The truel game. An approach based on Markov chains and game theory.

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Resumen

El juego del *truelo* fue estudiado inicialmente por C. Kinnaird alrededor del año 1946, [11]. Sin embargo, fue Kilgour, [9], quien desarrolló en mayor profundidad las múltiples formas que puede tomar el juego y el interés que tiene en cuanto a la modelización del comportamiento de una población cuando se trata de la transmisión de opiniones.

La manera más intuitiva de entender el juego es plantearlo como una extensión del duelo tradicional, donde dos personas se enfrentan con el objetivo de que finalmente haya un único superviviente, pero en este caso con tres jugadores. En particular, en el truelo encontramos a tres jugadores, A, B y C, cuya puntería podemos cuantificar como las probabilidades a, b y c, respectivamente, entre 0 y 1 que tienen de acertar al contrincante al que decidan disparar.

El juego comienza con tres jugadores y, dependiendo de la variante del juego, se elige a uno de ellos utilizando un cierto criterio (orden aleatorio, primero el más débil, etc.) para que dispare. A continuación, éste decide cuál es su objetivo (o incluso puede decidir apuntar al aire) y dispara, pudiendo eliminarlo o fallar y que ambos vuelvan a participar en la siguiente ronda. Independientemente de si el jugador consigue acertar o no, otro es elegido para que dispare y así sucesivamente hasta que sólo quede vivo uno de los tres.

En este trabajo se utilizan, por un lado, la teoría de juegos y, por otro lado, las cadenas de Markov para deducir la estrategia óptima que deben tomar los participantes del truelo así como para estudiar ciertos resultados paradójicos que surgen cuando los participantes adoptan dicha estrategia.

El primer capítulo, que es el más extenso, recopila en primer lugar una amplia terminología de teoría de juegos necesaria para definir los conceptos de *estrategia pura* y *equilibrio de Nash*. En segundo lugar, se proponen y demuestran diferentes resultados sobre cadenas de Markov que más adelante se utilizan para el estudio de la probabilidad de supervivencia que posee cada jugador y el tiempo esperado de la duración de una partida. La idea principal es que se plantean cadenas cuyos estados son las diversas combinaciones de participantes del juego que puede haber vivos en cada momento (y en alguno de los casos también incluyen a qué jugador le toca disparar). Seguidamente, se plantea de forma breve la modelización de un duelo con cadenas de Markov. Por último, la parte más extensa del capítulo 1 se dedica al estudio del truelo en dos variantes diferentes: aleatorio, donde en cada ronda se decide aleatoriamente qué jugador tiene el turno para disparar, y secuencial, donde mediante un orden preestablecido (en este trabajo disparan en orden ascendente de puntería, es decir, empieza el más débil y acaba el más hábil) los jugadores van tomando su turno en cada ronda.

Mientras que en el duelo es obvio que a cada jugador lo que más le conviene es disparar a su contrincante para que su probabilidad de supervivencia sea máxima, en el juego del truelo no está tan claro a quién es más ventajoso disparar. Aquí es donde adquiere relevancia el concepto del equilibrio de Nash, pues es el que proporciona una estrategia óptima para cada jugador teniendo en cuenta que los tres toman decisiones racionales y que no se van a dar alianzas ni van a cooperar entre ellos. El equilibrio de Nash, que es único, puede definirse como el conjunto de las estrategias que debe tomar cada uno de los participantes de un juego de forma que para cada uno ésta sea la óptima considerando que el resto de los jugadores no va a cambiar la suya. Es decir, en el equilibrio de Nash se encuentra la estrategia óptima para cada jugador en el sentido de que si alguno de ellos decidiera cambiarla unilateralmente nunca saldría beneficiado. En particular, para el caso del truelo el equilibrio de Nash indica el jugador al cual le conviene más atacar a cada uno para proporcionarle una probabilidad de supervivencia que nunca van a mejorar si cambia de forma individual de estrategia.

Para el truelo aleatorio, dicho equilibrio es el conjunto de estrategias que llamaremos *atacar al más fuerte*, ya que cada jugador apunta en su turno a aquel que tenga mayor puntería (y no sea él mismo). Para el truelo secuencial, el punto de equilibrio varía según el valor que toma una cierta función g dependiente de los valores de las punterías a, b y c, [9]. Cuando $g > 0$, el punto de equilibrio es la estrategia de atacar al más fuerte ya mencionada, mientras que cuando $g < 0$, el punto de equilibrio se encuentra cuando A y B se atacan el uno al otro y C dispara al cielo.

Ahora, el resultado paradójico que aparece en el juego del truelo y que queda ilustrado y ejemplificado en el trabajo, consiste en que en el punto de equilibrio correspondiente a cada caso se pueden encontrar valores de a, b y c para los cuales el jugador con menor probabilidad de supervivencia es el más hábil y el que tiene mayor probabilidad de ganar el juego es, precisamente, el que posee la peor puntería. El truelo secuencial es donde se hace más notable este resultado contra intuitivo, pues ocurre en mayor proporción que en el truelo aleatorio, aunque en éste último también tiene una relevancia notable.

En el segundo capítulo del trabajo se ve cómo puede llevarse el modelo propuesto para el truelo aleatorio a la modelización de la transmisión de opiniones en un grupo de tres individuos, cada uno con una opinión inicial diferente, A, B o C, y una cierta capacidad de convicción, a, b o c, asociada a cada opinión. Por ejemplo, aquel que mantiene la opinión A tiene una probabilidad a entre 0 y 1 de convencer al individuo que elija. Además, en este capítulo también queda explicado un modelo propuesto por Amengual y Toral en [3] que inicialmente utilizan para modelizar cómo se desarrollaría un truelo aleatorio colectivo, pero que puede emplearse para estudiar la transmisión de tres opiniones diferentes que inicialmente se encuentran en una proporción dada en una población de N personas.

Este modelo resulta muy interesante por todas sus posibles aplicaciones a situaciones cotidianas (propagación de una opinión política, etc.). Sin embargo, hay que destacar que los casos en los que se obtienen resultados contraintuitivos ocurren en mucha menor medida en comparación con el truelo aleatorio o secuencial.

Finalmente, en el tercer capítulo se utiliza *Maxima* para, con el código ya empleado para hallar las probabilidades de supervivencia, calcular estas probabilidades para unos ciertos valores concretos de a, b y c y para los posibles conjuntos de estrategias puras elegidas por los participantes del truelo. El estudio de casos concretos facilita la explicación del razonamiento a seguir para llegar a la conclusión de cuál es el punto de equilibrio tanto en el caso aleatorio como en el secuencial.

Contents

Chapter 1

The truel game

1.1 Introduction

A truel game may be defined from a game theory perspective as a series of one-to-one confrontations where, at each one, every player has the objective of eliminating his or her chosen rival. It can be seen as a generalization of the classic duel but instead of two players there are three who take turns in trying to eliminate their objective at the moment. More specifically, the game takes place following these steps:

1. At each round, a player is chosen with a certain criteria to shoot.
2. Said player chooses his objective according to a strategy that allows him to maximize his chance of survival.
3. He shoots with a determined marksmanship and either he succeeds in eliminating the target or he does not and thus both the shooter and the person shot at continue in the competition.
4. Regardless of the outcome in the previous step, the steps above are repeated until there is only one player remaining alive.

