(E_{\mathcal{A}})$.

Corollary 3.2.5. *For $i = 1, \ldots, n$, let α_i be the meridians of the lines L_i. For any singular point $P = L_{i_1} \cap L_{i_2} \cap \cdots \cap L_{i_m}$ with $i_1 = \nu(P)$, let*

$$\mathcal{R}_P = [\alpha_{i_m}^{c_{i_m}}, \cdots, \alpha_{i_2}^{c_{i_2}}, \alpha_{i_1}], \textit{ where } c_{i_j} = \left(\delta^l_{i_1,i_j}\right)^{-1} \mu_{i_1,i_j} \left(\delta^r_{i_1,i_j}\right)^{-1} \textit{ for all } j = 2, \cdots, m.$$

The fundamental group of $E_{\mathcal{A}}$ admits the following presentation:

$$\pi_1(E_{\mathcal{A}}) = \langle \alpha_1, \cdots, \alpha_n \mid \bigcup_{P \in \mathcal{P}} \mathcal{R}_P \rangle.$$

Proof. For each $\varepsilon_{s,t}$, let $r_{s,t}$ be the relation $\varepsilon_{s,t} = \left(\delta^l_{s,t}\right)^{-1} \mu_{s,t} \left(\delta^r_{s,t}\right)^{-1}$, and for each point $P \in \mathcal{P}$ (with $P = L_{i_1} \cap \cdots \cap L_{i_m}$ and $i_1 = \nu(P)$), we define the relation $\mathcal{R}'_P :\ [\alpha_{i_m}^{\varepsilon_{i_1,i_m}}, \cdots, \alpha_{i_2}^{\varepsilon_{i_1,i_2}}, \alpha_{i_1}]$. Then, Theorem 3.2.4 implies that we have the following presentation:

$$\pi_1(E_{\mathcal{A}}) \;=\; \langle \alpha_0, \alpha_1, \cdots, \alpha_n, \varepsilon_{s_1,t_1}, \cdots, \varepsilon_{s_l,t_l} \mid \bigcup_{P \in \mathcal{P}} \mathcal{R}'_P,\ \bigcup_{i=1}^{l} r_{s_i,t_i},\ \alpha_0 \cdots \alpha_n \rangle.$$

Consider the total order on the set $\{\varepsilon_{s,t}\}$: $\left(\varepsilon_{s,t} < \varepsilon_{s',t'}\right) \Leftrightarrow (s \leq s' \textit{ and } t < t')$. By construction, $\delta^l_{s,t}$ and $\delta^r_{s,t}$ depend on $\varepsilon_{s',t'}$ if and only if $\varepsilon_{s',t'} < \varepsilon_{s,t}$. Since $\mu_{s,t}$ is a product of meridians, then the smallest $\varepsilon_{s,t}$ is a product of meridians. And by induction, the relation $r_{s,t}$ expresses any $\varepsilon_{s,t}$ as a product of α_i.

Finally, using the relation $\alpha_0, \cdots, \alpha_n = 1$, the meridian α_0 can be removed from the set of generators of $\pi_1(E_{\mathcal{A}})$. Indeed no other relation contains α_0. □

Proposition 3.2.6. *The 2-complex modeled on the minimal presentation given in Corollary 3.2.5 is homotopy equivalent to $E_{\mathcal{A}}$.*

Proof. The proof of Theorem 3.2.4 shows in particular that the relations of the presentation in Corollary 3.2.5 are equivalent to the Zariski-Van Kampen relations, based on the braid monodromy. It is shown in [43] that the 2-complex modeled on a minimal presentation equivalent to the Zariski-Van Kampen presentation is homotopy equivalent to $E_{\mathcal{A}}$. □

3.2.5 Simplification

In the construction of the boundary manifold (see Section 3.1), the cycle $\varepsilon_{s,t}$ appear as stable letters in HNN-extension. Then the choice of such cycle is arbitrary. In this section, the cycles $\varepsilon_{s,t}$ correspond to the injection of $\Gamma_{\mathcal{A}}$ in $B_{\mathcal{A}}$ (i.e. the framed cycles). But we also consider the geometrical cycles defined from the projection of $W_{\mathcal{A}}$ in $B_{\mathcal{A}}$ (i.e. the $\delta(\varepsilon_{s,t})$, noted $\mathcal{E}_{s,t}$). The construction of this geometric cycles allows to consider them as stable letters for the HNN-extension too (i.e. in Proposition 3.1.16, $\mathfrak{e}_{s,t} = \mathcal{E}_{s,t}$). With this presentation of $\pi_1(B_{\mathcal{A}})$, we obtain a simpler version of Theorem 3.2.2:

Theorem 3.2.7. *For $i = 0, \ldots, n$, let α_i be the meridians of the lines and let $\{\mathcal{E}_{s,t}\}$ be a set of cycles indexed by a generating system $\mathscr{E}$ of cycles of the incidence graph $\Gamma_{\mathcal{A}}$. Then the fundamental group of $B_{\mathcal{A}}$ is generated by $\{\alpha_1, \ldots, \alpha_n, \mathcal{E}_{s_1,t_1}, \cdots, \mathcal{E}_{s_l,t_l}\}$, and the map $i_* : \pi_1(B_{\mathcal{A}}) \to \pi_1(E_{\mathcal{A}})$ induced by the inclusion is described as follows :*

$$i_* : \begin{cases} \alpha_i & \longmapsto & \alpha_i, \\ \mathcal{E}_{s,t} & \longmapsto & \mu_{s,t}, \end{cases}$$

Then we can also simplify the corollary of Theorem 3.2.2, and then:

Corollary 3.2.8. *For $i = 1, \ldots, n$, let α_i be the meridians of the lines L_i. For any singular point $P = L_{i_1} \cap L_{i_2} \cap \cdots \cap L_{i_m}$ with $i_1 = \nu(P)$, let*

$$\mathcal{R}_P = [\alpha_{i_m}^{\mu_{i_1,i_m}}, \cdots, \alpha_{i_2}^{\mu_{i_1,i_2}}, \alpha_{i_1}].$$

The fundamental group of $E_{\mathcal{A}}$ admits the following presentation:

$$\pi_1(E_{\mathcal{A}}) = \langle \alpha_1, \cdots, \alpha_n \mid \bigcup_{P \in \mathcal{P}} \mathcal{R}_P \rangle.$$

Remark 3.2.9. In fact, Theorem 3.2.7 is useful for Corollary 3.2.8. Because we do not control the position of the cycles $\mathcal{E}_{s,t}$ relative to the arrangement. In contrast to the cycles $\varepsilon_{s,t}$ of Theorem 3.2.2.

Proposition 3.2.10. *The presentation of the fundamental group of $E_{\mathcal{A}}$ is minimal, and the 2-complex built on it has also the same homotopy type with the exterior of $\mathcal{A}$.*

As consequence of this corollary, we found the theorem of Randell in the case of real complexified arrangement.

Theorem 3.2.11 (Randell, [58])**.** *Let $\mathcal{A}$ be a real complexified line arrangement. For $i = 1, \cdots, n$, let α_i be the meridians of the lines L_i. The fundamental group of $E_{\mathcal{A}}$ admits the following presentation:*

$$\pi_1(E_{\mathcal{A}}) = \langle \alpha_1, \cdots, \alpha_n \mid \bigcup_{P_i \in \mathcal{P}} [\alpha_{i_m}, \cdots, \alpha_{i_2}, \alpha_{i_1}], \text{ with } P = L_{i_1} \cap \cdots \cap L_{i_m} \rangle,$$

with $L_{i_1}, \cdots, L_{i_m}$ are taken from top to bottom at the left of P_i in $W_{\mathcal{A}}$.

3.3 Example: MacLane

In this section, we illustrate Theorem 3.2.2 with MacLane's arrangement Q^+ defined in Subsection 2.3.1. The incidence graph Γ of Q^+ is given in Figure 3.5.

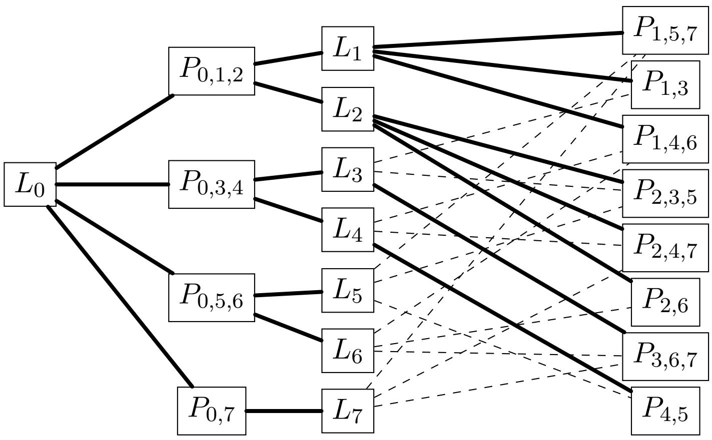

Figure 3.5: Incidence graph of the MacLane arrangement Q^+

It is worth mentioning that Q^+ is one of the only two ordered realizations of this combinatorial data by an arrangement in $\mathbb{CP}^2$.

3.3.1 Generating set of cycles of Γ_{Q^+}

Consider the maximal tree $\mathcal{T}$ in Γ_{Q^+} indicated with thick lines in Figure 3.5. Let $\mathscr{E}$ be the generating system of cycles induced by $\mathcal{T}$ (it is in one-to-one correspondence with the dotted lines in Figure 3.5):

$$\mathscr{E} = \{\xi_{2,3}, \xi_{2,5}, \xi_{2,4}, \xi_{2,7}, \xi_{2,6}, \xi_{4,5}, \xi_{3,6}, \xi_{3,7}, \xi_{1,5}, \xi_{1,7}, \xi_{1,3}, \xi_{1,4}, \xi_{1,6}\}.$$

3.3.2 Group of the boundary manifold

By Section 3.2.1, the images ε of the cycles ξ by the application σ provide a family of cycles in $\pi_1(B_{Q^+})$. Proposition 3.1.16 applies to this explicit family, and $\pi_1(B_{Q^+})$ admits a presentation with generators:

$$\{\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7\}\cup\{\varepsilon_{2,3},\varepsilon_{2,5},\varepsilon_{2,4},\varepsilon_{2,7},\varepsilon_{2,6},\varepsilon_{4,5},\varepsilon_{3,6},\varepsilon_{3,7},\varepsilon_{1,5},\varepsilon_{1,7},\varepsilon_{1,3},\varepsilon_{1,4},\varepsilon_{1,6}\},$$

and relations:

$$[\alpha_7^{\varepsilon_{1,7}},\alpha_5^{\varepsilon_{1,5}},\alpha_1],[\alpha_3^{\varepsilon_{1,3}},\alpha_1],[\alpha_6^{\varepsilon_{1,6}},\alpha_4^{\varepsilon_{1,4}},\alpha_1],[\alpha_5^{\varepsilon_{2,5}},\alpha_3^{\varepsilon_{2,3}},\alpha_2],$$
$$[\alpha_7^{\varepsilon_{2,7}},\alpha_4^{\varepsilon_{2,4}},\alpha_2],[\alpha_6^{\varepsilon_{2,6}},\alpha_2],[\alpha_7^{\varepsilon_{3,7}},\alpha_6^{\varepsilon_{3,6}},\alpha_3],[\alpha_5^{\varepsilon_{4,5}},\alpha_4].$$

3.3.3 Geometric cycles and unknotting map

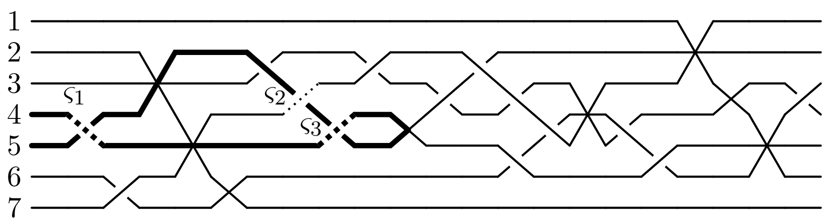

Figure 3.6: Wiring diagram of the positive MacLane arrangement

Let W_{Q^+} be the braided wiring diagram of Q^+ given in Figure 3.6. Note that W_{Q^+} differs from the wiring diagram considered in [24] by an axial symmetry and a local move on the wires corresponding to L_3, L_5, L_7.

The diagram W_{Q^+} is used to compute the unknotting map δ, and the images of the cycles ε in terms of geometric cycles, see Proposition 3.2.1. The thick lines in Figure 3.6 represent the cycle $\xi_{4,5}$, divided into two arcs of L_4 and L_5.

- The first arc L_4 meets the triple point $v_{P_{2,4,7}}$. This gives $\delta^l_{4,5}=\varepsilon_{2,4}^{-1}\alpha_2^{-1}\varepsilon_{2,4}$.

- The second arc L_5 meets $v_{P_{2,3,5}}$, and $\delta^r_{4,5}=\varepsilon_{2,5}^{-1}(\varepsilon_{2,3}\alpha_3\varepsilon_{2,3}^{-1})\alpha_2\varepsilon_{2,5}$.

This implies that

$$\delta(\varepsilon_{4,5})=\left(\varepsilon_{2,4}^{-1}\alpha_2^{-1}\varepsilon_{2,4}\right)\ .\ \varepsilon_{4,5}\ .\ \left[\varepsilon_{2,5}^{-1}\left(\varepsilon_{2,3}\alpha_3\varepsilon_{2,3}^{-1}\right)\alpha_2\varepsilon_{2,5}\right].$$

Similarly, one computes:

$$
\begin{aligned}
&\delta(\varepsilon_{2,3}) = \varepsilon_{2,3},\\
&\delta(\varepsilon_{2,5}) = \varepsilon_{2,5},\\
&\delta(\varepsilon_{2,4}) = \varepsilon_{2,4},\\
&\delta(\varepsilon_{2,7}) = \varepsilon_{2,7},\\
&\delta(\varepsilon_{2,6}) = \varepsilon_{2,6},\\
&\delta(\varepsilon_{4,5}) = \left(\varepsilon_{2,4}^{-1}\alpha_2^{-1}\varepsilon_{2,4}\right)\ .\ \varepsilon_{4,5}\ .\ \left[\varepsilon_{2,5}^{-1}\left(\varepsilon_{2,3}\alpha_3\varepsilon_{2,3}^{-1}\right)\alpha_2\varepsilon_{2,5}\right]\\
&\delta(\varepsilon_{3,6}) = \left(\varepsilon_{2,3}^{-1}\alpha_2^{-1}\varepsilon_{2,3}\right)\ .\ \varepsilon_{3.6}\ .\ \left(\varepsilon_{2,6}^{-1}\alpha_2\varepsilon_{2,6}\right)\\
&\delta(\varepsilon_{3,7}) = \left(\varepsilon_{2,3}^{-1}\alpha_2^{-1}\varepsilon_{2,3}\right)\ .\ \varepsilon_{3,7}\ .\ \left[\varepsilon_{2,7}^{-1}\left(\varepsilon_{2,4}\alpha_4\varepsilon_{2,4}^{-1}\right)\alpha_2\varepsilon_{2,7}\right]\\
&\delta(\varepsilon_{1,5}) = \varepsilon_{1,5}\ .\ \left[\left(\varepsilon_{4,5}^{-1}\alpha_4\varepsilon_{4,5}\right)\left(\varepsilon_{2,5}^{-1}\left(\varepsilon_{2,3}\alpha_3\varepsilon_{2,3}\right)^{-1}\alpha_2\varepsilon_{2,5}\right)\right]\\
&\delta(\varepsilon_{1,7}) = \varepsilon_{1,7}\ .\ \left[\left(\varepsilon_{3,7}^{-1}\left(\varepsilon_{3,6}\alpha_6\varepsilon_{3,6}^{-1}\right)\alpha_3\varepsilon_{3,7}\right)\left(\varepsilon_{2,7}^{-1}\left(\varepsilon_{2,4}\alpha_4\varepsilon_{2,4}^{-1}\right)\alpha_2\varepsilon_{2,7}\right)\right]\\
&\delta(\varepsilon_{1,3}) = \varepsilon_{1,3}\ .\ \left(\varepsilon_{2,3}^{-1}\alpha_2\varepsilon_{2,3}\right)\\
&\delta(\varepsilon_{1,4}) = \varepsilon_{1,4}\ .\ \left(\varepsilon_{2,4}^{-1}\alpha_2\varepsilon_{2,4}\right)\\
&\delta(\varepsilon_{1,6}) = \varepsilon_{1,6}\ .\ \left[\left(\varepsilon_{3,6}^{-1}\alpha_3\varepsilon_{3,6}\right)\left(\varepsilon_{2,6}^{-1}\alpha_2\varepsilon_{2,6}\right)\right]
\end{aligned}
$$

3.3.4 Retractions of geometric cycles

We now compute the family of $\mu_{s,t}$, required to obtain the inclusion map, see Subsection 3.2.2. The arcs of the wiring diagram W_{Q^+} are labeled by the algorithm of Arvola, see Subsection 2.1.2.

The case of $\mu_{4,5}$ is drawn in thick in Figure 3.6. The over arcs ς_1, ς_2 and ς_3 are dotted in Figure 3.6. Arvola's labellings of these arcs are respectively : $a_{\varsigma_1} = \alpha_4$, $a_{\varsigma_2} = \alpha_7$ and $a_{\varsigma_3} = \alpha_7^{-1}\alpha_4\alpha_7$. Furthermore, $\mathrm{sgn}(\varsigma_1) = -1$, $\mathrm{sgn}(\varsigma_2) = 1$ and $\mathrm{sgn}(\varsigma_3) = 1$. We obtain: $\mu_{4,5} = \left(\alpha_7^{-1}\alpha_4\alpha_7\right).\alpha_7.\alpha_4^{-1}$, which gives

$$\mu_{4,5} = \left(\alpha_7^{-1}\alpha_4\alpha_7\right)\ .\ \alpha_7\ .\ \alpha_4^{-1}.$$

Similarly:

$$
\begin{aligned}
&\mu_{2,3} = 1,\\
&\mu_{2,5} = -\alpha_4,\\
&\mu_{2,4} = 1,\\
&\mu_{2,7} = 1,\\
&\mu_{2,6} = \alpha_7,\\
&\mu_{4,5} = \left(\alpha_7^{-1}\alpha_4\alpha_7\right)\ .\ \alpha_7\ .\ \alpha_4^{-1},
\end{aligned}
$$

$$
\begin{aligned}
\mu_{3,6} &= \left[\left(\alpha_4^{-1}\alpha_5\alpha_4\right)\left(\alpha_7^{-1}\right)\left(\alpha_7^{-1}\alpha_4\alpha_7^2\alpha_4^{-1}\ \alpha_5^{-1}\ \alpha_4\alpha_7^{-2}\alpha_4^{-1}\alpha_7\right)(\alpha_7)\right]\ .\ \left[\left(\alpha_7^{-1}\right)\left(\alpha_7^{-1}\alpha_4^{-1}\alpha_7\right)(\alpha_7)\right],\\
\mu_{3,7} &= \left[\left(\alpha_4^{-1}\alpha_5\alpha_4\right)\left(\alpha_7^{-1}\right)\left(\alpha_7^{-1}\alpha_4\alpha_7^2\alpha_4^{-1}\ \alpha_5^{-1}\ \alpha_4\alpha_7^{-2}\alpha_4^{-1}\alpha_7\right)(\alpha_7)\right],\\
\mu_{1,5} &= \left(\alpha_7^{-1}\right)\left(\alpha_7^{-1}\alpha_4\alpha_7\right)(\alpha_7)\left(\alpha_4^{-1}\right),\\
\mu_{1,7} &= 1,\\
\mu_{1,3} &= \left(\alpha_7^{-1}\alpha_4^{-1}\alpha_7^2\ \alpha_6^{-1}\ \alpha_7^{-2}\alpha_4\alpha_7\right)\left(\alpha_7^{-1}\right)\left(\alpha_7^{-1}\alpha_4\alpha_7^2\alpha_4^{-1}\ \alpha_5\ \alpha_4\alpha_7^{-2}\alpha_4^{-1}\alpha_7\right)(\alpha_7)\left(\alpha_4^{-1}\alpha_5^{-1}\alpha_4\right),\\
\mu_{1,4} &= 1,\\
\mu_{1,6} &= \left(\alpha_7^{-1}\alpha_4\alpha_7\right)\left(\alpha_7^{-1}\right)\left(\alpha_7^{-1}\alpha_4^{-1}\alpha_7\right)(\alpha_7).
\end{aligned}
$$

3.3.5 Images in the group of the complement

Following Theorem 3.2.2, we can compute $i_* : \pi_1(B_{Q^+}) \twoheadrightarrow \pi_1(E_{Q^+})$. The computations above describe the relations induced by the images of the cycles ε in $\pi_1(E_{Q^+})$. By the previous computations, $\varepsilon_{2,3}, \varepsilon_{2,4}, \varepsilon_{2,7}$ are equal to 1 (i.e. they are contractible in E_{Q^+}). They are the relations $r_{2,3},\ r_{2,4}$ and $r_{2,7}$. Without additional computation, we obtain:

$$r_{2,5}\ :\ \varepsilon_{2,5} = \alpha_4^{-1}, \qquad\qquad r_{2,6}\ :\ \varepsilon_{2,6} = \alpha_7.$$

The case of $r_{4,5}$:

$$r_{4,5}\ :\ (\varepsilon_{2,4}^{-1}\alpha_2^{-1}\varepsilon_{2,4})\ .\ \varepsilon_{4,5}\ .\ [\varepsilon_{2,5}^{-1}(\varepsilon_{2,3}\alpha_3\varepsilon_{2,3}^{-1})\alpha_2\varepsilon_{2,5}] = \left(\alpha_7^{-1}\alpha_4\alpha_7\right)\ .\ \alpha_7\ .\ \alpha_4^{-1}.$$

Then using $r_{2,4}$, $r_{2,5}$ and $r_{2,3}$, we obtain that:

$$r_{4,5}\ :\ \varepsilon_{4,5} = (\alpha_2)\ .\ \left(\left(\alpha_7^{-1}\alpha_4\alpha_7\right)\ .\ \alpha_7\ .\ \alpha_4^{-1}\right)\ .\ \left(\alpha_4\alpha_2^{-1}\alpha_3^{-1}\alpha_4^{-1}\right).$$

The others relations can be computed by the same way. And from the proof of Corollary 3.2.5, we obtain:

Property 3.3.1. *The fundamental group of E_{Q^+} admits the following presentation:*

$$
\begin{aligned}
\pi_1(E(\mathcal{A}^+)) = \langle &\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7,\\
&\varepsilon_{2,3},\varepsilon_{2,5},\varepsilon_{2,4},\varepsilon_{2,7},\varepsilon_{2,6},\varepsilon_{4,5},\varepsilon_{3,6},\varepsilon_{3,7},\varepsilon_{1,5},\varepsilon_{1,7},\varepsilon_{1,3},\varepsilon_{1,4},\varepsilon_{1,6}\ \mid\\
&r_{2,3},r_{2,5},r_{2,4},r_{2,7},r_{2,6},r_{4,5},r_{3,6},r_{3,7},r_{1,5},r_{1,7},r_{1,3},r_{1,4},r_{1,6},\\
&[\alpha_7^{\varepsilon_{1,7}},\alpha_5^{\varepsilon_{1,5}},\alpha_1],[\alpha_3^{\varepsilon_{1,3}},\alpha_1],[\alpha_6^{\varepsilon_{1,6}},\alpha_4^{\varepsilon_{1,4}},\alpha_1],[\alpha_5^{\varepsilon_{2,5}},\alpha_3^{\varepsilon_{2,3}},\alpha_2],\\
&\qquad[\alpha_7^{\varepsilon_{2,7}},\alpha_4^{\varepsilon_{2,4}},\alpha_2],[\alpha_6^{\varepsilon_{2,6}},\alpha_2],[\alpha_7^{\varepsilon_{3,7}},\alpha_6^{\varepsilon_{3,6}},\alpha_3],[\alpha_5^{\varepsilon_{4,5}},\alpha_4],\alpha_0\cdots\alpha_n\rangle.
\end{aligned}
$$

Chapter 4

Inclusion in homology

The homology groups of a topological space are important invariants. It seems natural to compute the application induced by the inclusion of $B_{\mathcal{A}}$ in $E_{\mathcal{A}}$ on the first homology group. This computation is done in Section 4.1, and will reveal to be very useful in the results of E. Artal [5] to compute a part of the depth of a character (see Chapter 5). On the other hand, it allows to construct a topological invariant of the blow-up of an arrangement.

In Section 4.2, as examples, we explicitly compute the map $\mathrm{H}_1(B_{\mathcal{A}};\mathbb{Z}) \to \mathrm{H}_1(E_{\mathcal{A}};\mathbb{Z})$ in the case of extended Ceva's arrangement and MacLanes's arrangements.

In Section 4.3, we present a canonical decomposition of the first homology group of the boundary manifold into the 1-homology and the 2-cohomology of the complement. This result goes in the sense of the decomposition of the generators of $\pi_1(E_{\mathcal{A}})$ done in Chapter 3. Using Poincaré's residue, we obtain an explicit description of an isomorphism between $\mathrm{H}^2(E_{\mathcal{A}};\mathbb{C})$ and $\mathrm{H}_1(\Gamma_{\mathcal{A}};\mathbb{C})$.

