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An Optimal Property of B-Bases for the Modified Richardson Method

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Abstract: A space with a normalized totally positive basis has a unique normalized B-basis. In computer-aided geometric design, normalized B-bases present optimal shape-preserving properties. More optimal properties of normalized B-bases were proved previously. This paper provides a new optimal property concerning the modified Richardson iterative method when applied to collocation matrices of normalized B-bases. Moreover, a similar optimal property is proved for the tensor product of normalized B-bases.

Keywords: modified Richardson method; B-bases; totally positive matrices

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1. Introduction

Spaces generated by univariate functions often possess totally positive bases, that is, bases whose collocation matrices satisfy the property of being totally positive and, therefore, having all their minors nonnegative (see Section 2). Totally positive matrices are also called totally nonnegative matrices and they belong to the theory of total positivity. This theory has more than one century of history (see [1–4]). Several applications of totally positive matrices can be seen in the following books or surveys [5–11]. These applications arise in many different fields. For instance, in approximation theory (e.g., [12]), differential equations (e.g., [11]), chemistry (e.g., [13]), combinatorics (e.g., [14]), mechanics (e.g., [11]), statistics (e.g., [15]), computer-aided geometric design (e.g., [16]), Markov chains (e.g., [17]), numerical analysis (e.g., [18,19]), quantum groups (e.g., [9,20]), accurate computations (e.g., [21]), economics (e.g., [22]) and biomathematics (e.g., [8]). A space with a totally positive basis always has a special totally positive basis that generates all remaining totally positive bases of the space by using totally positive matrices (see [23]). This special basis is called a B-basis.

In the field of computer-aided geometric design, normalized totally positive bases are the bases leading to representations with shape-preserving properties. If a space has a normalized totally positive basis, then it has a unique normalized B-basis. In the case of the space of polynomials of degree less than or equal to n on $[0, +\infty)$, the monomial polynomials $1, t, , t^2, \ldots, t^n$ form a B-basis. For the same space of polynomials on a compact interval, the Bernstein polynomials (see Section 2) form the normalized B-basis. A conjecture in the field of computer-aided geometric design proposed by Goodman and Said in [24] claims that the Bernstein basis has optimal shape-preserving properties in the sense that the control polygon is the closest in shape to the generated curve. In [25], Carnicer and Peña proved it, and in [23], they proved that the normalized B-basis is always the normalized totally positive basis with optimal shape-preserving properties. Other optimal properties of B-bases and normalized B-bases are recalled in Section 2, and this paper provides a new optimal property of normalized B-bases.

Another remarkable property of several subclasses of totally positive matrices (e.g., [26–31]) comes from the fact that for them, high relative accuratcy algorithms were found to solve fundamental linear algebra computations, such as the computation of their inverses, their eigenvalues, their singular values or the solutions of some systems of linear equations $Ax = b$ (where b has alternating signs). However, even for these subclasses of totally positive matrices, there are no accurate methods for solving *all* linear systems. Hence, the minimal conditioning property of collocation matrices of normalized B-bases is still important, as well as the use of alternative methods, such as iterative methods. Among the classical iterative methods used for totally positive linear systems, we can outline the Richardson method (cf. Example 4.1 of [32–34]), which has already played a key role in the progressive iterative approximation (PIA) method. The PIA (see [35]) has been frequently used in computer-aided geometric design and it converges when the bases are normalized totally positive. In this paper, we prove that the nonsingular collocation matrices of normalized B-bases provide the fastest convergence rate of the modified Richardson method among the corresponding collocation matrices of all normalized totally positive bases of the space.

Let us now present the paper's organization. Section 2 introduces some notations and auxiliary results for totally positive matrices, totally positive bases and B-bases. In particular, some optimal properties of B-bases and normalized B-bases are recalled, as well as many examples of B-bases and normalized B-bases. The modified Richardson iterative method is presented in Section 3, where the mentioned result on its fastest convergence speed for nonsingular collocation matrices of normalized B-bases is proved. In this section, it is also shown that the optimal parameter for the modified Richardson iterative method of a nonsingular collocation matrix of the normalized B-basis of a space U is the minimum among the optimal parameters of the corresponding collocation matrices of all NTP bases of U.

