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ON PAIRS OF ANTAGONISTIC SUBGROUPS AND THEIRS INFLUENCE ON THE STRUCTURE OF GROUPS

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Abstract. In this survey we collect some results on the influence on the structure of a group of some families of its subgroups satisfying conditions related to normality. In particular we focus on groups whose subgroups have two antagonistic properties.

1. Introduction

Let G be a group and let S be a subgroup of G. We denote by $N_G(S)$

$$N_G(S) = \{x \mid x \in G : S^x = x^{-1}Sx = S\}$$

the normalizer of the subgroup S in G. Two particularly interesting extreme situations could occur:

$$N_G(S) = G \text{ or } N_G(S) = S.$$

In the first case, S is a normal subgroup; in the second case, S is a **selfnormalizing subgroup** of G. Normal and selfnormalizing subgroups are antagonistic subgroups, in fact a subgroup S is simultaneously normal and selfnormalizing in G only if $S = G$.

Other particular types of subgroups which also are antagonistic to selfnormalizing subgroups are defined as follows. Let S be a subgroup of a group G. Starting from the normalizer, we can construct the following canonical series:

$$N_{0,G}(S) = S, N_{1,G}(S) = N_G(S),$$

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and recursively

$$N_{\alpha+1,G}(S) = N_G(N_{\alpha,G}(S)),$$

for all ordinals α, and

$$N_{\lambda,G}(S) = \bigcup_{\beta<\lambda} N_{\beta,G}(S),$$

for all limit ordinals λ.

Then, we construct the upper normalizer series

$$S = N_{0,G}(S) \leq N_{1,G}(S) \leq \cdots \leq N_{\alpha,G}(S) \leq N_{\alpha+1,G}(S) \leq \cdots N_{\gamma,G}(S) = D$$

of the subgroup S in G. By construction, $N_{\alpha,G}(S)$ is a normal subgroup of $N_{\alpha+1,G}(S)$, for all ordinals $\alpha < \gamma$. The last term D of this series has the property $N_G(D) = D$, and is called the **hypernormalizer** (or **supernormalizer**) of the subgroup S in the group G.

Two particularly interesting extreme situations could occur: $N_{\gamma,G}(S) = G$, for some γ, or $N_G(S) = S$. In the first situation the subgroup S is ascendant in G, in the second situation the subgroup S is selfnormalizing. Thus, every subgroup S is ascendant in its hypernormalizer in G, and the hypernormalizer of S in G is a selfnormalizing subgroup. Particular cases of ascendant subgroups are subnormal subgroups, thus subnormal and selfnormalizing subgroups also form a pair of antagonistic subgroups.

If H is a selfnormalizing subgroup of G and S is a subgroup containing H, S is not always selfnormalizing. This leads to the following special types of selfnormalizing subgroups.

A subgroup H is called **weakly abnormal in** G if every subgroup containing H is selfnormalizing in G. The following interesting characterization of weakly abnormal subgroups was obtained by M. S. Ba and Z. I. Borevich in the paper [3].

Theorem 1.1. *Let G be a group and H be a subgroup of G. Then H is weakly abnormal in G if and only if $x \in H^{\langle x \rangle}$, for all $x \in G$.*

A slightly stronger property leads to the following concept.

A subgroup H of a group G is called **abnormal in** G if $g \in \langle H, H^g \rangle$, for all $g \in G$.

Abnormal subgroups have been first studied by P. Hall in the paper [24], while the term "abnormal subgroup" had previously been introduced by R. Carter in his paper [6].

M. S. Ba and Z. I. Borevich in the paper [3] obtained the following characterization of abnormal subgroups.

Theorem 1.2. *Let G be a group and A a subgroup of G. Then A is abnormal in G if and only if it satisfies the following conditions:*

(i) *if S is a subgroup of G containing A, then S is selfnormalizing;*
(ii) *if S, K are two conjugate subgroups of G containing A, then $S = K$.*

If G is a finite soluble group then the condition (ii) could be omitted (see, for example [29, p. 733, Theorem 11.17]). In other words, in finite soluble groups every weakly abnormal subgroup is abnormal. V. V. Kirichenko, L. A. Kurdachenko and I. Ya. Subbotin (see [30, Theorem 1.2]) extended this result to some classes of infinite groups.

First, we introduce the following definition.

A group G is called an **Ñ-group** if it satisfies the following condition: if M, L are subgroups of G such that M is maximal in L, then M is a normal subgroup of L.

We note that every locally nilpotent group is an Ñ-group, but the converse is not true as J. S. Wilson proved in [65].

Now we can state the following results from the paper [30].