4.1 Inclusion map

4.1.1 Homotopy vs homology

Consider the homology groups of the topological space, and more precisely, the first homology group. It is well known that it can be defined as the abelianised of the fundamental group. Then it contains some information present in the fundamental group, and considering the homology with integer coefficients, we have an abelian group; and with complex coefficients, we obtain vector spaces. In both cases, they admit better properties than finitely presented groups.

To conserve a lot of properties coming from the fundamental group, we study the presentation of the first group of homology inherited from the presentation of the fundamental group. Indeed these presentations have geometrical interpretation.

4.1.2 The map

To describe the inclusion map in homology, we compute the framed cycles and the geometric cycles in $\mathrm{H}_1(E_{\mathcal{A}})$. In this section, we consider the homology with integer coefficients.

Notation. Let $x \in \pi_1(E_{\mathcal{A}})$, the homological class of x is denoted by $\widetilde{x} \in \mathrm{H}_1(E_{\mathcal{A}};\mathbb{Z})$.

Let $\widetilde{\delta} : \mathrm{H}_1(B_{\mathcal{A}};\mathbb{Z}) \to \mathrm{H}_1(B_{\mathcal{A}};\mathbb{Z})$ be the application induced by the unknotting maps (see Subsection 3.2.1) on the homology groups. We have:

$$\widetilde{\delta} : \left\{ \begin{array}{ccc} \mathrm{H}_1(B_{\mathcal{A}};\mathbb{Z}) & \longrightarrow & \mathrm{H}_1(B_{\mathcal{A}};\mathbb{Z}) \\ \widetilde{\varepsilon}_{s,t} & \longmapsto & \widetilde{\delta}^l_{s,t} + \widetilde{\varepsilon}_{s,t} + \widetilde{\delta}^r_{s,t}. \end{array} \right.$$

Notation. To simplifiy the notation of the homological unknotting map we denote:

$$\widetilde{\delta}_{s,t} = \widetilde{\delta}^l_{s,t} + \widetilde{\delta}^r_{s,t}.$$

As in Chapter 3, by considering framed or geometric cycles (i.e. $\varepsilon_{s,t}$ or $\mathcal{E}_{s,t}$), we obtain two possible descriptions of the application induced by the inclusion on the first homology groups. Let $\varepsilon_{s,t} \in \pi_1(B_{\mathcal{A}})$ be framed cycles of $B_{\mathcal{A}}$ as described in Section 3.2. Since, the $\varepsilon_{s,t}$ with the α_i's generate $\pi_1(B_{\mathcal{A}})$, then the $\widetilde{\varepsilon}_{s,t}$'s with $\widetilde{\alpha}_i$'s generate $\mathrm{H}_1(B_{\mathcal{A}};\mathbb{Z})$.

Theorem 4.1.1. *The map induced by the inclusion of the boundary manifold in the complement of the arrangement $\mathcal{A}$ on the first homology groups can be described by:*

$$\begin{array}{ccc} \mathrm{H}_1(B_{\mathcal{A}};\mathbb{Z}) & \longrightarrow & \mathrm{H}_1(E_{\mathcal{A}};\mathbb{Z}) \\ \widetilde{\alpha}_i & \longmapsto & \widetilde{\alpha}_i \\ \widetilde{\varepsilon}_{s,t} & \longmapsto & -\widetilde{\delta}_{s,t} + \widetilde{\mu}_{s,t} \end{array}$$

where $\widetilde{\delta}_{s,t}$ and $\widetilde{\mu}_{s,t}$ are described in Proposition 4.1.5 and Proposition 4.1.6, respectively.

Proof. It is a direct consequence of Theorem 3.2.2. □

Corollary 4.1.2. *If $\mathcal{A}$ is a real complexified lines arrangement, then we have:*

$$\begin{array}{ccc} \mathrm{H}_1(B_{\mathcal{A}};\mathbb{Z}) & \longrightarrow & \mathrm{H}_1(E_{\mathcal{A}};\mathbb{Z}) \\ \widetilde{\alpha}_i & \longmapsto & \widetilde{\alpha}_i \\ \widetilde{\varepsilon}_{s,t} & \longmapsto & -\widetilde{\delta}_{s,t} \end{array}$$

Let $\mathcal{E}_{s,t} \in \pi_1(B_{\mathcal{A}})$ be the geometric cycles as describe in Section 3.2, by definition we have $\widetilde{\mathcal{E}}_{s,t} = \widetilde{\varepsilon}_{s,t} + \widetilde{\delta}_{,t}$. By Subsection 3.2.5, the $\widetilde{\mathcal{E}}_{s,t}$'s with the $\widetilde{\alpha}_i$'s generate $\mathrm{H}_1(B_{\mathcal{A}})$.

Theorem 4.1.3. *The map induced by the inclusion of the boundary manifold in the complement of the arrangement $\mathcal{A}$ on the first homology groups can be described by:*

$$\begin{array}{ccc} \mathrm{H}_1(B_{\mathcal{A}};\mathbb{Z}) & \longrightarrow & \mathrm{H}_1(E_{\mathcal{A}};\mathbb{Z}) \\ \widetilde{\alpha}_i & \longmapsto & \widetilde{\alpha}_i \\ \widetilde{\mathcal{E}}_{s,t} & \longmapsto & \widetilde{\mu}_{s,t} \end{array}$$

where $\widetilde{\mu}_{s,t}$ is described in Proposition 4.1.6.

Proof. It is a direct consequence of Theorem 3.2.7. □

Remark 4.1.4. Since we control the position of the $\varepsilon_{s,t}$ (in contrast with the $\mathcal{E}_{s,t}$), Theorem 4.1.1 is the most useful of the both Theorems (4.1.1 and 4.1.3). It plays a fundamental role in the following Chapters.

Let $\varepsilon_{s,t}$ be a framed cycle of $B_{\mathcal{A}}$, coming from a maximal tree $\mathcal{T}_0$ of the incidence graph $\Gamma_{\mathcal{A}}$, as described in Section 3.2. We consider the set of points on $\varepsilon_{s,t}$ and we extract from $\varepsilon_{s,t}$ two sets: the set of singular points on L_s and the set of singular points on L_t.

$$\mathcal{P}_s^t = \{P_i \in \varepsilon_{s,t} \mid P_i \in L_s\} \quad \text{and} \quad \mathcal{P}_t^s = \{P_i \in \varepsilon_{s,t} \mid P_i \in L_t\}.$$

Let $P \in \mathcal{P}$ and $L \in \mathcal{A}$ such that $P \in L$. We define the index of L in P (denoted by $Ind(L,P)$):

$$Ind(L,P) = m(P) - ord(L,P),$$

where $m(P)$ is the multiplicity of P, and $ord(L,P)$ the position of the line L at the left of the singular point P on the picture of $W_{\mathcal{A}}$. We denote $\widetilde{\alpha}_{P,j}$ the homological meridian around the j[st] line at the left of P.

Proposition 4.1.5. *The homological unknotting map $\widetilde{\delta}$ can be described as follows:*

$$\widetilde{\delta} : \left\{ \begin{array}{ccc} \mathrm{H}_1(B_{\mathcal{A}};\mathbb{Z}) & \longrightarrow & \mathrm{H}_1(B_{\mathcal{A}};\mathbb{Z}) \\ \widetilde{\varepsilon}_{s,t} & \longmapsto & \widetilde{\varepsilon}_{s,t} + \widetilde{\delta}_{s,t} \end{array} \right.$$

where $\widetilde{\delta}_{s,t}$ is:

$$\left(\sum_{P \in P_t^s}^{k_t} \sum_{j=1}^{Ind(L_t,P)-1} \widetilde{\alpha}_{P,j} \right) - \left(\sum_{P \in P_s^t}^{k_s} \sum_{j=1}^{Ind(L_s,P)-1} \widetilde{\alpha}_{P,j} \right).$$

Proof. By Proposition 3.2.1, we know that the contribution of P_l to $\delta_{s,t}^l$ in homotopy is:

$$\varepsilon_{l_1,l_h}^{-1} \left(\alpha_{l_1}^{-1} \left(\varepsilon_{l_1,l_2} \alpha_{l_2}^{-1} \varepsilon_{l_1,l_2}^{-1} \right) \cdots \left(\varepsilon_{l_1,l_{h-1}} \alpha_{l_{h-1}}^{-1} \varepsilon_{l_1,l_{h-1}}^{-1} \right) \right) \varepsilon_{l_1,l_h},$$

where $h = Ind(L_s, P_l)$, and l_j is the j[st] index at the left of P_l. Since the homology is the abelianised of the homotopy, then the previous expression becomes:

$$\sum_{j=1}^{Ind(L_s,P_i)-1} \widetilde{\alpha}_{P_i,j}.$$

And by the definition of the $\delta^l_{s,t}$, we obtain that:

$$\widetilde{\delta}^l_{s,t} = -\sum_{P\in P^t_s}^{k_s} \sum_{j=1}^{Ind(L_s,P)-1} \widetilde{\alpha}_{P,j}.$$

By the same way, we compute:

$$\widetilde{\delta}^r_{s,t} = \sum_{P\in P^s_t}^{k_t} \sum_{j=1}^{Ind(L_t,P)-1} \widetilde{\alpha}_{P,j}.$$

□

Let $\varepsilon_{s,t}$ be a framed cycle of $B_{\mathcal{A}}$, as in Subsection 3.2.2, we denote $\mathrm{sgn}(\varsigma) \in \{\pm 1\}$ the sign of the base $\{\varsigma, \varepsilon_{s,t}\}$ (in this order). Let α_ς be the meridian associated with the line supporting the arc ς.

Proposition 4.1.6. *Let $S_{s,t}$ be the set of the arcs over going $\varepsilon_{s,t}$ in $W_{\mathcal{A}}$, then:*

$$\widetilde{\mu}_{s,t} = \sum_{\varsigma\in S_{s,t}} \mathrm{sgn}(\varsigma)\widetilde{\alpha}_\varsigma.$$

Proof. In the definition of $\mu_{s,t}$, we consider the Arvola's word associated to the arc ς. But, by construction, it is a conjugate of the meridian associated to the line supporting the arc ς. Then, in homology, the Arvola's word becomes a meridian. □

We define $\overline{\mu_{s,t}} = \sum_{\varsigma\in\overline{S_{s,t}}} \mathrm{sgn}(\varsigma)\widetilde{\alpha}_\varsigma$, where $\overline{S_{s,t}}$ is the set of the arcs under going $\varepsilon_{s,t}$ (in $W_{\mathcal{A}}$). Let ϕ the isomorphism between $\mathrm{H}_1(E_{\mathcal{A}};\mathbb{Z})$ and $\mathrm{H}_1(E_{\overline{\mathcal{A}}};\mathbb{Z})$ defined by:

$$\phi : \left\{ \begin{array}{ccc} \mathrm{H}_1(E_{\mathcal{A}};\mathbb{Z}) & \to & \mathrm{H}_1(E_{\overline{\mathcal{A}}};\mathbb{Z}), \\ \widetilde{\alpha}_i & \mapsto & \widetilde{\alpha}_i \end{array} \right. .$$

Using the result of Subsection 2.1.4 on the complex conjugate of an arrangement, we have:

Proposition 4.1.7. *Let $\mathcal{A}$ be a complex line arrangement, and $\overline{\mathcal{A}}$ its conjugate. The following diagram is commutative:*

$$\begin{tikzcd} \mathrm{H}_1(B_{\overline{\mathcal{A}}};\mathbb{Z}) \arrow[r, "i_*"] \arrow[d, "\mathrm{Id}"'] & \mathrm{H}_1(E_{\overline{\mathcal{A}}};\mathbb{Z}) \\ \mathrm{H}_1(B_{\mathcal{A}};\mathbb{Z}) \arrow[r, "j_*"] & \mathrm{H}_1(E_{\mathcal{A}};\mathbb{Z}) \arrow[u, "\phi"'] \end{tikzcd}$$

where j_* is defined by:

$$j_* : \left\{ \begin{array}{ccc} \mathrm{H}_1(B_{\mathcal{A}};\mathbb{Z}) & \longrightarrow & \mathrm{H}_1(E_{\mathcal{A}};\mathbb{Z}) \\ \widetilde{\alpha_i} & \longmapsto & \widetilde{\alpha_i} \\ \widetilde{\varepsilon}_{s,t} & \longmapsto & -\widetilde{\delta}_{s,t} + \overline{\mu_{s,t}} \end{array} \right.$$

Proof. Since $\mathcal{A}$ and $\overline{\mathcal{A}}$ have the same combinatorics, then $B_{\mathcal{A}} \simeq B_{\overline{\mathcal{A}}}$. The map i_* sends $\widetilde{\alpha}_i$ on $\widetilde{\alpha}_i$, the composition map $\phi \circ j_* \circ \mathrm{Id}$ too. The map i_* sends $\widetilde{\varepsilon}_{s,t}$ on $-\widetilde{\delta}_{s,t} + \widetilde{\mu}_{s,t}$. To obtain the braided wiring diagram of $\bar{\mathcal{A}}$ we inverse the virtual crossings of the braided wiring diagram of $\mathcal{A}$. Then, the unknotting map is not changed when we pass from $\mathcal{A}$ to $\bar{\mathcal{A}}$. Since the arcs underpassing $\varepsilon_{s,t}$ in $W_{\mathcal{A}}$ are the arcs overpassing $\varepsilon_{s,t}$ is $W_{\bar{\mathcal{A}}}$, then:

$$\phi \circ j_* \circ \mathrm{Id}(\widetilde{\varepsilon}_{s,t}) = -\widetilde{\delta}_{s,t} + \widetilde{\mu}_{s,t}. \qquad \square$$

4.2 Examples

By abuse of notation, the meridians and the cycles will be denoted in the same way whether they are in the fundamental group or in the first homology group.

4.2.1 Ceva-7

Consider four points in $\mathbb{CP}^2$ in a generic position. The Ceva's arrangement is the arrangement composed of the six lines of $\mathbb{CP}^2$ passing through the four points. This arrangement admits four triple points and three double points. The extended Ceva's arrangement, also called Ceva-7, is the usual Ceva's arrangement with one additional line passing through two of the double points. Its real part is pictured in Figure 4.1.

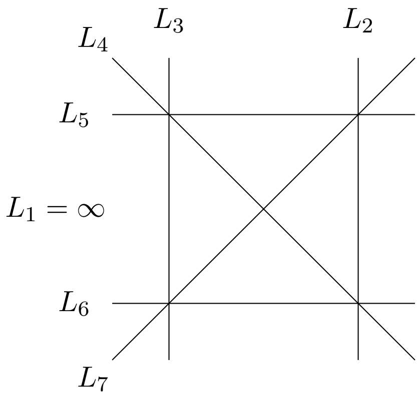

Figure 4.1: The extended Ceva's arrangement

As the Ceva's arrangement, the Ceva-7 arrangement is a real complexified arrangement. Then the inclusion map in homology can be computed from Corollary 4.1.2. Considering L_1 as the line at the infinity, we obtain that $\varepsilon_{3,4}$, $\varepsilon_{3,5}$, $\varepsilon_{3,6}$ and $\varepsilon_{3,7}$ are 0 in $\mathrm{H}_1(E_{\mathcal{A}};\mathbb{Z})$; and that:

$$\varepsilon_{4,7} = -\alpha_6, \qquad \varepsilon_{2,5} = -\alpha_3 - \alpha_4, \qquad \varepsilon_{2,7} = -\alpha_4 - \alpha_3 - \alpha_6,$$
$$\varepsilon_{2,4} = -\alpha_3, \qquad \varepsilon_{2,6} = -\alpha_3.$$

4.2.2 MacLane

The wiring diagrams of the MacLane's arrangements Q^+ and Q^- are pictured in Figure 2.6 and Figure 2.7. To compute the application induced in homology, we only consider the equations obtained in Section 3.3, which will be abelianised. Then for the positive MacLane arrangement Q^+, the map induced by the inclusion is:

$$\begin{aligned}
&\varepsilon_{2,3} = 0, && \varepsilon_{2,5} = -\alpha_4,\\
&\varepsilon_{2,4} = 0, && \varepsilon_{2,7} = 0,\\
&\varepsilon_{2,6} = \alpha_7, && \varepsilon_{4,5} = -\alpha_3 + \alpha_7,\\
&\varepsilon_{3,6} = -\alpha_4, && \varepsilon_{3,7} = -\alpha_4,\\
&\varepsilon_{1,5} = -\alpha_4 - \alpha_3 - \alpha_2, && \varepsilon_{1,7} = -\alpha_6 - \alpha_4 - \alpha_3 - \alpha_2,\\
&\varepsilon_{1,3} = -\alpha_2 - \alpha_6, && \varepsilon_{1,4} = -\alpha_2,\\
&\varepsilon_{1,6} = -\alpha_3 - \alpha_2.
\end{aligned}$$

To compute the inclusion map in the case of Q^-, we use Proposition 4.1.7, and then:

$$\begin{aligned}
&\varepsilon_{2,3} = 0, && \varepsilon_{2,5} = 0,\\
&\varepsilon_{2,4} = \alpha_5, && \varepsilon_{2,7} = -\alpha_6,\\
&\varepsilon_{2,6} = 0, && \varepsilon_{4,5} = 0,\\
&\varepsilon_{3,6} = 0, && \varepsilon_{3,7} = -\alpha_4,\\
&\varepsilon_{1,5} = -\alpha_4 - \alpha_3 - \alpha_2, && \varepsilon_{1,7} = -\alpha_6 - \alpha_4 - \alpha_3 - \alpha_2,\\
&\varepsilon_{1,3} = -\alpha_2, && \varepsilon_{1,4} = -\alpha_2,\\
&\varepsilon_{1,6} = -\alpha_3 - \alpha_2 - \alpha_6.
\end{aligned}$$

4.3 The 2-cohomology

In this section, we work with homology and cohomology with complex coefficients. Since $\mathrm{H}_1(E_{\mathcal{A}};\mathbb{Z})$ and $\mathrm{H}_1(B_{\mathcal{A}};\mathbb{Z})$ are torsion free, by passing to complex coefficients, we conserve essentially all the information present in the homology with integer coefficients. The 2-cohomology of line arrangement is studied, among others, by J.I. Cogolludo [19] and M. Yoshinaga [70].

4.3.1 Canonical decomposition of the complement

Let us recall some notations of Chapter 1. Let X be the minimal blow-up of $\mathbb{CP}^2$ such that the blow-up $\hat{\mathcal{A}}$ of $\mathcal{A}$ is a normal crossing divisor. And consider U a regular neighborhood of $\hat{\mathcal{A}}$. By construction, it is homeomorphic to the regular neighborhood of $\mathcal{A}$ constructed in Subsection 1.1.1.

Lemma 4.3.1. *There is a natural injection of* $\mathrm{H}_1(E_{\mathcal{A}};\mathbb{C})$ *in* $\mathrm{H}_1(B_{\mathcal{A}};\mathbb{C})$.

Proof. Let L be a line generic to $\mathcal{A}$. Since U is compact, we may assume that L is contained in U. Then we have the following diagram:

$$\begin{tikzcd} & U \setminus \hat{\mathcal{A}} \arrow[d] & \\ L \cap E_{\mathcal{A}} \arrow[ur, hook, "j"] \arrow[r, hook, "\phi"] & E_{\mathcal{A}} & B_{\mathcal{A}} \arrow[ul, hook', "\psi"'] \arrow[l, hook'] \end{tikzcd}$$

By Zariski-Lefschetz theorem, the map ϕ_* induced by ϕ on first homology groups is an isomorphism, and that ψ is a homotopy equivalence. The application $\psi_*^{-1} \circ j_* \circ \phi_*^{-1}$ from $\mathrm{H}_1(E_{\mathcal{A}})$ to $\mathrm{H}_1(B_{\mathcal{A}})$ is injective. This map is natural in the sense that it sends the meridian of L_i in $E_{\mathcal{A}}$ on the meridian of L_i in $B_{\mathcal{A}}$, and it is not depending on the choice of L. □

Theorem 4.3.2. *Let $E_{\mathcal{A}}$ be the complement of $\mathcal{A}$ and $B_{\mathcal{A}}$ its boundary manifold, then we have a split exact sequence:*

$$0 \to \mathrm{H}^2(E_{\mathcal{A}}; \mathbb{C}) \to \mathrm{H}_1(B_{\mathcal{A}}; \mathbb{C}) \to \mathrm{H}_1(E_{\mathcal{A}}; \mathbb{C}) \to 0.$$

Proof. The excision theorem imply that:

$$\mathrm{H}_2(E_{\mathcal{A}}, B_{\mathcal{A}}) \simeq \mathrm{H}_2(X, U).$$

But, $\mathrm{H}_2(X, U) \simeq \mathrm{H}_2(X, \hat{\mathcal{A}})$, then $\mathrm{H}_2(E_{\mathcal{A}}, B_{\mathcal{A}}) \simeq \mathrm{H}_2(X, \hat{\mathcal{A}})$. So we have:

$$\begin{tikzcd} \mathrm{H}_2(\hat{\mathcal{A}}) \arrow[r] \arrow[d, hook] & \mathrm{H}_2(X) \arrow[r] \arrow[d, equal] & \mathrm{H}_2(X, \hat{\mathcal{A}}) \arrow[r] \arrow[d, hook] & \mathrm{H}_1(\hat{\mathcal{A}}) \arrow[r] \arrow[d, hook] & \mathrm{H}_1(X) = 0 \\ \mathrm{H}_2(U) \arrow[r] & \mathrm{H}_2(X) \arrow[r] & \mathrm{H}_2(X, U) \arrow[r] & \mathrm{H}_1(U) \arrow[r] & \mathrm{H}_1(X) = 0 \end{tikzcd}$$

Remarks that the three previous maps (induced by embeddings) are in fact isomorphisms. As $\mathrm{H}_2(X)$ is generated by a line of $\mathbb{CP}^2$ and the exceptional lines, and as $\hat{\mathcal{A}}$ contains at least one line of $\mathbb{CP}^2$, and all the exceptional lines of X, then the map $\mathrm{H}_2(\hat{\mathcal{A}}) \to \mathrm{H}_2(X)$ is onto. Thus we have the following factorization:

$$\begin{tikzcd} \mathrm{H}_2(X) \arrow[rr] \arrow[dr] & & \mathrm{H}_2(X, U) \\ & 0 \arrow[ur] & \end{tikzcd}$$

Furthermore, since B is contained in U, and M in X, we have:

$$\begin{tikzcd} \mathrm{H}_2(U) \arrow[r] & \mathrm{H}_2(X) \arrow[rr] \arrow[dr] & & \mathrm{H}_2(X, U) \arrow[r] & \mathrm{H}_1(U) \arrow[r] & \mathrm{H}_1(X) = 0 \\ & & 0 \arrow[ur] & & & \\ \mathrm{H}_2(B_{\mathcal{A}}) \arrow[r] \arrow[uu] & \mathrm{H}_2(E_{\mathcal{A}}) \arrow[rr] \arrow[uu] & & \mathrm{H}_2(E_{\mathcal{A}}, B_{\mathcal{A}}) \arrow[r] \arrow[uu, equal] & \mathrm{H}_1(B_{\mathcal{A}}) \arrow[r] \arrow[uu] & \mathrm{H}_1(E_{\mathcal{A}}) \end{tikzcd}$$

This commutative diagram implies that the application $\mathrm{H}_2(E_{\mathcal{A}}) \to \mathrm{H}_2(E_{\mathcal{A}}, B_{\mathcal{A}})$ can be factorized by zero. Then:

$$0 \to \mathrm{H}_2(E_{\mathcal{A}}, B_{\mathcal{A}}) \to \mathrm{H}_1(B_{\mathcal{A}}) \to \mathrm{H}_1(E_{\mathcal{A}}) \to \mathrm{H}_1(E_{\mathcal{A}}, B_{\mathcal{A}})$$

By Poincaré's duality $\mathrm{H}_1(E_{\mathcal{A}}, B_{\mathcal{A}}) \simeq \mathrm{H}^3(E_{\mathcal{A}})$. Since $E_{\mathcal{A}}$ has the homotopical type of a 2 dimensional CW-complex, then $\mathrm{H}^3(E_{\mathcal{A}}) = 0$, and the map $\mathrm{H}_1(B_{\mathcal{A}}) \to \mathrm{H}_1(E_{\mathcal{A}})$ is onto. This implies that the previous exact sequence is short:

$$0 \to \mathrm{H}_2(E_{\mathcal{A}}, B_{\mathcal{A}}) \to \mathrm{H}_1(B_{\mathcal{A}}) \to \mathrm{H}_1(E_{\mathcal{A}}) \to 0. \tag{4.1}$$

Using the section obtained in the proof of Lemma 4.3.1, we obtain a splitting of the short exact sequence 4.1, so:

$$\mathrm{H}_2(E_{\mathcal{A}}, B_{\mathcal{A}}) \oplus \mathrm{H}_1(E_{\mathcal{A}}) \xrightarrow{\sim} \mathrm{H}_1(B_{\mathcal{A}}).$$

To conclude, using Poincaré duality, we obtain $\mathrm{H}_2(E_{\mathcal{A}}, B_{\mathcal{A}})$ as the dual of $\mathrm{H}_2(E_{\mathcal{A}})$ which is isomorphic to $\mathrm{H}^2(E_{\mathcal{A}})$. □

Corollary 4.3.3. *Let $E_{\mathcal{A}}$ be the complement of $\mathcal{A}$ and let $B_{\mathcal{A}}$ be its boundary manifold, we have:*

$$\mathrm{H}_1(B_{\mathcal{A}}; \mathbb{C}) \simeq \mathrm{H}_1(E_{\mathcal{A}}; \mathbb{C}) \oplus \mathrm{H}^2(E_{\mathcal{A}}; \mathbb{C}).$$

4.3.2 Relationship with the dual graph

In this Subsection, we give geometric interpretation of different isomorphism with $\mathrm{H}_1(\Gamma_{\mathcal{A}})$. We assume that $\mathcal{A}$ is not a pencil.