Although the concept of total positivity cannot be extended to multivariate spaces, in Section 4, we extend the result on the fastest convergence speed of the modified Richardson method for nonsingular collocation matrices of normalized B-bases to the case of tensor product spaces. Collocation matrices of tensor product totally positive bases lead to the Kronecker product of their corresponding collocation matrices. We extended the result in spite of the fact that the Kronecker product of totally positive matrices is not a totally positive matrix. Finally, Section 5 summarizes the main conclusions of the paper.

2. B-Bases and Optimal Properties

Let us introduce some matrix notations and basic definitions. A matrix is *totally positive* (TP) if all its minors are nonnegative (see [5,10]). As already commented in the introduction, TP matrices are also called totally nonnegative matrices in the literature and they arise in many scientific fields, such as approximation theory, differential equations, combinatorics, mechanics, statistics, computer-aided geometric design and biomathematics (e.g., [5,8,10]).

The previous matrix concept leads to a corresponding concept for systems of univariate functions. Let $u_0, \dots, u_n$ be functions of a space U of functions defined on $I \subseteq \mathbb{R}$. Given a sequence of arbitrary parameters $t_1 < \cdots < t_{n+1}$ in I, the corresponding collocation matrix at these parameters is given by

$$M(t_1, \dots, t_{n+1}) := \big(u_{j-1}(t_i)\big)_{1 \le i,j \le n+1}. \tag{1}$$

The system $(u_0, \dots, u_n)$ of functions defined on $I \subseteq \mathbb{R}$ is *totally positive* (TP) if all its collocation matrices $M(t_1, \dots, t_{n+1})$ are TP. A TP system of functions on I is called *normalized* (NTP) if $\sum_{i=0}^{n} u_i(t) = 1$ for all $t \in I$. NTP systems are frequently used in computer-aided geometric design [36] due to their nice shape-preserving properties (see [23,25]) in the sense that the curve imitates the shape of their control polygon. Observe that the collocation matrix of an NTP basis is *row stochastic* because it is nonnegative and the entries of each row sum to 1.

B-bases are TP bases that generate all TP bases of a space by means of TP matrices. The next characterization of a B-basis is a consequence of Corollary 3.10 of [23] and Proposition 3.11 of [23].

Theorem 1. *Let $(u_0, \ldots, u_n)$ be a TP basis of a given space U of univariate functions on I. Then, $(u_0, \ldots, u_n)$ is a B-basis if and only if for another TP basis $(v_0, \ldots, v_n)$ of U, the matrix K of the change in basis such that $(v_0, \ldots, v_n) = (u_0, \ldots, u_n)K$ is TP.*

The following result shows the existence of a B-basis in a space with a TP basis (which was proved in Remark 3.8 of [23]) and the existence and uniqueness of a normalized B-basis if the space has an NTP basis (which was proved in Theorem 4.2 (i) of [23]).

Theorem 2. *Let U be a space of univariate functions on an interval I. If U has a TP basis, then it has a B-basis, and, if U has an NTP basis, then it has a unique normalized B-basis.*

As we shall recall later, among all NTP bases of a given space, the normalized B-basis is the basis satisfying the optimal shape-preserving properties (cf. [23]). As first examples, we can mention that the Bernstein bases and the B-spline bases are the normalized B-bases of their corresponding spaces. As an example of a B-basis that is not normalized, we can mention the monomial basis of the space of polynomials of degree at most n on $I = [0, +\infty)$ (cf. [25]). At the end of the section, a list of B-bases and normalized B-basis is included.

