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SOME GROUP-THEORETICAL APPROACHES TO SKEW LEFT BRACES

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Abstract. The algebraic structure of skew left brace has become a useful tool to construct set-theoretic solutions of the Yang-Baxter equation. In this survey we present some descriptions of skew left braces in terms of bijective derivations, triply factorised groups, and regular subgroups of the holomorph of a group, as well as some applications of these descriptions to the study of substructures, nilpotency, and factorised skew left braces.

1. Introduction

The Yang-Baxter equation (YBE for short) introduced in seminal works of Yang [21] and Baxter [7] is one of the basic equations in mathematical physics and led to the foundations of the theory of quantum groups. It also appears in topology and algebra above all for its connections with braid groups and Hopf algebras.

In order to find new solutions of the YBE, Drinfeld [10] posed the question of studying the set-theoretic solutions. This paper stimulated a lot of interest in developing some algebraic tools.

Recall that a *set-theoretic solution of the YBE* is a pair (X, r), where X is a non-empty set and $r\colon X \times X \to X \times X$ is a map such that

$$r_{12}r_{23}r_{12} = r_{23}r_{12}r_{23},$$

with the maps, $r_{12} = r \times \mathrm{id}_X$ and $r_{23} = \mathrm{id}_X \times r$.

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An important class of set-theoretic solutions consists of the involutive non-degenerate ones, i.e., solutions (X, r) such that $r^2 = \mathrm{id}_{X\times X}$ and the first and the second projections are bijective. It was in determining this kind of solutions that Rump introduced in [17] the left brace structure. It turns out that left braces characterise completely involutive non-degenerate solutions of YBE (see [3]).

Guarnieri and Vendramin [13] generalised left braces to skew left braces and this structure was used to produce and study non-degenerate bijective solutions, not necessarily involutive. Skew braces are useful for studying regular subgroups and Hopf-Galois extensions, rings and triply factorised groups (see [19]).

Recall that a *skew left brace* $(B, +, \cdot)$ is defined to be a set B endowed with two group structures $(B, +)$ (*the additive group*) and $(B, \cdot)$ (*the multiplicative group*) satisfying the following property:

$$a(b + c) = ab - a + ac, \quad \text{for every } a, b, c \in B. \tag{1.1}$$

Let $\mathfrak{X}$ be a class of groups. If $(B, +)$ belongs to $\mathfrak{X}$, then B is called a skew left brace *of* $\mathfrak{X}$*-type*. Rump's braces are exactly the skew left braces of abelian type. We call them simply left braces.

Here we use the convention of omitting the multiplication sign and, in absence of parentheses, the products are evaluated before the sums.

In this survey we will present descriptions of skew left braces in terms of bijective derivations and in terms of triply factorised groups and regular subgroups of the holomorph of their additive group.

We describe substructures of skew left braces and obtain some results about nilpotency of skew left braces in terms of nilpotency properties of the multiplicative group. In particular, we obtain an analogue for the Fitting subgroup with respect to the left nilpotency for left braces. Some of these results have been proved by the first and the second author in [4].

For a class of groups $\mathfrak{X}$, we say that a group is an $\mathfrak{X}$YB-group if it is the multplicative group of a skew left brace of $\mathfrak{X}$-type. We use the description in terms of regular subgroups of the holomorph to obtain sufficient conditions for a product of two $\mathfrak{X}$YB-groups to be an $\mathfrak{X}$YB-group. These results form part of [6].

2. **Skew Left Braces**

First of all, we note that in a skew left brace, the neutral elements for $(B, +)$ (call it 0) and $(B, \cdot)$ (call it 1) coincide:

$$\begin{aligned} 1 &= 0 + 1 = (0 + 0) + 1 = 1(0 + 0) + 1 \\ &= 1 \cdot 0 - 1 + 1 \cdot 0 + 1 = 0 - 1 + 0 + 1 = -1 + 1 = 0. \end{aligned}$$

Proposition 2.1. [13, Proposition 1.9] *If B is a skew left brace, then $\lambda\colon (B, \cdot) \longrightarrow \mathrm{Aut}(B, +)$ defined by $\lambda(a) = \lambda_a$, $a \in B$, where $\lambda_a(b) = -a + ab$ for all a, $b \in B$, is an action of $(B, \cdot)$ on $(B, +)$. This action is called the* lambda map *of B.*

Each operation of a skew left brace can be recovered from the other one and the lambda map. More precisely, if a, $b \in B$, then $ab = a + \lambda_a(b)$ and $a + b = a \cdot \lambda_{a^{-1}}(b) = a \cdot \lambda_a^{-1}(b)$.