Denote $\hat{\mathcal{A}} = \bigcup D_i$, and $D_i^{[1]}$ the line of $\hat{\mathcal{A}}$ without the singular points of $\hat{\mathcal{A}}$. using the Poincaré residues, we have:

$$\begin{tikzcd} \mathrm{H}^2(E_{\mathcal{A}}; \mathbb{C}) \arrow[r, "Res."] \arrow[rr, bend left, "Res^{[2]}"] & \bigoplus_i \mathrm{H}^1(D_i^{[1]}; \mathbb{C}) \arrow[r, "Res."] & \bigoplus_{i<j} \mathrm{H}^0(D_i \cap D_j; \mathbb{C}). \end{tikzcd}$$

Using De Rham cohomology, $\mathrm{H}^2(E_{\mathcal{A}}; \mathbb{C})$ can be viewed as classes of 2-forms, and [38, Proposition 5.14] states that the global differential logarithmic forms on $X \setminus \hat{\mathcal{A}}$ with poles along $\hat{\mathcal{A}}$ are enough to generate the cohomology of $E_{\mathcal{A}}$. Let ω be such a 2-form, and $P = D_i \cap D_j$ be a singular point of $\hat{\mathcal{A}}$. Consider local coordinates centered in P, such that $D_i : \{x = 0\}$ and $D_j : \{y = 0\}$. Locally, ω can be written $f(x, y)\frac{dx \wedge dy}{xy}$, where f is differentiable. The residue of ω on D_i is

$$Res^{[1]}(\omega) = f(0, y)\frac{dy}{y},$$

and the residue of this 1-form on P is

$$Res^{[2]}(\omega) = f(0, 0).$$

Remark 4.3.4.

- If we switch D_i and D_j then the residue of the 2-form ω on P is $-f(0,0)$.
- The sum of the residues on a line D_i of $\hat{\mathcal{A}}$ is zero.

From the proof of Theorem 4.3.2, we consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^2(E_{\mathcal{A}};\mathbb{C}) \arrow[r] \arrow[d, "r"'] & \mathrm{H}_1(B_{\mathcal{A}};\mathbb{C}) \arrow[r] \arrow[d, equal] & \mathrm{H}_1(E_{\mathcal{A}};\mathbb{C}) \arrow[r] & 0 \\
0 & \mathrm{H}_1(\hat{\Gamma}_{\mathcal{A}};\mathbb{C}) \arrow[l] & \mathrm{H}_1(B_{\mathcal{A}};\mathbb{C}) \arrow[l] & K \arrow[l] \arrow[u] & 0 \arrow[l]
\end{tikzcd}$$

with $r:$ $\begin{tikzcd} \mathrm{H}_2(E_{\mathcal{A}};\mathbb{C}) \arrow[r] \arrow[rr, bend left] & \mathrm{H}_1(B_{\mathcal{A}};\mathbb{C}) \arrow[r] & \mathrm{H}_1(\hat{\Gamma}_{\mathcal{A}};\mathbb{C}) \end{tikzcd}$ defined as the composition.

Theorem 4.3.5. *The application r is an isomorphism.*

Proof. It can be seen as a consequence of the snake lemma. □

In the oriented dual graph $\hat{\Gamma}_{\mathcal{A}}$, D_i and D_j are vertices and $P_{i,j} = D_i \cap D_j$ is an edge (noted $E_{i,j}$). If we associate to each 1-chain of $\hat{\Gamma}_{\mathcal{A}}$ a complex number (i.e. the residue of the 2-form on P), then the Remark 4.3.4 implies that the sum of complex numbers associated to the edges incoming in a vertex is zero. Then we can define:

Definition 4.3.6.

$$\begin{array}{rccc} \mathscr{R}: & \mathrm{H}^2(E_{\mathcal{A}};\mathbb{C}) & \longrightarrow & \mathrm{H}_1(\hat{\Gamma}_{\mathcal{A}};\mathbb{C}) \\ & \omega & \longmapsto & \sum\limits_{i<j} Res^{[2]}_{P_{i,j}}(\omega) E_{i,j} \end{array}.$$

The image of ω by $\mathscr{R}$ is a cycle of $\mathrm{H}_1(\hat{\Gamma}_{\mathcal{A}};\mathbb{C})$. Indeed, let $r_{i,j}$ be the residue of ω on $P_{i,j} = D_i \cap D_j$. The boundary of $\gamma = \sum\limits_{i,j} r_{i,j} P_{i,j}$ is $\sum\limits_{i} D_i (\sum\limits_{j} r_{i,j})$, then using the previous remark, this boundary is null.

Proposition 4.3.7. *The application $\mathscr{R}$ is an isomorphism.*

Proof. By Remark 4.3.4, the image of $Res^{[2]}$ is contained in:

$$\mathcal{I} = \left\{ (t_{i,j})_{i<j} \in \bigoplus_{i<j} \mathrm{H}^0(D_i \cap D_j) \mid \forall i \sum_j t_{i,j} = 0 \right\}.$$

It is known that $\bigoplus\limits_{i<j} \mathrm{H}^0(D_i \cap D_j)$ is isomorphic to $C_1(\hat{\Gamma}_{\mathcal{A}})$, then $\mathcal{I}$ is isomorphic to $Z_1(\hat{\Gamma}_{\mathcal{A}})$. Since $\hat{\Gamma}_{\mathcal{A}}$ has no boundary, then $\mathcal{I} \simeq \mathrm{H}_1(\hat{\Gamma}_{\mathcal{A}})$. By [19, Proposition 2.5], this application is onto. By a dimensional argument the result holds. □

Lemma 4.3.8. *Let (u, v) be local coordinates near $P_{i,j}$, and assume that $D_i : \{u = 0\}$ and $D_j : \{v = 0\}$. Then the tori $T_{i,j} = \{|u| = |v| = \varepsilon\}$ form a system of generators of $H_2(E_{\mathcal{A}}; \mathbb{C})$.*

Proof. Consider the set indexed by $\mathscr{E}$ of tori $T_{i,j}$ defined as follows. For each geometric cycle $\mathcal{E}_{s,t}$ we consider the tori around the intersection of the exceptional component over $L_s \cap L_t$ with L_t. We denote $\mathscr{T}$ the set of these tori. Let $m_{s,t}$ be the 2-cell glue along $\mathcal{E}_{s,t}$ in the proof of Theorem 3.2.2.

The proof is decomposed in two steps:
Step 1: The set $\{m_{s,t}\}$ indexed by $\mathscr{E}$ is a basis of $\mathrm{H}_2(E_{\mathcal{A}}, B_{\mathcal{A}})$.

Proof. Remark that the boundary of $M = \bigcup m_{s,t}$ is contained in $B_{\mathcal{A}}$. Then we have the inclusion:

$$(M, \partial M) \hookrightarrow (E_{\mathcal{A}}, B_{\mathcal{A}}),$$

By Corollary 3.2.6 $E_{\mathcal{A}} \simeq M \cup B_{\mathcal{A}}$, then the excision theorem implies that the previous inclusion induced an isomorphism: $\mathrm{H}_2(M, \partial M) \simeq \mathrm{H}_2(E_{\mathcal{A}}, B_{\mathcal{A}})$. Since $\{m_{s,t}\}$ is a basis of $\mathrm{H}_2(M, \partial M)$, then it is also true in $\mathrm{H}_2(E_{\mathcal{A}}, B_{\mathcal{A}})$. □

Step 2: The set $\mathscr{T}$ indexed by $\mathscr{E}$ is the dual basis of $\{m_{s,t}\}$ for the intersection form: $\mathrm{H}_2(E_{\mathcal{A}}) \times \mathrm{H}_2(E_{\mathcal{A}}, B_{\mathcal{A}}) \to \mathbb{C}$.

Proof. Consider the intersection $T_{i,j} \cdot m_{s,t}$. Since the intersection points are in the boundary manifold, then it is equal to $T_{i,j} \cdot \mathcal{E}_{s,t}$. By construction of $B_{\mathcal{A}}$, see Section 3.1, it is constructed as the gluing of S^1-bundles, and the gluing loci are exactly the tori considered. In each piece of the gluing, $\mathcal{E}_{s,t}$ is transverse to the fiber. Then if it intersects $T_{i,j}$, it is transversely. There are three kinds of intersections between $T_{i,j}$ and $\mathcal{E}_{s,t}$:

- If $(s, t) \neq (i, j)$ and the intersection is empty then $T_{i,j} \cdot \mathcal{E}_{s,t} = 0$.
- If $(s, t) \neq (i, j)$ and the intersection is non empty then $\mathcal{E}_{s,t}$ stays in the tubular neighborhood of the same line, and intersects $T_{i,j}$ in two points with opposite orientation. This implies that $T_{i,j} \cdot \mathcal{E}_{s,t} = 0$.
- If $(s, t) = (i, j)$ then $\mathcal{E}_{s,t}$ change of tubular neighborhood, and intersect $T_{i,j}$ in on point. This imply that $T_{i,j} \cdot \mathcal{E}_{s,t} = 1$.

Finally, the set $\mathscr{T}$ indexed by $\mathscr{E}$ is the dual basis of $\{m_{s,t}\}$ since:

$$T_{i,j} \cdot m_{s,t} = \begin{cases} 0 & \text{if } (s,t) \neq (i,j) \\ 1 & \text{if } (s,t) = (i,j) \end{cases} .$$ □

Since $\mathscr{T}$ is a basis of $\mathrm{H}_2(E_{\mathcal{A}})$, then the set of all the tori $T_{i,j}$ generates $\mathrm{H}_2(E_{\mathcal{A}})$. □

Theorem 4.3.9. *The applications r and $\mathscr{R}$ are equal.*

Proof. Let $\omega \in \mathrm{H}^2(E_{\mathcal{A}})$. As previous we denote $t_{i,j} = Res^{[2]}_{P_{i,j}}(\omega)$, for $P_{i,j} = D_i \cap D_j$. We have:

$$\mathscr{R} = \sum_{i<j} t_{i,j} E_{i,j}.$$

Let us construct the image of ω by the application: $\mathrm{H}^2(E_{\mathcal{A}}) \to \mathrm{H}_2(E_{\mathcal{A}}, B_{\mathcal{A}})$. In a first step, consider (u, v) as in Lemma 4.3.8. Then ω admits the following equation:

$$\omega = (t_{i,j} + \alpha(u, v, \bar{u}, \bar{v})) \frac{\mathrm{d}\, u}{u} \wedge \frac{\mathrm{d}\, v}{v} + A,$$

where A is without pole of order 2. Let us remark that

$$\left(\frac{1}{2i\pi}\right)^2 \int_{T_{i,j}} \omega = t_{i,j}.$$

In a second step, let us construct a particular lift of $\mathscr{R}(\omega)$. Let γ' an arbitrary lift of $\mathscr{R}(\omega)$ in $\mathrm{H}_1(B_{\mathcal{A}})$. By Theorem 4.1.1, it exists a sum of meridians M such that $\gamma' + M$ is in the kernel of the application $\mathrm{H}_1(B_{\mathcal{A}}) \to \mathrm{H}_1(E_{\mathcal{A}})$, denote γ this cycle. Consider an element $\tau \in \mathrm{H}_2(E_{\mathcal{A}}, B_{\mathcal{A}})$ such that $\partial\tau = \gamma$. Remark the following fact on the intersection with $T_{i,j}$:

$$\tau \cdot T_{i,j} = t_{i,j}.$$

Finally, we have constructed two applications:

$$\Phi : \left\{ \begin{array}{ccc} \mathrm{H}^2(E_{\mathcal{A}}) & \longrightarrow & (\mathrm{H}_2(E_{\mathcal{A}}))^* \\ \omega & \longmapsto & ([\sigma] \mapsto \int_\sigma \omega) \end{array} \right. \quad \text{and} \quad \Psi : \left\{ \begin{array}{ccc} \mathrm{H}_1(E_{\mathcal{A}}, B_{\mathcal{A}}) & \longrightarrow & (\mathrm{H}_2(E_{\mathcal{A}}))^* \\ \tau & \longmapsto & ([\sigma] \mapsto \tau \cdot \sigma) \end{array} \right. .$$

Remarks that $\Phi(\omega) = \Psi(\tau)$ for the set of generators $\{T_{i,j}\}_{i<j}$ of $\mathrm{H}_2(E_{\mathcal{A}})$ (see Lemma 4.3.8). Then τ is the image of ω by the application $\mathrm{H}^2(E_{\mathcal{A}}) \to \mathrm{H}_1(E_{\mathcal{A}}, B_{\mathcal{A}})$.

Since $\partial\tau = \gamma$ and since we can assume that $T_{i,j}$ is in $B_{\mathcal{A}}$ then $\gamma \cdot T_{i,j} = t_{i,j}$. And the image of ω by the application $\Psi^{-1} \circ \Phi : \mathrm{H}^2(E_{\mathcal{A}}) \to \mathrm{H}_1(B_{\mathcal{A}})$ is γ. By construction the image of γ in $\mathrm{H}_1(\hat{\Gamma}_{\mathcal{A}})$ is $\mathscr{R}(\omega)$, then the result holds. $\square$

Chapter 5

Cycles & characters

The characteristic varieties are classical invariants of the fundamental group of a topological space. They first appeared in a more general context than line arrangements in [54, 32, 44], and have been recently studied in [1, 39, 45]. Some important questions are still opened about this subject.

Question. *Are the characteristic varieties of a line arrangement of a combinatorial nature ?*

A. Libgober [44] gives a method to compute almost all irreducible components of characteristic varieties; this method has a high computational complexity. E. Artal [5] found a method to compute the remaining components (which are only isolated torsion points). He obtains a description of the quasi-projective terms of the depth. We show in this work that his method, useful for the computation of characteristic varieties, is interesting in its own and provides topological invariants.

In Section 5.1, after defining the twisted cohomology, we give definitions about characteristic varieties and some structural properties.

In Section 5.2, we detail the method developed by E. Artal in [5] to compute the quasi-projective depth of a character, in the aim of using it in Chapter 6. This section is concluded by several results about the combinatoriality (or the non combinatoriality) of the quasi-projective depth.

In Section 5.3, we introduce the notion of *inner-cyclic characters*, and illustrate their importance. Then we show that the image of particular cycles of the complement of $\mathcal{A}$ by an associated character is a topological invariant of the pair $(X, \hat{\mathcal{A}})$.

5.1 Characteristic varieties

5.1.1 Definitions

As mentioned at the beginning of Chapter 4, the fundamental group of an arrangement is a very important invariant, but it is very difficult in general to get properties directly from a presentation of this group. Several weaker, but effective, invariants can be used. We focus our attention on the characteristic varieties.

By the work of A. Libgober [43] the CW-complex $K(\mathcal{A})$ construct on the presentation given in [9], [21] or [34] have the same fundamental group as $E_{\mathcal{A}}$ (we may use any other CW-complex with the same fundamental group). Choosing the dual basis B for the cells of $K(\mathcal{A})$ of the associated cochain complex:

$$C^*(\mathcal{A}) : 0 \longrightarrow C^0(\mathcal{A}) \xrightarrow{A_1} C^1(\mathcal{A}) \xrightarrow{A_2} C^2(\mathcal{A}) \longrightarrow 0,$$

the matrices (with integer coefficients) of A_1 and A_2 completely determine $C^*(\mathcal{A})$.

Remark 5.1.1. The rank of $\mathrm{H}^1(E_{\mathcal{A}}; \mathbb{Z})$ is n.

The abelianisation map $\pi_{ab} : \pi_1(E_{\mathcal{A}}) \to \mathrm{H}_1(E_{\mathcal{A}}; \mathbb{Z})$ induces an infinite abelian cover $\check{E}(\mathcal{A}) \to E_{\mathcal{A}}$; the manifold $\check{E}(\mathcal{A})$ is analytic but in general it will not be algebraic. Lifting the cells of $K(\mathcal{A})$, we construct an infinite CW-complex $\check{K}(\mathcal{A})$ with the same homotopy type than $\check{E}(\mathcal{A})$. The group $H_1(E_{\mathcal{A}}; \mathbb{Z})$ acts on these cells and the cochain complex of $\check{K}(\mathcal{A})$ acquires a structure of $\Lambda_{\mathbb{C}}$-module for which a basis is determined by an arbitrary choice of liftings for any cell in B, where $\Lambda_{\mathbb{C}} = \mathbb{C}\left[\alpha_1^{\pm 1}, \cdots, \alpha_n^{\pm 1}\right]$.

$$\check{C}^*(\mathcal{A}) : 0 \longrightarrow \check{C}^0(\mathcal{A}) \xrightarrow{\check{A}_1} \check{C}^1(\mathcal{A}) \xrightarrow{\check{A}_2} \check{C}^2(\mathcal{A}) \longrightarrow 0.$$

The $\Lambda_{\mathbb{C}}$-matrices $\check{A}_1$ and $\check{A}_2$ determine the cochain complex $\check{C}^*(\mathcal{A})$.

Remark 5.1.2.

- $\dim_{\mathbb{C}} C^*(\mathcal{A}) = \mathrm{rk}_{\Lambda_{\mathbb{C}}} \check{C}^*(\mathcal{A})$, with respect of the graduation.
- The evaluation of $\check{A}_1$ (resp. $\check{A}_2$), with $x_i = 1$, $\forall i \in \{1, \cdots, n\}$, is A_1 (resp. A_2).

Definition 5.1.3. The *character torus* of $\mathcal{A}$ is defined as:

$$\mathbb{T}(\mathcal{A}) = \mathrm{H}^1(E_{\mathcal{A}}; \mathbb{C}^*) = \mathrm{Hom}(\mathrm{H}_1(E_{\mathcal{A}}; \mathbb{Z}), \mathbb{C}^*) = \mathrm{Hom}(\pi_1(E_{\mathcal{A}}), \mathbb{C}^*).$$

From a character $\xi \in \mathbb{T}(\mathcal{A})$ we define a twisted complex as follows. The character ξ induces an evaluation map $ev_\xi : \Lambda_{\mathbb{C}} \to \mathbb{C}$, defining a structure of $\Lambda_{\mathbb{C}}$-module on $\mathbb{C}$, denoted by $\mathbb{C}_\xi$. Consider the cochain complex:

$$\check{C}^*(\mathcal{A}) \otimes_{\Lambda_{\mathbb{C}}} \mathbb{C}_\xi.$$

Definition 5.1.4. The *twisted cohomology* of $E_{\mathcal{A}}$ is the cohomology of the previous cochain complex.

Remark 5.1.5. The twisted cohomology is the cohomology of $E_{\mathcal{A}}$ on the local system of coefficient $\mathcal{L}_\xi$ defined by ξ.

Definition 5.1.6. The characteristic varieties of $\mathcal{A}$ are:

$$\mathcal{V}_k(\mathcal{A}) = \left\{ \xi \in \mathbb{T}(\mathcal{A}) \mid \dim_{\mathbb{C}} \left(\mathrm{H}^1(E_{\mathcal{A}}; \mathcal{L}_\xi) \right) \geq k \right\}.$$

Definition 5.1.7. The depth of a character $\xi \in \mathbb{T}(\mathcal{A})$ is:

$$\operatorname{depth}(\xi) = \max \{ k \in \mathbb{N} \mid \xi \in \mathcal{V}_k \} = \dim_{\mathbb{C}} \mathrm{H}^1(E_{\mathcal{A}}; \mathbb{C}_\xi).$$

5.1.2 Properties

The very construction of the characteristic varieties implies that they are algebraic sub-varieties of $\mathbb{T}(\mathcal{A})$ defined over $\mathbb{Q}$.

Theorem 5.1.8 ([1, 8])**.** *Let $\mathcal{A}$ be an arrangement. The irreducible components of $\mathcal{V}_k(\mathcal{A})$ are sub-tori translated by torsion elements.*

This Theorem implies that we can reduce the study of characteristic varieties to the study of torsion character.

Remark 5.1.9. This Theorem holds for quasi-projective varieties. For more references on Theorem 5.1.8, see [14, 29, 47].

Corollary 5.1.10. *The varieties $\mathcal{V}_k(\mathcal{A})$ are determined by their torsion points.*

Proof. A subtorus is the closure of its torsion points. And it is also true for torsion-translated tori. □

Moreover, the work of D. Cohen and A. Suciu [23] or A. Libgober and S. Yuzvinski [49] implies that all the components of the characteristic torus containing 1 have a combinatorial description.

Using Corollary 5.1.10, we focus our study only on the torsion characters. Since the meridians generate $\mathrm{H}_1(E_{\mathcal{A}})$, let t_i be the image of the meridian associated with L_i by ξ.

Definition 5.1.11. The subset of $\mathbb{T}(\mathcal{A})$ composed of the torsion character is denoted by $\mathbb{T}^{tors}(\mathcal{A})$.

Let $\xi \in \mathbb{T}^{tors}(\mathcal{A})$ be a torsion character of order N, and let $\rho : E_{\mathcal{A}}^{\xi} \to E_{\mathcal{A}}$ be the unramified N-fold cyclic cover associated to ρ. It is explicitly described in Subsection 5.2.1. We put $\check{\xi}$ a generator of the group of deck automorphisms of $E_{\mathcal{A}}^{\xi}$, and $\check{\xi}^*$ the induced map on the cohomology groups. We denote:

$$\mathrm{H}^{\bullet,\xi} = \ker \left(\check{\xi}^* - \exp\left(2i\pi/N\right) . \mathrm{Id}_{\mathrm{H}^\bullet(E_{\mathcal{A}}^{\xi};\mathbb{C})} \right).$$

Proposition 5.1.12 ([62])**.** *There is an isomorphism:*

$$\mathrm{H}^{1,\xi} \simeq \mathrm{H}^1(E_{\mathcal{A}}; \mathcal{L}_\xi).$$

5.2 Depth & quasi-projective depth

Consider a character ξ on the first homology group of the complement of $\mathcal{A}$:

$$\xi \in \mathrm{H}^1(E_{\mathcal{A}}; \mathbb{C}^*) = \mathrm{Hom}(\mathrm{H}_1(E_{\mathcal{A}}, \mathbb{Z}); \mathbb{C}^*).$$

It is completely determined by the images $t_i \in \mathbb{C}^*$ of the meridians associated to L_i, where t_0 is $t_n^{-1} \dots t_1^{-1}$.

Definition 5.2.1. A torsion character ξ is said to be *fully-ramified*, if all the t_i are different of 1. Let $\mathcal{A}_\xi^0 = \{L_i \in \mathcal{A} \mid t_i = 1\}$. If ξ is fully-ramified then $\mathcal{A}_\xi^0 = \emptyset$.

5.2.1 Covers

If ξ is a torsion character of order N then the application $\rho : \pi_1(E_{\mathcal{A}}) \to \mu_N \subset \mathbb{C}^*$ induced by ξ is onto. We construct a smooth model of the branched cyclic cover of X –the blow-up of $\mathbb{CP}^2$ on the more than double points of $\mathcal{A}$, see Section 1.2– induced by ξ. Let us recall that $E_{\mathcal{A}} \simeq X \setminus \hat{\mathcal{A}}$.

Definition 5.2.2. Let $\boldsymbol{v} = (v_1, \cdots, v_k) \in \mathbb{N}^k$, with the v_i coprime, the weighted projective complex space $\mathbb{P}_{\boldsymbol{v}}^{k-1}$ is defined by the natural structure of normal variety defined on $\mathbb{C}^k \setminus \{0\}/\sim$, where $\sim$ is the following equivalence relation:

$$x \sim y \Leftrightarrow [x_1 : \cdots : x_k] = [t^{v_1} x_1 : \cdots : t^{v_k} x_k].$$

The equivalence class will be denoted by $[x_1 : \ldots : x_k]_{\boldsymbol{v}}$ (quasi-homogeneous coordinates). We have $\xi(x_i) = t_i$ and let us denote k_j the element of $\{0, 1, \cdots, N-1\}$ such that

$$t_j = \exp(2i\pi k_j / N).$$

Since $\prod x_i = 1$ then there exists k such that $\sum k_j = kN$. Let $\boldsymbol{v} = (1, 1, 1, k)$, we consider the hypersurface $\mathbb{P}^\xi \subset \mathbb{P}_{\boldsymbol{v}}^3$ defined by a quasi-homogeneous polynomial:

$$\mathbb{P}^\xi = \left\{ [x : y : z : T] \mid T^N - \prod_{j=0}^{n} F_j^{k_j}(x, y, z) = 0 \right\}.$$

Remark 5.2.3. The space $\mathbb{P}_{\boldsymbol{v}}^3$ has a unique singular point $[0 : 0 : 0 : 1]$ and it is not in $\mathbb{P}^\xi$. Hence, $\mathbb{P}^\xi$ is contained in the smooth part of $\mathbb{P}_{\boldsymbol{v}}^3$.