Theorem 1.3. *Let G be a hyper-Ñ-group and A be a subgroup of G. Then A is abnormal in G if and only if every subgroup S of G containing A is selfnormalizing.*

Corollary 1.4. *Let G be a radical group and A be a subgroup of G. Then A is abnormal in G if and only if every subgroup S of G containing A is selfnormalizing.*

Corollary 1.5. *Let G be a hyperabelian group and A be a subgroup of G. Then A is abnormal in G if and only if every subgroup S of G containing A is selfnormalizing.*

Corollary 1.6. *Let G be a soluble group and A be a subgroup of G. Then A is abnormal in G if and only if every subgroup S of G containing A is selfnormalizing.*

Now the following question arises.

Are there weakly abnormal subgroups which are not abnormal?

The answer is positive even for finite groups, an example due to M. S. Ba and Z. I. Borevich can be found in [3, §7].

From $x \in H^{\langle x \rangle}$, we obtain that $x \in H^G$. It follows that $G = H^G$ whenever H is a weakly abnormal (in particular, abnormal) subgroup of G.

A subgroup H of a group G is called **contranormal in** G if $H^G = G$.

The term "contranormal subgroup" has been introduced by J. S. Rose in the paper [59].

Abnormal and weakly abnormal subgroups are contranormal, while the converse is not true.

Contranormal subgroups can be constructed in the following way.

Let G be a group and H be a subgroup of G. Starting from the normal closure, we can construct the following canonical series:

$$\nu_{0,G}(H) = G, \nu_{1,G}(H) = H^G,$$

and, recursively,

$$\nu_{\alpha+1,G}(H) = H^{\nu_{\alpha,G}(H)},$$

for every ordinal α,

$$\nu_{\lambda,G}(H) = \bigcap_{\beta<\lambda} \nu_{\beta,G}(H),$$

for all limit ordinals λ.

Thus we construct the **lower normal closure series**

$$G = \nu_{0,G}(H) \geq \nu_{1,G}(H) \geq \cdots \geq \nu_{\alpha,G}(H) \geq \nu_{\alpha+1,G}(H) \geq \cdots \geq \nu_{\gamma,G}(H) = D$$

of the subgroup H in the group G. By construction $\nu_{\alpha+1,G}(H)$ is a normal subgroup of $\nu_{\alpha,G}(H)$ for all ordinals $\alpha < \gamma$. The last term D of this series has the property $H^D = D$ and is called the **lower normal closure of the subgroup H in the group G**.

Two opposite particular situations can happen: $\nu_{\gamma,G}(H) = H$ or $H^G = G$. In the first situation, the subgroup H is descendant in G, in the second situation, the subgroup H is contranormal in G. Hence every subgroup H is contranormal in its lower normal closure, and the lower normal closure of H in G is a descendant subgroup.

Notice that subnormal subgroups are simultaneously ascendant and descendant.

Thus we obtain the following pairs of mutually opposite subgroups:

$$ascendant\ subgroup \longleftrightarrow selfnormalizing\ subgroup,$$
$$subnormal\ subgroup \longleftrightarrow selfnormalizing\ subgroup,$$
$$normal\ subgroup \longleftrightarrow selfnormalizing\ subgroup,$$
$$ascendant\ subgroup \longleftrightarrow weakly\ abnormal\ subgroup,$$
$$subnormal\ subgroup \longleftrightarrow weakly\ abnormal\ subgroup,$$
$$normal\ subgroup \longleftrightarrow weakly\ abnormal\ subgroup,$$
$$ascendant\ subgroup \longleftrightarrow abnormal\ subgroup,$$
$$subnormal\ subgroup \longleftrightarrow abnormal\ subgroup,$$
$$normal\ subgroup \longleftrightarrow abnormal\ subgroup,$$
$$descendant\ subgroup \longleftrightarrow contranormal\ subgroup,$$
$$subnormal\ subgroup \longleftrightarrow contranormal\ subgroup,$$
$$normal\ subgroup \longleftrightarrow contranormal\ subgroup.$$

Many authors studied groups whose subgroups all belong to some of the above pairs of antagonistic subgroups, see for example [19], [47], [38], [10], [16], [13], [33], [40], [12]. As we can see from the results of these papers, whether a group G has only two types of antagonistic subgroups and the family of each of these subgroups is not empty, very often it is possible to obtain the full description of the structure of G. Full description is possible not only in this case (when a group has only two types of subgroups), but also in the case when a group has a very big family of subgroups of one type and the others are antagonistic; for example, results for groups having a big family of normal, subnormal or pronormal subgroups are in the papers [8], [28], [44], [46], [16], [17], [9], [43], [45] [20], [21], [62]. However, if a group G has the subgroups of only one type, then the situation is different. In some cases we can still get the full description of such groups. For example, groups whose subgroups are all normal have been described by R. Dedekind (see [57, 5.3.7]); groups whose subgroups are all permutable were characterized by K. Iwasawa (see [61, §2]). The theory of groups whose subgroups are all subnormal is one of the best developed theory in infinite group theory, see the book [52] by J. C. Lennox and S.E. Stonehewer, the important paper [53] by W. Möhres, and the survey article [7]) by C. Casolo. But quite different is the situation of groups, whose subgroups are ascendant or descendant. Groups whose subgroups are all ascendant satisfy the normalizer condition, meaning their subgroups do not coincide with normalizer. These groups are also called **N-groups**.