Now we recall some *optimal* properties of B-bases from several points of view:

1. Normalized B-bases present optimal shape-preserving properties in the sense that their control polygon is closer in shape than the control polygon with respect to another NTP basis. In the case of the space of polynomials of degree at most n on $[0, 1]$, this property was first conjectured by Goodman and Said for the Bernstein basis in [24] and later proved by Carnicer and Peña in [25]. In [23], this optimal property was proved for all normalized B-bases.
2. B-bases are optimally stable with respect to the evaluation among all bases of nonnegative functions. This property was first proved for the Bernstein basis in [37] and later in [38], as well as for many more B-bases in [39].
3. B-bases are the least supported bases, as shown in [40].
4. Nonsingular collocation matrices of nomalized B-bases have a minimal condition number for $\|\cdot\|_\infty$ among the corresponding matrices of all NTP bases of the space, as shown in [41].

In Section 3, we show an additional property of the collocation matrices of normalized B-bases: for them, the modified Richardson method is the fastest among the corresponding matrices of all the NTP bases of the space.

A new field where TP matrices have shown their importance is that of accurate computations. Up to now, algorithms with a high relative accuracy for linear algebra problems have been found only for a few classes of structured matrices. Among them, we have some subclasses of TP matrices (e.g., [26–29,31]) for which there are high relative accuracy algorithms for computing all their eigenvalues, all their singular values, their inverses and some associated linear systems. In all cases, the starting point is the bidiagonal decomposition of the matrix in order to use the accurate algorithms of [42,43]. The fact that even for these subclasses of TP matrices, we do not have accurate algorithms for solving *all* linear systems shows the importance of the optimal condition property (mentioned above) of collocation matrices associated with normalized B-bases, as well as the possible convenience of using iterative methods, as shown in the next sections.

We now finish this section by including a list of B-bases and normalized B-bases that illustrate the many types of examples that arise in theoretical and practical applications, as well as the potential applicability of the results of this paper:

1. In the space $\mathcal{P}_n([a, +\infty))$ of polynomials of degree less than or equal to n on $[a, +\infty)$, the *monomial* basis $(1, (t-a), (t-a)^2, \ldots, (t-a)^n)$ is a B-basis (cf. Section 6 of [25]).

2. In the space $\mathcal{P}_n([a,b])$ of polynomials of degree less than or equal to n on a compact interval $[a,b]$, the normalized B-basis is the corresponding *Bernstein* basis of polynomials $(b_0^n(t;a,b),\dots,b_n^n(t;a,b))$, with

$$b_i^n(t;a,b)(t) := \binom{n}{i}\frac{(b-t)^{n-i}(t-a)^i}{(b-a)^n}, \quad i=0,1,\dots,n$$

(cf. [23,25]). The standard Bernstein basis used in computer-aided design is the Bernstein basis on $[0,1]$:

$$b_i^n(t) := \binom{n}{i}(1-t)^{n-i}(t)^i, \quad i=0,1,\dots,n.$$

3. Given a sequence $(w_i)_{0\le i\le n}$ of positive weights, we can define the following system of rational functions $(r_0^n,\dots,r_n^n)$ on the compact interval $[a,b]$ by

$$r_i^n(t) = \frac{w_i b_i^n(t;a,b)}{\sum_{j=0}^n w_j b_j^n(t;a,b)}, \quad i=0,1,\dots,n,$$

which is the normalized B-basis of the corresponding spanned space of functions. It is the rational Bernstein basis of its space. When the weight $w_i = 1$ for all i, then $(r_0^n,\dots,r_n^n) = (b_0^n(t;a,b),\dots,b_n^n(t;a,b))$ is the Bernstein basis on $[a,b]$.
4. In the space of exponential functions

$$\mathcal{E} := \{\sum_{i=0}^n c_i \exp(\lambda_i t) \,|\, c_i \in, i=0,\dots,n\}, \quad \lambda_0 < \cdots < \lambda_n, \quad t \in,$$

the basis $(\exp(\lambda_0 t),\dots,\exp(\lambda_n t))$ is a B-basis.
5. In the space of *Müntz polynomials* on $[0,+\infty)$, where

$$\mathcal{M} := \{\sum_{i=0}^n c_i t^{\lambda_i} \,|\, c_i \in, i=0,\dots,n\}, \quad \lambda_0 < \cdots < \lambda_n,$$