Proposition 5.2.4. *The space $\mathbb{P}^\xi$ is a ramified cyclic cover of $\mathbb{CP}^2$ associated with ξ; it is unramified outside $\mathcal{A}$.*

Remark 5.2.5. We consider $\mathbb{P}_{\boldsymbol{v}}^3$ instead of $\mathbb{CP}^3$ to obtain a ramified cover.

We have the stereographic projection of center $[0:0:0:1]$ of $\mathbb{P}^3_v$ in $\mathbb{CP}^2$ (i.e. $[x:y:z:T] \to [x:y:z]$), which restricts as a morphism of $\mathbb{P}^\xi$:

$$\begin{tikzcd} \mathbb{P}^3_v \arrow[d, dotted, "\pi"'] & \mathbb{P}^\xi \arrow[l, hook'] \arrow[dl] \\ \mathbb{CP}^2 & \end{tikzcd}$$

We define $\mathcal{A}^\xi$ as the inverse image of $\mathcal{A}$ by the restriction of the projection π to $\mathbb{P}^\xi$ (i.e. $\mathcal{A}^\xi = \pi^{-1}(\mathcal{A}) \cap \mathbb{P}^\xi$). By construction, $\mathcal{A}^\xi \simeq \mathcal{A}$ is a reduced divisor of $\mathbb{P}^\xi$.

Proposition 5.2.6. *The space $\mathbb{P}^\xi \setminus \mathcal{A}^\xi$ is a model for $E^\xi_{\mathcal{A}}$.*

Consider the fibered product X^ξ_{fb} of X and $\mathbb{P}^\xi$ over $\mathbb{CP}^2$:

$$\begin{tikzcd} X^\xi_{fb} \arrow[r] \arrow[d, "\pi_{fb}"'] & \mathbb{P}^\xi \arrow[d, "\pi"] \\ X \arrow[r, "\sigma"] & \mathbb{CP}^2 \end{tikzcd}$$

We define the inverse image $\mathcal{A}^\xi_{fb}$ of $\hat{\mathcal{A}}$ by π_{fb}. We have $\mathcal{A}^\xi_{fb} \simeq \hat{\mathcal{A}}$, and $X^\xi_{fb} \setminus \mathcal{A}^\xi_{fb}$ is smooth and still isomorphic to $E^\xi_{\mathcal{A}}$. However, the projective variety X^ξ_{fb} has singularities only along $\mathcal{A}^\xi_{fb}$.

The points of $\hat{\mathcal{A}}$ are of two types:

a) The point $P \in \hat{\mathcal{A}}$ has $\{u = 0\} \subset \mathbb{C}^2$ as local equation (i.e. P is a smooth point of $\hat{\mathcal{A}}$).

b) The point $P \in \hat{\mathcal{A}}$ has $\{uv = 0\} \subset \mathbb{C}^2$ as local equation (i.e. P is a singular point of $\hat{\mathcal{A}}$).

We denote by Q the inverse image of P in X^ξ_{fb} by the application π_{fb}. Let us give the equations of X^ξ_{fb}, $\mathcal{A}^\xi_{fb}$ and Q in both cases a) and b).

Case a)

- $X^\xi_{fb} : \left\{T^N = u^l\right\} \subset \mathbb{C}^3$, with:

  $l = k_j$ if P is on the component D_j coming from a line of $\mathcal{A}$.

  $l = \sum k_j$ if P is on the exceptional component of $\hat{\mathcal{A}}$ coming from the point $\bigcap L_j$.

- $\mathcal{A}^\xi_{fb} : \left\{T^N = 0, u = 0\right\}$.

- $Q : (0, 0)$.

Case b)

- $X^{\xi}_{fb} : \left\{ T^N = u^{l_1} v^{l_2} \right\} \subset \mathbb{C}^3$, with similar notations as in case a).
- $\mathcal{A}^{\xi}_{fb} : \left\{ T^N = 0, uv = 0 \right\}$.
- $Q : (0,0)$.

We normalize X^{ξ}_{fb} to construct $\overline{X}^{\xi}$. For each singular point, its preimage is as many points as local branches. The normalization splits the d branches and resolves each one by parametrizations. The normalization has a second step: adding holomorphic functions. In case a), the germs become smooth, in case b), we need to quotient by a cyclic group to obtain smooth germs. The space $\overline{X}^{\xi}$ then obtained is a normal manifold with quotient singularities over the double points of $\hat{\mathcal{A}}$. Let us construct $\overline{X}^{\xi}$:

Case a) We denote $d = \gcd(N, l)$, $N' = N/d$ and $l' = l/d$. The equation of X^{ξ}_{fb} previously given becomes: $\prod_{i=0}^{d-1} (T^{N'} - \zeta^i u^{l'})$, where ζ is a d-primitive root of unity. The normalization is defined over the d preimages of P (with $d = N$ if ξ unramified over P). Its parametrization is defined by the following d local maps:

$$(t, s) \mapsto (c_1 t^{N'}, s, c_2 t^{l'}),$$

where c_i are appropriate constants.

Case b) We denote $d = \gcd(N, l_1, l_2)$, $N' = N/d$, $l'_1 = l_1/d$ and $l'_2 = l_2/d$. The equation of X^{ξ}_{fb} previously given becomes: $\prod_{i=0}^{d-1} (T^{N_1} - \zeta^i u^{l'_1} v^{l'_2})$, where ζ is a d-primitive root of unity. The parametrization of the normalization is:

$$(t, s) \mapsto (c_1 t^{N'}, s^{N'}, c_2 t^{l'_1} s^{l'_2}),$$

where c_i are appropriate constants. The map defines a *parametrization* as far as the source is seen as the quotient of a neighborhood of 0 in $\mathbb{C}^2$ by the action of a suitable cyclic group (of order the quotient of N' by the gcd's with l'_i). This gives d non-smooth ramified points over P (unramified and then smooth if $d = N$). It is easily seen that the map is unramified if and only if $d = N$; in that case, the source is smooth.

Finally, in order to obtain X^{ξ}, we resolve the singular points of $\overline{X}^{\xi}$ to obtain smooth points over the double points of $\hat{\mathcal{A}}$. The inverse image of a double point of $\hat{\mathcal{A}}$ is given by d finite linear sequences of at most $N'' = N'/(\gcd(N', l'_1)\gcd(N', l'_2))$ copies of $\mathbb{CP}^1$ (if $N'' = 1$ then the point is already smooth and no resolution is needed). Then we have:

$$\begin{tikzcd}
E^{\xi}_{\mathcal{A}} \arrow[r, hook, "i_N"] \arrow[d, "N:1"', "\rho"] & X^{\xi} \arrow[d, "N:1"', "\overline{\rho}"] \\
E_{\mathcal{A}} \arrow[r, hook, "i"] & X
\end{tikzcd}$$

5.2.2 Quasi-projective depth

The map i_N induces an application in cohomology:

$$i_N^* : \mathrm{H}^1(X^\xi;\mathbb{C}) \longrightarrow \mathrm{H}^1(E_{\mathcal{A}}^\xi;\mathbb{C}). \tag{5.1}$$

Notation. Let ζ be a primitive root of unity of order N, we denote with an exponent $\check{\xi}^*$ the eigenspace associated with the eigenvalue ζ.

The map i_N^* restricts to the eigenspace associate to $\check{\xi}^*$, and we define:

$$i_{N,\xi}^* : \mathrm{H}^1(X^\xi;\mathbb{C})^{\check{\xi}^*} \longrightarrow \mathrm{H}^1(E_{\mathcal{A}}^\xi;\mathbb{C})^{\check{\xi}^*} = \mathrm{H}^{1,\xi}$$

Theorem 5.2.7. *([42]) If ξ is fully ramified (i.e. $\mathcal{A}_\xi^0 = \emptyset$) then, $i_{N,\xi}^*$ is an isomorphism.*

When ξ is not fully ramified, the default of isomorphism of $i_{N,\xi}^*$ (i.e. the dimension of $\operatorname{coker}(i_{N,\xi}^*)$) and the dimension of $\mathrm{H}^1(X^\xi;\mathbb{C})^{\check{\xi}^*}$ determine the depth of ξ. Indeed, by Proposition 5.1.12, the map $i_{N,\xi}^*$ can be considered with values in $\mathrm{H}^1(E_{\mathcal{A}},\mathcal{L}_\xi)$.

Let $K_{\mathcal{A}}$ be the cokernel of i_N^*, and $K_{\mathcal{A}}^{\check{\xi}^*}$ the restriction to the eigenspace of $\check{\xi}^*$associated with the eigenvalue ζ, then we have:

$$0 \longrightarrow \mathrm{H}^1(X^\xi;\mathbb{C})^{\check{\xi}^*} \xrightarrow{i_{N,\xi}^*} \mathrm{H}^1(E_{\mathcal{A}}^\xi;\mathbb{C})^{\check{\xi}^*} \longrightarrow K_{\mathcal{A}}^{\check{\xi}^*} \longrightarrow 0,$$

where the exactness at the left comes from the injectivity of i_N.

Definition 5.2.8. Let $\xi \in \mathbb{T}^{tors}(\mathcal{A})$, we define the *quasi-projective depth* of ξ by:

$$\overline{\mathrm{depth}}(\xi) = \dim K_{\mathcal{A}}^{\check{\xi}^*}.$$

Proposition 5.2.9. *Let $\xi \in \mathbb{T}^{tors}(\mathcal{A})$, then we have:*

$$\mathrm{depth} = \overline{\mathrm{depth}} + \dim \mathrm{H}^1(X^\xi;\mathbb{C})^{\check{\xi}^*}.$$

Definition 5.2.10. The dimension of $\mathrm{H}^1(X^\xi;\mathbb{C})^{\check{\xi}^*}$ is the *projective depth* of ξ.

Corollary 5.2.11. *If ξ is fully-ramified then:*

$$\overline{\mathrm{depth}}(\xi) = 0.$$

Proof. By Theorem 5.2.7, $\dim \mathrm{H}^1(X^\xi;\mathbb{C})^{\check{\xi}^*} = \mathrm{H}^1(E_{\mathcal{A}}^\xi;\mathbb{C})^{\check{\xi}^*}$, then Proposition 5.2.9 implies that $\overline{\mathrm{depth}}(\xi) = 0$. □

Consider $\hat{\mathcal{A}}^\xi = X^\xi \setminus E_{\mathcal{A}}^\xi = \overline{\rho}^{-1}(\hat{\mathcal{A}})$, and remark that $\hat{\mathcal{A}}^\xi$ is a normal crossing divisor (n.c.d.). An irreducible component D of $\hat{\mathcal{A}}^\xi$ can be of two types:

$$(1) \quad \begin{cases} \overline{\rho}(D) = \left\{\text{double point of } \hat{A}\right\}, \text{ then } D \text{ comes from a double point of } \mathcal{A}, \\ \overline{\rho}(D) = L_i, \text{ then } D \text{ comes from a line of } \mathcal{A}, \end{cases}$$

(2) $\overline{\rho}(D) = E_P$, then D comes from an exceptional component of $\hat{\mathcal{A}}$.

Definition 5.2.12. A component $L \in \hat{\mathcal{A}}^\xi$ is of *single type* if $\overline{\rho}(L) = \{P\}$. If $\overline{\rho}(L)$ is a component $H \in \hat{\mathcal{A}}$ then L is a *H-component.*

A divisor $D \in \hat{\mathcal{A}}^\xi$ induces a 2-cycle in homology. By the previous description of the divisors of $\hat{\mathcal{A}}^\xi$, they are sent to $\overline{\rho}$ on a singular point of $\hat{\mathcal{A}}$ or to an irreducible component of $\hat{\mathcal{A}}$. Then we have the map:

$$\bigoplus_{D \in \hat{\mathcal{A}}^\xi} \mathbb{C}\langle D\rangle \xrightarrow{\Phi} \mathrm{H}_2(X^\xi, \mathbb{C}).$$

Using Poincaré duality, it is equivalent to:

$$\bigoplus_{D \in \hat{\mathcal{A}}^\xi} \mathbb{C}\langle D\rangle \longrightarrow \mathrm{H}^2(X^\xi, \mathbb{C}).$$

Proposition 5.2.13. *The kernel of Φ is $K_{\mathcal{A}}$.*

Proof. A classical result of Pure Hodge Theory (see for example [37]) implies that:

$$\mathrm{H}^1(X^\xi; \mathbb{C}) \simeq \mathrm{H}^1(X^\xi; \mathcal{O}_{X^\xi}) \oplus \mathrm{H}^0(X^\xi; \Omega^1_{X^\xi}). \tag{5.2}$$

In the other hand, the Deligne's Mixed Hodge Theory for quasi-projective varieties (see [26]) implies that:

$$\mathrm{H}^1(E^\xi_{\mathcal{A}}; \mathbb{C}) \simeq \mathrm{H}^1(X^\xi; \mathcal{O}_{X^\xi}) \oplus \mathrm{H}^0(X^\xi; \Omega^1_{X^\xi} \log(\hat{\mathcal{A}}^\xi)). \tag{5.3}$$

As the first term of the both decompositions are the same, then:

$$K_{\mathcal{A}} = \operatorname{coker}\left(\mathrm{H}^0(X^\xi; \Omega^1_{X^\xi}) \to \mathrm{H}^0(X^\xi; \Omega^1_{X^\xi} \log(\hat{\mathcal{A}}^\xi))\right). \tag{5.4}$$

In [61], K. Saito shows that:

$$0 \to \Omega^1_{X^\xi} \to \Omega^1_{X^\xi} \log(\hat{\mathcal{A}}^\xi) \to \bigoplus_{D \in \hat{\mathcal{A}}^\xi} i_* \mathcal{O}_D \to 0. \tag{5.5}$$

Since the functor $\mathrm{H}^*(X^\xi; \bullet)$ is covariant, then we have:

$$0 \to \mathrm{H}^0(X^\xi; \Omega^1_{X^\xi}) \to \mathrm{H}^0(X^\xi; \Omega^1_{X^\xi} \log(\hat{\mathcal{A}}^\xi)) \to \bigoplus_{D \in \hat{\mathcal{A}}^\xi} i_* \mathcal{O}_D \to \mathrm{H}^1(X^\xi; \Omega^1_{X^\xi}).$$

From equation 5.4, we can deduce the following exact sequence:

$$0 \to K_{\mathcal{A}} \to \bigoplus_{D \in \hat{\mathcal{A}}^\xi} i_* \mathcal{O}_D \to \mathrm{H}^1(X^\xi; \Omega^1_{X^\xi}).$$

As each $D \in \hat{\mathcal{A}}^\xi$ is an irreducible divisor then $\mathrm{H}^0(D; \mathcal{O}_D) \simeq \mathbb{C}$, and as $\mathrm{H}^1(X^\xi; \Omega^1_{X^\xi}) \subset \mathrm{H}^2(X^\xi; \mathbb{C})$, then:

$$K_{\mathcal{A}} \simeq \ker\left(\bigoplus_{D \in \hat{\mathcal{A}}^\xi} \mathbb{C}\langle D\rangle \to \mathrm{H}^2(X^\xi; \mathbb{C})\right).$$

□

Since the analytic structure is not needed, we have also the following possible proof.

Proof. (A. Degtyarev)
Let U be a regular tubular neighborhood of $\hat{\mathcal{A}}$. We have the following diagram:

$$\begin{tikzcd} \mathrm{H}^1(X) \arrow[r, "i^*"] & \mathrm{H}^1(E_{\mathcal{A}}) \arrow[r] & \mathrm{H}^2(X, E_{\mathcal{A}}) \arrow[r] \arrow[d, equal] & \mathrm{H}^2(X) \arrow[d, equal] \\ & & \mathrm{H}_2(U) \arrow[r] & \mathrm{H}_2(X) \end{tikzcd},$$

where the two vertical isomorphism come from PoincarÃ©-Lefschetz duality. Then, we have:

$$\operatorname{coker}\left(\mathrm{H}^1(X) \to \mathrm{H}^1(E_{\mathcal{A}})\right) \simeq \ker\left(\mathrm{H}_2(U) \to \mathrm{H}_2(X)\right).$$

Passing to the eigenspace and since $\hat{\mathcal{A}} \sim U$, we obtain the result. □

Remark 5.2.14. The previous Proposition can be restricted to the eigenspace of $\check{\xi}$:

$$K_{\mathcal{A}}^{\check{\xi}^*} \simeq \ker\left(\left(\bigoplus_{D \in \hat{\mathcal{A}}^\xi} \mathbb{C}\langle D\rangle\right)^{\check{\xi}^*} \to (\mathrm{H}^2(X^\xi;\mathbb{C}))^{\check{\xi}_*}\right),$$

Proposition 5.2.15. *Let $\xi \in \mathbb{T}_{\mathcal{A}}$, then:*

$$\overline{\mathrm{depth}}(\xi) = \dim\ker\left(\left(\bigoplus_{D \in \hat{\mathcal{A}}^\xi} \mathbb{C}\langle D\rangle\right)^{\check{\xi}^*} \to (\mathrm{H}^2(X^\xi;\mathbb{C}))^{\check{\xi}_*}\right),$$

Proof. It is a direct consequence of Proposition 5.2.8 and the previous remark. □

5.2.3 Twisted intersection form

We compute the quasi-projective depth of a character using Proposition 5.2.15.

Definition 5.2.16. A component $H \in \hat{\mathcal{A}}$ is *unramified* if $\xi(x_H) = 1$. It is *inner unramified* if ξ takes value 1 for all the meridians of its neighbors in $\hat{\Gamma}_{\mathcal{A}}$. The set of the inner unramified components of $\hat{\mathcal{A}}$ is denoted by $\mathcal{U}_\xi \subset \hat{\mathcal{A}}$.

Lemma 5.2.17. *Let $U_\xi = \mathbb{C}\langle\mathcal{U}_\xi\rangle$. For each $H \in \mathcal{U}_\xi$, fix a lift D_0 of H in $\hat{\mathcal{A}}^\xi$. And we denote $D_j = (\check{\xi}^*)^j(D_0)$, with $j \in \{1, \cdots, N-1\}$, the H-components. Then, the map:*

$$\Psi : \begin{cases} U_\xi & \longrightarrow & \left(\bigoplus_{D \in \hat{\mathcal{A}}^\xi} \mathbb{C}\langle D\rangle\right)^{\check{\xi}^*} \\ H & \longmapsto & \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \exp\left(\frac{2i\pi j}{N}\right) D_j, \end{cases}$$

is an isomorphism.

Proof. As H is in U_ξ, then the cover $X^\xi \to X$ is not ramified over H. Furthermore, at the singular points of H the inner unramification implies that the cover is not ramified at the singular point of H. Since H is unramified, then outside the singular the cover is also unramified on H. Then $\overline{\rho}^{-1}(H)$ admits N connected components: $D_0, \cdots, D_{N-1}$. The action of $\check{\xi}^*$ is transitive on $\overline{\rho}^{-1}(H)$. The definition of Ψ implies that $\Psi(H)$ is in the eigenspace:

$$\left(\bigoplus_{D \in \hat{\mathcal{A}}^\xi} \mathbb{C}\langle D \rangle \right)^{\check{\xi}^*}.$$

Conversely the elements of U_ξ are the only ones such that X^ξ is not ramified over them. □

Consider the intersection form on $\mathrm{H}^2(X^\xi;\mathbb{Z})$, it induces a non-degenerate hermitian form in $\mathrm{H}^2(X^\xi;\mathbb{C})$, and it induces an hermitian form on $\bigoplus_{D \in \hat{\mathcal{A}}^\xi} \mathbb{C}\langle D \rangle$.

Proposition 5.2.18.

(i) For the previous hermitian form, the decomposition of $\mathrm{H}^2(X^\xi;\mathbb{C})$ *in eigenspaces for* $\check{\xi}^*$ *is orthogonal.*

(ii) For the previous hermitian form, the decomposition of $\bigoplus_{D \in \hat{\mathcal{A}}^\xi} \mathbb{C}\langle D \rangle$ *in eigenspaces for* $\check{\xi}^*$ *is orthogonal.*

Definition 5.2.19. Using the isomorphism of Lemma 5.2.17, we defined an intersection form on U_ξ noted $\cdot_\xi$. The matrix of this form for an order in the basis $\mathcal{U}_\xi$ is denoted by A_ξ.

Theorem 5.2.20. *The twisted intersection form determines* $\overline{\mathrm{depth}}(\xi)$*:*

$$\overline{\mathrm{depth}}(\xi) = \operatorname{corank} A_\xi.$$

Proof. The Hodge Index Theorem implies that the signature of the intersection form on $\mathrm{H}^{1,1}(X^\xi;\mathbb{C})$ is $(1, b_2(X^\xi) - 1)$, i.e. it can be diagonalized with only one 1. Since the part of $\mathrm{H}^2(X^\xi;\mathbb{C})$ coming from divisors is contained in $\mathrm{H}^{1,1}(X^\xi;\mathbb{C})$, then it is also true in $\mathrm{H}^2(X^\xi;\mathbb{C})$.

Let L be a generic line in $\mathbb{CP}^2$, then it self intersection in $\mathbb{CP}^2$ is: $(L \cdot L)_{\mathbb{CP}^2} = 1$. The strict transformation $\check{L}$ of L in X does not pass through the centers of the blow-ups. Then the transform self-intersection $(\check{L} \cdot \check{L})_X$ is also 1. Fix a preimage $\check{L}^\xi$ of $\check{L}$ in X^ξ. By the Poincaré duality, it defines an element in the 1-eigenspace of $\mathrm{H}^2(X^\xi;\mathbb{C})$, and $(\check{L}^\xi \cdot \check{L}^\xi)_{X^\xi} = h > 0$.

By Proposition 5.2.18, the decomposition in eigenspaces of $\mathrm{H}^2(X^\xi;\mathbb{C})$ is orthogonal for the intersection form. Then its restriction to $\mathrm{H}^2(X^\xi;\mathbb{C})^{\check{\xi}^*}$ is negative definite, and so non-degenerate.

This implies that the corank of A_ξ is the dimension of the kernel of

$$\left(\bigoplus_{D\in\hat{\mathcal{A}}^\xi} \mathbb{C}\langle D\rangle\right)^{\check{\xi}^*} \to \left(\mathrm{H}^2(X^\xi;\mathbb{C})\right)^{\check{\xi}_*}.$$

□

5.2.4 Computation of A_ξ

First, remark that the twisted intersection form on U_ξ is not well-defined. Indeed, different arbitrary choices of a H-component in Lemma 5.2.17 imply some differences between the matrices A_ξ then obtained. This difference is a multiplication of the H-row by a square-root of unity, and a multiplication of the H-column by its conjugate. But the dimension of the kernel of A_ξ does not change with this different choices.

Let $\hat{\Gamma}_{\mathcal{U}_\xi}$ be the dual graph of $\mathcal{U}_\xi$ (see Definition 5.2.16). Its vertices are the inner unramified components, and the edges are the double points.

Consider the following choices for the H-components:

- Let $\mathcal{T}_{\mathcal{U}_\xi}$ be a maximal tree for $\hat{\Gamma}_{\mathcal{U}_\xi}$.

- Choose a vertex in $\hat{\Gamma}_{\mathcal{U}_\xi}$. It corresponds to an irreducible component H of $\mathcal{U}_\xi$.

- Choose arbitrarily a H-component H_0.

- By induction on the distance-graph r inside $\mathcal{T}_{\mathcal{U}_\xi}$ of a vertex to H, we choose the other components: fix a vertex associated to a component H of $\mathcal{U}_\xi$. If $r = 0$, the choice is done. If $r > 0$, there is a unique component H' such that the distance to H is $r-1$ and it is connected to H (both conditions are necessary to the uniqueness).

The previous description is done in the case where $\hat{\Gamma}_{\mathcal{U}_\xi}$ is connected. If it is not, we apply the same choice for each connected component.

Such a choice of H-components is called a *tree choice*. From this tree choice we construct A_ξ. Consider $E(\hat{\Gamma}_{\mathcal{U}_\xi}, \mathcal{T}_{\mathcal{U}_\xi})$ the set of the edges of $\hat{\Gamma}_{\mathcal{U}_\xi}$ not in $\mathcal{T}_{\mathcal{U}_\xi}$. For any oriented edge e in $E(\hat{\Gamma}_{\mathcal{U}_\xi}, \mathcal{T}_{\mathcal{U}_\xi})$, consider O the origin of e and E its end. Let $p \in B_{\mathcal{A}}$ be close to the double point associated to e. We denote by $\widetilde{\gamma}_e$ the cycle in $B_{\mathcal{A}}$ defined as a lift very close to $\hat{\mathcal{A}}$, of the unique polygonal path g_e in $\mathcal{T}_{\mathcal{U}_\xi}$ joining O and E. In other terms, $\widetilde{\gamma}_e$ is contained in the boundary of a regular neighborhood of $\bigcup_{L\in g_e} L$.