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Skew left braces are closely related to the Yang-Baxter equation: if $(B,+,\cdot)$ is a skew left brace, then the map $r\colon B^2 \longrightarrow B^2$ defined by $r(x,y) = (-x+xy,(x^{-1}+y)^{-1}y)$ provides a solution of the YBE, the *solution of the YBE associated to the skew left brace* B. Futhermore, r is involutive if and only if B is a left brace (see [13, Theorem 3.1]).

3. Skew left braces and derivations

Suppose that $(B,+,\cdot)$ is a skew left brace. Let us denote by $K=(B,+)$ its additive group and by $C=(B,\cdot)$ its multiplicative group. Denote by $\delta\colon C \longrightarrow K$ the identity map of B. Recall that we have the action $\lambda\colon C \longrightarrow \operatorname{Aut}(K)$. With this notation, the equality $ab = a+\lambda_a(b)$ for a, $b \in B$ can be written as $\delta(ce) = \delta(c)+\lambda_c(\delta(e))$ for c, $e \in C$. Hence we have that δ is a bijective derivation or 1-cocycle with respect to λ. More in general, we have:

Theorem 3.1. *Suppose that $(A,+)$ and $(B,\cdot)$ are groups and that there exists an action $\lambda\colon (B,\cdot) \longrightarrow \operatorname{Aut}(A,+)$ and that $\delta\colon (B,\cdot) \longrightarrow (A,+)$ is a bijective derivation with respect to λ. Then we can define an addition on B via $b+c = \delta^{-1}(\delta(b)+\delta(c))$ such that $(B,+,\cdot)$ becomes a skew left brace.*

If $(C,\cdot)$ and $(K,+)$ are two groups and $\delta\colon C \longrightarrow K$ is a bijective derivation associated to an action λ of C on K, the image $\delta(E)$ of a subgroup E of C is not a subgroup of K in general, and if L is a subgroup of K, in general we do not have that $\delta^{-1}(L)$ is a subgroup of C. We have a positive result for preimages of subgroups of K that are invariant under the action of C.

Lemma 3.2. *Let $(C,\cdot)$ and $(K,+)$ be two groups. Suppose that $\delta\colon C \longrightarrow K$ is a derivation associated to an action λ of C on K and L is a C-invariant subgroup of K (for instance, this happens when L is a characteristic subgroup of K). Then $\delta^{-1}(L)$ is a subgroup of C.*

As an application of this result, suppose that $(B,+,\cdot)$ is a finite skew left brace with $K=(B,+)$ nilpotent. For every set of primes π, K has a characteristic Hall π-subgroup K_π. Then $C_\pi = \delta^{-1}(K_\pi)$ is by, Lemma 3.2, a Hall π-subgroup of $C=(B,\cdot)$. As a consequence of a well-known theorem of Hall (see, for instance, [9, Chapter I, Theorem 3.6]), C is soluble. This generalises the following result of Etingof, Schedler, and Soloviev.

Theorem 3.3. [12] *The multiplicative group $(B,+)$ of a finite left brace is soluble.*

4. Triply Factorised Groups

The approach to skew left braces via triply factorised groups follows an idea of Sysak [20] for left braces. Let $(B,+,\cdot)$ be a skew left brace and, as before, $K=(B,+)$ and $C=(B,\cdot)$. We have the action $\lambda\colon C \longrightarrow \operatorname{Aut}(K)$ and we can construct the semidirect product

$$G = [K]G = \{(k,c) \mid k \in K, c \in C\}$$

with respect to this action. The operation in this group is

$$(k_1,c_1)(k_2,c_2) = (k_1+\lambda_{c_1}(k_2), c_1c_2)$$

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for k_1, $k_2 \in K$, c_1, $c_2 \in C$.