the generalized monomial basis $(t^{\lambda_0},\dots,t^{\lambda_n})$ is a B-basis. The restriction of Müntz polynomials to $[0,1]$ has a B-basis: the generalization of the Bernstein basis given by

$$p_i(t) = (-1)^{n-i}(\lambda_{i+1}-\lambda_0)(\lambda_{i+2}-\lambda_0)\cdots(\lambda_n-\lambda_0)t^{\lambda}[\lambda_i,\dots,\lambda_n],$$

where $t^{\lambda}[\lambda_i,\dots,\lambda_n]$ is the usual notation of divided differences of the function t^{λ}, that is, $t^{\lambda}[\lambda_i] := t^{\lambda_i}$ and

$$t^{\lambda}[\lambda_i,\dots,\lambda_{i+j}] := \frac{t^{\lambda}[\lambda_{i+1},\dots,\lambda_{i+j}] - t^{\lambda}[\lambda_i,\dots,\lambda_{i+j-1}]}{\lambda_{i+j}-\lambda_i}.$$

6. The space of trigonometric polynomials

$$\mathcal{T}_n = \{1, \cos t, \sin t, \cos 2t, \sin 2t, \dots, \cos nt, \sin nt\}$$

on $I = [-A, A]$ with $A < \frac{\pi}{2}$ has the normalized B-basis $(v_0,\dots,v_m)$, $m = 2n$, given by

$$v_i(t) = d_i\left(\frac{\sin\left(\frac{A+t}{2}\right)}{\sin A}\right)^i\left(\frac{\sin\left(\frac{A-t}{2}\right)}{\sin A}\right)^{m-i}, \quad i=0,1,\dots,m$$

where
$$d_i = \sum_{k=0}^{[i/2]} \binom{m/2}{i-k}\binom{i-k}{k}(2\cos A)^{i-2k}, \quad i = 0, 1, \ldots, m.$$

7. The space of even trigonometric functions given by
$$\mathcal{C}_n = span\{1, \cos\, t, \cos\, 2t, \ldots, \cos\, nt\}$$
on the compact interval $[0, \pi]$ has the normalized B-basis $(u_0^n, \ldots, u_n^n)$ defined by
$$u_i^n(t) = \binom{n}{i} \cos^{2(n-i)}(t/2) \sin^{2i}(t/2), \quad i = 0, 1, \ldots, n.$$

8. Given a sequence of positive weights $(w_i)_{0 \le i \le n}$ and a knots vector $(t_0, \ldots, t_{n+d})$ with $t_i \le t_{i+1}$ for all $i = 0, 1, \ldots, n+d-1$, we can define the B-spline basis $(N_{0,d}, N_{1,d}, \ldots, N_{n,d})$ over the previous knots vector by
$$N_{i,0}(t) = \begin{cases} 1, & \text{if } t_i \le t < t_{i+1}, \\ 0, & \text{otherwise,} \end{cases}$$
$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k} - t_i} N_{i,k-1}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} N_{i+1,k-1}(t), \quad k = 1, \ldots, d,$$
which is the normalized B-basis of the corresponding splines space (see [23]). The basis $(r_0, \ldots, r_n)$ defined by
$$r_i(t) = \frac{w_i N_{i,d}(t)}{\sum_{j=0}^n w_j N_{j,d}(t)}, \quad i = 0, 1 \ldots, n,$$
is the normalized B-basis of the corresponding NURBS space (see Section 4 of [23]).