Groups whose nontrivial subgroups are all contranormal are obviously simple, so their study in the infinite case is difficult. The following question is still open: what can we say about groups whose subgroups are among one of the antagonistic subgroups indicated in the pairs above? For example, looking at the first pair, we have the following result: if a group G does not contain selfnormalizing subgroups, then every subgroup of G is ascendant (see, for example, [57, 12.2.1]).

Let G be a finite group and suppose that G does not contain proper abnormal subgroups. Since the normalizer of every Sylow p-subgroup is abnormal, we obtain that every Sylow p-subgroup of G is normal, thus the group G is nilpotent. Conversely, every nilpotent group does not contain proper abnormal subgroups. Moreover, using the arguments from the paper [51] we can prove the following result:

Theorem 1.7. *Let G be an $\tilde{N}$-group. Then G does not contain proper abnormal subgroups. In particular, every locally nilpotent group does not contain proper abnormal subgroups.*

Therefore we can ask:

What can we say about groups, which do not contain proper abnormal subgroups?
In particular, in which case groups without proper abnormal subgroups are locally nilpotent?

In other words, it would be interesting to obtain some criteria of locally nilpotency in terms of abnormality. L. A. Kurdachenko, J. Otal and I. Ya. Subbotin in the paper [34] show the first of them. We start with a definition. Let G be a group and A a normal subgroup of G, then A is **G-minimax** if A has a series of G-invariant subgroups, in which every infinite factor is abelian and satisfies either the maximal or the minimal condition for G-invariant subgroups. A group G is called **generalized minimax**, if G itself is G-minimax.

Theorem 1.8. *Let G be a generalized minimax group. If G does not contain proper abnormal subgroups, then G is hypercentral.*

Let G be a group. If x is an element of G, write $x^G = \{x^g \mid g \in G\}$, where $x^g = g^{-1}xg$. The **FC-center of** G is the set $FC(G) = \{x \mid x^G \text{ is finite}\}$, and is a characteristic subgroup of G. A group

G is called an FC**-group** if $G = FC(G)$. Starting from the FC-center it is possible to construct the **upper** FC**-central series of a group** G:

$$\{1\} = FC_0(G) \leq FC_1(G) \leq \cdots FC_\alpha(G) \leq FC_{\alpha+1}(G) \leq \cdots \leq FC_\gamma(G),$$

defining $FC_1(G) = FC(G)$ and, recursively, $FC_{\alpha+1}(G)/FC_\alpha(G) = FC(G/FC_\alpha(G))$ for all ordinal α, and $FC_\lambda(G) = \bigcup_{\beta<\lambda} FC_\beta(G)$ for all limit ordinals λ. The last term $FC_\gamma(G)$ is called the **upper** FC**-hypercenter of** G and we have $FC(G/FC_\gamma(G)) = \{1\}$. A group G is called FC**-hypercentral** if the upper FC-hypercenter of G coincides with G. G is FC**-nilpotent** if G has a finite FC-central series.

The following criteria of hypercentrality have been proved by L. A. Kurdachenko, A. Russo and G. Vincenzi in their paper [41].

Theorem 1.9. *Let G be a group without proper abnormal subgroups.*

(i) *Every subgroup $F_i(G)$ with i finite is hypercentral. In particular, if G is FC-nilpotent, then G is hypercentral.*

(ii) *Let $C = FC_i(G)$, with i finite, and S be a normal subgroup of G containing C. If the quotient S/C is hypercentral, then S is hypercentral.*

Let G be a finite group and suppose that G does not contain proper contranormal subgroups, then every maximal subgroup is normal, and G is nilpotent (see, for example [29, Satz III.3.11]). In infinite groups the situation is different. If G is an infinite locally nilpotent group, then, as we have seen above, G does not contain proper abnormal subgroups, howewer a locally nilpotent group can contain proper contranormal subgroups. There are also examples of hypercentral groups with proper contranormal subgroups, as the following example shows.