Lemma 4.1. [4, Lemma 3.1] *Let $\delta\colon C \longrightarrow K$ be the bijective derivation associated to λ. The set $D = \{(\delta(c), c) \mid c \in C\}$ is a subgroup of the semidirect product $G = [K]C$ such that $G = KD = DC$ and $K \cap D = D \cap C = \{(0,1)\}$.*

This means that the semidirect product $G = [K]C$ becomes a triply factorised group or trifactorised group. Note that $\alpha\colon C \longrightarrow D$ given by $\alpha(c) = (\delta(c), c)$, $c \in C$, defines a group isomorphism. This allows us to use results about factorised and trifactorised groups, like the ones appearing in [2] or [5]. The following result is a sample.

Theorem 4.2. *Suppose that the multiplicative group of a skew left brace is nilpotent. Then its additive group is soluble.*

Proof. Note that since $C \cong D$, C and D are nilpotent. By a result of Kegel and Wielandt, $G = CD$ is soluble. Hence $K \leqslant G$ is soluble. □

We can describe images and preimages by δ in this setting.

Lemma 4.3. [4, Lemma 3.3] *Let $G = [K]C = KD = DC$ with $K \cap D = D \cap C = \{(0,1)\}$.*

(1) *If $L \subseteq K$, then $\delta^{-1}(L) = (-L)D \cap C$.*
(2) *If $E \subseteq C$, then $\delta(E) = DE^{-1} \cap K$.*

In the sequel, we will use multiplicative notation in the semidirect product. In particular, given k, $l \in K$ and $c \in C$, $(k+l, 1)$ will be denoted as kl, $(-k, 1)$ as k^{-1}, and $(\lambda_c(k), 1)$ by $ckc^{-1} = k^{c^{-1}}$ (here we denote the conjugate by $u^g = g^{-1}ug$).

The following elementary property of commutators is crucial in our treatment. Here the commutator $[g, h]$ denotes $g^{-1}h^{-1}gh$ for g, $h \in G$.

Lemma 4.4. [4, Lemma 3.4] *Let G be a group and let k, l, c, $e \in G$. Then*

$$[kc, le] = [k, e]^c [k, l]^{ec} [c, l]^{c^{-1}ec} [c, e].$$

Lemma 4.4 is especially interesting when we consider $G = [K]C$, c, $e \in C$, $k = \delta(c)$, and $l = \delta(e)$.

Lemma 4.5. [4, Lemma 3.5] *Let $G = [K]C = KD = DC$ iwth $D \leqslant G$, $K \cap D = D \cap C = \{1\}$, and let $\delta\colon C \longrightarrow K$ be the corresponding derivation. Let c, $e \in C$, $k = \delta(c)$, $l = \delta(e)$. Then*

$$\delta([c, e]) = [k, e]^c [k, l]^{ec} [c, l]^{c^{-1}ec}.$$

Proof. In this case, kc, $le \in D$, and so

$$[kc, le] = [k, e]^c [k, l]^{ec} [c, l]^{c^{-1}ec} [c, e] \in D.$$

Since $[k, e]^c [k, l]^{ec} [c, l]^{c^{-1}ec} [c, e] \in K$ and $[c, e] \in C$, we obtain the conclusion. □

The following result appears as an immediate consequence of this fact.

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Theorem 4.6. [4, Theorem 3.6] *Let $G = [K]C = KD = DC$ with $D \leqslant G$, $K \cap D = D \cap C = \{1\}$, and let $\delta\colon C \longrightarrow K$ be the corresponding derivation. Suppose that H is a subgroup of K such that H is normalised by C. Let c, $e \in C$, $k = \delta(c)$, $l = \delta(e)$. Suppose that three of the elements $[k, e]$, $[k, l]$, $[c, l]$, and $\delta([c, e])$ belong to H. Then so does the other one.*

The following result will be handy when in the study of the ideal substructure of a brace.

Lemma 4.7. [4, Lemma 3.7] *Let $G = [K]C = KD = DC$ with $D \leqslant G$, $K \cap D = D \cap C = \{1\}$, and let $\delta\colon C \longrightarrow K$ be the corresponding derivation. Suppose that $E \leqslant C$ and $L = \delta(E) \trianglelefteq G$. Then the following are equivalent:*

(1) $E \trianglelefteq C$.
(2) $[E, C] \subseteq E$.
(3) $[K, E] \subseteq L$.