9. The space given by
$$\mathcal{P}_1 = \text{span}\{1, t, \cos t, \sin t\}$$
on $[0, p]$, with $p < 2\pi$, has the normalized B-basis $(Z_0(t), Z_1(t), Z_2(t), Z_3(t))$. To give the expression of these functions, we use the following cycloidal functions:
$$\sin \mathrm{C}(\mathrm{t}) := \mathrm{t} - \sin \mathrm{t}, \quad \cos \mathrm{C}(\mathrm{t}) := 1 - \cos \mathrm{t}, \quad \tan \mathrm{C}(\mathrm{t}) := \frac{\sin \mathrm{C}(\mathrm{t})}{\cos \mathrm{C}(\mathrm{t})}.$$
Using these cycloidal functions, we now define the functions of the normalized B-basis:
$$\begin{aligned} Z_3(t) &= \frac{\sin \mathrm{C}(\mathrm{t})}{S}, \quad Z_0(t) = Z_3(p-t), \\ Z_2(t) &= M\left(\frac{\cos \mathrm{C}(\mathrm{t})}{C} - Z_3(t)\right), \quad Z_1(t) = Z_2(p-t), \end{aligned} \tag{2}$$
where
$$S = \sin \mathrm{C}(\mathrm{p}), \quad \mathrm{C} = \cos \mathrm{C}(\mathrm{p}), \quad \mathrm{T} = \tan \mathrm{C}(\mathrm{p}),$$
and
$$M = \begin{cases} 1 & \text{if } p = \pi, \\ \frac{\sin p}{p - 2T} & \text{otherwise.} \end{cases}$$

Finally, let us mention that for many spaces of the form
$$\mathcal{P}_n = \text{span}\{1, t, \ldots, t^n \cos t, \sin t\},$$

as well as for other mixed spaces generated by algebraic, trigonometric and even hyperbolic polynomials, the normalized B-bases were obtained, as can be seen in papers published by Carnicer, Mainar and Peña.

3. A New Optimal Property of Normalized B-Bases

This section considers Richardson iteration. Before presenting the method, we come back to its connection with the PIA method, which was already announced in the Introduction and for which Richardson iteration plays a key role (cf. [44]). The PIA method was used for curve and spline curve fitting 50 years ago (see [45,46]) and was later extended to curves generated by NTP bases. Other extensions of the method were derived (cf. [47,48]). Nowadays, the PIA is very powerful in computer-aided geometric design for curves and surface fitting.

Let us start by recalling the Richardson iterative method, which is a classical iterative method for solving linear systems of equations. If A is a nonsingular matrix, then the *Richardson method* for solving the linear system $Ax = b$ is given through the recurrence

$$x_{m+1} = x_m - Ax_m + b, \quad m = 0, 1, 2, \ldots \tag{3}$$

If we denote by y the new iteration coming from x, the method can be written in the form

$$y = (I - A)x + b, \tag{4}$$

where I denotes the identity matrix.

If one wants to accelerate the convergence of the method, then a relaxation of the method is often performed, replacing the x_{m+1} by $(1 - w)x_m + wx_{m+1}$. In the particular case of the Richardson method, this approach derives the *modified Richardson method*

$$y = (I - wA)x + wb, \tag{5}$$

The following result is contained in Theorem 2.3 of [44] and analyzes the convergence of the modified Richardson method.

Theorem 3. *If A is a nonsingular TP row-stochastic matrix, then the modified Richardson iterative method (5) converges to the solution of the system $Ax = b$ if and only if $w \in (0, 2)$.*

The following result shows that normalized B-bases satisfy an optimal property for the modified Richardson method in the sense that it converges faster for their nonsingular collocation matrices than for the corresponding collocation matrices of other NTP bases.

Theorem 4. *Let U be a given space of functions defined on an interval I with an NTP basis, let $(b_0, \ldots, b_n)$ be a normalized B-basis of U and let $w \in (0, 2)$. Then, the modified Richardson method for w and for any nonsingular collocation matrix of $(b_0, \ldots, b_n)$ has the fastest convergence rate among all the corresponding collocation matrices of all NTP bases of U.*

Proof. The existence of the normalized B-basis $(b_0, \ldots, b_n)$ is guaranteed by Theorem 1. If B is a nonsingular collocation matrix of $(b_0, \ldots, b_n)$, let A be the corresponding collocation matrix at the same points of another NTP basis $(u_0, \ldots, u_n)$.