Let D be a divisible abelian 2-group. Then D has an automorphism φ such that $\varphi(d) = d^{-1}$ for each element $d \in D$. Define the semidirect product $G = D \rtimes \langle b \rangle$ such that $d^b = \varphi(d) = d^{-1}$ for each element $d \in D$. Let a be an arbitrary element of D. Since D is divisible, there exists an element $d \in D$ such that $d^2 = a$. We have $[b, d] = b^{-1}d^{-1}bd = d^2 = a$. It follows that $[b, D] = D$. From $[b, D] \leq \langle b \rangle^G$ and $\langle b \rangle \leq \langle b \rangle^G$ we obtain that $\langle b \rangle^G = \langle b \rangle [b, D] = \langle b \rangle D = G$, so that the subgroup $\langle b \rangle$ is contranormal in G. We notice that the group G is not nilpotent, however the series

$$\{1\} \leq \Omega_1(D) \leq \cdots \leq \Omega_n(D) \leq \Omega_{n+1}(D) \leq \cdots \leq D \leq G$$

is central, hence G is hypercentral and abelian-by-finite. In particular, G is locally nilpotent.

Another example can be constructed in the following way. Let D be a Prüfer 2-group and α be a non-trivial automorphism of D of infinite order, then the semidirect product $G = D \rtimes \langle \alpha \rangle$ is a hypercentral group and $\langle \alpha \rangle$ is contranormal in G. As in the previous example the subgroup $\langle \alpha \rangle$ is ascendant in G.

These examples motivate the following question.

What can we say about groups having no proper contranormal subgroups?

Some properties of contranormal subgroups are summarized below.

Theorem 1.10. *Let G be a group. Then:*

(i) *If C is a contranormal subgroup of G and K is a subgroup containing C, then K is a contranormal subgroup of G.*

(ii) *If C is a contranormal subgroup of G and H is a normal subgroup of G, then CH/H is a contranormal subgroup of G/H.*

(iii) *If H is a normal subgroup of G and C is a subgroup of G such that $H \leq C$ and C/H is a contranormal subgroup of G/H, then C is a contranormal subgroup of G.*

(iv) *If C is a contranormal subgroup of G and D is a contranormal subgroup of C, then D is a contranormal subgroup of G.*

(v) *If M is a maximal subgroup of G and M is not normal, then M is a contranormal subgroup of G.*

However, the intersection of two contranormal subgroups is not always contranormal. For example, in the alternating group A_4 every Sylow 3-subgroup is contranormal, but the intersection of two Sylow 3-subgroups is trivial, thus it is not contranormal.

We say that a group G is **contranormal-free** if G does not contain proper contranormal subgroups.

A periodic group G is called **Sylow-nilpotent** if G is locally nilpotent and a Sylow p-subgroup of G is nilpotent for every prime p.

First results about contranormal-free groups have been obtained by L. A. Kurdachenko and I. Ya. Subbotin in their paper [48].

Theorem 1.11. *Let G be a contranormal-free group. Then we have:*

(i) If G is a Chernikov group, then G is nilpotent.

(ii) If G is a locally finite group and every Sylow p-subgroup of G is Chernikov, for every prime p, then G is Sylow-nilpotent.

Notice that groups without contranormal subgroups can be very far from being nilpotent.

Indeed, groups whose subgroups are subnormal do not contain proper contranormal subgroups. Such groups are locally nilpotent, but in general not nilpotent. In the paper [27] H. Heineken and A. Mohamed constructed a p-group H with the following properties: H contains a normal elementary abelian p-subgroup A such that H/A is a Prüfer p-group, every proper subgroup of H is subnormal and the center of H is trivial.

Other examples of contranormal-free groups are Sylow-nilpotent groups. It is not hard to see that every Sylow-nilpotent group does not contain proper contranormal subgroups, but in general Sylow-nilpotent groups are not nilpotent.

A group G is called **hyperfinite**, if G has an ascending series of normal subgroups whose factors are finite.

The following results have been obtained by L. A. Kurdachenko, P. Longobardi and M. Maj in the paper [31].

Theorem 1.12. *Let G be a group, H a locally nilpotent normal subgroup of G such that G/H is hyperfinite. If G is contranormal-free, then G is locally nilpotent.*

As corollaries we have:

Corollary 1.13. *Let G be a periodic group, H a normal locally nilpotent subgroup of G such that G/H is nilpotent. If G is contranormal-free, then G is locally nilpotent.*

Corollary 1.14. *Let G be a periodic group and H be a normal locally nilpotent subgroup such that G/H is a Chernikov group. If G is contranormal-free, then G is locally nilpotent.*

Corollary 1.15. *Let G be a locally finite group and H be a normal locally nilpotent subgroup such that the Sylow p-subgroups of G/H are Chernikov for every prime p. If G is contranormal-free, then G is locally nilpotent.*

Corollary 1.16. *Let G be a hyperfinite group. If G is contranormal-free, then G is hypercentral.*

Corollary 1.17. *Let G be a periodic group, H a normal nilpotent subgroup of G such that G/H is nilpotent and $\pi(H) \cap \pi(G/H) = \emptyset$. If G is contranormal-free, then G is nilpotent.*

Corollary 1.17 had already been obtained by B. A. Wehrfritz in the paper [64] under the additional hypothesis that G/H is a Chernikov group.