There are different ways in which the order they shoot in might be chosen, it could be random, so at each time-step the player shooting is selected arbitrarily, or it could be sequential, meaning that the three players agree on who the first, second and third shooter is and maintain that order throughout the whole game. There are other variables to be considered, such as the number of bullets available and the possibility of a player shooting into the air rather than aiming at another player. This work focus on the case where the players have an infinite number of bullets. Also, the benefit of a player abstaining from shooting is studied for certain cases.

What is now known as the *truel game* was first studied by C. Kinnaird around 1946 [11], however, this was not the term used at the time, as it is Shubik who coins the name around the 1960s. Kinnaird describes it as one of many mathematical puzzles with a somewhat paradoxical result, as will be demonstrated later in this work. After that, Kilgour [9] studies truel games deeply in various forms, whether that be simultaneous truels, sequential, etc. In [9], he states that, even though at the beginning the truel was "originally regarded as an intriguing puzzle" then became "both a significant problem of the game theory and an important paradigm of political behaviour". He was one of the first to study the convenience of different strategies for the players. However, other authors before him also took a deep look into the truel game, such as Larson and Moser [12] and Gardner [6].

Now, apart from the game theory approach, the truel can be modelled as a Markov chain with the players alive at each time-step as its states. This approach will be further developed here following the line of work of Amengual and Toral, as it provides explicit formulas for the the winning probabilities of each player. Those formulas allow us to study the possible different cases and will reveal that not necessarily is the best player the one who has the greatest probability of survival. In fact, he is the one with the lowest chance of winning at certain cases. To exemplify this, we will use similar graphs to those that can be found in [1] and [2].

As we mentioned earlier, the players of the truel are considered rational beings and, as a result, each of them will pursue a strategy that maximizes his chance of being the only one left alive at the end of the game. Such strategy is the one given by the Nash equilibrium point. It is assumed that the players will not adopt strategies in which they cooperate with each other or form alliances, they will play individually.

The game can be seen from another perspective where the players do not try to kill their opponents, but instead they are simply eliminated from the game, so that the results obtained may be applied to other situations. So this representation is merely an illustration of the idea of the game.

The game could be reinterpreted in a way that considers that each of the three players (now it is not necessary that they are players, they can be individuals of a population of three) has a different opinion on a certain topic and a probability of convincing another player of their opinion. An individual being convinced of another individual's opinion means that there are now two who share the same viewpoint. This leaves the remaining person in the situation with an unfavorable position, as it is "one against two". This will advantage the person with the highest convincing ability. That is why we may find less of a paradoxical result of which opinion is the most likely to prevail among the three of them. However, we still find that in some cases the opinion initially with the least of convincing ability can be the most likely to prevail. The idea of how opinion spreads could be further developed with scale-free networks, as it is analogously done in [13] or [3] with the truel. That way, the model could be used to study larger populations with three different opinions held by its components in a certain proportion and to find out which opinion is the most likely to outlast the others.

1.2 Strategy followed: Nash equilibrium point

Let us consider from now on three different players: A, B and C, and let a, b and c be their respective marksmanships, with $1 \geq a > b > c > 0$.
As mentioned above, the players are rational, which means that they try to find the strategy that maximizes their probability of survival. It was found by Kilgour that this strategy is the one found at what is called the Nash equilibrium point.

Types of games

John von Neumann defines an N-person game in [15] as follows:

Definition 1.1. A (N-person) **game of strategy** consists of a certain series of events each of which may have a finite number of distinct results. In some cases, the outcome depends on chance (...). All other events depend on the free decision of the players $1,2,\ldots,N$. In other words, for each of these events it is known which player, i, determines its outcome and what is his state of information with respect to the results of other events at the time when he makes his decision. Eventually, after the outcome of all events is known, one can calculate according to a fixed rule what payments the players $1,2,\ldots,N$ must make to each other.

Elements of a game

The following are basic elements of a game, based off of the theory found in [5].

Definition 1.2. A **player** is a participant of the game.

Let us denote by $I = \{1,2,\ldots,k\}, k \geq 2$, the set of players of the game.

Definition 1.3. The representation of a possible situation of the game is called a **node**.

Let X be the **set of nodes** of a game. Three different types of nodes are identified:

(i) The starting point of the game is called the **origin node**, $O \in X$.

(ii) If a node has no subsequent nodes, it is called a **terminal node**.

(iii) If a node has at least one subsequent node, it is called a **decision node**.

Let us define the function

$$\sigma : X \longrightarrow X$$
$$x \longmapsto \begin{cases} \sigma(x), & x \neq O \\ 0, & x = O \end{cases},$$

where $\sigma(x)$ is the node inmediately preceding x. Also, for every $x \in X$ let us denote the set of nodes inmediately after x as $s(x) := \{\sigma^{-1}(x) | x \in X\}$. Then, we can also deduce that:

- $T(X) = \{x \in X | \nexists\, x_0 \in X$ such that $x_0 \in s(x)\}$ is the set of terminal nodes.
- $D(X) = \{x \in X | \exists\, x_0 \in X$ such that $x_0 \in s(x)\}$ is the set of decision nodes.

Observation. We obtain the following partition of the set of nodes of a game: $X = T(X) \cup D(X)$.

Definition 1.4. An **action** of a player is one of the possible decisions they can make in their turn (regarding their options in the game at the moment).

Let us define as

$$\alpha : X \backslash \{O\} \longrightarrow \mathcal{A}$$
$$x \longmapsto \alpha_x$$

the function that maps a certain node x different from the origin to an action α_x that maps $\sigma(x)$ to x, i.e. $\alpha_x(\sigma(x)) = x$. Note that, if $x_0, x_1 \in s(x)$ are such that $x_0 \neq x_1$, then $\alpha_{x_0} \neq \alpha_{x_1}$.

Now, for every $x \in D(X)$, the set of available actions from x is

$$\mathcal{A}(x) = \{a \in \mathcal{A} | \exists\, x_0 \in s(x) \text{ such that } a = \alpha_{x_0}\}.$$

Definition 1.5. A **set of information** is a set of decision nodes for a certain player that satisfies the following condition: when the player is in one of the nodes within this set, they ignore which one it is.

Let us define as

$$h : X \longrightarrow H$$
$$x \longmapsto h(x),$$

the function that maps a certain node to the set of information $h(x)$ where it belongs.

The set of *available actions* in the set of information h is

$$\mathcal{A}(h) = \{a \in \mathcal{A} | a \in \mathcal{A}(x), \text{ where } x \in h\}.$$

Definition 1.6. A **payoff** is the product of a game that each player receives depending on his actions and the others chosen by the rest of the players.

Definition 1.7. The **payoff function** is defined as

$$r : T(X) \longrightarrow \mathbb{R}^n$$
$$x \longmapsto r(x) := (r_1(x), \ldots, r_k(x)),$$

where $r_i(x)$, $i \in I$, is the payoff of the player i if x is the terminal node reached.

Thus, the payoff obtained by each player will depend on the terminal node x reached at the end of the game.