By Corollary 6.6 of [5], a nonsingular TP matrix has all positive eigenvalues. Therefore, let us denote by $\lambda_{\min}(B)(> 0)$ and $\lambda_{\min}(A)(> 0)$ the minimal eigenvalues of the matrices B and A, respectively. Let us also take into account the fact that since both bases $(b_0, \ldots, b_n)$ and $(u_0, \ldots, u_n)$ are NTP, the matrices B and A are both row stochastic. Therefore, 1 is their maximal eigenvalue. Since A and B are nonsingular row-stochastic TP matrices and $w \in (0, 2)$, the modified Richardson method converges for both matrices by Theorem 3.

Looking at Equation (5), we conclude that the convergence speed of the modified Richardson methods for matrices B and A are related to $\rho(I - wB)$ and $\rho(I - wA)$, respectively. Let us see that $\rho(I - wB) \leq \rho(I - wA)$ and the result is proved.

Clearly, the eigenvalues of matrices $I - wB$ and $I - wA$ are of the form $1 - w\lambda$, where λ is an eigenvalue of B or A. Since $w > 0$, we have that $[1 - w, 1 - w\lambda_{\min}(B)]$ and $[1 - w, 1 - w\lambda_{\min}(A)]$ are the least intervals containing the spectrum of $I - wB$ and the spectrum of $I - wA$, respectively. Since A is a collocation matrix of an NTP basis and B is the corresponding collocation matrix of the normalized B-basis of the space, Corollary 2 of [41] implies that

$$\lambda_{\min}(A) \leq \lambda_{\min}(B). \tag{6}$$

Then, since $w > 0$, we deduce from (6) that

$$1 - w\lambda_{\min}(A) \geq 1 - w\lambda_{\min}(B). \tag{7}$$

On the other hand, we also have that

$$\rho(I - wA) = \max(|w - 1|, |1 - w\lambda_{\min}(A)|), \quad \rho(I - wB) = \max(|w - 1|, |1 - w\lambda_{\min}(B)|). \tag{8}$$

Let us assume first that $1 - w\lambda_{\min}(B) < 0$. Since $\lambda_{\min}(B) \leq 1$, we can deduce that $1 - w\lambda_{\min}(B) < 0$ implies that $w > 1$ and $|1 - w\lambda_{\min}(B)| = w\lambda_{\min}(B) - 1$. Hence, again taking into account the fact that $\lambda_{\min}(B) \leq 1$ and (8), we can deduce that

$$|1 - w\lambda_{\min}(B)| \leq w - 1 (= \rho(I - wB)) \leq \max(|w - 1|, |1 - w\lambda_{\min}(A)|) = \rho(I - wA)$$

and the result follows in this case.

It remains the case that $1 - w\lambda_{\min}(B) \geq 0$. Therefore, $|1 - w\lambda_{\min}(B)| = 1 - w\lambda_{\min}(B)$. By (7), we also have that $1 - w\lambda_{\min}(A) \geq 0$, and again, $|1 - w\lambda_{\min}(A)| = 1 - w\lambda_{\min}(A)$. Hence, the result follows from (7) and (8):

$$\rho(I - wA) = \max(|w - 1|, |1 - w\lambda_{\min}(A)|) \geq \max(|w - 1|, |1 - w\lambda_{\min}(B)|) = \rho(I - wB).$$

□

In the second part of Theorem 2.3 of [44], the parameter corresponding to the optimal (fastest) convergence speed for the modified Richardson method applied to a nonsingular totally nonnegative row-stochastic matrix is obtained, as the following result shows.

Theorem 5. *Let A be a nonsingular TP row-stochastic matrix. Then, the optimal convergence speed for the modified Richardson iterative method with $w \in (0, 2)$ corresponds to $\rho_{\text{opt}} = \rho(I - w_{\text{opt}}A)$ and is achieved for*

$$w_{\text{opt}} = \frac{2}{1 + \lambda_{\min}(A)}, \quad \rho_{\text{opt}} = \frac{1 - \lambda_{\min}(A)}{1 + \lambda_{\min}(A)}. \tag{9}$$

We finish this section with an observation derived from Theorem 5.