Some classes of non-periodic contranormal-free groups have also been investigated.

Let A be a torsion-free abelian normal subgroup of a group G. We say that A **is rationally irreducible with respect to G or that A is G-rationally irreducible** if, for every nontrivial G-invariant subgroup B of A, the quotient A/B is periodic.

The following result has been proved by L.A. Kurdachenko, P. Longobardi and M. Maj in the paper [31].

Theorem 1.18. *Let G be a group and let A be a normal nilpotent subgroup of G such that G/A is a Chernikov π-group. Assume that A has a finite series of G-invariant subgroups*

$$A = A_0 \geq A_1 \geq \cdots \geq A_j \geq A_{j+1} \geq \cdots \geq A_t = \{1\}$$

such that every factor A_j/A_{j+1}, $0 \leq j \leq t-1$ satisfies one of the following conditions:

A_j/A_{j+1} is torsion-free and G-rationally irreducible;

A_j/A_{j+1} is a periodic π'-group;

A_j/A_{j+1} is a Chernikov π-group.

If G is contranormal-free, then G is nilpotent.

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The result of Theorem 1.18 is a generalization of the main result in the above mentioned paper of B. A. F. Wehrfritz [64].

One more result of this nature has been obtained by L.A. Kurdachenko, J. Oral and I. Ya. Subbotin in the paper [37]. They considered the following generalization of generalized minimax groups.

A group G is called a **generalized A_3-group** if G has a finite series of normal subgroups

$$\{1\} = H_0 \leq H_1 \leq \cdots \leq H_n = G$$

such that every infinite factor H_j/H_{j-1} is abelian and satisfies one of the following conditions:

(i) H_j/H_{j-1} is a torsion-free group of finite 0-rank;

(ii) H_j/H_{j-1} satisfies the maximal condition for G-invariant subgroups;

(iii) H_j/H_{j-1} satisfies the minimal condition for G-invariant subgroups.

Here the group G has **finite 0-rank** r if G has a finite series of subgroups:

$$\{1\} = H_0 \leq H_1 \leq \cdots \leq H_n = G$$

such that H_{j-1} is normal in H_j, every factor H_j/H_{j-1} is periodic or infinite cyclic, $1 \leq j \leq n$, and the number of infinite cyclic sections is exactly r.

Next two results have been proved in [37]:

Theorem 1.19. *Let G be a generalized A_3-group. If G is contranormal-free, then G is a nilpotent A_3-group (in the sense of A.I. Maltsev).*

Theorem 1.20. *Let G be a group and C a normal G-minimax subgroup of G such that G/C is a nilpotent group of finite 0-rank, whose Sylow p-subgroups are Chernikov for every prime p. If G is contranormal-free, then G is nilpotent.*

The last result is a generalization of the main result in the above mentioned paper of B. A. F. Wehrfritz [64].

In the same paper the following dual situation was investigated.

Theorem 1.21. *Let G be a group and C a normal nilpotent subgroup having finite 0-rank, whose Sylow p-subgroups are Chernikov for every prime p. Suppose that G/C satisfies the minimal condition for normal subgroups. If G is contranormal-free, then G is nilpotent.*

Notice that the above theorem implies that the quotient G/C is a central-by-finite Chernikov group.

In the paper [64] B. A. F. Wehrfritz also proved the following result.

Theorem 1.22. *Let G be a nilpotent-by-finite group. If G is contranormal-free, then G is nilpotent.*

M. R. Dixon, L. A. Kurdachenko and I. Ya. Subbotin studied in [11] a more general situation. The main result of the paper is the following.

Theorem 1.23. *Let G be a group and H be a nilpotent normal subgroup of G such that G/H is finitely generated and soluble-by-finite. If G is contranormal-free, then G is hypercentral.*

As a corollary we obtain:

Corollary 1.24. *Let G be a finitely generated group and suppose that G has an ascending series of normal subgroups whose factors are abelian or finite. If G is contranormal-free, then G is nilpotent.*

Recall that a group G is **generalized radical** if it has an ascending series whose factors are locally nilpotent or locally finite.