Definition 1.8. A **pure strategy** for a player $i \in I$ is a function

$$\begin{aligned} s_i : H_i &\longrightarrow \mathcal{A} \\ h &\longmapsto s_i(h), \end{aligned}$$

where $s_i(h) \in \mathcal{A}(h)$.

Let us denote by S_i the set of every pure strategy of a player $i \in I$.

Given a pure strategy $s_i \in S_i, \forall i \in I$, a complete development of the game is determined, leading to a terminal node.

A combination of pure strategies or **strategy profile** may be denoted by

$$s := (s_1, s_2, \dots, s_k) \in S_1 \times S_2 \times \cdots \times S_k =: S. \tag{1.1}$$

Definition 1.9. A **mixed strategy** $\mu_i = (\mu_i^1, \mu_i^2, \dots, \mu_i^k)$ for a player $i \in I$ is defined as the strategy consisting of playing the pure strategy s_i^1 with probability μ_i^1, the pure strategy s_i^2 with probability μ_i^2, $\dots$, and the pure strategy s_i^k with probability μ_i^k, where $\mu_i^j \geq 0, j = 1, \dots, k$ and $\sum_{j=1}^k \mu_i^j = 1$.

Observation. A pure strategy for a player $i \in I$ is a particular case of a mixed strategy $\mu_i = (\mu_i^1, \mu_i^2, \dots, \mu_i^k)$ where $\exists! m \in \{1, \dots, k\}$ such that $\mu_i^m = 1$ and $\mu_i^l = 0, \forall l \neq m$. Hence, μ_i verifies $\mu_i^j \geq 0, \forall j$ and $\sum_{j=1}^k \mu_i^j = \mu_i^m = 1$.

Let us denote by $P_i^j, i \in \{A, B, C\}, j \in \{A, B, C, 0\}$ the probability of a player i choosing to shoot at j, where 0 represents shooting into the air and, clearly, no player is going to shoot at themselves. The set of every pure strategy of each player are the following:

- Player A, $P_A := (P_A^B, P_A^C, P_A^0)$.
- Player B, $P_B := (P_B^A, P_B^C, P_B^0)$.
- Player C, $P_C := (P_C^A, P_C^B, P_C^0)$.

P_i are what was denoted by s in (1.1).

Definition 1.10. Let $t(s)$ be the terminal node reached when the players use the strategy profile s. In that case, the **utility function** for player $i \in I$ is given by

$$\begin{aligned} u_i : S &\longrightarrow \mathbb{R} \\ s &\longmapsto u_i(s) := r_i(t(s)). \end{aligned}$$

There are two possible terminal nodes: one that represents winning and one that represents losing. We will consider that the utility function takes a value of 1 when a player reaches the terminal node of winning (being the only survivor) and a value of 0 if a player reaches the terminal node of losing (being eliminated by another player).

Definition 1.11. The **normal form** (or strategic form) of a game is specified by the set of elements

$$N = \{I, (S_i)_{i \in I}, (u_i)_{i \in I}\}.$$

Strategy of the players in the truel

The players choose a strategy to maximize their utility (the outcome obtained from the game). There are various ways to approach the resolution of a game, such as the dominant strategy, that has as its objective to obtain the highest possible payoff of a player, regardless of what the rest of the players do; the cooperative strategy, where subsets of I form coalitions which seek the maximum joint payoff for all of the players in it; and the Nash equilibrium, which is a set of strategies, one for each player, such that the payoff for each one cannot be improved by unilaterally changing the strategy of one of them.

Let us examine now a formal definition of the Nash equilibrium, as it will be the strategy used by the players in the truel.

Definition 1.12. Let $N = \{I, (S_i)_{i\in I}, (u_i)_{i\in I}\}$ be the normal representation of a game. A pure strategy profile $(s_1^*, \ldots, s_{i-1}^*, s_i^*, s_{i+1}^*, \ldots, s_k^*)$ is called a **Nash equilibrium** if, for every $i \in I$, it verifies that:

$$u_i(s_1^*, \ldots, s_{i-1}^*, s_i^*, s_{i+1}^*, \ldots, s_k^*) \geq u_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_k^*), \quad \forall s_i \in S_i.$$

We will see that, as the players are rational, they pursue the Nash equilibrium and therefore they will play pure strategies. As an illustration, for the Nash equilibrium in the random firing truel (first column) and sequential truel (first or second column, depending on a certain factor explained in section 1.5.2), the players take the following pure strategies (Kilgour and Brams, [4]):

- $\mu_A = (1,0,0)$
- $\mu_B = (1,0,0)$
- $\mu_C = (1,0,0)$

- $\mu'_A = (1,0,0)$
- $\mu'_B = (1,0,0)$
- $\mu'_C = (0,0,1)$

1.3 Markov chains based modelling of the problem

As presented by R. Toral and P. Amengual [1], Markov chains can be employed as a means of modelling the problem of finding the ultimate survivor in the truel game, and thus, also in the duel. For this matter, the sets of players remaining alive at each time-step are seen as the different states.
The following are several results regarding Markov chains that will be of use when particularized to the truel game. These results are based off of the theory in [7] and [8].

Definition 1.13. A **discrete-time Markov chain** is a stochastic process $\{X_t\}_{t\geq 1}$ defined on a finite set of states, S, such that

$$P(X_{n+1} = x_{n+1} | X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) = P(X_{n+1} = x_{n+1} | X_n = x_n), \quad n = 0, 1, \ldots.$$

Definition 1.14. A Markov chain is said to be **homogeneous** if $P(X_{n+1} = j | X_n = i)$ is independent of n, that is,

$$P(X_{n+1} = j | X_n = i) = P(X_1 = j | X_0 = i), \quad n = 0, 1, \ldots.$$

Let us denote by $p_{ij} := P(X_{n+1} = j | X_n = i)$, the transition probability between states i and j. The transition probabilities can be arranged to form a matrix called the **transition probability matrix**, P.

Let us also denote by $p_{ij}^{(n)}$ the transition probability between states i and j in n steps.

Definition 1.15. The **Chapman-Kolmogorov equation** is given by

$$p_{ij}^{(n)} = \sum_{k\in S} p_{ik}^{(m)} p_{kj}^{(n-m)}$$

for every $i, j \in S$ and for any $m = 1, 2, \ldots, n-1, \quad n = m+1, m+2, \ldots.$

This equation offers a recursive way to compute the transition probabilities in n steps.

Proposition 1.1. *Let P be the transition probability matrix of a Markov chain. Each element of the matrix $P^{(n)} = \underbrace{P \times \cdots \times P}_{n} = P^n$ is the transition probability from state i to state j in n steps.*

Proof. If $n = 2$, then

$$p_{ij}^{(2)} = \sum_{k \in S} p_{ik} p_{kj}, \text{for every } i, j \in S,$$

where $p_{ij}^{(2)}$ are the elements in the matrix $P^{(2)}$. Therefore note that these elements can be obtained by multiplying the transition probability matrix in one step by itself:

$$P^{(2)} = PP = P^2.$$

Let us study the cases when $m = 1$ and $m = n-1$, for some $n \in \mathbb{N}$. We have, respectively, the following Chapman-Kolmogorov equations:

$$p_{ij}^{(n)} = \sum_{k \in S} p_{ik} p_{kj}^{(n-1)}, \quad p_{ij}^{(n)} = \sum_{k \in S} p_{ik}^{(n-1)} p_{kj}.$$

In the same way as the case for $n = 2$, the equations above for $p_{ij}^{(n)}$ indicate that:

$$\begin{aligned} P^{(n)} =& PP^{(n-1)} = P^{(n-1)} P \\ =& P^{n-1} P = PP^{n-1} = P^n. \end{aligned}$$

□

Definition 1.16. (i) A state j is called **transient** if there is a positive probability that, starting in state j, the stochastic process does not return to such state.