Remark 1. *If we denote by $w_{\text{opt}}(C)$ the parameter corresponding with the optimal convergence speed for the modified Richarson iterative method of a nonsingular totally nonnegative row-stochastic matrix C, by B a nonsingular collocation matrix of the normalized B-basis of a space U and by A the collocation matrix of another NTP basis of U, we can deduce from (6) and (9) that*

$$w_{\text{opt}}(B) \leq w_{\text{opt}}(A),$$

that is, the optimal parameter for the modified Richarson iterative method of a nonsingular collocation matrix of the normalized B-basis of a space U is the minimum among the optimal parameters of the corresponding collocation matrices of all NTP bases of U.

In spite of the fact that the concept of total positivity cannot be extended to multivariate spaces, in the following section, we extend Theorem 4 to the case of tensor product spaces.

4. Tensor Product Case

The practical interest of considering the Richardson iteration method for the Kronecker product of matrices can be illustrated again by the PIA method. For instance, in [49], the PIA method, and thus, the Richardson iteration method for the Kronecker product of matrices, was used for the approximation of tensor product surfaces.

We start by introducing the tensor product of systems of functions and the Kronecker product of matrices.

Let us consider two systems $u^1 = (u_0^1, \dots, u_m^1)$ and $u^2 = (u_0^2, \dots, u_n^2)$ of univariate functions defined on $[a,b]$ and $[c,d]$, respectively. Then, the system $u^1 \otimes u^2 := (u_i^1(x) \cdot u_j^2(y))_{i=0,\dots,m}^{j=0,\dots,n}$ is named a tensor product system and generates a tensor product surface. Now, we introduce the *Kronecker product* of two square matrices $A = (a_{ij})_{1\le i,j\le m}$ and $B = (b_{ij})_{1\le i,j\le n}$ as the matrix $A \otimes B$ defined as the $mn \times mn$ block matrix

$$A \otimes B := \begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mm}B \end{pmatrix}.$$

If we consider the collocation matrices $B_1 := (u_j^1(x_i))_{0\le i,j\le m}$ and $B_2 := (u_j^2(y_i))_{0\le i,j\le n}$ of u^1 and u^2, then $B_1 \otimes B_2$ is the collocation matrix of $u^1 \otimes u^2$ at $((x_i, y_j)_{j=0,\dots,n})_{i=0,\dots,m}$.

The following result shows that the tensor product of normalized B-bases also satisfies an optimal property for the modified Richardson method in the sense that it converges faster for their nonsingular collocation matrices than for the corresponding collocation matrices of other tensor products of NTP bases.

Theorem 6. *Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be two spaces of univariate functions on $[a,b]$ and $[c,d]$, respectively, both with NTP bases. Let $b^1 = (b_0^1, \dots, b_m^1)$ and $b^2 = (b_0^2, \dots, b_n^2)$ be the normalized B-bases of $\mathcal{U}_1$ and $\mathcal{U}_2$, respectively. Let $w \in (0,2)$. Then, the modified Richardson method for w and for any nonsingular collocation matrix of $b^1 \otimes b^2$ has the fastest convergence rate among all the corresponding collocation matrices of the tensor product of NTP bases of $\mathcal{U}_1$ and $\mathcal{U}_2$.*

Proof. Let $u^1 = (u_0^1, \dots, u_m^1)$ be any NTP basis on $[a,b]$ of $\mathcal{U}_1$ and let $u^2 = (u_0^2, \dots, u_n^2)$ be any NTP basis on $[c,d]$ of $\mathcal{U}_2$. The existence of the normalized B-bases b^1 and b^2 is guaranteed by Theorem 1. If B is a nonsingular collocation matrix of $b^1 \otimes b^2$, then $B = B_1 \otimes B_2$, where B_1 and B_2 are collocation matrices of b^1 and b^2, respectively. Let A be the corresponding collocation matrix at the same points of $u^1 \otimes u^2$, and thus, $A = A_1 \otimes A_2$, where A_1 and A_2 are collocation matrices of u^1 and u^2, respectively.