Let p be a prime. We say that G **has finite section p-rank** $sr_p(G) = r$ if every elementary abelian p-section of G is finite of order at most p^r and there is an elementary abelian p-section A/B of G such that $|A/B| = p^r$.

We say that the group G has **finite section rank** if the section p-rank of G is finite for every prime p.

The other main result obtained by M. Dixon, L. A. Kurdachenko and I. Ya. Subbotin in the paper [11] is the following.

Theorem 1.25. *Let G be a locally generalized radical group, having finite section rank. If G is contranormal-free, then G is hypercentral, with hypercentral length at most $\omega + k$ for some positive integer k (here ω is the first infinite ordinal). Moreover, every quotient G/H with $\pi(G/H)$ finite is nilpotent.*

In particular, if the set $\pi(G)$ is finite, then G is nilpotent. This assertion had already been proved by L. A. Kurdachenko, J. Otal and I. Ya. Subbotin in the paper [37] and by B. A. F. Wehrfritz in the paper [64].

Until now we have considered pairs of subgroups which are antipodes one to other. However there exist some types of subgroups, which combine in some sense the two properties. For example, the following class of subgroups contains both abnormal and normal subgroups.

A subgroup H of a group G is called **pronormal in** G if for all $g \in G$ the subgroups H and H^g are conjugate in $\langle H, H^g \rangle$.

Normal and abnormal subgroups are pronormal, Sylow p-subgroups and Hall π-subgroups of normal soluble subgroups of a finite group are other examples of pronormal subgroups.

As for abnormality, we shall consider a weak variant of pronormality.

Let G be a group. A subgroup H is called **weakly pronormal in** G if the subgroups H and H^x are conjugate in $H^{\langle x \rangle}$, for all $x \in G$ (see [3]).

The inclusion $\langle H, H^x \rangle \leq H^{\langle x \rangle}$ shows that every pronormal subgroup is weakly pronormal. The converse statement is not true, M. S. Ba and Z. I. Borevich constructed an example in the paper [3].

Let G be a group and H be a subgroup of G. We say that H has **the Frattini property** if for all subgroups K, L such that $H \leq K$ and K is normal in L we have the equality $L = N_L(H)K$.

M. S. Ba and Z. I. Borevich proved in [3] the following result.

Theorem 1.26. *Let G be a group. A subgroup H is weakly pronormal in G if and only if H has the Frattini property.*

Theorem 1.26 has the following interesting corollaries.

Corollary 1.27. *Let G be a group and H be a subgroup of G. If H is pronormal in G, then H has the Frattini property.*

Corollary 1.28. *Let G be a group and H be a subgroup of G. Then H is abnormal (respectively weakly abnormal) in G if and only if H is pronormal (respectively weakly pronormal) in G and H is selfnormalizing.*

The relations between weakly pronormal and pronormal subgroups are outlined in the following result proved by M. S. Ba and Z. I. Borevich in [3].

Theorem 1.29. *A subgroup H of a group G is pronormal in G if and only if it satisfies the following conditions:*

(i) H is weakly pronormal;

(ii) if L is a subgroup including H and g is an element of G such that $H \leq L^g$, then there exists $x \in N_G(H)$ such that $L^g = L^x$.

Corollary 1.30. *Let G be a group, H a subgroup of G. If H is pronormal (respectively weakly pronormal) in G, then its normalizer is abnormal (respectively weakly abnormal) in G.*

As an application of Theorem 1.29 we have.

Corollary 1.31. *Let G be an $\tilde{N}$-group. Then every pronormal subgroup of G is normal. In particular, every pronormal subgroup of a locally nilpotent group is normal.*

In the paper [55] T. A. Peng obtained the following characterization of pronormal subgroups in finite groups.

Theorem 1.32. *Let G be a finite group and S be a subgroup of G. S is pronormal in G if and only if it has the Frattini property.*

For infinite groups the following more general result was obtained by L. A. Kurdachenko, J. Otal and I. Ya. Subbotin in the paper [35].

Theorem 1.33. *Let G be a hyper-N-group and S be a subgroup of G. Then S is pronormal in G if and only if S has the Frattini property.*

If S is a pronormal subgroup of a group G and L is a subgroup containing S, then S is pronormal in L and in this case $N_L(S)$ is abnormal in L. In particular, $N_L(S)$ is contranormal in L. This leads to the following concept.

A subgroup S of a group G is called **nearly pronormal in** G if $N_L(S)$ is contranormal in L for every subgroup L of G contaning S.

Hence every pronormal subgroup is nearly pronormal. The converse is not true, as V. V. Kirichenko, L. A. Kurdachenko and I. Ya. Subbotin proved in [30] with a counterexample.