(ii) A state j is called **recurrent** if it is certain that, starting in state j, the stochastic process returns to state j (not necessarily in one step).

(ii.1) A recurrent state j such that $p_{jj} = 1$ is called **absorbing**.

Note that absorbing states are a particular case of recurrent states, to which the process does return in one step.
It is also noteworthy that transient and recurrent states are excluding, which means that a certain state is transient if and only if it is not recurrent and vice versa.

Considering that there are t transient states and r recurrent states in a Markov chain, the matrix P can be rearranged in a **canonical form** such as

$$P = \begin{pmatrix} Q & R \\ 0 & U \end{pmatrix}$$

where:

- Q is a $t \times t$ matrix, containing the transition probabilities between transient states.
- R is a $t \times r$ matrix, containing the transition probabilities from transient to recurrent states.
- U is a $r \times r$ matrix, containing the transition probabilities between recurrent states.

From now on we assume that the Markov chains we refer to are written in the canonical form.
Observe that, just as earlier stated in Proposition 1.1, Q^n, R^n, U^n have an analogous meaning, since the proposition is applied *element-wisely*.

Definition 1.17. A Markov chain is called **absorbing** if it has at least one absorbing state and it is possible to reach an absorbing state from every state.

Observation. When considering an absorbing chain where every recurrent state is absorbing, U is the identity matrix of order r, I_r.

Theorem 1.1. *An absorbing Markov chain verifies that:*

(i) $\lim_{n\to\infty} Q^n = 0$.

(ii) $I - Q$ *has an inverse.*

(iii) $(I-Q)^{-1} = \sum_{n=0}^{\infty} Q^n$.

Proof. (i) By the definition of an absorbing Markov chain, it is possible to reach an absorbing state from every nonabsorbing state j. Let m_j be the minimum number of steps needed to attain an absorbing state when starting from j. Also, let p_j be the probability that the chain does not attain an absorbing state in m_j steps when starting from j (in that case, $p_j < 1$).

Let us define $m := max_{j\in S}\{m_j\}$ and $p := max_{j\in S}\{p_j\}$.

The probability that the chain does not get absorbed in m states is lower or equal to p, therefore the probability of it not getting absorbed in nm states is p^n. As $p < 0$, $p^n \xrightarrow{n\to\infty} 0$.

Given that $\{p^n\}_{n\in\mathbb{N}}$ is monotone decreasing, $\lim_{n\to\infty} Q^n = 0$.

(ii) The matrix $I-Q$ has an inverse if and only if the vector $x = 0$ is the only solution to $(I-Q)x = 0$. We have

$$(I-Q)x = 0 \iff x = Qx.$$

Also,

$$x = Qx = Q^2x = \cdots = Q^n x = \ldots.$$

Now, since $Q^n \xrightarrow{n\to\infty} 0$, then $Q^n x \xrightarrow{n\to\infty} 0$. Thus $x = 0$.

(iii) It should be noted that

$$(I-Q)(I+Q+Q^2+\cdots+Q^n) = [I+Q+Q^2+\cdots+Q^n] - [Q+Q^2+\cdots+Q^{n+1}] = I - Q^{n+1}.$$

We obtain multiplying both sides by $(I-Q)^{-1}$:

$$(I-Q)^{-1}(I-Q)(I+Q+Q^2+\cdots+Q^n) = (I-Q)^{-1}(I-Q^{n+1}).$$

Then,

$$I+Q+Q^2+\cdots+Q^n = (I-Q)^{-1} - (I-Q)^{-1}Q^{n+1}.$$

Lastly, letting $n\to\infty$, because $Q^n x \xrightarrow{n\to\infty} 0$, we reach the result we wanted to prove:

$$(I-Q)^{-1} = I+Q+Q^2+\cdots+Q^n+\cdots = \sum_{n=0}^{\infty} Q^n.$$

□

Remark. 1. Item (i) in the theorem above implies that the Markov chain will be absorbed with probability 1. In other words, there exists a sufficiently large number of steps in which the probability of the chain reaching a transient state is 0.

2. The convergence of the series in item (iii) is element-wise.

Definition 1.18. The matrix $N := (I-Q)^{-1}$ is called the **fundamental matrix**.

Let **T** be the set of transient states and let **A** be the set of absorbing sates of a Markov chain.

Theorem 1.2. *Let $i, j \in \mathbf{T}$. The entry n_{ij} of N is the expected number of times the absorbing chain is in state j, considering that it starts in state i.*

Proof. Let t_i and t_j be two transient states, where i, j are fixed. Also, let $X^{(k)}$ denote a stochastic variable that assumes a value of 1 if the chain is in state t_j in k steps and assumes a value of 0 otherwise.

For every k, $X^{(k)}$ depends on both i and j. We have

$$P(X^{(k)} = 1) = q_{ij}^{(k)},$$
$$P(X^{(k)} = 0) = 1 - q_{ij}^{(k)},$$

where $q_{ij}^{(k)}$ is the ijth entry of Q^k. The equations hold when $k = 0$, as $Q^0 = I$.

Now, note that $E(X^{(k)}) = q_{ij}^{(k)}$ since this is a 0-1 random variable. The expected frequency of occurrences during which the chain occupies state t_j within the initial n steps, considering that it starts in state t_i, clearly is

$$E(X^{(0)} + X^{(1)} + \cdots + X^{(n)}) = q_{ij}^{(0)} + q_{ij}^{(1)} + \cdots + q_{ij}^{(n)}.$$

Thus, letting $n \to \infty$, by Theorem 1.1 (iii):

$$E(X^{(0)} + X^{(1)} + \dots) = q_{ij}^{(0)} + q_{ij}^{(1)} + \cdots = n_{ij}.$$

□

Corollary 1.1. *Considering that the absorbing chain starts in state i, the expected number of steps until it is absorbed is*

$$t_i := \sum_{j \in \mathbf{T}} n_{ij}.$$

Proof. Let $I_n := I(X_n = j)$ be an indicator random variable of $\{X_n = j\}$, $n \in \mathbb{N}$. Then, $T_j := \sum_{n=0}^{\infty} I_n$ is the number of times the chain is in state j.

Thus we obtain the following identities:

$$E(T_j | X_0 = i) = E\left(\sum_{n=0}^{\infty} I(X_n = j)\ | X_0 = i\right) = \sum_{n=0}^{\infty} E(I(X_n = j) | X_0 = i) =$$

$$= \sum_{n=0}^{\infty} P(X_n = j | X_0 = i) = \sum_{n=0}^{\infty} r_{ij}(n) = n_{ij}.$$

Note that, for $n = 0$, we have $r_{ij}(0) = 1$ if $i = j$ and $r_{ij}(0) = 0$ otherwise.