5. Substructures of Skew Left Braces

In this section we will recall the definitons of some substructures of skew left braces and we will interpret them in the semidirect product $G = [K]C$ of its additive group by its multiplicative group. These interpretations appear in [4].

Some substructures of skew left braces are defined in terms of the *star* operation of the brace.

Definition 5.1. *Let B be a skew left brace. Given a, $b \in B$, we define $a * b = -a + ab - b = \lambda_a(b) - b$.*

If, in this definition, a is regarded as an element of $C = (B, \cdot)$ and b is regarded as an element of $K = (B, +)$, then $a * b$ corresponds in $G = [K]C$ to the element $aba^{-1}b^{-1} = [a^{-1}, b^{-1}] \in [C, K] \subseteq K$.

Definition 5.2. *Given X, $Y \subseteq B$, we denote by $X * Y$ the subgroup of K generated by $\{x * y \mid x \in X, y \in Y\}$.*

If X corresponds to a subgroup E of C and Y to a subgroup H of K, this can be identified with the subgroup

$$\langle\{[e^{-1}, h^{-1}] \mid e \in E, h \in H\}\rangle = [E, H] \leqslant K.$$

Definition 5.3. *Let $(B, +, \cdot)$ be a skew left brace. A subgroup I of K is said to be a* left ideal *of B if $B * I$ is a subgroup of I, that is, $\lambda_a(I) \subseteq I$ for all $a \in I$. Furthermore, a left ideal I is called* strong left ideal *if I is a normal subgroup of K.*

Note that if I is a left ideal of B corresponding to $L \leqslant K$, then L is C-invariant and so $E = \delta^{-1}(L) \leqslant C$ and $[L, C] \subseteq L$. Moreover, if I is a strong left ideal of B, then $L \trianglelefteq G$.

Definition 5.4. *An* ideal *of the skew left brace $(B, +, \cdot)$ is a left ideal I of B such that $aI = Ia$ and $a + I = I + a$ for all $a \in B$.*

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Ideals of skew left braces are true analogues of normal subgroups in groups and ideals in rings. In fact, if I is an ideal of B, we can construct the quotient skew left brace B/I. Moreover, suppose that the left ideal I corresponds to $L \leqslant K$ and to $E = \delta^{-1}(L) \leqslant C$. Then I is an ideal of B if, and only if, $LE \trianglelefteq G$.

6. Factorisations of Braces and Nilpotency

Let X, Y be two subsets of a left brace $(B, +, \cdot)$. We define inductively

$$\mathrm{L}_0(X,Y) = Y; \qquad \mathrm{L}_n(X,Y) = X * \mathrm{L}_{n-1}(X,Y) \quad (n \geqslant 1);$$
$$\mathrm{R}_0(X,Y) = X, \qquad \mathrm{R}_n(X,Y) = \mathrm{R}_{n-1}(X,Y) * Y \quad (n \geqslant 1).$$

We have that, in the semidirect product $G = [K]C$,

$$\mathrm{L}_n(X,Y) = [[Y,X],X],\dots,X] = [Y,X,\dots,X] \quad (X \text{ appears } n \text{ times})$$

where X is regarded as a subgroup of C and Y as a subgroup of K. We also note that $\mathrm{L}_n(B,B) = B^{n+1}$ for all n, the terms of the radical series of B defined by Rump in [17].

Definition 6.1. *A left brace $(B, +, \cdot)$ is called* left nilpotent *if $\mathrm{L}_n(B,B) = 0$ for some $n \in \mathbb{N}$.*

A theorem of Smoktunowicz [18] states that a finite left brace is left nilpotent if, and only if, the multiplicative group $(B, \cdot)$ is nilpotent. We prove that for skew left braces of nilpotent type (that is, with nilpotent additive group), Smoktunowicz's result can be generalised. We denote by B_π the left ideal of B corresponding to the Hall π-subgroup of the additive group of B.

Definition 6.2. *Let π be a set of primes. We say that a skew left brace $(B, +, \cdot)$ is* left π-nilpotent *if for some n we have that $\mathrm{L}_n(B, B_\pi) = 0$.*

The following result extends [15, Theorem 14] and [1, Theorem 6.4] to a set of primes π.