However for some classes of generalized soluble groups the two concepts are equivalent, as L. A. Kurdachenko, A. A. Pypka and I. Ya. Subbotin proved in the paper [39] with the following results.

Theorem 1.34. *Let G be a hyper-N-group and S be a subgroup of G. Then S is pronormal in G if and only if S is nearly pronormal. In particular, every nearly pronormal subgroup of a soluble group is pronormal.*

Corollary 1.35. *Let G be a soluble group and S be a subgroup of G such that $N_L(S)$ is abnormal in L for any subgroup L containing S. Then S is pronormal in G.*

For finite groups this result had been proved by J. G. Wood in [66].

Now we introduce another property, which generalizes both the properties "to be a contranormal subgroup" and "to be a normal subgroup".

A subgroup H of a group G is called **conormal in** G if H is contranormal in H^G.

Obviously, every contranormal subgroup is conormal; and every normal subgroup is conormal, because if H is normal in G, then $H = H^G$. More generally, every subgroup satisfying the Frattini property is conormal. In fact, let G be a group and suppose that S is a subgroup of G satisfying the Frattini property; write $K = S^G$, then $G = KN_G(S)$ implies that $K = S^K$, so that S is a conormal subgroup.

In particular, every pronormal subgroup is conormal.

But the two conditions are not equivalent. For example, let $G = D \rtimes \langle b \rangle$, where D is a divisible abelian 2-group and $d^b = d^{-1}$ for all $d \in D$. Then the subgroup $\langle b \rangle$ is contranormal, hence it is conormal. But $\langle b \rangle$ can not be pronormal, because pronormal subgroups in a locally nilpotent group are normal.

We list some basic elementary properties of conormal subgroups.

Theorem 1.36. *Let G be a group. Then*

(i) *if C is a conormal subgroup of G and H is a normal subgroup of G, then CH/H is a conormal subgroup of G/H;*
(ii) *if H is a normal subgroup of G and C is a subgroup of G containing H and C/H is a conormal subgroup of G/H, then C is a conormal subgroup of G;*
(iii) *if M is a maximal subgroup of G, then M is a conormal subgroup of G. In particular, if M is not normal, then M is a contranormal subgroup of G.*
(iv) *Let $\mathcal{S}$ be a family of conormal subgroups of G, then the subgroup C generated by all subgroups in $\mathcal{S}$ is a conormal subgroup of G.*

Notice that usually the intersection of conormal subgroups is not conormal, as the following easy example shows. Let $G = Sym(4)$, then $G = AB$ where A is the normal subgroup of order 4 and $B \simeq Sym(3)$. Let $b = (12)$, then $C_A(b)$ contains the element $a = (12)(34)$. The subgroup $\langle a, b \rangle$ is maximal and not normal in G, hence it is contranormal. Let c be an element of B of order 3, then $\langle c \rangle$

is a normal subgroup of B, therefore the subgroup $A\langle c\rangle$ is normal in G, hence is conormal in G. We have $\langle a, b\rangle \cap A\langle c\rangle = \langle a\rangle$. But $\langle a\rangle^G = A$ and $\langle a\rangle^A = \langle a\rangle$, therefore the subgroup $\langle a\rangle$ is not conormal in G.

Conormal subgroups have the following important property.

Theorem 1.37. *Let G be a group and C be a conormal subgroup of G. If C is subnormal in G, then C is normal in G.*

From Theorem 1.37 we get the following interesting result.

Theorem 1.38. *Let G be a group. Every subgroup of G is conormal in G if and only if the relation "to be a normal subgroup" is transitive in G.*

It is well-known that usually normality is not transitive, groups in which this relation is transitive are called T**-groups**.

T-groups have been widely studied in recent years (see, for example, [1], [2], [5], [4], [14], [15], [21], [22], [26], [25], [56], [63], [60]). Considering normality not only in the whole group, but in its subgroups, L.A. Kurdachenko and I.Ya Subbotin introduced in [49] the following definition. A subgroup H of a group G is called **transitively normal in** G if H is normal in every subgroup S such that H is subnormal in S. Transitively normal subgroups are also called **pseudonormal subgroups** and studied by F. de Giovanni and G. Vincenzi in the paper [22].

Obviously, if H is a transitively normal subgroup of G, then $N_G(N_G(H)) = N_G(H)$. It turn out that there is an interesting connection with conormal subgroups.

Theorem 1.39. *Let G be a group and C be a subgroup of G. If C is conormal in every subgroup H containing C, then C is transitively normal in G.*

A group G is called a $\bar{T}$-group if every subgroup of G is a T-group. Clearly G is a $\bar{T}$-group if and only if every subgroup of G is transitively normal in G. The structure of finite soluble T-groups has been described by W. Gaschütz in [18]. In particular, he proved that every finite soluble T-group is a $\bar{T}$-group.