Now, let us define

$$T = \min\{n \geq 1 | X_n \in \mathbf{A}\} \quad \text{and} \quad t_i = E(T | X_0 = i).$$

Finally, because $T = \sum_{j \in \mathbf{T}} T_j$, we have the result that we wanted to prove:

$$t_i = E(T | X_0 = i) = \sum_{j \in \mathbf{T}} E(T_j | X_0 = i) = \sum_{j \in \mathbf{T}} n_{ij}.$$

□

Theorem 1.3. *The entries of the $t \times r$ matrix $B := NR$, b_{ij}, are the probabilities of the absorbing Markov chain being absorbed in an absorbing state j considering that it starts in a transient state i.*

Proof. Let us fix a certain state $i \in \mathbf{T}$ and assume that the absorbing chain starts in i. Then, for every $j \in \mathbf{A}$:

$$b_{ij} = p_{ij} + \sum_{k \in \mathbf{T}} p_{ik} b_{kj}.$$

Note that p_{ij} is an element of the matrix R if $i \in \mathbf{T}$ and $j \in \mathbf{A}$ and p_{ik} is an element of the matrix Q if $i, k \in \mathbf{T}$. Taking this into account,

$$B = R + QB \iff (I - Q)B = R \iff B = (I - Q)^{-1} R = NR.$$

□

1.4 Original problem: the duel

Let us consider two players for the moment, A and B, with respective marksmanships a and b ($0 < b < a \leq 1$). Also, let us assume that, at the start of the game, every player is given a set of m bullets ($0 \leq m \leq \infty$). The objective is to compute the probability of each player winning the game in both of the following scenarios: random firing and sequential firing. It is worth noting that the states that correspond to the cases when only A or B remain alive, respectively, are absorbing, as it is impossible for a player who was eliminated to return to the game at any point. Also, from every transient state both of those states are reachable, therefore we can apply the previous results regarding Markov chain theory for absorbing chains.
In this situation, it becomes evident that none of the players have any motivation to shoot into the air, since it would mean the loss of an opportunity to eliminate their opponent and, of course, as there are only two players, A will shoot at B and vice versa. Later, we will see that this is not always the case when there are three players.

1.4.1 Random firing

First, we consider that the person who shoots at each time-step is chosen arbitrarily, with an equal probability assigned to each of them. As the game progresses, we may find that the situation is one of the following: both of the players are still alive, player A has eliminated player B or vice versa. This implies that the set of states is $S = \{AB, A, B\} = \{0, 1, 2\}$, where AB is the state in which both of the players remain alive, A is the state in which only player A is still alive and in state B only player B is still alive. This is the transition probability matrix obtained for the random firing duel:

$$P = \left(\begin{array}{c|cc} 1 - \frac{a+b}{2} & \frac{1}{2}a & \frac{1}{2}b \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right). \tag{1.2}$$

Proposition 1.2. *An absorbing chain with a transition probability matrix of the form* (1.2) *has the subsequent absorption probabilities at states 1 and 2, respectively:*

$$b_{01} = \frac{a}{a+b}, \quad b_{02} = \frac{b}{a+b}.$$

Proof. When the transition probabilities are given by (1.2), we obtain the matrices

$$Q = \left(1 - \frac{a+b}{2}\right), \qquad R = \left(\frac{1}{2}a \quad \frac{1}{2}b\right), \qquad U = I_2.$$

Then, the fundamental matrix is

$$N = (I - Q)^{-1} = \left(\frac{2}{a+b}\right).$$

Therefore, applying Theorem 1.3, we calculate

$$B = NR = \begin{pmatrix} \frac{a}{a+b} & \frac{b}{a+b} \end{pmatrix},$$

which gives us the absorption probabilities of the chain at its absorbing states, 1 and 2, considering that it starts at state 0, since that is the state where both of the players are still alive. Each of them corresponds to the probability of players A and B of winning, respectively. □

Observation. In this case, we do not encounter a paradoxical result, since it is easy to see that the player with the highest marksmanship is also the player with the greatest probability of winning:

$$a > b \iff \frac{a}{a+b} > \frac{b}{a+b} \iff b_{01} > b_{02}.$$

1.4.2 Sequential firing

In the sequential firing duel, there is a fixed order in which the two players shoot and they maintain it throughout the whole game until one of them wins. That order can either be A shooting first and then B, or vice versa. Unlike the case of the random duel, we know for sure who is shooting at each time-step. Then we just have to take into account the fixed order at the beginning to know in which state the chain starts.

The set of states is $S = \{\underline{A}B, A\underline{B}, A, B\} = \{0, 1, 2, 3\}$, where the underlined player represents the player who shoots at the time. For instance, we consider the chain to be in state 0, which means that both players remain alive and it is player A's turn. If the chain is in state 1, this has a similar interpretation, except that it is player B's turn.

The transition probability matrix that represents the sequential firing duel is:

$$P = \left(\begin{array}{cc|cc} 0 & 1-a & a & 0 \\ 1-b & 0 & 0 & b \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right). \tag{1.3}$$

As we will explain later, our interest is that the players follow an order that favours the weakest one, that is, the player with the lowest marksmanship, so that the paradox (if existing) is noticeable. In this case, it is player B. Therefore, B is the first one to shoot (thus the chain starts in state 1).

Proposition 1.3. *An absorbing chain with a transition probability matrix of the form* (1.3) *has the following absorption probabilities at states 2 and 3, respectively:* [1]

$$b_{12} = \frac{a(1-b)}{(a-1)(1-b)+1}, \quad b_{13} = \frac{b}{(a-1)(1-b)+1}.$$

Proof. Having (1.3) as the transition probability matrix, we obtain:

$$Q = \begin{pmatrix} 0 & 1-a \\ 1-b & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \qquad U = I_2.$$

The fundamental matrix is

$$N = (I-Q)^{-1} = \begin{pmatrix} \frac{1}{(a-1)(1-b)+1} & \frac{1-a}{(a-1)(1-b)+1} \\ \frac{1-b}{(a-1)(1-b)+1} & \frac{1}{(a-1)(1-b)+1} \end{pmatrix}.$$

[1] Note that in [1] a mistake is found in the given survival probabilities of players A and B in the sequential firing duel. The probabilities mentioned in the article which are stated to correspond to the scenario where player B initiates the shot, in fact align with the scenario where player A takes the first shot, that is, when the chain starts in state 0. The correct probabilities are presented in Proposition 1.3.