Theorem 6.3. [4, Theorem 5.4] *Let $(B, +, \cdot)$ be a skew left brace of nilpotent type. Suppose that $C = (B, \cdot)$ has a nilpotent Hall π-subgroup. Then B is left π-nilpotent if, and only if, C is π-nilpotent.*

The proof of this result depends on Theorem 4.6 and on the following theorem about trifactorised groups that appears as a consequence of Theorem 6.5.4 and the remarks after its proof in [2].

Theorem 6.4. *Let $\mathfrak{F}$ be a saturated formation of finite groups, and let the group $G = AB = AK = BK$ be the product of three subgroups A, B, and K, where K is normal in G. If A and G are $\mathfrak{F}$-groups and K is nilpotent, then G is an $\mathfrak{F}$-group.*

The next result can be regarded as an analogue for left braces of the classical theorem of Fitting that asserts that the product of two nilpotent normal subgroups is again nilpotent.

Theorem 6.5. [4, Theorem 6.12] *Suppose that a left brace $(B, +, \cdot)$ can be decomposed as the sum of two ideals that are left nilpotent as left braces. Then B is left nilpotent.*

This result allows to define a left-Fitting ideal of a finite left brace.

Definition 6.6. [4, Definition 6.14] *Given a finite left brace $(B,+,\cdot)$, the* left-Fitting ideal $\mathrm{l\text{-}F}(B)$ *of B is the largest ideal that, as a left brace, is left nilpotent. It coincides with the ideal generated by all ideals of B that, as left braces, are left nilpotent.*

Now we give an interpretation of $\mathrm{R}_n(X,Y)$ in terms of iterated commutators. Assume that X corresponds to $E \leqslant C$ and that Y corresponds to $H \leqslant K$. Then $\mathrm{R}_2(X,Y) = X * Y$ corresponds to $[E,H] \leqslant K$. However, to compute $\mathrm{R}_3(X,Y) = (X*Y)*Y$, it is convenient to consider $X*Y$ as a subgroup of C, namely $\delta^{-1}([E,H])$. Hence we can identify $\mathrm{R}_3(X,Y)$ with $\delta^{-1}([\delta^{-1}([E,H]),H])$. By induction, $\mathrm{R}_n(X,Y)$ can be identified with

$$\delta^{-1}([\cdots[\delta^{-1}([E,H]),H],\dots,H]),$$

with n commutators (and $\mathrm{R}_0(X,Y) = X$ with $E \leqslant C$).

Definition 6.7. *Let $(B,+,\cdot)$ be a skew left brace. We say that B is* right nilpotent *if $\mathrm{R}_n(B,B) = 0$ for some $n \in \mathbb{N}$.*

We generalise this concept to a set of primes π as follows.

Definition 6.8. *If π is a set of primes, we say that the brace B is* right π-nilpotent *when for some n we have that $\mathrm{R}_n(B_\pi,B) = 0$.*

We prove the following result.

Theorem 6.9. [4, Theorem 5.6] *Suppose that $(B,+,\cdot)$ is a skew left brace of nilpotent type, the Hall π-subgroup $G_\pi = K_\pi C_p i$ of the trifactorised group associated with B is nilpotent, and that C_π is an abelian normal Hall π-subgroup of C. Then B is right π-nilpotent.*

Our proof of this result depends strongly on Theorem 4.6.

We have not been able to prove or disprove the existence of a right Fitting-like ideal, more precisely, whether or not a brace generated by two ideals that are right nilpotent as left braces is again right nilpotent. However, we have a positive answer when one of the ideals is trivial as a left brace. We have the following slightly more general result:

Theorem 6.10. [4, Theorem 6.15] *Let $(B,+,\cdot)$ be a left brace that can be factorised as the product of an ideal I_1 that is trivial as a left brace and a strong left ideal I_2 that is rigth nilpotent as a left brace. Then B is right nilpotent.*

Other results about factorisations of skew left braces that appear in [14] have been revisited in [4] in terms of trifactorised groups.