A basic result on soluble T-groups is that every soluble group is metabelian. Infinite soluble T-groups have been studied in detail in a fundamental paper by D. J. S. Robinson (see [56]). In particular, he proved that locally soluble $\bar{T}$-groups have the following structure.

Theorem 1.40. *Let G be a locally soluble $\bar{T}$-group.*

(i) *If G is not periodic, G is abelian.*
(ii) *If G is periodic and L is the locally nilpotent residual of G, then we have:*
  (a) *G/L is a Dedekind group;*
  (b) *$\pi(L) \cap \pi(G/L) = \emptyset$;*
  (c) *$2 \notin \pi(L)$;*

(*d*) *every subgroup of L is G-invariant.*

In particular, if L is non-trivial, then $L = [L, G]$.

We notice that in general the locally nilpotent residual has no complement. In the paper [23] Yu. M. Gorchakov presented a related sophisticated construction. This construction, in particular, supplies examples of periodic groups that are non-splitting extensions of their abelian Hall derived subgroup by an uncountable elementary abelian 2-group.

A subgroup S of a group G is called **polynormal in** G if for every subgroup H containing S, S is contranormal in S^H (see [3]). In other words, S is a polynormal subgroup if it is conormal in every subgroup containing it.

The following characterization of polynormal subgroups has been obtained by M. S. Ba and Z. I. Borevich in the paper [3].

Theorem 1.41. *Let G be a group. A subgroup S is polynormal in G if and only if S is contranormal in $S^{\langle g \rangle}$, for every element $g \in G$.*

Using Theorem 1.39 we obtain the following interesting characterization of $\bar{T}$-groups.

Theorem 1.42. *A group G is a $\bar{T}$-group if and only if every subgroup of G is polynormal in G.*

For finite groups T. A. Peng in [54] proved another characterization of $\bar{T}$-groups.

Theorem 1.43. *A finite group G is a $\bar{T}$-group if and only if every (cyclic) subgroup of G is pronormal in G.*

This result is not true for infinite group, as Theorem 1.40 together with the following result proved by N. F. Kuzennyi and I. Ya. Subbotin in [50] show.

Theorem 1.44. (1) *Let G be a periodic locally graded group. Then every subgroup of G is pronormal if and only if G is a soluble $\bar{T}$-group satisfying the following condition:*

if L is the locally nilpotent residual of G and S is an arbitrary Sylow $\pi(G/L)$-subgroup of G, then $G = LS$.

(2) *Let G be a non-periodic locally soluble group. Then every subgroup of G is pronormal if and only if G is abelian.*

Here a group G is **locally graded** if every non-trivial finitely generated subgroup of G has a proper subgroup of finite index.

We notice that non-periodic locally graded groups whose subgroups are pronormal are abelian, as D. J. S. Robinson, A. Russo and G. Vincenzi proved in [58].

Groups whose subgroups are nearly pronormal were studied by Kurdachenko, Russo and Vincenzi. In [42] they obtained another characterization of $\bar{T}$-groups.

Theorem 1.45. *Let G be a locally radical group.*

(i) *If every cyclic subgroup of G is nearly pronormal, then G is a $\bar{T}$-group.*

(ii) *If every subgroup of G is nearly pronormal, then every subgroup of G is pronormal in G.*

We end this survey with a description of groups whose cyclic subgroups are transitively normal. These groups have been considered by L. A. Kurdachenko and J. Otal in the paper [32]. The main results are the following.

Theorem 1.46. *Let G be a locally finite group whose cyclic subgroups of order either a prime or 4 are transitively normal. Then G is hypercyclic, the locally nilpotent residual L of G is an abelian Hall subgroup of G, and every subgroup of L is G-invariant. In particular, a locally finite group whose cyclic subgroups are transitively normal is a $\bar{T}$-group.*

In non-periodic groups the situation is rather different.

Theorem 1.47. *Let G be a locally generalized radical group whose cyclic subgroups are transitively normal. Suppose that G is non-abelian and non periodic. Then G contains a normal abelian subgroup L and an element b such that $G = L\langle b\rangle$, $b^2 \in L$, $x^b = x^{-1}$ for every $x \in L$. Furthermore:*

- if $b^2 = 1$, then the Sylow 2-subgroup D of L is elementary abelian;

- if $b^2 \neq 1$, then either the Sylow 2-subgroup D of L is elementary abelian or $D = E \times \langle v\rangle$ where E is elementary abelian and $\langle b, v\rangle$ is a quaternion group. In both cases, $\langle D, b\rangle$ is a Dedekind group.

Conversely, if G has the above structure, then every cyclic subgroup of G is transitively normal.

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