And, by Theorem 1.3,

$$B = NR = \begin{pmatrix} \frac{a}{(a-1)(1-b)+1} & \frac{(1-a)b}{(a-1)(1-b)+1} \\ \frac{a(1-b)}{(a-1)(1-b)+1} & \frac{b}{(a-1)(1-b)+1} \end{pmatrix}.$$

The absorption probabilities of interest now are in the second row of the matrix B, since the entries in the second row correspond to the absorption probabilities of the chain at states 2 and 3, respectively, considering that it starts in state 1. □

From the result in Proposition 1.3, it can be deduced that A is not necessarily the survivor despite having a better marksmanship, as

$$b_{12} < b_{13} \iff a < \frac{b}{1-b}.$$

If we contemplate the optimal scenario for B, the weakest player, it would be the sequential duel when he is the one to initiate the shooting. In the figure 1.1, it is represented in two regions, a purple one and a green one, when A or B respectively have the highest probability of surviving the duel. We can see how on the axes of the graph we have the values that a (horizontal axis) and b (vertical axis) can take (verifying $0 < b < a \leq 1$). The green region occupies a considerable area of the graph, which means that, despite being the least skilled player, B still possesses a very favorable probability of winning the duel under these conditions.
Secondly, B would be most benefited playing the random truel, given that he at least has the chance of randomly initiating the shooting, which grants him a certain advantage.
Lastly, the worst case scenario for B would be the sequential duel when player A makes the first shot.

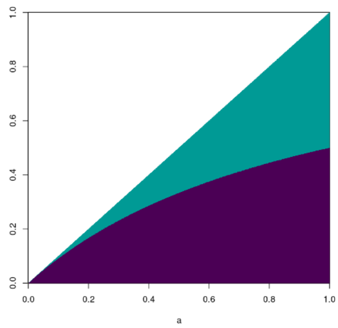

Figure 1.1: Survival probabilities in the sequential duel (having B as the first shooter).

1.5 The truel paradox

In this section, we study the probability of each player being the last survivor in the generalization of the duel to three players: A, B and C. Each of them has a marksmanship of a, b and c, respectively (satisfying $0 < c < b < a \leq 1$).

Here, the paradoxical result that shows that in some cases the weakest player is the fittest to win is more noticeable as this occurs in a bigger proportion compared to that in the duel. For analogous reasons to those mentioned in the previous section, in both of the following cases the absorbing states are the ones that correspond to players A, B or C winning. The rest of the states are transient and from all of them it is possible to reach at least one of the absorbing states. This makes both of the markovian approaches for the random firing and the sequential firing truel to be absorbing chains, and that allows to follow a similar procedure to that in the duel.

However, in the truel it is not so clear who is in the best interest for each player to shoot at, hence the players must consider all the possible moves their opponents might make and the potential outcomes of those moves. This leads them to take the strategy of the Nash equilibrium, which is a *utility-maximizing strategy* that allows each player to make the best response with respect to the strategies used by their competitors.

Following an analogous procedure to the one in the proofs of Propositions 1.2 and 1.3 and using the computer software *Maxima* to calculate matrices N and B, we reach the results mentioned below in Propositions 1.5 and 1.6 as a conclusion.

1.5.1 Random firing

The Nash equilibrium for the random firing truel is the set of strategies (Kilgour and Brams, [4]):

$$\begin{cases} P_A^B = P_B^A = P_C^A = 1 \\ P_A^C = P_B^C = P_C^B = P_A^0 = P_B^0 = P_C^0 = 0 \end{cases} \tag{1.4}$$

This is what we refer to as the ***strongest opponent strategy***, as each player aims at the remaining player with the highest marksmanship (excluding oneself).

We have $S = \{ABC, AB, AC, BC, A, B, C\} = \{0, 1, \ldots, 6\}$ as the set of states.

In figure 1.2 we find the graphic representation of the Markov chain which shows the allowed transitions between the states mentioned before. The values of the transition probabilities q_{ij} are presented in Proposition 1.4.

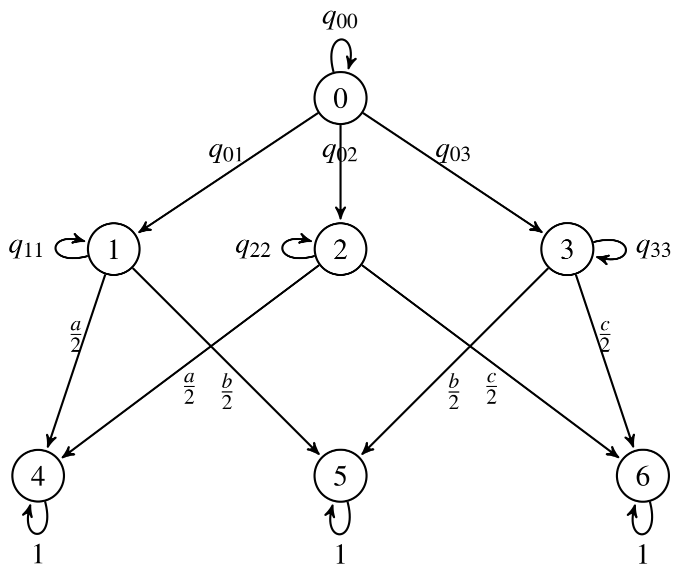

Figure 1.2: Diagram of the Markov chain representing the random firing truel. Own elaboration.

Proposition 1.4. *The following are the values of the elements in the submatrix Q of the canonical form of the transition probability matrix* (A.1).

- $q_{00} = 1 - \dfrac{(1-P_C^0)c + (1-P_B^0)b + (1-P_A^0)a}{3}$
- $q_{01} = \dfrac{bP_B^C + aP_A^C}{3}$
- $q_{02} = \dfrac{cP_C^B + aP_A^B}{3}$
- $q_{11} = 1 - \dfrac{a+b}{2}$
- $q_{22} = 1 - \dfrac{a+c}{2}$
- $q_{33} = 1 - \dfrac{b+c}{2}$

Proof. What follows are the steps taken to calculate the value of each element in matrix Q:

q_{00} In order to compute the value of q_{00} it must be taken into account that this is the transition probability from state 0 to itself, that is, after one randomly selected player fires, he fails to hit anyone, resulting in all three players remaining alive. Then, we compute the complementary case of one of them being eliminated.

For this matter, we use the law of total probability. On the one hand, the probability of a certain player actually eliminating another, knowing which player is going to shoot, is $(1-P_m^0)n$, where $(m,n) \in \{(A,a),(B,b),(C,c)\}$, as shooting into the air would have no effect in eliminating another player. On the other hand, the probability that each player $m \in \{A,B,C\}$ is chosen to shoot is of $1/3$, as all participants of the game possess an equivalent probability of being selected.

q_{01} The value of q_{01} is the transition probability of going from the three players being alive to only players A and B, namely, having either A or B eliminate C. Therefore, we use again the law of total probability. Note that the probability of either A or B being chosen to shoot is of $1/3$ and that the probability of player A eliminating C (once we know it is his turn to shoot) is aP_A^C and of B eliminating C is bP_B^C.

q_{02} This is an analogous case to the previous one, however, here it is the probability of either player A or C eliminating player B.

q_{11} In this case, we compute the value of q_{11} as the complementary case of either A or B eliminating each other when only them remain alive. Again, we use the law of total probability.

q_{22}, q_{33} These are analogous cases to q_{11}.