Proposition 5.2.21. *For $e \in E(\hat{\Gamma}_{\mathcal{U}_\xi}, \mathcal{T}_{\mathcal{U}_\xi})$, the value of $\xi(\widetilde{\gamma}_e)$ only depends on e.*

Proof. First, remark that $\widetilde{\gamma}_e$ is not well-defined. Indeed the lift of the polygonal path in $B_{\mathcal{A}}$ admits some indeterminacy (This difference is well illustrated by the difference between Theorem 3.2.2 and Theorem 3.2.7). The indeterminacy is of two types:

(1) The lift can turn around a component A of the polygonal path. This implies a multiplication by a power of t_A. As the components of the polygonal path are in $\mathcal{U}_\xi$, then $\xi(\alpha_A) = 1$.

(2) The lift can also turn around A' a neighbor of the components of the polygonal path. This implies a multiplication by $t_{A'}$, but by definition of inner unramified, $\xi(\alpha_{A'}) = t_{A'} = 1$.

□

Remark 5.2.22. The generators used in Theorem 3.2.2 are in the boundary of a regular neighborhood of the blow-up of $L_0 \cup L_s \cup L_t$, then we can compute $\widetilde{\gamma}_e$ as a product of meridians, and then determine its image by ξ. However, generators $\mathcal{E}_{s,t}$ of Theorem 3.2.7 are not in the boundary of a regular neighborhood of the blow-up of $L_0 \cup L_s \cup L_t$. Then they cannot be used to compute $\xi(\widetilde{\gamma}_e)$.

There is a *non-twisted* intersection form $\cdot$ on $\bigoplus_{D \in \hat{\mathcal{A}}^\xi} \mathbb{C}\langle D\rangle$, which corresponds to the one in X. Then $\cdot$ and $\cdot_\xi$ are linked by:

Theorem 5.2.23. *Let us consider a tree choice, and let E and F two components of $\mathcal{U}_\xi$. Then:*

$$E \cdot_\xi F = \begin{cases} 0 & \text{if } E \cdot F = 0, \\ E \cdot F & \text{if } E = F \text{ or } (EF) \subset \mathcal{T}_{\mathcal{U}_\xi}, \\ \xi(\widetilde{\gamma}_{(EF)}) & \text{if } (EF) \in E(\hat{\Gamma}_{\mathcal{U}_\xi}, \mathcal{T}_{\mathcal{U}_\xi}) \end{cases}$$

Proof.

- If E and F are disjoint, then the E-component and the F-component too. And then $E \cdot F = E \cdot_\xi F = 0$.

- If $(EF) \subset \mathcal{T}_{\mathcal{U}_\xi}$ then $(E_i \cdot F_j)_{X^\xi} = \delta_{i,j}$. The intersection form in X^ξ implies:

$$\begin{aligned} E \cdot_\xi F &= \left(\frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \exp\left(\frac{2i\pi j}{N}\right) E_j\right) \cdot \left(\frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \exp\left(\frac{2i\pi j}{N}\right) F_j\right) \\ &= \frac{1}{N} \sum_{j=0}^{N-1} \exp\left(\frac{2i\pi j}{h}\right) \exp\left(-\frac{2i\pi j}{N}\right) = 1. \end{aligned}$$

- If $(EF) \in E(\hat{\Gamma}_{\mathcal{U}_\xi}, \mathcal{T}_{\mathcal{U}_\xi})$, let $k \in \{0, 1, \cdots, h-1\}$ such that $\xi(\widetilde{\gamma}_{EF}) = \exp\left(-\frac{2i\pi k}{N}\right)$. Then the lifting covering property implies that $E_0 \cdot F_k = 0$.

□

Corollary 5.2.24. *If $\hat{\Gamma}_{\mathcal{U}_\xi}$ is a tree then, for a tree choice, $\cdot_\xi$ and $\cdot$ coincides.*

Property 5.2.25. *Let E be an element of $\mathcal{U}_\xi$, then the self-intersection $E \cdot E$ is:*

- *If E comes from a line L of $\mathcal{A}$, then $E \cdot E = 1 - \#\{P \in \mathcal{Q} \mid P \in L,\ m_P > 2\}$.*

- *If E is an exceptional component, then $E \cdot E = -1$.*

Remark 5.2.26. Theorem 5.2.23 determines which coefficients of A_ξ are combinatorial.

To conclude, by Remark 5.2.22, we have that Theorem 3.2.2 and Theorem 5.2.23 allow to compute the quasi-projective depth of a character.

Corollary 5.2.27. *Let $\xi \in \mathbb{T}_{\mathcal{A}}$, if $\hat{\Gamma}_{\mathcal{U}_\xi}$ is a tree, then A_ξ is combinatorial, and* $\overline{\mathrm{depth}}(\xi)$ *too.*

5.3 Topological invariant

5.3.1 Inner-cyclic combinatorics

In this part we use Theorem 4.1.1 to compute the non combinatorial part of A_ξ. First, remark that the construction of $\Gamma_{\mathcal{U}_\xi}$ only depends on the combinatorics.

Definition 5.3.1. Let C be a combinatorics.

1. A character $\xi \in \mathbb{T}_{\mathcal{A}}$ is *inner-cyclic* if it is torsion and $b_1(\hat{\Gamma}_{\mathcal{U}_\xi}) > 0$.
2. C is *inner-cyclic* if $\mathcal{A}$ admits an inner-cyclic character.
3. If γ is a cycle of $\hat{\Gamma}_{\mathcal{U}_\xi} \subset \hat{\Gamma}_{\mathcal{A}}$, then γ is *inner-cyclic*.
4. An arrangement $\mathcal{A}$ is *inner-cyclic* if its combinatorics is also.

Remark 5.3.2. In the second definition, if a realization admits an inner-cyclic character, then it is true for all realizations. In fact, the notion of inner-cyclic is combinatorial.

Example 5.3.3. The Ceva-7 arrangement is the first known example of inner-cyclic arrangement. Indeed consider the following character:

$$\xi : (\alpha_1, \cdots, \alpha_7) \longmapsto (1, -1, -1, 1, -1, -1, 1),$$

with indexation of Figure 4.1. Before this thesis, it was the only one example known of inner-cycle arrangement.

Consider a realization $\mathcal{A}$ of C, and let $\tilde{\gamma}$ be a shift of $\gamma \in \Gamma_{\mathcal{A}}$ in the first homology group of the boundary manifold $B_{\mathcal{A}}$ of $\mathcal{A}$. By the inclusion of $B_{\mathcal{A}}$ in $E_{\mathcal{A}}$, we can see $\tilde{\gamma}$ in $E_{\mathcal{A}}$. The application from $\mathrm{H}_1(\Gamma_{\mathcal{A}})$ to $\mathrm{H}_1(E_{\mathcal{A}})$ is divided in two parts:

$$\begin{array}{ccccc} \mathrm{H}_1(\Gamma_{\mathcal{A}}) & \longrightarrow & \mathrm{H}_1(B_{\mathcal{A}}) & \longrightarrow & \mathrm{H}_1(E_{\mathcal{A}}) \\ \gamma & \longmapsto & \tilde{\gamma} & \longmapsto & \sum\limits_{L \in \mathcal{A}} l_L \alpha_L \end{array},$$

where α_L is the meridian associated to L, and $l_L \in \mathbb{Z}$.

Definition 5.3.4. The number l_L is the *linking number* of $\tilde{\gamma}$ around L.

Remark 5.3.5. The map $\mathrm{H}_1(\Gamma_{\mathcal{A}}) \to \mathrm{H}_1(B_{\mathcal{A}})$ is not uniquely defined.

Definition 5.3.6. Let γ be a cycle of $\hat{\Gamma}_{\mathcal{A}}$ (or equivalently in $\Gamma_{\mathcal{A}}$).

1. The vertices $v_1, \cdots, v_r \in \gamma$ are the *inners*.
2. The vertices $w_1, \cdots, w_s$ connected by a single edge to an inner are the *neighbors*.
3. The other vertices $u_1, \cdots, u_l$ are the *distants*.

To partially skip the previous remark, we assume that $\widetilde{\gamma}$ is contained in a regular neighborhood of $\bigcup_{v_L \in \gamma} L$. With this hypothesis, we have:

Proposition 5.3.7. *The linking number l_L of $\widetilde{\gamma}$ is well defined if v_L is distant of γ.*

Proof. The arguments of the proof of Proposition 5.2.21 are sufficient to obtain the result. □

From this Proposition, we obtain that:

Proposition 5.3.8. *If γ is an inner-cyclic cycle of $\hat{\Gamma}_{\mathcal{A}}$, then $\xi(\widetilde{\gamma})$ is well defined.*

Definition 5.3.9. Let γ be an inner-cyclic cycle, the set $\left\{L \in \hat{A} \mid v_L \in \gamma\right\}$ is the *support* of γ.

5.3.2 Nearby cycles

Consider an inner-cyclic character $\xi \in \mathbb{T}(\mathcal{A})$ on an arrangement $\mathcal{A}$ with cycle γ. We construct the set $\mathcal{S}_\gamma$ of nearby cycles associated with γ.

Consider $U_{\xi,\gamma}$ a tubular neighborhood of the inner-unramified line L of $\hat{\mathcal{A}}$ such that $v_L \in \gamma$.

Definition 5.3.10. A *nearby cycle* $\widetilde{\gamma}$ associated with γ, is a lift in $U_{\xi,\gamma} \setminus \bigcup_{L \in \mathcal{U}_\xi} L$ of γ. The set of all such cycles is $\mathcal{S}_\gamma$.

Remark 5.3.11. Let $\mathcal{A} = \{L_0, \cdots, L_n\}$ be a line arrangement. Assume that L_0, L_s and L_t (with $L_0 \cap L_s \cap L_t = \emptyset$) are inner-unramified for an inner-cyclic character ξ, with the notation of Subsection 3.2.1, $\varepsilon_{s,t}$ is a nearby cycle, however $\mathcal{E}_{s,t}$ is not.

Lemma 5.3.12. *Let C be an inner-cyclic combinatorics, and let γ be a cycle of $\mathrm{H}_1(\hat{\Gamma}_{\mathcal{U}_\xi})$. Assume that C admits two realizations $\mathcal{A}_1$ and $\mathcal{A}_2$, with sets of nearby cycles $\mathcal{S}^1_\gamma$ and $\mathcal{S}^2_\gamma$. If there exists a topological equivalence ϕ between $\hat{A}_1$ and $\hat{A}_2$ then:*

$$\widetilde{\gamma} \in \mathcal{S}^1_\gamma \Rightarrow \phi(\widetilde{\gamma}) \in \mathcal{S}^2_\gamma.$$

Proof. For $i = 1, 2$, let F_i be a fundamental system of regular neighborhoods of the lines of the support $\bigcup_{v_L \in \gamma} L \subset X_i$. Consider $\widetilde{\gamma} \in \mathcal{S}^1_\gamma$, then it is $V_1 \in F_1$ such that $\widetilde{\gamma} \subset V_1$. There exists $V_2 \in F_2$ such that $\phi(V_1) \subset V_2$. Then $\phi(\widetilde{\gamma})$ is contained in an element of F_2.

By definition, $\widetilde{\gamma}$ is a slight deformation of a path contained in $\hat{\mathcal{A}}_1$, then its image by ϕ is also a slight deformation of the same path but in $\hat{\mathcal{A}}_2$. Thus $\widetilde{\gamma}$ and $\phi(\widetilde{\gamma})$ are lift of γ in $U_{\xi,\gamma} \setminus \bigcup_{L \in \mathcal{U}_\xi} L$. This implies that $\phi(\widetilde{\gamma}) \in \mathcal{S}^2_\gamma$. □

Remark 5.3.13. The nearby cycles were constructed to compute the depth of a character using Theorem 5.2.23, and then the quasi-projective part of the characteristic varieties. But the next Theorem shows that they give invariants of the ordered and oriented topology of $(X, \hat{\mathcal{A}})$.

Theorem 5.3.14. *Let ξ be an inner-cyclic character on an arrangement $\mathcal{A}$, with inner-cyclic cycle γ. If $\widetilde{\gamma} \in \mathcal{S}_\gamma$ then $\xi(\widetilde{\gamma})$ is a topological invariant of the ordered oriented pair $(X, \hat{\mathcal{A}})$.*

Proof. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two ordered arrangements. Let ϕ be a homeomorphism from X_1 in X_2 sending $\hat{\mathcal{A}}_1$ on $\hat{\mathcal{A}}_2$, preserving the orientation and the ordered of the arrangements.

Let ξ be a character on $\mathrm{H}_1(X_1 \setminus \hat{\mathcal{A}}_1)$. By the homeomorphism ϕ, ξ defines a character ξ' on $\mathrm{H}_1(X_2 \setminus \hat{\mathcal{A}}_2)$. Then we have:

$$\begin{tikzcd} \mathrm{H}_1(X_1 \setminus \hat{\mathcal{A}}_1) \arrow[r, "\phi_*"] \arrow[d, "\xi"'] & \mathrm{H}_1(X_2 \setminus \hat{\mathcal{A}}_2) \arrow[d, "\xi'"] \\ \mathbb{C}^* \arrow[r, "\overline{\phi}"] & \mathbb{C}^* \end{tikzcd}$$

Since ξ is of combinatorial nature, and since ϕ preserves the order and the orientation (i.e. $\phi_*(x_i^1) = x_i^2$, where x_i^j is the meridian around L_i in X_j), then $\overline{\phi} = \mathrm{Id}$.

By Lemma 5.3.12, $\xi(\widetilde{\gamma}) = \xi'(\phi_*(\widetilde{\gamma}))$. Then $\xi(\widetilde{\gamma})$ is a topological invariant of the oriented and ordered pair $(X_1, \hat{\mathcal{A}}_1)$. □

The power of this invariant is illustrated in Theorem 6.3.2, Theorem 6.4.8 and Theorem 6.4.10 with the detection of new examples of NC-Zariski pairs.

Remark 5.3.15. The slight perturbations considered in the proof of Lemma 5.3.12, need to be nearby the exceptional components met along γ. As it is, these deformations do not work in the projective plane.

Let c' the map induced on X by the complex conjugation of $\mathbb{CP}^2$, and let c be the complex conjugation on $\mathbb{C}^*$.

Proposition 5.3.16. *Let ξ be an inner-cyclic character for a combinatorics C with two complex conjugate realizations $\mathcal{A}$ and $\overline{\mathcal{A}}$. Let $\widetilde{\gamma}$ and $c'(\widetilde{\gamma})$ be the corresponding nearby cycles, then*

$$c \circ \xi(\widetilde{\gamma}) = \xi \circ c'(\widetilde{\gamma}).$$

Proof. Since the following diagram is commutative, then the result holds.

$$\begin{tikzcd}
\mathrm{H}_1(X \setminus \hat{\mathcal{A}}_1) \arrow[r, "c'"] \arrow[d, "\xi"'] & \mathrm{H}_1(X \setminus \hat{\mathcal{A}}_2) \arrow[d, "\xi"] \\
\mathbb{C}^* \arrow[r, "c"] & \mathbb{C}^*
\end{tikzcd}$$

□

Corollary 5.3.17. *Let $\mathcal{A}$ be a complexified real arrangement, and let ξ be a inner-cyclic character on $\mathcal{A}$, with inner-cyclic cycle γ. Then we have:*

$$\xi(\widetilde{\gamma}) \in \{-1, 1\}\,.$$

Proof. If $\mathcal{A}$ is a complexified real arrangement, then $\overline{\mathcal{A}} = \mathcal{A}$ and $c'(\widetilde{\gamma}) = \widetilde{\gamma}$. Using Proposition 5.3.16, we obtain that:

$$\xi(\widetilde{\gamma}) = c \circ \xi(\widetilde{\gamma}).$$

Since $\xi(\widetilde{\gamma})$ is a root of unity, then $\xi(\widetilde{\gamma}) \in \{-1, 1\}$. □

Chapter 6

Prime combinatorics, computation & examples

Now that we know how to compute the quasi-projective depth of a character, and that we have constructed a topological invariant, the time has come to apply it. But on which arrangements shall we test our computations? The first ones are naturally the classical arrangements, like Ceva-7, MacLane, Rybnikov, Fan, etc... And then? To find more interesting arrangements, we consider *prime combinatorics*. The non prime combinatorics provide arrangement with characteristic varieties computable from an arrangement with fewer lines. This definition greatly reduces the combinatorics to test, and they provide a lot of interesting examples.

In Section 6.1, we first discuss about the two methods presented in Chapter 5 and consider conditions on a character to have a positive depth, see Proposition 6.1.1. In the second part of this section, we introduce the notion of *prime combinatorics* in order to reduce the number of computations. The last subsection concerns other properties.

In Section 6.2, we present a Sage program to select inner-cyclic or prime realizable combinatorics. A Maple program is also described, that computes realizations.

In Section 6.3, we give a selection of examples coming from our computations:

- three prime arrangements with nine lines,
- an ordered, oriented NC-Zariski pair with nine lines,
- a pair of complexified real prime arrangement conjugate in $\mathbb{C}[\sqrt{5}]$ with ten lines and admitting three inner-cyclic cycles,
- a prime combinatorics admitting four realizations with eleven lines.

In the last section, we present the detailed computation of the prime combinatorics with eleven lines given in the previous section. From this example we construct explicit examples of NC-Zariski pairs, and a NC-Zariski 4-tuplet.

6.1 Prime combinatorics

6.1.1 Depth & characters

The characteristics varieties are algebraic invariants. We do not know an algebraic method to compute this quasi-projective depth. But the algorithm developed in Subsection 5.2.2, is topological.

The ultimate result is to discover a Zariski pair of line arrangements of a different nature than the previous [60, 7]. It means to detect them with a new invariant. To do this, we use the methods developed in Chapter 5. The goal of the first approach is to distinguish two arrangements with the same combinatorics by finding a character with distinct quasi-projective depths. To detect a pair with such a method would allow to solve the problem of the combinatoriality of the characteristics varieties. Unfortunately, the following proposition is often sufficient to show that this method does not give a positive answer.

Proposition 6.1.1. *Let $\mathcal{A}$ be an arrangement, and ξ be a torsion character of $\mathcal{A}$. Assume that $\hat{\Gamma}_\xi$ admit only three vertices L_1, L_2, L_3 with a self intersection $L_i \cdot L_i \leq -2$. If we denote γ the cycle of $\hat{\Gamma}_\xi$ then:*

$$\overline{\operatorname{depth}}(\xi) > 0 \ \Leftrightarrow\ L_i \cdot L_i = -2 \text{ and } \xi(\gamma) = 1.$$

Proof. In such a situation, the intersection matrix is:

$$A_\xi = \begin{pmatrix} L_1 \cdot L_1 & 1 & 1 \\ 1 & L_2 \cdot L_2 & \xi(\gamma) \\ 1 & \xi(\gamma)^{-1} & L_3 \cdot L_3 \end{pmatrix}.$$

Then, we have:

$$\det(A_\xi) = \xi(\gamma) + \xi(\gamma)^{-1} + L_2 \cdot L_2 + L_3 + L_3 - L_1 \cdot L_1[(L_2 \cdot L_2)(L_3 \cdot L_3) - 1].$$

A simple computation shows that $\det(A_\xi) = 0$ if and only if we have $L_i \cdot L_i = -2$ and $\xi(\gamma) = 1$. ☐

Nevertheless, with the second approach, we differentiate two combinatorially equivalent arrangements by different images for a same nearby cycle. Such a pair gives an example of oriented ordered NC-Zariski pair, and shows that the maps induced by the inclusion on the fundamental groups (or the homology) are not of combinatorial nature. In Section 6.3, some positive cases are obtained with this method.

6.1.2 Prime combinatorics

Let us remark the following fact: consider an inner-cyclic combinatorics C' with realization $\mathcal{A}'$. If we add a line L to $\mathcal{A}'$ (and then obtain a new arrangement $\mathcal{A}$), then $\mathcal{A}$ is also inner-cyclic (keep the same character on the common lines of $\mathcal{A}$ and $\mathcal{A}'$, and send the meridian associated with L on 1). Furthermore, the inner-cyclic cycles of $\mathcal{A}'$ correspond to some inner-cyclic cycles of $\mathcal{A}$.

Definition 6.1.2. Let C be an inner-cyclic combinatorics, and ξ an inner-cyclic character of a realization of C. The pair (C, ξ) is said *prime* if all the unramified lines of C are inner unramified (i.e. $\mathcal{U}_\xi = \mathcal{A}^0_\xi$).

Remark 6.1.3. By an abuse of language, a combinatorics C is said to be prime if it does admit a character ξ such that (C, ξ) is prime.

If (C, ξ) (with realization $\mathcal{A}$) is not prime then there is a pair (C', ξ') prime (with realization $\mathcal{A}'$) and a set of lines $\{L_1, \cdots, L_k\}$ such that $\mathcal{A} = \mathcal{A}' \bigcup L_i$, L_i is unramified for ξ but not inner unramified. Then the inner-cyclic cycles of C are in correspondence with the inner-cyclic cycles of C', and their image by ξ are completely determined by $\mathcal{A}'$.

Proposition 6.1.4. *Let C be a combinatorics, and $\mathcal{A}$ a realization of C. If C is not prime, then the characteristics varieties of $\mathcal{A}$ are completely determined by realizations of prime combinatorics with less lines than $\mathcal{A}$.*

Proof. Let ξ a character of $\mathcal{A}$. Consider the construction doing before (i.e. $\mathcal{A} = \mathcal{A}' \bigcup L_i$). Here, remark that $\mathcal{A}'$ depends of the character ξ. For each L_i three cases are possible:

1. L_i is generic with the lines of the support of the inner-cyclic cycle. Then $A_\xi = A_{\xi'}$, so $\overline{\mathrm{depth}}(\xi) = \overline{\mathrm{depth}}(\xi')$.

2. Adding L_i created a singular point with multiplicity greater than 2 on a line of the support. Then, the support of the inner-cyclic cycle admits an additional line (i.e. the exceptional line coming from the new point). Then usual operations on the lines and the rows of A_ξ show that:
$$\mathcal{A}_\xi = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & A_{\xi'} \end{pmatrix}.$$
Then $\overline{\mathrm{depth}}(\xi) = \overline{\mathrm{depth}}(\xi')$.

3. The line passes through the intersection of two lines of the support, then once again we add a line to the support of the inner-cyclic cycle. By the same argument as in the previous case we have $\overline{\mathrm{depth}}(\xi) = \overline{\mathrm{depth}}(\xi')$.

Since the ramified lines do not appear in the computation of the projective depth, then adding or deleting such lines does not change the projective depth. This proves the result. $\square$

Remark 6.1.5. It is enough to study the prime character of a combinatorics in order to compute its characteristic varieties.

6.1.3 Combinatorics & realizations

To simplify the computation, we are interested by particular combinatorics. The best case is the combinatorics such that Aut_C is trivial. Indeed, if it is not, there exists a non trivial $\sigma \in \mathrm{Aut}_C$; and then we can find to different realizations $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\sigma(\mathcal{A}_1) = \mathcal{A}_2$. This implies that $E_{\mathcal{A}_1} = E_{\mathcal{A}_2}$.

For the definition of Σ_C and $\mathcal{M}(C)$, see Subsection 1.1.2.

Definition 6.1.6. An *automorphism of the incidence graph* Γ is an automorphism of the bipartite graph Γ respecting the two sets of vertices. The set of such automorphisms is denoted by Aut_Γ.

Property 6.1.7. *Let* Aut_Γ *be the automorphism group of the incidence graph* Γ *of a combinatorics* C*, then* Aut_C *and* Aut_Γ *are isomorphic.*

Proposition 6.1.8. *Let* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *be two realizations of the same combinatorics* C*. If they are in the same connected component of* Σ_C*, then they have the same topological type.*

Proof. It is a direct consequence of Property 1.1.9 and [59]. □

Proposition 6.1.9. *Let* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *be two realizations of* C*. Assume that they are in distinct connected components of* Σ_C*. If there exists an element* $\sigma \in \mathrm{Aut}_C$ *inducing a linear application of* $\mathbb{CP}^2$ *sending* $\mathcal{A}_1$ *on* $\mathcal{A}_2$*, then all the realizations of the two connected components have the same topological type.*

Proof. If σ induces a linear application ϕ of $\mathbb{CP}^2$ sending $\mathcal{A}_1$ on $\mathcal{A}_2$, then ϕ is a homeomorphism. By Definition 1.1.5, $\mathcal{A}_1$ and $\mathcal{A}_2$ have the same topological type. □

Remark 6.1.10. If there exists an element of Aut_C realizable by a linear application of $\mathbb{CP}^2$ then $\mathcal{A}_1$ and $\mathcal{A}_2$ are in the same connected component of $\mathcal{M}(C)$.

Proposition 6.1.11. *Let* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *be two realizations of* C*. Assume that they are in the same connected component of* $\mathcal{M}(C)$*. Then all the realizations of the connected component have the same topological type.*

Let us define the rigidity of a combinatorics:

Definition 6.1.12. Let $\mathcal{A}$ be an arrangement and C its combinatorics. The *rigidity* of $\mathcal{A}$ is the dimension of a connected component of Σ_C containing $\mathcal{A}$. An arrangement is *rigid* if its rigidity is zero.