7. Regular Subgroups of the Holomorph

Let $(G,+)$ be a group. The holomorph $\mathrm{Hol}(G) = [G]\mathrm{Aut}(G)$ is the semidirect product of G with its automorphism group. We will let automorphism act on the left. The operation in $\mathrm{Hol}(G)$ is given by

$$(g,\varphi)(h,\psi) = (g+\varphi(h), \varphi\psi)$$

for every (g,φ), $(h,\psi) \in \mathrm{Hol}(G,+)$. The group $\mathrm{Hol}(G)$ acts on G in the following way: $(h,\psi) * g = h+\psi(g)$ for $(h,\psi) \in \mathrm{Hol}(G)$, $g \in G$. We say that a subgroup $H \leqslant \mathrm{Hol}(G)$ is *regular* if it is regular with respect to this action. This is equivalent to saying that for every $g \in G$ there exists a unique $\varphi_g \in \mathrm{Aut}(G)$ such that $(g,\varphi_g) \in H$. The following result was proved by Guarnieri and Vendramin [13, Theorem 4.2].

Proposition 7.1. *If $(B,+,\cdot)$ is a skew left brace, then $H = \{(b,\lambda_b) \mid b \in B\}$ is a regular subgroup of $\mathrm{Hol}(B,+)$ isomorphic to $(B,\cdot)$.*

Conversely, suppose that for a group $(B,+)$ we have a regular subgroup $H \leqslant \mathrm{Hol}(B,+)$. Then we can define on B a binary operation $bc := b+\varphi_b(c)$, with $(b,\varphi_b) \in H$, such that $(B,+,\cdot)$ becomes a skew left brace and $(B,\cdot)$ is isomorphic to H.

We observe that in this case the map $\pi\colon H \longrightarrow (B,+)$ given by $\pi(b,\lambda_b) = b$, $b \in B$, is a bijective derivation with respect to the action $\bar{\lambda}\colon H \longrightarrow \mathrm{Aut}(B,+)$ given by $\bar{\lambda}(b,\lambda_b) = \lambda_b$, the projection on the second component. The computation of the regular subgroups of the holomorph of a group $(B,+)$ is a way of obtaining all skew left braces with this additive group, as shown by Guarnieri and Vendramin in [13, Algorithm 5.1].

8. Yang-Baxter Groups

The fact that the multiplicative group of a skew left brace of nilpotent type is soluble motivates the study of the groups that can appear as the multiplicative group of a skew left brace of nilpotent type. More precisely, a skew left brace of nilpotent type can be expressed as the sum of skew left subbraces associated to the Sylow subgroups. In general, if $\mathfrak{X}$ is a class of groups and the additive group $(B,+)$ of the skew left brace $(B,+,\cdot)$ belongs to $\mathfrak{X}$, we say that $(B,+,\cdot)$ is a skew left brace of $\mathfrak{X}$-type. In particular, Rump's braces are exactly the skew left braces of abelian type. The groups that appear as multiplicative groups of left braces (of abelian type) are called *involutive Yang-Baxter groups*, or *IYB-groups*, and have received a lot of attention in the literature. For instance, these are the groups that can appear as the permutation group of an involutive, non-degenerate, set-theoretic solution of the YBE. Cedó, Jespers, and del Río [8] and Eisele [11] have made interesting contributions in this topic. A common extension of the reuslts of these authors has been presented in [16]. We present in [6] the following extension of the notion of IYB-group.

Definition 8.1. *Let $\mathfrak{X}$ be a class of groups. We say that a group G is an* $\mathfrak{X}$-Yang-Baxter group *($\mathfrak{X}$YB-group, for short) if G is isomorphic to the multiplicative group of a skew left brace of $\mathfrak{X}$-type.*

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The trivial skew braces show that every $\mathfrak{X}$-group is an $\mathfrak{X}$YB-group, and IYB-groups are exactly the $\mathfrak{A}YB$-groups for the class $\mathfrak{A}$ of all abelian groups.

When we deal with $\mathfrak{X}$YB-groups, actions of groups on braces become crucial. Here we will use juxtaposition for the product in the brace and $\cdot$ for the action.