□

Now, in particular, we are interested in the transition probability matrix resulting from the evaluation of P_i^j, with $i \in \{A,B,C\}, j \in \{A,B,C,0\}$ in (A.1) according to the Nash equilibrium (1.4).

$$P = \left(\begin{array}{cccc|ccc} 1-\dfrac{a+b+c}{3} & 0 & \dfrac{a}{3} & \dfrac{b+c}{3} & 0 & 0 & 0 \\ 0 & 1-\dfrac{a+b}{2} & 0 & 0 & \dfrac{a}{2} & \dfrac{b}{2} & 0 \\ 0 & 0 & 1-\dfrac{a+c}{2} & 0 & \dfrac{a}{2} & 0 & \dfrac{c}{2} \\ 0 & 0 & 0 & 1-\dfrac{b+c}{2} & 0 & \dfrac{b}{2} & \dfrac{c}{2} \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right). \tag{1.5}$$

Remark. If we examine the matrix (1.5), it follows that state 1 cannot be attained when the chain starts in state 0. This is because players A and B will shoot each other, and no one will shoot C, since he has the lowest marksmanship and the priority of every player is to eliminate the one with the optimal one. (As explained earlier, the Nash equilibrium for this problem is achieved when each player aims at the one with the highest marksmanship.)

For that reason, we remove the row and column corresponding to state 1 from matrix (1.5) to obtain:

$$P = \left(\begin{array}{ccc|ccc} 1-\dfrac{a+b+c}{3} & \dfrac{a}{3} & \dfrac{b+c}{3} & 0 & 0 & 0 \\ 0 & 1-\dfrac{a+c}{2} & 0 & \dfrac{a}{2} & 0 & \dfrac{c}{2} \\ 0 & 0 & 1-\dfrac{b+c}{2} & 0 & \dfrac{b}{2} & \dfrac{c}{2} \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right). \tag{1.6}$$

We find in the figure below its graphic representation.

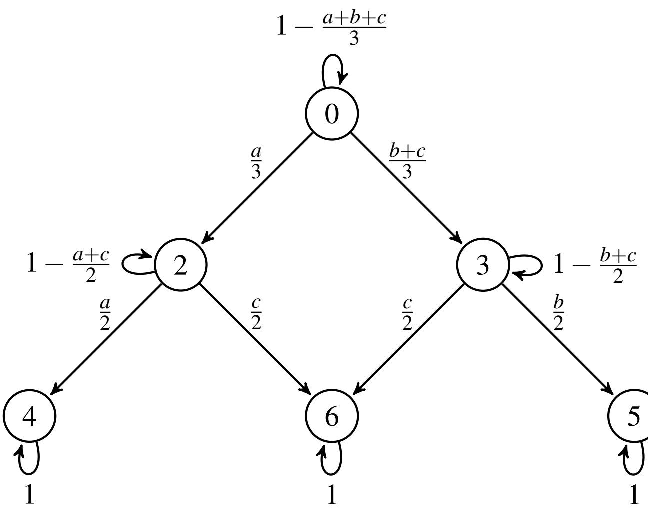

Figure 1.3: Diagram of the Markov chain representing the random firing truel when following the strongest opponent strategy. Own elaboration.

Proposition 1.5. *An absorbing chain with a transition probability matrix of the form* (1.6) *has the following absorption probabilities at states 4, 5 and 6, respectively:*

$$b_{04} = \frac{a^2}{(c+a)(c+b+a)}, \quad b_{05} = \frac{b}{c+b+a}, \quad b_{06} = \frac{c(2a+c)}{(c+a)(c+b+a)}.$$

Proof. Having (1.6) as the transition probability matrix, the following are the submatrices that make up the canonical form:

$$Q = \begin{pmatrix} 1-\dfrac{a+b+c}{3} & \dfrac{a}{3} & \dfrac{b+c}{3} \\ 0 & 0 & 1-\dfrac{a+c}{2} \\ 0 & 0 & 1-\dfrac{b+c}{2} \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 0 & 0 \\ \dfrac{a}{2} & 0 & \dfrac{c}{2} \\ 0 & \dfrac{b}{2} & \dfrac{c}{2} \end{pmatrix}, \qquad U = I_3.$$

The fundamental matrix is

$$N = (I-Q)^{-1} = \begin{pmatrix} \dfrac{3}{c+b+a} & \dfrac{2a}{(c+a)(c+b+a)} & \dfrac{2}{c+b+a} \\ 0 & \dfrac{2}{c+a} & 0 \\ 0 & 0 & \dfrac{2a}{c+b} \end{pmatrix}.$$

And, by Theorem 1.3,

$$B = NR = \begin{pmatrix} \dfrac{a^2}{(c+a)(c+b+a)} & \dfrac{b}{c+b+a} & \dfrac{c(c+2a)}{(c+a)(c+b+a)} \\ \dfrac{a}{c+a} & 0 & \dfrac{c}{c+a} \\ 0 & \dfrac{b}{c+b} & \dfrac{c}{c+b} \end{pmatrix}.$$

The first row of matrix B corresponds to the probability of the chain being absorbed at each absorbing state starting from state 0.

□

To illustrate the paradox that arises in the survival probability of the players, the value $a = 1$ is fixed and all possible values of b and c are considered (as long as they satisfy $0 < c < b < a = 1$). In this way, using the *RStudio* software, figure 1.4 is obtained. The code used to build the graph can be found in Appendix C.

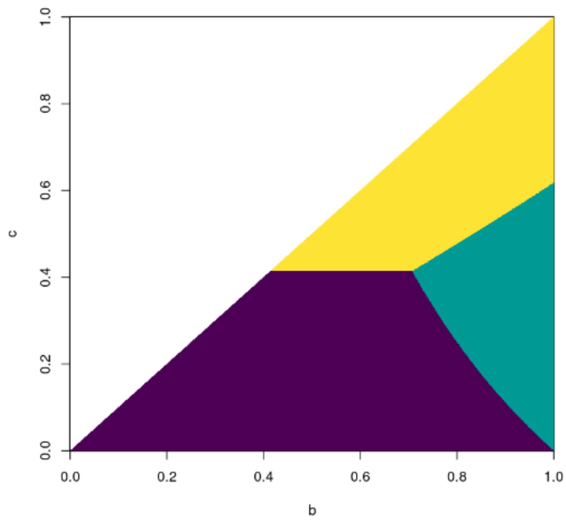

Figure 1.4: Most probable winner of the random truel when $a = 1$ and $0 < c < b < 1$. Purple corresponds to player A, green to player B and yellow to player C.

Fixing $a = 1$, we mark out different regions in purple, green and yellow in the parameter space (b, c) that correspond to where players A, B and C are, respectively, the most likely to win the game.

As can be observed in the graph, the three players occupy substantially large regions in which each of them is the most likely to be the sole survivor and, therefore, win the game. Despite the fact that the largest region in the space is the one where player A has the highest probability of winning, the ones of B and C also have quite a significance. Anticipating of what is elaborated in Chapter 3, we can see that, for example, when $b = 0.85$ and $c = 0.6$, the survival probabilities of players A, B, and C are (approximately) $P(A \text{ wins}) = 0.255$, $P(B \text{ wins}) = 0.346$, $P(C \text{ wins}) = 0.397$, which is precisely in the reverse order of their marksmanship.