We focus on combinatorics with a small (by the cardinality) automorphism group of the combinatorics. Indeed if it is trivial then the notion of ordered topology is equivalent to the topology. If it is not, then we search for combinatorics where $\mathcal{M}(C)$ is not connected, so that there is no evident obstacle to have different topological types.

6.2 Computation & output

6.2.1 Sage program

a) Inner-cyclic & prime combinatorics program

In Annexe A.2, we give a program to determine if a combinatorics is inner-cyclic. The entry data of this program is a combinatorics, and the algorithm can be describe as follows:

- (function *our_cycle_basis*) In the data entries of this function, we choose a line at infinity. And we compute the basis of cycle $\mathscr{E}$ defined in Sub-section 3.1.3.
- (functions *cycle_matrix_color*, *corner_intersection*) We compute here the matrix of conditions associated to the combinatorics, and a cycle E of $\mathscr{E}$.
- (function *is_a_cycle*) This short function distinguishes the true cycle and the pencil of lines.
- (function *inner-cyclic_cycle*) Using the color matrix defined with the function *cycle_matrix_color*, we check if the cycle E is an inner-cyclic cycle for some characters of order a prime number less than 50.
- (function *is_a_prime*) We check if a character is inner-cyclic (only in the case of a single cycle).
- (function *inner-cyclic_combinatorics*) For a combinatorics C, we test every cycle, for all infinity lines, to know if it is an inner-cyclic cycle. If not, then the function returns false. Else it returns the set of inner-cyclic cycles with the corresponding character and its order, as additional information.
- (function *check*) This function was written with M.A. Marco-Buzunáriz: we use the Gröbner basis to know if a combinatorics is realizable, see [67] for details about the Gröbner basis. To optimize the computation, we suppose that the combinatorics admit four lines in generic position.
- (function *generic_lines*) As previously explained, we need to find four lines intersecting in six different points. This function finds (if it exists) such a 4-tuplet of lines in C.
- (function *is_actual*) This function uses the two previously described functions to know if the combinatorics C admits a realization. We use the Sage function *fork* to stop the computation if it is too long.
- (functions *industrial_test* and *display_combinatorics*) These functions permit to check a lot of combinatorics, and display the result. The parameter s permits to initialize the counter at a precised value. It was useful in the case of combinatorics with nine and ten lines.

Remark 6.2.1. Since the program can forget some combinatorics (due to the complexity of computation of the realizability), we need to study this combinatorics with another method than the Gröbner bases. To do that, we have written a `Maple` program (see Appendix A.3) to obtain directly a set –as smaller as possible– containing Σ_C from a given combinatorics.

b) Realization program (computation of Σ_C)

The program, given in Appendix A.3, returns the possible equations to realize a combinatorics C. It is developed on `Maple` to make formal computation on $\mathbb{C}$. It is composed as follows:

- (function *equations*) Returns the set of polynomial equations defining the set Σ_C.
- (function *fixed_lines*) Fixes the equations of five lines to rigidify the arrangement.
- (function *rigidification*) Permits to add the rigidification previously defined to the set of polynomials obtained by the function *equations*.
- (function *lines_number*) Returns the index of five lines in a rigid position.
- (function *reduc*) Deletes the result with double lines on the rigid lines previously defined.
- (function *realization*) Regroups the previous functions.

c) Possible ameliorations

In the first program, the function *is_a_prime*, only finds the case of inner-cyclic character with a single cycle. Thus it is possible to apply Proposition 6.1.1. But with it, we omit the case with several cycles. Then we need to develop a function to detect such combinatorics and apply it to the case of combinatorics with eleven lines.

Remark 6.2.2. For the case of combinatorics with seven, eight, nine and ten lines, we have computed all the primes combinatorics because they were extracted from lists containing all the inner-cyclic combinatorics.

A possible amelioration of the `Maple` program is to delete also the results with double lines. With such a improvement this program will return exactly the set Σ_C

6.2.2 Output

As our program was constructed to make a lot of examples, we have used the lists of all possible combinatorics with less than eleven lines given by M.A. Marco-Buzunàriz:

http://riemann.unizar.es/combinatorias/

The results of the computations are present at:

https://www.benoit-guervilleballe.com/publications/

By this computation on all the possible combinatorics, we have shown that the extended Ceva arrangement is the smallest arrangement (i.e. with a minimal number of lines) admitting a non trivial inner-cyclic character, and then its combinatorics is the first prime one.

With all the programs present in this thesis, we can know if a combinatorics is inner-cyclic, and compute the quasi-projective depth of its inner-cyclic characters. Indeed, with the previous Sage program, we can determine if a combinatorics is inner-cyclic; with the Maple program, we obtain the equations of the realizations; and the Sage program on the wiring diagram permits to compute the image by the inner-cyclic character of the inner-cyclic cycles. Thus we have computed the quasi-projective depth if the inner-cyclic character.

6.3 Examples

6.3.1 First tests

Before making a complete analysis, we have studied some particular examples. First, we have tested MacLane's arrangements. Indeed they are the smallest (in terms of number of lines) without real realizations. And we have also tested Fan's arrangements. But they do not admit an inner-cyclic character.

In a second time, we have tested the two arrangements obtained by G. Rybnikov in [60]. Indeed, it is a Zariski pair, therefore we would like to test them in the prospect of showing that our method is able to distinguish a Zariski pair. But once again they do not admit some inner-cyclic character.

The first non completely negative test comes from the pair of arrangements obtained in [7]. Indeed they admit an inner-cyclic character, but the images by the character of the inner-cyclic cycle are exactly the same.

Then we have used the lists obtained by the `Sage` program. We have tested some examples admitting a non-connected realization space. But in all the cases the quasi-projective depth of the inner-cyclic character was similar.

After all these tests, we have tried to construct from the known Zariski pairs an inner-cyclic combinatorics (without success). It is here that the notion of prime combinatorics has appeared.

6.3.2 Prime combinatorics

a) Ceva-7

The extended arrangement of Ceva (pictured in Figure 4.1), is the first known case of arrangement with an inner-cyclic character. As previously said in Subsection 6.2.2, it is the smallest (in terms of number of lines) admitting an inner-cyclic character ξ, and then also the first prime pair (C, ξ). Remark also that it is the first example of arrangement with essential component (see [23]). Furthermore, the computation of its inner-cyclic cycle γ by ξ, shows that $\xi(\gamma) = -1$ then:

$$A_\xi = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix},$$

this implies that $\overline{\text{depth}}(\xi) = 2$. This is in harmony with the result of D. Cohen and A. Suciu [23]. But the combinatorics of Ceva-7 admits a single realization, so it cannot give any information about the combinatoriality of i_*.

Property 6.3.1. *All the inner-cyclic combinatorics with eight lines come from Ceva-7 with an additional unramified line. And no one is prime.*

b) Nine lines

There are exactly four prime combinatorics with nine lines. The first three are:

$$C_1 = [[1,2],[1,3,4],[1,5,6],[1,7,8],[1,9],[2,3,5],[2,4,7],[2,6,8],\\ [2,9],[3,6],[3,7],[3,8,9],[4,5],[4,6,9],[4,8],[5,7,9],[5,8],[6,7]]$$

$$C_2 = [[1,2],[1,3,4],[1,5,6],[1,7,8],[1,9],[2,3,5],[2,4,7],[2,6,8],\\ [2,9],[3,6,9],[3,7],[3,8],[4,5],[4,6],[4,8,9],[5,7,9],[5,8],[6,7]]$$

$$C_3 = [[1,2,3],[1,4,5],[1,6,7],[1,8,9],[2,4,6],[2,5,8],[2,7,9],\\ [3,4,9],[3,5,7],[3,6,8],[4,7],[4,8],[5,6],[5,9],[6,9],[7,8]]$$

They all admit a single realization pictured in Figure 6.1.

Furthermore, the fourth is:

$$C_4 = [[1,2,3,7],[1,4,8],[1,5,9],[1,6],[2,4],[2,5,8],[2,6,9],\\ [3,4,9],[3,5],[3,6,8],[4,5,6,7],[7,8],[7,9],[8,9]]$$

and it admits two different realizations conjugated in $\mathbb{C}$. The equations of the realizations are:

$$\begin{array}{ccc} L_1: \ z = 0 & L_2: \ x = 0 & L_3: \ x - z = 0 \\ L_4: \ y = 0 & L_5: \ \alpha x - y + z = 0 & L_6: \ \alpha x + \alpha^2 y + z = 0 \\ L_7: \ x + \alpha^2 = 0 & L_8: \ y - z = 0 & L_9: \ x - \alpha^2 y - z = 0 \end{array}$$

with $\alpha = \frac{-1 \pm i\sqrt{3}}{2}$. The character ξ such that (C_4, ξ) is prime is $(\zeta, \zeta, \zeta, \zeta^2, \zeta^2, \zeta^2, 1, 1, 1)$ with ζ a primitive 3-root of unity. Its inner-cyclic cycle is then supported by the lines L_7, L_8, L_9. The graph $\hat{\Gamma}_{\mathcal{U}_\xi}$ is pictured in Figure 6.2.

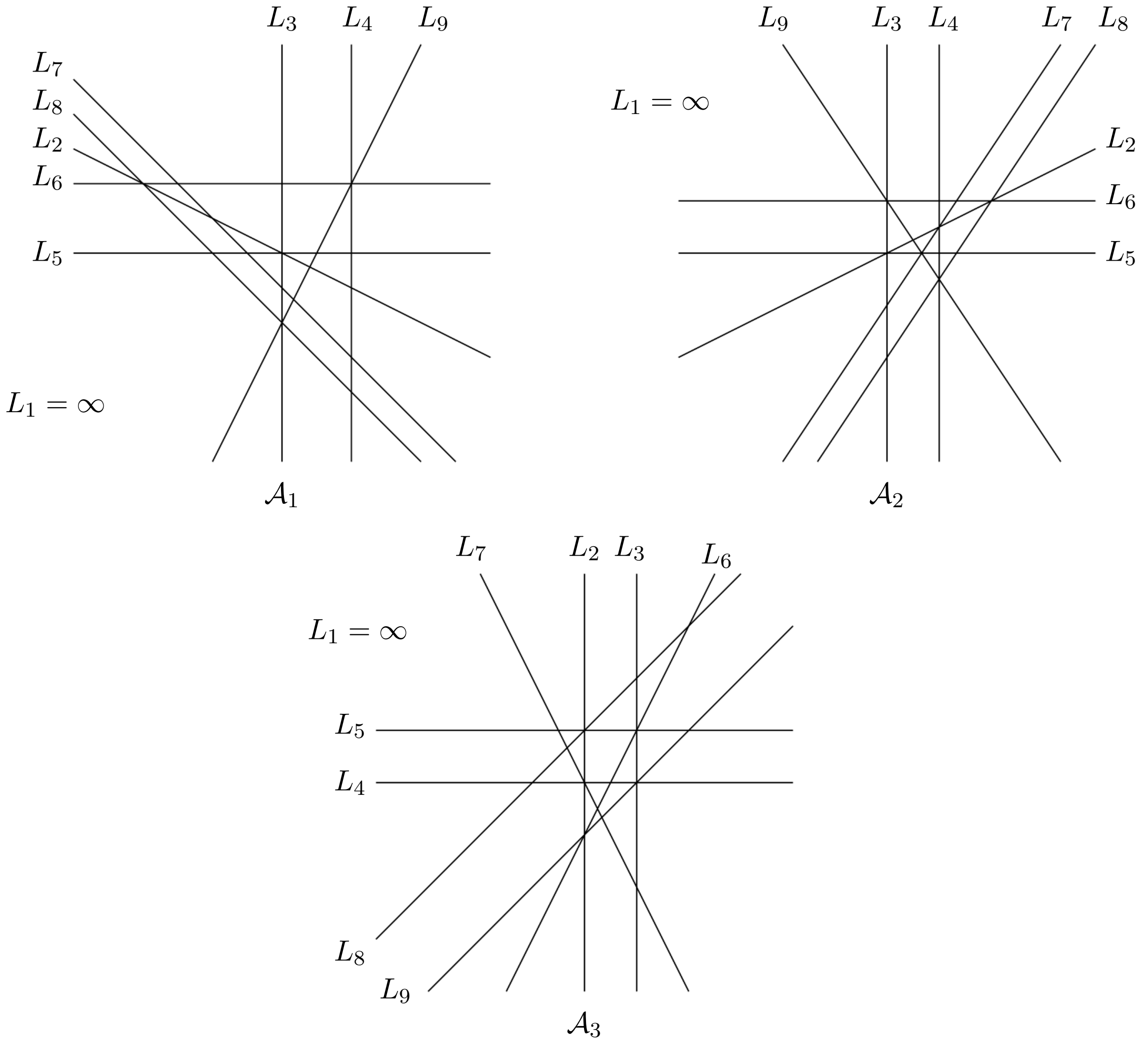

Figure 6.1: Real picture of the first three prime arrangements with nine lines

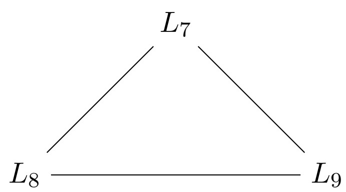

Figure 6.2: The graph $\hat{\Gamma}_{\mathcal{U}_\xi}$ for the fourth prime combinatorics with nine lines: C_4

The automorphism group Aut_Γ of the incidence graph is of order 12, and it is gen-

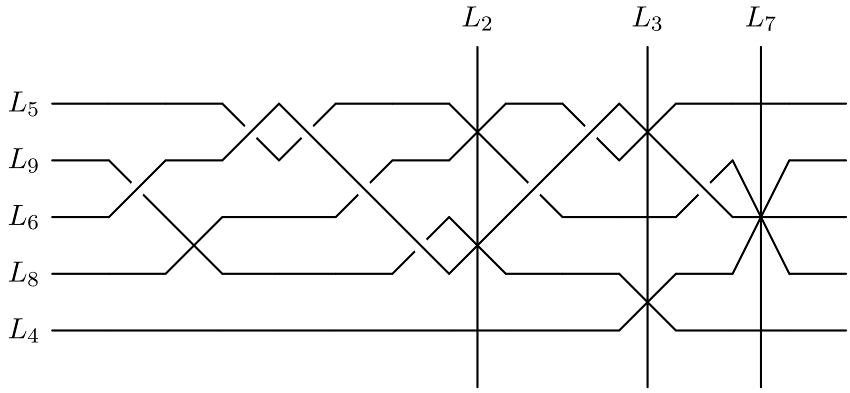

Figure 6.3: Braided wiring diagram of the positive realization $\mathcal{A}^+$ of C_4

erated by:

$$\{ \ (2,3)(4,5)(8,9), \ (1,2,3)(4,5,6), \ (1,5)(2,4)(3,6) \ \}.$$

The braided wiring diagram of $\mathcal{A}^+$ is pictured in Figure 6.3. We obtain the image of the inner-cyclic cycle $\gamma_\pm$ in the two cases (using Proposition 4.1.7or Proposition 5.3.16 for γ_-):

$$\xi(\gamma_+) = \xi(-\alpha_5 - \alpha_6 + \alpha_6) = \zeta, \qquad \xi(\gamma_-) = \xi(-\alpha_5 - \alpha_6) = \zeta^2 = \overline{\zeta}.$$

It is the first case with different values by the character of a same cycle in two different realizations. But they are conjugates, then they do not permit to distinguish the two realizations.

Theorem 6.3.2. *There is no ordered and oriented homeomorphism from* $(\mathbb{CP}^2, \mathcal{A}^+)$ *to* $(\mathbb{CP}^2, \mathcal{A}^-)$. *In other terms* $(\mathcal{A}^+, \mathcal{A}^-)$ *is an ordered and oriented* NC*-Zariski pair.*

Proof. By Theorem 5.3.14, $\xi(\gamma)$ is an invariant of the oriented and ordered topological type of an arrangement. Then the previous computation shows the result. □

Remark 6.3.3. We may apply the construction of G. Rybnikov [60] to this example to this pair of arrangements as future work.

c) Ten lines

With ten lines, there is no prime combinatorics admitting only three lines in $\mathcal{U}_\xi$, with ξ inner-cyclic. But, it is an example of two arrangements admitting four lines inner unramified and conjugate in $\mathbb{Q}[\sqrt{5}]$. Their combinatorics is:

$$C_5 = [[1,2,3],[1,4,7],[1,6,9],[2,4,8],[2,5,7],[2,9,10],[3,5,9],[3,6,8],[3,4,10],[4,5,6], \\ [7,8,9],[6,7,10],[1,5],[1,8],[1,10],[2,6],[3,7],[4,9],[5,8],[5,10],[8,10]].$$

And the equations realizing this combinatorics are:

$$
\begin{array}{ccc}
L_1 :\ z = 0 & L_2 :\ x = 0 & L_3 :\ x - z = 0 \\
L_4 :\ y = 0 & L_5 :\ (\alpha+1)x + y - z = 0 & L_6 :_x -y - \alpha z = 0 \\
L_7 :\ y - z = 0 & L_8 :\ x - (2+\alpha)y = 0 & L_9 :\ \alpha x - \alpha y - z = 0 \\
L_{10} :\ x - \alpha y - z = 0 & &
\end{array}
$$

with $\alpha = \frac{-1\pm\sqrt{5}}{2}$. This two realizations are denoted by $\mathcal{R}^+$ and $\mathcal{R}^-$. They are both pictured in Figure 6.5. The prime character of this arrangement is $(1,-1,-1,-1,1,-1,-1,1,-1,1)$, and the lines of the support are L_1, L_5, L_8 and L_{10}. The automorphism group of the incidence graph is of order 24 and it is generated by:

$$\{\ (2,3)(4,9)(5,8)(6,7),\ (2,4)(3,7)(5,10)(6,9),\ (1,5)(3,7)(4,9)(8,10)\ \}.$$

The graph $\hat{\Gamma}_{\mathcal{U}_\xi}$, is pictured in Figure 6.4. It admits a basis of cycles composed of γ_1, γ_2 and γ_3, where:

- γ_1 contains L_1, L_5 and L_{10},
- γ_2 contains L_1, L_5 and L_8,
- γ_3 contains L_1, L_8 and L_{10}.

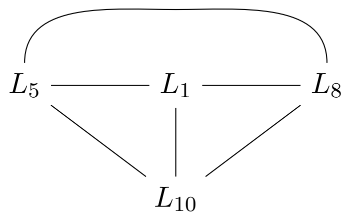

Figure 6.4: The graph $\hat{\Gamma}_{\mathcal{U}_\xi}$ for the combinatorics C_5

Using the real parts of $\mathcal{Q}^\pm$ pictured in Figure 6.5, we compute the lift $\gamma_i^+ \in E_{\mathcal{R}^+}$ and $\gamma_i^- \in E_{\mathcal{R}^-}$. Then we obtain their images by ξ:

$$
\begin{array}{ll}
\gamma_1^+ = \alpha_9 \text{ then } \xi(\gamma_1) = -1, & \gamma_1^- = \alpha_4 \text{ then } \xi(\gamma_1) = -1, \\
\gamma_2^+ = \alpha_4 \text{ then } \xi(\gamma_2) = -1, & \gamma_2^- = \alpha_9 \text{ then } \xi(\gamma_1) = -1, \\
\gamma_3^+ = \alpha_9 \text{ then } \xi(\gamma_3) = -1, & \gamma_3^- = \alpha_4 \text{ then } \xi(\gamma_1) = -1.
\end{array}
$$

Finally, we found that in both cases:

$$A_\xi = \begin{pmatrix} -2 & 1 & 1 & 1 \\ 1 & -2 & -1 & -1 \\ 1 & -1 & -2 & -1 \\ 1 & -1 & -1 & -2 \end{pmatrix}.$$

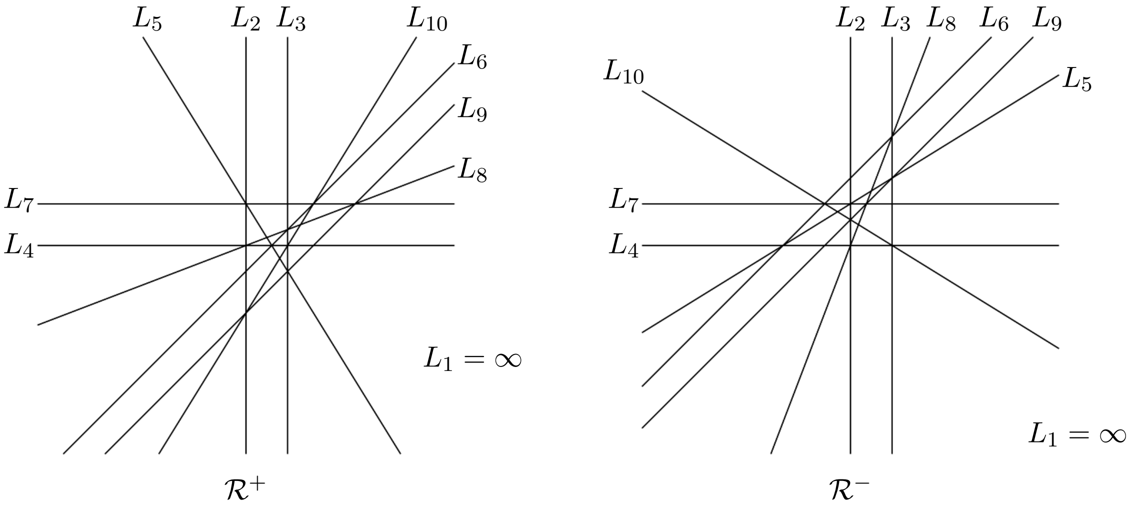

Figure 6.5: A prime pair of arrangements with ten lines

d) Eleven lines

In the list of prime combinatorics with eleven lines, only one of them (denoted by C) admits an inner-cyclic character of order 5. This combinatorics is:

$$\begin{aligned}C = [&[1,2],[1,3,5,7],[1,4,6,8],[1,9],[1,10,11],[2,3,6,9],[2,4,5,10],\\ &[2,7,11],[2,8],[3,4],[3,8,11],[3,10],[4,7],[4,9,11],[5,6],\\ &[5,8,9],[5,11],[6,7,10],[6,11],[7,8],[7,9],[8,10],[9,10]].\end{aligned}$$

It is realized by four arrangements defined by the following equations:

$$\begin{array}{cc}
L_1 : \ z = 0 & L_2 : \ x + y - z = 0 \\
L_3 : \ x = 0 & L_4 : \ y = 0 \\
L_5 : \ x - z = 0 & L_6 : \ y - z = 0 \\
L_7 : \ -\alpha^3 x + z = 0 & L_8 : \ y - \alpha z = 0 \\
L_9 : \ (\alpha - 1)x - y + z = 0 & L_{10} : \ -\alpha(\alpha - 1)x + y + \alpha(\alpha - 1)z = 0 \\
L_{11} : \ -\alpha(\alpha - 1)x + y - \alpha z = 0 &
\end{array}$$

with α a roots of the polynomial $X^4 - X^3 + X^2 - X + 1$, the cyclotomic polynomial of order 10. Approximately, the values of α are $-0.3 + 0.9i$, $-0.3 - 0.9i$, $0.8 + 0.6i$ and $0.8 - 0.6i$; the different realizations are denoted respectively by $\mathcal{A}^+$, $\mathcal{A}^-$, $\mathcal{B}^+$ and $\mathcal{B}^-$. This gives two pairs of conjugate arrangements. A complete study of this arrangement is done in Section 6.4.

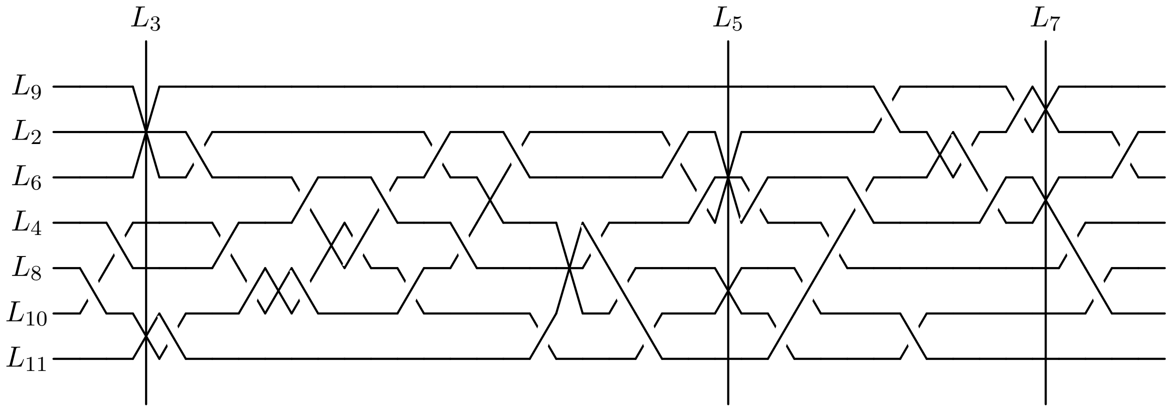

Figure 6.6: Braided wiring diagram of positive arrangement $\mathcal{A}^+$

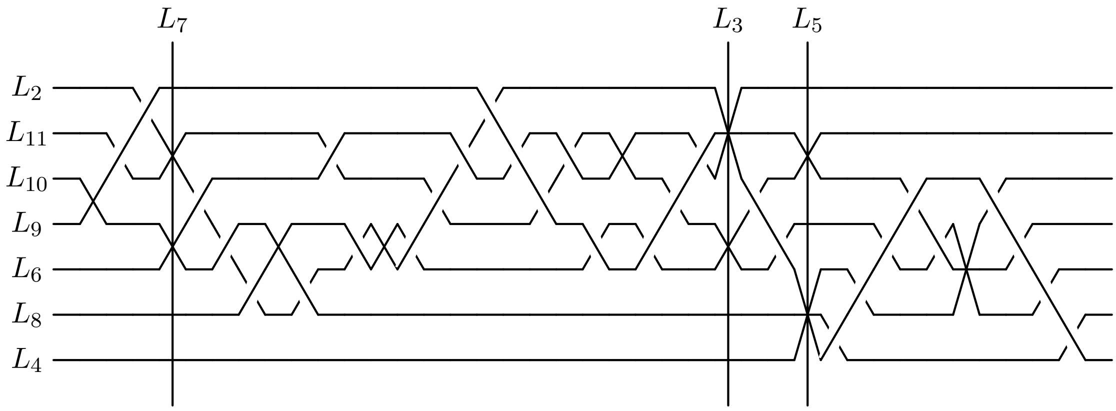

Figure 6.7: Braided wiring diagram of positive arrangement $\mathcal{B}^+$

6.4 NC-Zariski pairs

In this section, we study the example with eleven lines previously given to obtain a NC-Zariski pair.