Definition 8.2. *Let A be a group and let $(B,+,\cdot)$ be a skew left brace of $\mathfrak{X}$-type. We say that A* acts on the skew left brace $(B,+,\cdot)$ *if there is an action of A on the set B such that $a\cdot(g+h)=a\cdot g+a\cdot h$ and $a\cdot(gh)=(a\cdot g)(a\cdot h)$ for all $g,\ h\in B$, in other words, the action of A on the set B is actually an action of A on the group $(B,+)$ and an action of A on the group $(B,\cdot)$.*

An action of a group A on an $\mathfrak{X}$YB-group G for which it is understood that the associated skew left brace is $(G,+,\cdot)$ is said to be equivarint *if A acs on the skew left brace $(G,+,\cdot)$.*

The main result of [6] is the following one, that is also valid for infinite groups.

Theorem 8.3. *Assume that $\mathfrak{X}$ is a class of groups closed under taking quotients and direct products. Let the group $G=NH$ be the product of its subgroups N and H with $N\trianglelefteq G$ and $N\cap H\leqslant \mathrm{Z}(N)$. Suppose that N and H are both $\mathfrak{X}$YB-groups with, respectively, associated skew left braces $(N,+,\cdot)$ and $(H,+,\cdot)$ satisfying the following conditions:*

1. $N\cap H\leqslant \mathrm{Ker}(N)\cap\mathrm{Ker}(H)$.
2. $(N\cap H,+)\leqslant \mathrm{Z}(N,+)\cap\mathrm{Z}(H,+)$.
3. *The action of H on N by conjugation in G is equivariant.*

The proof of this result uses the description of skew left braces in terms of regular subgroups of the holomorph of their additive group. A consequence of Theorem 8.3 is the following one.

Corollary 8.4. [6] *Let N and H be $\mathfrak{X}$YB-groups and let $G=NH$ be a group satisfying the conditions of Theorem 8.3. Suppose that a group A acts on G so that the actions on N and H are equivariant. Then the action of A on G is also equivariant.*

We can obtain the following general version of [16, Theorem A].

Corollary 8.5. [6] *Let the group $G=NH$ be the product of (not necessarily finite) subgroups N and H with $N\trianglelefteq G$ and $N\cap H\leqslant \mathrm{Z}(N)$. Suppose that N and H are both IYB-groups with, respectively, associated skew left braces $(N,+,\cdot)$ and $(H,+,\cdot)$ satisfying the following conditions:*

1. $N\cap H\leqslant \mathrm{Ker}(N)\cap\mathrm{Ker}(H)$.
2. *The action of H on N by conjugation in G is equivariant.*

Then G is an IYB-group such that $\mathrm{Ker}(N)\mathrm{C}_{\mathrm{Ker}(H)}(N)\leqslant\mathrm{Ker}(G)$.

As a consequence, [8, Theorem 3.3] and [11, Proposition 2.2] are also true for infinite IYB-groups.

Corollary 8.6. [6] *Let G be a (not necessarily finite) group such that $G=AH$, where A is an abelian normal subgroup of G and H is an IYB-subgroup of G such that $H\cap A\leqslant\mathrm{Ker}(H)$. Then G is an*

IYB-group. In particular, every semidirect product $[A]H$ of a finite abelian group A by an IYB-group H is also an IYB-group.

Corollary 8.7. [6] *Let $G = [N]H$ be a (not necessarily finite) group. If N and H are IYB-groups nad the action of H on N by conjugation in G is eqivariant, then G is an IYB-group.*

For arbitrary classes of groups $\mathfrak{X}$ that are closed under taking direct products and quotients, we have:

Corollary 8.8. [6] *Let G be a group such that $G = AH$, where $A \in \mathfrak{X}$ is a normal subgorup of G and H is an $\mathfrak{X}$YB-subgroup of G with associated skew left brace $(H, +, \cdot)$ such that $H \cap A \leqslant \mathrm{Ker}(H)$ and $H \cap A \leqslant \mathrm{Z}(A, \cdot) \cap (H, +)$. Then G is an $\mathfrak{X}$YB-group.*

Corollary 8.9. [6] *Let $G = [N]H$ be a group. If N and H are $\mathfrak{X}$YB-groups such that the action of H on N is equivariant, then G is an $\mathfrak{X}$YB-group. In particular, every semidirect product $[N]H$ of a group $N \in \mathfrak{X}$ by an $\mathfrak{X}$YB-group H is an $\mathfrak{X}$YB-group.*

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