Since B_1 and B_2 are row-stochastic matrices, it follows from the definition of the Kronecker product that $B = B_1 \otimes B_2$ is also row stochastic and, by the same reason, $A = A_1 \otimes A_2$ is, again, row stochastic.

Given C_1, which is an $n \times n$ matrix, and C_2, which is an $m \times m$ matrix, consider $C_1 \otimes C_2$. If we have an eigenvalue λ of C_1 and an eigenvalue μ of C_2, then by Theorem 4.2.12 of [50], $\lambda\mu$ is an eigenvalue of $C_1 \otimes C_2$ and every eigenvalue of $C_1 \otimes C_2$ arises as a product of eigenvalues of C_1 and C_2. Then, since $B = B_1 \otimes B_2$ is nonsingular, it does not have a zero eigenvalue, and thus, B_1 and B_2 do not have zero eigenvalues either and they are nonsingular. Hence, A_1 and A_2 are nonsingular and the same property holds for $A = A_1 \otimes A_2$. Taking into account the fact that A_1, A_2, B_1 and B_2 are all nonsingular TP matrices, they have all positive eigenvalues and, since they are also row stochastic, their maximal eigenvalue is 1. In conclusion, the eigenvalues of A and B are also positive and their maximal eigenvalue is also 1. In particular, $\lambda_{\min}(A) > 0$ and $\lambda_{\min}(B) > 0$. The eigenvalues of $I - wB$ and $I - wA$ are $1 - w\lambda$, where λ is an eigenvalue of B or A. Since $w > 0$, then $[1 - w, 1 - w\lambda_{\min}(A)]$ is the least interval containing the spectrum of $I - wA$, and the same holds for B. Since $w \in (0,2)$, $1 - w > -1$, and so $\rho(I - wA) < 1$ and $\rho(I - wB) < 1$. Thus, the modified Richardson method converges for

A and for B. It remains to compare the convergence speeds. We only need to prove that $\rho(I - wB) \leq \rho(I - wA)$.

Since $w \in (0, 2)$, $[1 - w, 1 - w\lambda_{\min}(B)]$ and $[1 - w, 1 - w\lambda_{\min}(A)]$ are the least intervals containing the spectrum of $I - wB$ and the spectrum of $I - wA$, respectively. Taking into account the fact that by Theorem 1 (ii) of [51], $\lambda_{\min}(A) \leq \lambda_{\min}(B)$, and reasoning as in the end of Theorem 4, we can deduce that

$$\rho(I - wA) = \max(|w - 1|, |1 - w\lambda_{\min}(A)|) \geq \max(|w - 1|, |1 - w\lambda_{\min}(B)|) = \rho(I - wB),$$

and the result follows. □

Observe that in the proof of the previous theorem, we guaranteed the convergence of the modified Richardson method for $w \in (0, 2)$ in spite of the fact that the Kronecker product of TP matrices is not necessarily TP. In fact, if we consider the TP matrices

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

then we have that

$$A \otimes B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

is not TP because, for instance, its submatrix M formed by rows 3 and 4 and columns 2 and 3:

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

has a negative determinant: $\det M = -1 < 0$.

5. Conclusions

In addition to several known optimal properties of normalized B-bases of spaces of univariate functions, this paper provides a new one. The modified Richardson method converges faster for their nonsingular collocation matrices than for the corresponding collocation matrices of other normalized totally positive bases of the space. We also prove that the optimal parameter associated with the fastest convergence is smaller for the collocation matrices of the normalized B-basis than the optimal parameter associated with the fastest convergence of the corresponding collocation matrices of the remaining normalized totally positive bases of the space.

The tensor product of normalized B-bases presents, for the generated spaces of bivariate functions, a similar optimal property. The modified Richardson method converges faster for their nonsingular collocation matrices than for the corresponding collocation matrices of other tensor products of normalized totally positive bases. These last collocation matrices are given by the Kronecker product of the collocation matrices of the normalized totally positive bases.

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Abbreviations

The following abbreviations are used in this manuscript:

PIA	progressive iterative approximation
TP	totally positive
NTP	normalized totally positive

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