6.4.1 Oriented NC-Zariski pair

The automorphism group of the combinatorics C is cyclic of order 4. It is generated by the permutation:

$$\sigma = (1, 3, 2, 4)(5, 6)(7, 9, 10, 8).$$

We compute the images of $\mathcal{A}^+$ by the linear maps induced by the power of σ then we obtain: $\sigma \cdot \mathcal{A}^+ = \mathcal{B}^+$, $\sigma^2 \cdot \mathcal{A}^+ = \mathcal{A}^-$ and $\sigma^3 \cdot \mathcal{A}^+ = \mathcal{B}^-$.

Property 6.4.1. *All the realizations of the combinatorics C admit the same ordered topological type.*

Proof. It is known that two conjugate arrangements have the same ordered topological type. Then Proposition 6.1.9 implies the result. □

Remark 6.4.2. All the realizations of C have isomorphic fundamental groups. In other terms:

$$\pi_1(E_{\mathcal{A}^\pm}) \simeq \pi_1(E_{\mathcal{B}^\pm}).$$

Note that conjugates arrangements may have distinct ordered and oriented topology, see Theorem 6.3.2.

Remark 6.4.3. In other terms, the moduli space $\mathcal{M}^{ord}(C)$ of C admits a unique connected component.

There is an inner-cyclic character ξ on the realizations: $(\zeta, \zeta^4, \zeta^3, \zeta^2, 1, 1, \zeta, \zeta^2, \zeta^3, \zeta^4, 1)$. Then the support of the inner-cyclic cycle is composed of the lines L_5, L_6 and L_{11}. Let us express the inner-cyclic cycles in terms of meridians. Using the braided wiring diagram pictured in Figure 6.6, we compute the image of $\gamma_{\mathcal{A}^+}$ by the unknotting map:

$$\widetilde{\delta}(\gamma_{\mathcal{A}^+}) = \gamma_{\mathcal{A}^+}.$$

We also deduce from the wiring diagram that:

$$i_* \circ \widetilde{\delta}(\gamma_{\mathcal{A}^+}) = i_*(\gamma_{\mathcal{A}^+}) = -\alpha_2.$$

Finally, $i_*(\gamma_{\mathcal{A}^+}) = -\alpha_2$. Since $\gamma_{\mathcal{A}^+} \simeq \gamma_{\mathcal{A}^-}$, then $\widetilde{\delta}(\gamma_{\mathcal{A}^-}) = \gamma_{\mathcal{A}^-}$. Using Proposition 4.1.7, we obtain from the wiring diagram of $\mathcal{A}^+$, see Figure 6.6, that:

$$i_* \circ \widetilde{\delta}(\gamma_{\mathcal{A}^-}) = i_*(\gamma_{\mathcal{A}^-}) = -\alpha_{10} - \alpha_4 - \alpha_8 - \alpha_9.$$

We compute the image of $\gamma_{\mathcal{B}^+}$ and $\gamma_{\mathcal{B}^-}$ from the braided wiring diagram pictured in Figure 6.7. Then we obtain that:

$$\widetilde{\delta}(\gamma_{\mathcal{B}^+}) = \gamma_{\mathcal{B}^+} + \alpha_2 + \alpha_9, \quad \text{and,} \quad \widetilde{\delta}(\gamma_{\mathcal{B}^-}) = \gamma_{\mathcal{B}^-} + \alpha_2 + \alpha_9.$$

We also deduce from the braided wiring diagram of $\mathcal{B}^+$, pictured in Figure 6.7, that $\widetilde{\delta}(\gamma_{\mathcal{B}^+})$ retracts on α_2, and that $\widetilde{\delta}(\gamma_{\mathcal{B}^-})$ is contractible.

To summarize, in their respective complements, we have:

$$\gamma_{\mathcal{A}^+} = -\alpha_2, \quad \gamma_{\mathcal{A}^-} = -\alpha_{10} - \alpha_4 - \alpha_8 - \alpha_9, \quad \gamma_{\mathcal{B}^+} = -(\alpha_2 + \alpha_9) + \alpha_2, \quad \gamma_{\mathcal{B}^-} = -(\alpha_2 + \alpha_9).$$

Then we obtain the image of the cycles by ξ:

$$\xi(\gamma_{\mathcal{A}^+}) = \zeta \quad \xi(\gamma_{\mathcal{A}^-}) = \zeta^4 \quad \xi(\gamma_{\mathcal{B}^+}) = \zeta^2 \quad \xi(\gamma_{\mathcal{B}^-}) = \zeta^3$$

Thus the matrix of the intersection form is:

$$A_\xi = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & \xi(\gamma) \\ 1 & \xi(\gamma)^{-1} & -3 \end{pmatrix}$$

Then for all the realizations $\overline{\operatorname{depth}}(\xi) = 0$.

Remark 6.4.4. This result can be obtained without computing the images of the inner-cyclic cycles, but using Proposition 6.1.1.

Furthermore, this computation permits to obtain the two following results:

Theorem 6.4.5. *There is no oriented and ordered homeomorphism between the couples* $(X^+_{\mathcal{A}}, \mathcal{A}^+)$, $(X^-_{\mathcal{A}}, \mathcal{A}^-)$, $(X^+_{\mathcal{B}}, \mathcal{B}^+)$ *and* $(X^-_{\mathcal{B}}, \mathcal{B}^-)$. *In other terms,* $(\mathcal{A}^+, \mathcal{A}^-, \mathcal{B}^+, \mathcal{B}^-)$ *form an oriented and ordered* NC*-Zariski 4-tuplet.*

Proof. It is a direct consequence of Theorem 5.3.14. □

Theorem 6.4.6. *Let* $\mathcal{A}$ *be a complex line arrangement, then the map:*

$$i_* : \mathrm{H}_1(B_{\mathcal{A}}; \mathbb{Z}) \to \mathrm{H}_1(E_{\mathcal{A}}; \mathbb{Z}),$$

induced by the inclusion of $B_{\mathcal{A}}$ *in* $E_{\mathcal{A}}$ *is not of combinatorial nature.*

Proof. If i_* is of combinatorial nature then for all realizations $\mathcal{A}_1$ and $\mathcal{A}_2$ of a same combinatorics then $B_{\mathcal{A}_1} = B_{\mathcal{A}_2} = B$ and the following diagram is commutative:

$$\begin{tikzcd} \mathrm{H}_1(B) \arrow[r, "i^1_*"] \arrow[d, "i^2_*"] & \mathrm{H}_1(E_{\mathcal{A}_1}) \arrow[d, "\xi"] \\ \mathrm{H}_1(E_{\mathcal{A}_2}) \arrow[r, "\xi"] & \mathbb{C}^* \end{tikzcd}$$

But the previous example is a counter-example, then i_* is not of combinatorial nature. □

Corollary 6.4.7. *Let* $\mathcal{A}$ *be an arrangement admitting an inner-cyclic character* ξ *with cycle* γ. *The topological invariant* $\xi(\gamma)$ *is not determined by the combinatorics of* $\mathcal{A}$.

6.4.2 NC-Zariski pair

Theorem 6.4.8. *The arrangements* $\mathcal{A}^+$ *and* $\mathcal{B}^+$ *form an ordered* NC*-Zariski pair.*

Proof. We decompose the proof in 3 steps.

STEP 1: There is no homeomorphism between $(X^+_{\mathcal{A}}, \mathcal{A}^+)$ and $(X^+_{\mathcal{B}}, \mathcal{B}^+)$ preserving the ordering, the orientations on the plane, and reversing the orientations on the lines.

Proof. Composing such a homeomorphism with the complex conjugation (which preserves the orientation of the plane, but reverses the orientations on the lines) one obtains a homeomorphism that is ruled out by Theorem 6.4.5. □

STEP 2: There is no homeomorphism between $(X^+_{\mathcal{A}}, \mathcal{A}^+)$ and $(X^+_{\mathcal{B}}, \mathcal{B}^+)$ preserving the ordering, the orientation on the plane, and the orientation on at least one line.

Proof. By Step 1 there should be at least one component L_1^+ whose orientation is reversed and, by hypothesis, at least one component L_2^+ whose orientation is preserved. Assume that both components intersect at one point. Let us consider their intersection: $L_1^+ \cdot L_2^+ = 1$. Let L_i^- the image of L_i^+ by the homeomorphism.

By hypothesis, the homeomorphism reverse the orientation of L_2^+ and not that of L_1^+, then $L_1^- \cdot L_2^- = -1$. But the homeomorphism does not reverse the orientation of the plane, then $L_1^- \cdot L_2^- = 1$. Then the orientation of L_2^+ cannot be inversed by the homeomorphism. The result is then obtained by connectivity of $\hat{\mathcal{A}}^+$. □

Step 3: There is no homeomorphism between $(X_{\mathcal{A}}^+, \mathcal{A}^+)$ and $(X_{\mathcal{B}}^+, \mathcal{B}^+)$ preserving the ordering and reversing the orientation on the plane.

Proof. Assume that there is a homeomorphism between $X_{\mathcal{A}}^+$ and $X_{\mathcal{B}}^+$ reversing the orientation. It induces a homeomorphism from $\mathbb{CP}^2$ in $\mathbb{CP}^2$ also reversing the orientation. Since signature of the intersection form is non null, then there is no homeomorphism from $\mathbb{CP}^2$ to $\mathbb{CP}^2$ reversing the orientation of the plane. □

This proves the result. □

To remove the hypothesis ordered, let us consider the same combinatorics with an additional line L_{12} passing through the point $L_1 \cap L_3 \cap L_5 \cap L_7$. We obtain as combinatorics:

$$
\begin{aligned}
\mathscr{C} = [\ &[1,2],[1,3,5,7,12],[1,4,6,8],[1,9],[1,10,11],[2,3,6,9],\\
&[2,4,5,10],[2,7,11],[2,8],[2,12],[3,4],[3,8,11],[3,10],[4,7],\\
&[4,9,11],[4,12],[5,6],[5,8,9],[5,11],[6,7,10],[6,11],[6,12],\\
&[7,8],[7,9],[8,10],[8,12],[9,10],[9,12],[10,12],[11,12]\ ].
\end{aligned}
$$

By construction it is an inner-cyclic combinatorics, for the character:

$$(\zeta, \zeta^4, \zeta^3, \zeta^2, 1, 1, \zeta, \zeta^2, \zeta^3, \zeta^4, 1, 1).$$

The support of the intersting cycle γ is L_5, L_6 and L_{11}. It admits four realizations denoted by $\mathscr{A}^+$, $\mathscr{A}^-$, $\mathscr{B}^+$ and $\mathscr{B}^-$ (in accordance with the example with eleven lines).

Proposition 6.4.9. *The automorphism group of the combinatorics $\mathscr{C}$ is trivial.*

Proof. By definition of L_{12} all automorphisms fix it. Then an automorphism of $\mathscr{C}$ fixes the unique point of multiplicity 5. But in the example with eleven lines, the action of the automorphism group permutes cyclically the points of multiplicity 4. Since one of them was transformed in the point of multiplicity 5, then the automorphism group of $\mathscr{C}$ is trivial. □

Theorem 6.4.10. *The arrangements $\mathscr{A}^+$ and $\mathscr{B}^+$ form a* NC*-Zariski pair.*

Proof. Since the line L_{12} is sent on 1 by ξ then it does not change the values of the inner-cyclic cycle. By Theorem 6.4.8, $\mathscr{A}^+$ and $\mathscr{B}^+$ form an ordered NC-Zariski pair. And Proposition 6.4.9 implies that there is no homeomorphism of $X_{\mathscr{A}}^+$ in $X_{\mathscr{B}}^+$ sending $\mathscr{A}^+$ on $\mathscr{B}^+$ and preserving the ordered. □

Proposition 6.4.11. *By analogue construction with $\mathcal{A}^-$ and $\mathcal{B}^-$ to obtain $\mathscr{A}^-$ and $\mathscr{B}^-$, the four pairs $(\mathscr{A}^\pm, \mathscr{B}^\pm)$ are* NC*-Zariski pairs.*

Lemma 6.4.12. *The pairs $(\mathscr{A}^+, \mathscr{A}^-)$ and $(\mathscr{B}^+, \mathscr{B}^-)$ are oriented* NC*-Zariski pairs.*

Proof. By Theorem 5.3.14, there is no oriented and ordered homeomorphism between the pairs $(X_{\mathscr{A}^+}, \mathscr{A}^+)$ and $(X_{\mathscr{A}^-}, \mathscr{A}^-)$. But by construction there is no automorphism of the combinatorics $\mathscr{C}$. Then there is no oriented homeomorphism from $(X_{\mathscr{A}^+}, \mathscr{A}^+)$ to $(X_{\mathscr{A}^-}, \mathscr{A}^-)$. □

Theorem 6.4.13. *The arrangements $\mathscr{A}^+$, $\mathscr{A}^-$, $\mathscr{B}^+$ and $\mathscr{B}^-$ form an oriented* NC*-Zariski 4-tuple.*

Proof. It is a consequence of Proposition 6.4.11 and Lemma 6.4.12. □

Conclusion and future works

And after...

During all this thesis, we have studied how the topology and the combinatorics of complex line arrangements are related. Even if it seems out of reach to describe explicitly the relationship between this two facets of line arrangements, we have obtained some results in this direction. The description of the map induced by the inclusion of the boundary manifold in the complement of an arrangement on the Poincaré's groups and on the first homology groups has been a first important step in the comprehension of this link. The study of the characteristic varieties, with the method due to E. Artal, was the second important step. Indeed it allows to develop a geometrical method to compute the quasi-projective part of the depth of any character. It also puts us on the right track to define the new invariant and permits us to discover NC-Zariski pairs.

A possible future work following this thesis, will be to carry on with the study of this relationship, and to focus on the non quasi-projective part of the depth. Another way, is to extend a part of the results obtained here to the case of algebraic plane curves, some reflexion has already started.

As mentioned in Chapter 2 and 6, some ameliorations are possible on the programs developed during this thesis. Optimizing the computation of the braided wiring diagram, developing a function to picture it, another to extract from it the braid monodromy, developing a better algorithm to detect all the prime combinatorics, completing the `Maple` program to compute the space realization of a combinatorics, this is as much work to do after this thesis.

Directions of the future works

The difference between a NC-Zariski pair and a usual pair of Zariski is very thin, then we may think that the arrangements $\mathscr{A}^+$ and $\mathscr{B}^+$ form a Zariski pair.

Some of the results presented in the last chapter give a hope to find an example of a pair of arrangements distinguished by different characteristic varieties. The present work and all the tested arrangements tend towards the fact that if $\mathcal{A}$ is a complexified real arrangement then the quasi-projective part of its characteristic varieties are of combinatorial nature.

Indeed, in all the examples of pairs of complexified real arrangements the image of the inner-cyclic cycle is always the same for each arrangement of the pair. In contrast to the complex line arrangements where we have found conjugate images (see the example with nine lines), or completely different (see example with eleven lines). Furthermore, each time we have constructed a pair of complexified real arrangements, it always appeared that to be inner-cyclic, the image of the cycle by the character in both cases needs to be the same.

For the case of complex line arrangements, no rule emerges to know if the characteristic varieties are of combinatorial nature. The examples previously presented give prospect that they are not combinatorial. But Proposition 6.1.1, gives very binding conditions on the combinatorics of such an example.

Conclusion et travaux futurs

Et après...

Durant cette thèse, nous avons étudié comment la topologie et la combinatoire d'un arrangement complexe sont reliées. Même s'il semble hors de portée de décrire explicitement la relation entre ces deux facettes, nous avons obtenu des résultats dans cette direction. La description de l'application induite par inclusion de la variété bord d'un arrangement dans son complémentaire sur les groupes de Poincaré et sur les premiers groupes d'homologies fut une étape importante dans la compréhension de ce lien. L'étude des variétés caractéristiques, avec la méthode due à E. Artal, a été la seconde étape importante. En effet, elle nous a permis de développer une méthode géométrique pour calculer la profondeur quasi-projective d'un caractère. Cela nous a aussi mis sur la bonne voie pour définir le nouvel invariant topologique et nous a donc permis de découvrir une NC-paire de Zariski.

Etudier cette relation, et s'intéresser à la partie projective de la profondeur pourraient être des travaux possibles pour la suite de cette thèse. Une autre extension possible serait de traduire nos résultats dans le cas encore plus général des courbes algébriques planes. Des premières réflexions ont déjà été faites à ce sujet.

Comme nous l'avons mentionné dans les Chapitres 2 et 6, des améliorations sont possibles sur les programmes développés durant cette thèse. Optimiser le calcul du diagramme de câblage, développer une fonction pour les dessiner directement, une autre pour extraire la monodromie de tresse, développer un meilleur algorithme pour détecter les combinatoires premières, compléter le programme sous `Maple` pour calculer l'espace des réalisations d'une combinatoire, voilà autant de travaux possibles à la suite de cette thèse.

Orientations des futurs travaux

La différence entre une NC-paire de Zariski et une paire de Zariski classique est très faible, il semblerait donc naturel de penser que $\mathscr{A}^+$ et $\mathscr{B}^+$ forment une paire de Zariski.

Certains des résultats présentés dans le dernier chapitre donnent un espoir de trouver un exemple de paire d'arrangements distingués par des variétés caractéristiques différentes. Les travaux développés dans cette thèse, ainsi que les nombreux exemples que nous avons testés tendent à dire que si $\mathcal{A}$ est un arrangement réel complexifié, alors la partie quasi-projective de ses variétés caractéristiques est déterminée par la combinatoire.

En effet, dans tous les exemples d'arrangements réels complexifiés, l'image des cycles inner-cyclic est toujours la même pour chaque arrangement de la paire. Ceci contraste avec le cas d'arrangement complexe (non réel complexifié) où l'on trouve des images complexes conjuguées, cf exemple à neuf droites, ou même complètement différentes, cf exemple à onze droites. De plus, à chaque tentative de construction d'un exemple de paire d'arrangements réels complexifiés, pour obtenir une combinatoire inner-cyclic, l'image des méridians par le caractère devenait toujours la même.

Pour le cas des arrangements complexes en général, aucune règle ne semble se dessiner sur la nature combinatoire des variétés caractéristiques (ni même sur leurs parties quasi-projectives). Les exemples donnés dans le dernier chapitre encouragent à croire en l'existence d'une paire distinguée par un caractère dont les profondeurs seraient différentes. En revanche, la Proposition 6.1.1 impose des conditions fortes sur la combinatoire d'un tel exemple.

Conclusiones y trabajo futuro

Y después ...

Durante la realización de esta tesis, hemos estudiado la forma en la que están relacionadas la topología y la combinatoria en las configuraciones de rectas complejas. Aunque parece completamente inalcanzable llegar a describir explícitamente la relación entre estas dos facetas de las configuraciones de rectas, hemos obtenido algunos resultados en esta dirección. La descripción de la aplicación inducida por la inclusión de la variedad borde en el complementario de una configuración sobre los grupos de Poincaré y los primeros grupos de homología ha sido un primer paso importante en la comprensión de esta relación. El estudio de variedades características con el método dado por E. Artal fue el segundo paso fundamental. De hecho, esto nos permitió desarrollar un método geométrico para calcular la parte cuasi-proyectiva de la profundidad de un carácter cualquiera y nos puso en el buen camino para definir el nuevo invariante así como para descubrir nuevos NC–pares de Zariski.

Un posible trabajo futuro a esta tesis pasa por la profundización del estudio de esta relación así como en el estudio de la parte no cuasi-proyectiva de la profundidad de un carácter. Otro posible camino es el de extender los resultados aquí obtenidos al caso de las curvas algebraicas planas, caso sobre el cual ya hemos comenzado a reflexionar.

Como se ha mencionado en los capítulos 2 y 6, existes múltiples mejoras a realizar en los programas de cálculo desarrollados durante esta tesis: optimizar el cálculo de los diagramas de cableado, programar una función para dibujarlo o para extraerlo directamente de la monodromía de trenzas, desarrollar un algoritmo más eficaz para detectar combinatorias primas o completar el programa en `Maple` para el cálculo del espacio de realizaciones de las combinatorias.

Orientaciones para un trabajo futuro

La diferencia entre un NC–par de Zariski y un par de Zariski usual es muy fina por lo que podemos pensar que las configuraciones $\mathscr{A}_+$ y $\mathscr{B}_+$ encontradas pueden formar muy posiblemente un par de Zariski.

Algunos de los resultados presentados en el último capítulo de la tesis nos dan la esperanza de encontrar un ejemplo de un par de configuraciones distinguidas por diferentes variedades características. El presente trabajo y todas las configuraciones estudiadas nos inclinan a pensar el hecho de que si $\mathcal{A}$ es una configuración real complexificada entonces la parte cuasi-proyectiva de su variedad característica tiene naturaleza combinatoria.

De hecho, en todos los ejemplos de pares de configuraciones reales complexificadas la imagen del ciclo *inner-cyclic* es siempre el mismo para toda configuración del par, en contraste con las configuraciones de rectas complejas, donde hemos encontrado o bien imágenes conjugadas (véase el ejemplo de nueve rectas) o bien imágenes completamente diferentes (véase el ejemplo de nueve rectas). Es mas, cada vez que hemos construido un par de configuraciones reales complexificadas, siempre ha sucedido que para que la imagen de un ciclo por el carácter fuera *inner-cyclic*, en ambos casos debía de ser la misma.

Para el caso de configuraciones de rectas complejas, no aparece ninguna regla que nos permita saber si las variedades características son de naturaleza combinatoria. Los ejemplos previamente presentados nos hacen pensar que no son combinatorias. Sin embargo, la Proposición 6.1.1 nos da una condiciones muy específicas sobre las combinatorias que deben poseer tales ejemplos.

Appendix A

Code

A.1 Wiring diagram

This program is coded in `Sage` v.5.8.

```
reset()
```

```
# definition of the arrangement A from the equations.
var('x,y')
var('a')
```

```
# Copy a vector in another
def copy(G):
    g=[]
    for i in range(0,len(G)):
        g.append(G[i])
    return g
```

```
# singularities computation
def sing(E):
    S=[]
    for i in range(0,len(E)-1):
        for j in range(i+1,len(E)):
            s=solve([E[i],E[j]],x,y)
            if s<>[]:
                if not(s[0] in S):
                    S.append([x==(s[0][0].rhs()),y==(s[0][1].rhs())])
    return S
```

```
# projection on the axes
```

```
def Xprojection(S):
    X=[]
    for i in range(0,len(S)):
        X.append(S[i][0].rhs())
    return X

# projection on real and imaginary part
def Rprojection(S):
    R=[]
    for s in S:
        R.append(real(s))
    return R

def Iprojection(S):
    I=[]
    for s in S:
        I.append(imag(s))
    return I

# ordering of the projection along x
def sort(R,L):
    S1=[]
    S2=[]
    r=copy(R)
    l=copy(L)
    for i in range(0,len(R)):
        min=0
        for j in range(0,len(r)):
            if r[j]<r[min]:
                min=j
        S1.append(r[min])
        S2.append(l[min])
        del r[min]
        del l[min]
    return S1,S2

def order(L):
    e=0.000001
    R=Rprojection(L)
    I=Iprojection(L)
    (I,R)=sort(I,R)      #ordering the y
    (R,I)=sort(R,I)      #ordering the x
    m=[]
    n=[]
    for i in range(0,len(R)-1):
        if abs(R[i]-R[i+1])>e or abs(I[i]-I[i+1])>e:
            m.append(R[i])
            n.append(I[i])
    m.append(R[len(R)-1])
```

```
    n.append(I[len(I)-1])
    return m,n
```

```
# computing of the cloud for the linear interpolation

# founding of the minimum delta between 2 xx
def MIN(xx):
    e=1
    for i in range(0,len(xx)-1):
        if 0<abs(xx[i]-xx[i+1])<e :
            e=abs(xx[i]-xx[i+1])
    return e/3

# creation of the cloud of points
def cloud(xx,xy):
    e=MIN(xx)
    XX=[]
    XY=[]
    for i in range(0,len(xx)):
        XX.append(xx[i]-e)
        XX.append(xx[i]+e)
        XY.append(xy[i])
        XY.append(xy[i])
    return XX,XY
```

```
# linear interpolation of a cloud
def linear_interp(XX,XY,draw=false):
    XX1=[]
    XY1=[]
    XX1.append(XX[0]-10)
    XY1.append(XY[0])
    for i in range(0,len(XX)):
        XX1.append(XX[i])
        XY1.append(XY[i])
    XX1.append(XX[len(XX)-1]+10)
    XY1.append(XY[len(XX)-1])
    C=[]
    p=[]
    for i in range(0,len(XX1)-1):
        f(t)=XY1[i]+(t-XX1[i])*(XY1[i+1]-XY1[i])/(XX1[i+1]-XX1[i])
        if XX1[i]-XX1[i+1]<0:
            C.append([f(t),XX1[i],XX1[i+1],1])
            p.append(plot(f,(XX1[i],XX1[i+1])))
        else :
            C.append([f(t),XX1[i+1],XX1[i],-1])
            p.append(plot(f,XX1[i+1],(XX1[i])))
    if draw:
        P=p[0]
        for i in range(1,len(p)):
```

```
            P=P+p[i]
        show(P)
    return C
```

```
# Computation of the lifts of C
def lift(A,C,draw=false):
    T=[]
    W=[]
    for j in range(0,len(C)):

        # lifting of the paths
        T.append([])
        for k in range(0,len(A)):
            Ak=A[k].lhs()
            Ak=Ak.subs(x=t+I*C[j][0])
            Sol=solve(Ak,y)
            if not(Sol==[]):
                T[j].append(Sol[0])

        # projection of the lift on the real part
        W.append([])
        for i in range(len(T[j])):
            W[j].append(real(T[j][i].rhs()))

        # displaying the Wiring diagram
        if draw :
            var('d')
            d=plot([x],[1,-1])
            for i in range(len(W[j])):
                s(t)=W[j][i]
                if j==0:
                    d=d+plot([s(t)],[C[j][1],C[j][2]],
                    rgbcolor=hue(i*0.1+0.1), legend_label=str(i))
                else:
                    d=d+plot([s(t)],[C[j][1],C[j][2]],
                    rgbcolor=hue(i*0.1+0.1))
            if j==0:
                D=d
            else :
                D=D+d
    if draw :
        show(D, axes=false)
    return (T,W)
```

```
# Founding the order of the lines at the beginning of the phase m
def position(W,m,C):
    P=[]
    for k in range(len(W)):
        if C[m][3]==1:
```

```
            P.append(W[k].subs(t=C[m][1]))
        else:
            P.append(W[k].subs(t=C[m][2]))
    L=range(len(P))
    (P,L)=sort(P,L)
    return L
```

```
# Founding the cross type
def found_cross(m,A,T,W,C,e):
    wd=[]
    for i in range(0,len(W[m])-1):
        for j in range(i+1,len(W[m])):
            s1(t)=W[m][i]
            s2(t)=W[m][j]
            SOL=solve([s1==s2],t)
            for k in SOL :
                K=(k.rhs())
                if (C[m][1]<=K) and (K<=C[m][2]):
                    r1(t)=T[m][i].rhs()
                    r2(t)=T[m][j].rhs()
                    h1=imag(r1(K))
                    h2=imag(r2(K))
                    if h1>h2+e :
                        wd.append([K.n(20),(i,-1),(j,1)])
                    else :
                        if h1<h2-e :
                            wd.append([K.n(20),(i,1),(j,-1)])
                        else :
                            wd.append([K.n(20),(i,0),(j,0)])
    return wd
```

```
# Founding the explicit Wiring diagram
def Wiring(A,T,W,C,draw):
    e=0.00001
    WD=[]

    # Founding the cross type
    for m in range(0,len(C)):
        wd=found_cross(m,A,T,W,C,e)

        # ordering the cross
        WD.append([])
        if C[m][3]==1:
            for k in range(0,len(wd)):
                mini=wd[0][0]
                l=0
                for i in range(1,len(wd)):
                    if wd[i][0]<=mini :
                        l=i
```

```
                    mini=wd[i][0]
                WD[m].append(wd[l])
                del wd[l]
        else:
            for k in range(0,len(wd)):
                maxi=wd[0][0]
                l=0
                for i in range(0,len(wd)):
                    if wd[i][0]>=maxi :
                        l=i
                        maxi=wd[i][0]
                WD[m].append(wd[l])
                del wd[l]

    # concatenation of the information
    diagram=[]
    for i in range(0,len(WD)):
        diagram=diagram+WD[i]

    return diagram
```

```
# Printing with a good presentation the result
def display(W):
    for i in range(0,len(W)):
        print(W[i])
```

```
# Global function giving the Wiring diagram
# with entering the equations of A
def Wiring_diagram(A,draw=false):
    S=sing(A)
    L=Xprojection(S)
    (xx,xy)=order(L)
    (XX,XY)=cloud(xx,xy)
    var('t')
    C=linear_interp(XX,XY,draw)
    P=position(A,0,C)
    (T,W)=lift(A,C,draw)
    diagram=Wiring(A,T,W,C,draw)
    #diagram=convert(diagram)
    if draw:
        print(P)
        display(diagram)
    return (diagram,P)

# Warning, enter the equations under the form f(x,y)=0 ! #
%# Attention à mettre les équations sous la forme f(x,y)=0 !! #
```

A.2 Inner-cyclic combinatorics

This program is coded in `Sage` v.5.8.

```
reset()
```

```
def our_cycle_basis(infini,n):
    R=[]
    for i in range(infini+1,n):
        for j in range(i+1,n+1):
            R.append([j,i])
    return R
```

```
def corner_intersection(C,E,infini):
    corner=[]
    e=set([E[0],E[1],infini])
    for i in range(len(C)):
        if len((set(C[i])).intersection(e))>1:
            corner.append(i)
    return corner
```

```
def cycle_matrix_color(C,E,infini,n):
    l=len(C)
    c=0
    corner=corner_intersection(C,E,infini)
    p=0
    for i in corner:
        p=p+len(C[i])-
                len((set(C[i])).intersection(set([E[0],E[1],infini])))
    Z=matrix(ZZ,1,n,sparse=true)
    M=Z.copy()
    for i in range(l):
        if ((E[0] in C[i]) or (E[1] in C[i]) or (infini in C[i])):
            M=M.stack(Z)
            for j in range(0,len(C[i])):
                M[c,C[i][j]-1]=1
            c=c+1
    for i in corner:
        for j in (set(C[i])).difference(set([E[0],E[1],infini])):
            M=M.stack(Z)
            M[c,j-1]=1
            c=c+1
    M[c,E[0]-1]=1
    M=M.stack(Z)
    M[c+1,E[1]-1]=1
    M=M.stack(Z)
    M[c+2,infini-1]=1
    return M
```

```
def is_a_cycle(C,E):
    R=true
    for i in range(len(C)):
        if (E[0] in C[i]) and (E[1] in C[i]) and (E[2] in C[i]):
            R=false
    return R
```

```
def inner-cyclic_cycle(C,E,infini,n):
    R=[]
    M=cycle_matrix_color(C,E,infini,n)
    for p in primes(50) :
            MM=M.change_ring(ZZ.quo(p*ZZ))
            B=(MM.right_kernel()).basis()
            if len(B)>0:
                R.append([p,B])
    if len(R)>0:
        return R
    else:
        return false
```

```
def is_a_prime(B):
    for b in B:
        c=0
        for k in b:
            if k==0:
                c=c+1
        if c>3:
            return false
        else:
            return true
```

```
def inner-cyclic_cycle(C,E,infini,n,prime):
    R=[]
    M=cycle_matrix_color(C,E,infini,n)
    for p in primes(10) :
            MM=M.change_ring(ZZ.quo(p*ZZ))
            B=(MM.right_kernel()).basis()
            if prime:
                test=is_a_prime(B)
            else:
                test=true
            if (len(B)>0) and test:
                R.append([p,B])
    if len(R)>0:
        return R
    else:
        return false
```

```
def inner-cyclic_combinatorics(C,prime=false):
    R=[]
    n=max(flatten(C))
    for infini in range(1,n-1):
        BE=our_cycle_basis(infini,n)
        for E in BE:
            e=copy(E)
            e.append(infini)
            if (infini<E[0]) and (infini<E[1]) and (is_a_cycle(C,e)) :
                IE=inner-cyclic_cycle(C,E,infini,n,prime)
                if not(IE==false):
                    e=[[infini,E[0],E[1]]]
                    e.append(IE)
                    R.append(e)
    if len(R)>0:
        return R
    else:
        return false
```

```
# Check for an arrangement with 4 "generic" lines
def check(comb,l1,l2,l3,l4):
    n=max(flatten(comb))
    R=PolynomialRing(QQ,3*n,'x')
    M=matrix(n,3,R.gens())
    relations=[R.gen(3*l1-3)-1,R.gen(3*l1-2),R.gen(3*l1-1),
        R.gen(3*l2-3),R.gen(3*l2-2)-1,R.gen(3*l2-1),R.gen(3*l3-3),
        R.gen(3*l3-2),R.gen(3*l3-1)-1,R.gen(3*l4-3)-1,
        R.gen(3*l4-2)-1,R.gen(3*l4-1)-1]
    M=M.apply_map(lambda a:a.reduce(relations))
    for i in comb:
        if len(i)>2:
            N=M.matrix_from_rows([j-1 for j in i])
            relations+=N.minors(3)
    I=R.ideal(relations)
    J=I.radical()
    for i in Set(range(n)).subsets(2):
        N=M.matrix_from_rows(list(i))
        Min=N.minors(2)
        if Min[0] in J and Min[1] in J and Min[2] in J:
            return False
    cond=R(1)
    for i in Set(range(n)).subsets(3):
        comprobar=True
        for k in comb:
            if i.issubset(Set([s-1 for s in k])):
                comprobar=False
        if comprobar:
            N=M.matrix_from_rows(list(i))
```

```
            cond=(cond*N.determinant()).reduce(J)
    return not cond in J
```

```
# Research of 4 "generic"' lines
def generic_lines(C):
    G=[]
    c=[]
    n=max(flatten(C))
    for i in range(len(C)):
        c.append(Set(C[i]))
    G.append(C[0][0])
    G.append(C[0][1])
    G=Set(G)
    test=true
    i=1
    while test :
        if (G.intersection(c[i])).cardinality()>0 :
            test=false
            A=(c[i].difference(G))[0]
            G=G.union(Set([A]))
        else :
            i=i+1
        if i>len(C) :
            test=false
    j=0
    test=false
    L=Set(range(1,n+1))
    L=L.difference(G)
    while (j<(L.cardinality()) and not(test)) :
        k=true
        for i in range(len(C)):
            if (G.union(Set([L[j]])).intersection(c[i])).cardinality()>2 :
                k=false
        if k :
            test=true
            G=G.union(Set([L[j]]))
        j=j+1
    G=G.list()
    if len(G)<4:
        return false
    else:
        return G
```

```
# Complete function
@fork(timeout=500,verbose=True)
def is_actual(C):
    finish=false
    G=generic_lines(C)
    if G==false:
```

```
        R='bad'
    else:
        R=check(C,G[0],G[1],G[2],G[3])
    finish=true
    return [R,finish]
```

```
def display_combinatorics(C,IC,i):
    print(str(i)+"/      "+str(C))
    for j in range(len(IC)):
        print("    Cycle : "+str(IC[j][0]))
        for k in range(len(IC[j][1])):
            print("         - Modulo: "+str(IC[j][1][k][0]) +
                        "    Character: "+str(IC[j][1][k][1][0]))
    print("  ")
```

```
def industrial_test(L,s):
    R=[]
    c=s
    Bug=[]
    for i in range(len(L)):
        n=max(flatten(L[i]))
        r=inner-cyclic_combinatorics(L[i],true)
        if not(r==false):
            realisable=false
            finish=false
            IA=is_actual(L[i])
            if not(IA=='NO DATA (timed out)'):
                realisable=IA[0]
                finish=IA[1]
            if realisable and finish:
                R.append(L[i])
                display_combinatorics(L[i],r,c)
                c=c+1
            if not(finish):
                Bug.append(i)
    return [R,Bug]
```

A.3 Space of realization Σ_C

This program is coded in `Maple` v.7.

```
restart: with(linalg):
```

```
equations:=proc(C)
> local i,j,k,m,EQ;
> k:=0;
> for i from 1 to nops(C) do
>     if nops(C[i])>2 then
>         for j from 3 to nops(C[i]) do
>             k:=k+1;
>             m:=matrix(3,3,[[a[C[i][1]],b[C[i][1]],c[C[i][1]]],
                 [a[C[i][2]],b[C[i][2]],c[C[i][2]]],
                 [a[C[i][j]],b[C[i][j]],c[C[i][j]]]]);
>             EQ[k]:=(det(m)=0);
>             od;
>     fi;
> od;
> RETURN(convert(EQ,set));
end:
```

```
fixed_lines:=proc(l1,l2,l3,l4,l5)
> local i,EQ;
> EQ[1]:=(a[l1]=0);EQ[2]:=(b[l1]=0);EQ[3]:=(c[l1]=1);
> EQ[4]:=(a[l2]=1);EQ[5]:=(b[l2]=0);EQ[6]:=(c[l2]=0);
> EQ[7]:=(a[l3]=1);EQ[8]:=(b[l3]=0);EQ[9]:=(c[l3]=-1);
> EQ[10]:=(a[l4]=0);EQ[11]:=(b[l4]=1);EQ[12]:=(c[l4]=0);
> EQ[13]:=(a[l5]=0);EQ[14]:=(b[l5]=1);EQ[15]:=(c[l5]=-1);
> RETURN(convert(EQ,set));
end:
```

```
rigidification:=proc(C)
> local i,j,R,L,J,K1,K2,Bool:
> R:=[]:
> L:=[];
> for i from 1 to nops(C) do
>   if nops(C[i])>2 then L:=[op(L),C[i]]: end if:
> end do:
> Bool:=true:
> for i from 1 to nops(L)-1 do
>   for j from (i+1) to nops(L) do
>     if Bool then
>       J:=convert(L[i],set) intersect convert(L[j],set);
>       if nops(J)>0 then
>         R:=[op(R),J[1]];
>         K1:=convert(L[i],set) minus J;
>         R:=[op(R),K1[1],K1[2]];
```

```
>           K2:=convert(L[j],set) minus J;
>           R:=[op(R),K2[1],K2[2]];
>           Bool:=false;
>           RETURN(R):
>         end if:
>       end if:
>    end do:
> end do:
end:
```

```
lines_number:=proc(C)
> local i,j,n:
> n:=0:
> for i from 1 to nops(C) do
>    for j from 1 to nops(C[i]) do
>       if C[i][j]>n then n:=C[i][j]: end if:
>    end do:
> end do:
> RETURN(eval(n)):
end:
```

```
reduc:=proc(S,l1,l2,l3,l4,l5,n)
> local i,R,t,good,j;
> t:=1;
> for i from 1 to nops(S) do
>       good:=true:
>       for j from 1 to n do
>           if (j<>l1 and j<>l2 and j<>l3 and j<>l4 and j<>l5) and
              (({a[j]=0,b[j]=0} subset S[i]) or
              ({a[j]=0,c[j]=0} subset S[i]) or
              ({b[j]=0,c[j]=0} subset S[i])) then
>                 good:=false:
>           end if;
>       end do;
>       if good then
>           R[t]:=S[i]:
>           t:=t+1:
>       end if;
> end do;
> for i from 1 to t-1 do
>       printf('Next_realization');
>       print(evalm(R[i]));
>       od;
> RETURN(evalm(R))
end:
```

```
realization:=proc(C)
> local EQ, EQ1, eq, v, V, S, K, i, R, l, n:
```

```
> n:=lines_number(C):
> EQ:=equations(C):
> l:=rigidification(C,10);
> EQ1:=fixed_lines(l[1],l[2],l[3],l[4],l[5]):
> eq:=EQ union EQ1:
> for i from 1 to n do
>    v[3*i-2]:=a[i]:
>    v[3*i-1]:=b[i]:
>    v[3*i]:=c[i]:
> od:
> V:=convert(v,set):
> S:=solve(eq,V):
> R:=reduc([S],l[1],l[2],l[3],l[4],l[5],n):
> RETURN(eval(R))
end:
```

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Abstract

This thesis is the intersection point between the two facets of the study of line arrangements: combinatorics and topology.

In the first part, we study the inclusion of the boundary manifold in the complement of an arrangement. We generalize the results of E. Hironaka to the case of any complex line arrangement. To get around the problems due to the case of non complexified real arrangement, we study the braided wiring diagram. We develop a `Sage` program to compute it from the equation of the complex line arrangement. This diagram allows to give two explicit descriptions of the map induced by the inclusion on the fundamental groups. From theses descriptions, we obtain two new presentations of the fundamental group of the complement. One of them is a generalization of the R. Randell Theorem to any complex line arrangement.

In the next step of this work, we study the map induced by the inclusion on the first homology group. Then we obtain two simple descriptions of this map. Inspired by ideas of J.I. Cogolludo, we give a canonical description of the homology of the boundary manifold as the product of the 1-homology with the 2-cohomology of the complement. Finally, we obtain an isomorphism between the 2-cohomology of the complement with the 1-homology of the incidence graph of the arrangement.

In the second part, we are interested by the study of character on the group of the complement. We start from the results of E. Artal on the computation of the depth of a character. This depth can be decomposed into a projective term and a quasi-projective term, vanishing for characters that ramify along all the lines. An algorithm to compute the projective part is given by A. Libgober. E. Artal focuses on the quasi-projective part and gives a method to compute it from the image by the character of certain cycles of the complement. We use our results on the inclusion map of the boundary manifold to determine these cycles explicitly. Combined with the work of E. Artal we obtain an algorithm to compute the quasi-projective depth of any character.

From the study of this algorithm, we obtain a strong combinatorial condition on characters to admit a quasi-projective depth potentially not determined by the combinatorics. With this property, we define the inner-cyclic characters. From their study, we observe a strong condition on the combinatorics of an arrangement to have only characters with null quasi-projective depth. Related to this, in order to reduce the number of computations, we introduce the notion of prime combinatorics. If a combinatorics is not prime, then the characteristics varieties of its realizations are completely determined by realization of a prime combinatorics with less line.

In parallel, we observe that the composition of the map induced by the inclusion with specific characters provide topological invariants of the blow-up of arrangements. We show that the invariant captures more than combinatorial information. Thereby, we detect two new examples of NC-Zariski pairs.

Résumé

Cette thèse est le point d'intersection entre deux facettes de l'étude des arrangements de droites : la combinatoire et la topologie.

Dans une première partie nous avons étudié l'inclusion de la variété bord dans le complémentaire d'un arrangement. Nous avons ainsi généralisé le résultat d'E. Hironaka au cas de tous les arrangements complexes. Pour contourner les problèmes provenant des arrangements non réels, nous avons étudié le diagramme de câblage, dit wiring diagram, qui code la monodromie de tresses sous forme de tresse singulière. Pour pouvoir l'utiliser, nous avons implémenté un programme sur `Sage` permettant de calculer ce diagramme en fonction des équations de l'arrangement. Cela nous a permis de d'obtenir deux descriptions explicites de l'application induite par l'inclusion de la variété bord dans le complémentaire sur les groupes fondamentaux. Nous obtenons ainsi deux nouvelles présentations du groupe fondamental du complémentaire d'un arrangement. L'une d'entre elle généralise le théorème de R. Randell au cas des arrangements complexes.

Pour continuer ces travaux, nous avons étudié l'application induite par l'inclusion sur le premier groupe d'homologie. Nous obtenons deux descriptions simples de cette application. En s'inspirant des travaux de J.I. Cogolludo, nous décrivons une décomposition canonique du premier groupe d'homologie de la variété bord comme produit de la 1-homologie et de la 2-cohomologie du complémentaire, ainsi qu'un isomorphisme entre la 2-cohomologie du complémentaire et la 1-homologie du graphe d'incidence.

Dans la seconde partie de notre travail nous nous sommes intéressés à l'étude des caractères du groupe fondamental du complémentaire. Nous partons des résultats obtenus par E. Artal sur le calcul de la profondeur d'un caractère. Cette profondeur peut être décomposée en un terme projectif et un terme quasi-projectif. Un algorithme pour calculer la partie projective a été donné par A. Libgober. Les travaux de E. Artal concernent la partie quasi-projective. Il a obtenu une méthode pour la calculer en fonction de l'image de certains cycles particuliers du complémentaire par le caractère. En utilisant les résultats obtenus dans la première partie, nous avons obtenu un algorithme complet permettant le calcul de la profondeur quasi-projective d'un caractère.

A travers l'étude de cet algorithme, nous avons obtenu une condition combinatoire pour admettre une profondeur quasi-projective potentiellement non combinatoire. Nous avons ainsi défini la notion de caractère inner-cyclic . Cette notion nous a permis de formuler des conditions fortes sur la combinatoire pour qu'un arrangement n'ait que des caractères de profondeur quasi-projective nulle. Enfin pour diminuer le nombre d'exemples à considérer nous avons introduit la notion de combinatoire première. Si une combinatoire ne l'est pas, alors les variétés caractéristiques de ses réalisations sont définies par celles d'un arrangement avec moins de droites.

En parallèle à cette étude, nous avons observé que la composition de l'application induite par l'inclusion sur le premier groupe d'homologie avec un caractère nous fournit un invariant topologique de l'arrangement obtenu en désingularisant les points multiples (blow-up). De plus, nous montrons que cet invariant n'est pas de nature combinatoire. Il nous a ainsi permis de découvrir deux nouvelles NC-paires de Zariski.

Resumen

Esta tesis estudia problemas que se encuentran en la intersección de dos facetas de la teoría de configuraciones de rectas: la combinatoria y la topología.

La primera contribución original de este trabajo es el estudio de la inclusión del borde de un entorno regular de la configuración en el complementario de la misma dentro del plano proyectivo complejo. Este estudio ya fue realizado por E. Hironaka en el caso de una configuración real complexificaday en este trabajo se ha generalizado para configuraciones complejas cualesquiera. Para resolver los problemas provenientes de configuraciones no reales, se ha estudiado el *diagrama de cableado* (*wiring diagram*) asociado a una configuración, quien codifica la monodromía de trenzas bajo la forma de una trenza singular. Para poder hacer uso de ella, se implementó un programa sobre `Sage` que permite calcular este diagrama en función de las ecuaciones de la configuración. Esto nos ha permitido obtener dos descripciones explícitas de la aplicación sobre grupos fundamentales inducida por la inclusión de la variedad borde en el complementario. De esta forma, se obtienen dos nuevas presentaciones del grupo fundamental del complementario de una configuración. Una de ellas generaliza un teorema de R. Randell en el caso de configuraciones complejas.

Continuando este trabajo, se estudió la aplicación inducida por la inclusión sobre los primeros grupos de homología, obteniendo dos descripciones simples de tal aplicación. Inspirándose en los trabajos de J.I. Cogolludo, se describe una descomposición canónica del primer grupo de homología de la variedad borde como producto de la 1–homología y la 2–cohomología del complementario, así como un isomorfismo entre la 2-cohomología el complementario y la 1–homología del grafo de incidencia.

En una segunda parte, nos hemos interesado en el estudio de caracteres del grupo fundamental del complementario, partiendo de resultados de E. Artal sobre el cálculo de la profundidad de un carácter. Esta profundidad puede ser descompuesta en dos sumandos que llamamos proyectivo y cuasi-proyectivo, pudiéndose calcular la parte proyectiva mediante un algoritmo de A. Libgober. La parte cuasi-proyectiva es tratada en los trabajos de E. Artal, donde obtiene un método para calcularla en función de la imagen por el carácter de ciertos ciclos particulares del complementario. Usando los resultados anteriores sobreaplicación inclusión en homología, se ha obtenido un algoritmo completo que permite el cálculo de la profundidad cuasi-proyectiva de un carácter.

A través del estudio de este algoritmo, se ha podido aislar qué parte del algoritmo no está determinada por la combinatoria y de esta manera se puedereducir el universo de ejemplos para los que la profundidad cuasi-proyectiva pudiera ser no combinatoria. Se ha estudiado también la noción de carácter *inner–cyclic*, la cual nos permite formular condiciones fuertes sobre la combinatoria para que una configuración pueda tener caracteres en los que la profundidadcuasi-proyectiva es no nula. Finalmente, para disminuir el número de ejemplos a considerar, se ha introducido la noción de *combinatoria prima*. Si una combinatoria no es prima, entonces las variedades características de sus realizaciones están definidas por las de una configuración con un número menor de rectas.

Paralelamente a este estudio, se ha observado que la composición de la aplicación inducida por la inclusión sobre el primer grupo de homología con un carácter proporciona un invariante topológico de la configuración de rectas obtenida por desingularización de puntos múltiples (*blow-up*); este invariante tiene interés *per se*, independientemente de su aplicación al algoritmo citado anteriormente. Se demuestra que este invariante no es de naturaleza combinatoria, lo cual a permitido descubrir dos nuevos NC–pares de Zariski.