Once we grasp the following point, we can understand the nature of this paradoxical outcome: as participants adopt the *strongest opponent strategy*, initially, A and B are the only players vulnerable to attack. It is only when one of them is eliminated that C also becomes a target. This dynamic gives player C a certain "immunity" during a portion of the game. Consequently, it becomes clear that, in certain scenarios, C possesses the highest likelihood of survival.

1.5.2 Sequential firing

The Nash equilibrium for the sequential truel depends on the value of the following function, found by Kilgour in his doctoral dissertation and explained further in [9] and [10]:

$$g(a,b,c) = a^2(1-b)^2(1-c) - b^2c - ab(1-bc). \tag{1.7}$$

If $g(a,b,c) > 0$, then the Nash equilibrium point is the strongest opponent strategy (1.4). If $g(a,b,c) < 0$, then the equilibrium point is the set of strategies

$$\begin{cases} P_A^B = P_B^A = P_C^0 = 1 \\ P_A^C = P_B^C = P_C^A = P_C^B = P_A^0 = P_B^0 = 0 \end{cases} \tag{1.8}$$

Using the notation given in section 1.3 for pure strategies, the players will follow these strategies

We have $S = \{\underline{A}BC, A\underline{B}C, AB\underline{C}, \underline{A}B, A\underline{B}, \underline{A}C, A\underline{C}, \underline{B}C, B\underline{C}, A, B, C\} = \{0, 1, \ldots, 11\}$. Similarly to the situation in the sequential firing duel, in this case, every state signifies the surviving players, with the underlined player being the one currently taking their turn.

Proposition 1.6. *An absorbing chain with a transition probability matrix of the form* (A.2) *has the following absorption probabilities at states 9, 10 and 11, respectively:* [2]

- *If* $g(a,b,c) > 0$*, when the values* P_i^j *are evaluated as in* (1.4)*:*

$$b_{2;9} = \frac{a^2(b-1)(c-1)^2}{(ac-c-a)(abc-bc-ac+c-ab+b+a)}, \quad b_{2;10} = -\frac{b(bc^2-2bc+c+b)}{(bc-c-b)(abc-bc-ac+c-ab+b+a)},$$

$$b_{2;11} = -\frac{c(ab^2c^2-4abc^2+2bc^2+2ac^2-c^2-2ab^2c+6abc-bc-2ac+ab^2-2ab)}{(ac-c-a)(bc-c-b)(abc-bc-ac+c-ab+b+a)}.$$

- *If* $g(a,b,c) < 0$*, when the values* P_i^j *are evaluated as in* (1.8)*:*

$$b_{2;9} = \frac{a^2(b-1)(c-1)}{(ab-b-a)(ac-c-a)}, \quad b_{2;10} = -\frac{b^2(c-1)}{(ab-b-a)(bc-c-b)},$$

$$b_{2;11} = -\frac{c(ab^2c-3abc+bc+ac-ab^2+2ab)}{(ab-b-a)(ac-c-a)(bc-c-b)}.$$

Proof. The proof can be found in the appendix, as the proof of Proposition A.1. □

As evident from figure 1.5, the paradox becomes more pronounced here than in the random truel scenario. The graph has been constructed following the same approach as in 1.4, with the value of $a = 1$ remaining constant and considering all possible combinations of b and c as long as they verify $0 < c < b < a = 1$.

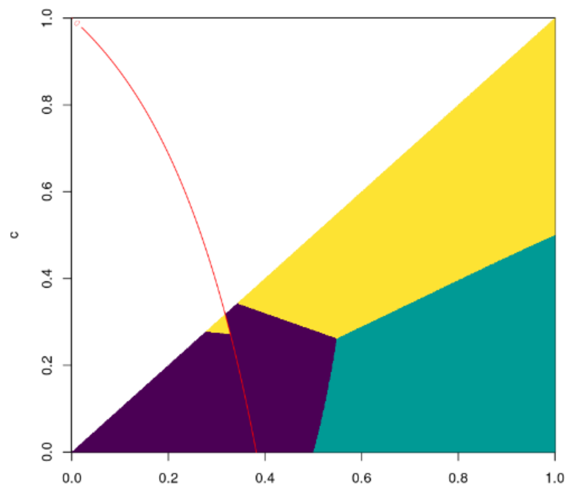

Figure 1.5: Most likely winner of the sequential truel when C is the first to shoot. The curve $g(1,b,c)$ is drawn in red. The color coding for the different regions is the same as in figure 1.4.

Once again, the counter intuitive result relates to the fact that the player initially having the lowest marksmanship among all competitors actually has the highest probability of winning in a great proportion

[2] It is worth mentioning that in [1] and [2] the absorption probabilities of the Markov chain the absorption probabilities meant for cases where $g(a,b,c) < 0$ and $g(a,b,c) > 0$ have been switched. The actual probabilities for the case of $g(a,b,c) < 0$ (when following strategy (1.8)) and the case of $g(a,b,c) > 0$ (following strategy (1.4)) are correctly presented in Proposition 1.6.

of the cases. Here, it is evident that the region occupied by C is notably the largest on the graph, followed by B and, finally, A.

When the function (1.7) is greater than 0, we have an analogous explanation for the paradoxical result. However, in the sequential truel scenario, the paradox is accentuated. This is because the most capable player, A, is the last to shoot, putting him at a disadvantage from the beginning compared to both B and C. On another note, when (1.7) is smaller than 0 and the players take the strategy in (1.8). This means that, until either A eliminates B or vice versa, C abstains from firing any of his opponents, expecting that it is B who takes A out of the game leaving C to fight B, who is the second most capable player.

The small yellow region and the portion of the purple one in figure 1.5 that lie below the curve $g(1,b,c)$ correspond to the values of (b,c) that make $g(1,b,c) < 0$, therefore here the equilibrium point is $BA0$. The rest of the graph represents the cases when $g(1,b,c) > 0$ and the equilibrium point is BAA.

Chapter 2

Extrapolation of the game model for opinion spreading

From the random firing truel an analogous model can be extrapolated and converted into a model to explain a certain type of opinion spread. In particular, we find a certain population of three people where each person holds a different opinion on a subject, A, B and C. Also, each person has a convincing ability of a,b and c, respectively, where $1 \geq a > b > c > 0$. The objective of each participant is to persuade others of their viewpoint. To achieve this, one-on-one conversations are conducted in which both parties attempt to convince each other.

As an illustration, an individual supporting perspective A possesses a probability of a for persuading another individual to adopt their standpoint.

The primary distinction that we observe in comparison to the random firing truel is that, in this scenario, the number of participants until the process concludes – that is, when all three present people share the same opinion – remains constant. Whereas, in the other case, the game concludes when only one player remains alive. However, both models share the same Nash equilibrium, shown in (1.4). In this given scenario, the Nash equilibrium is attained when every participant tries to convince of their own opinion the remaining participant with the highest convincing probability (different from themselves). Clearly, it is illogical for a participant to persuade another individual who shares the same opinion as him or to lose their opportunity to attempt to convince someone else, just as it was illogical in the random truel for a player to shoot into the air.

We can model the case of opinion spread as a Markov chain with the following set of states: $S = \{ABC, BCC, BBC, ACC, AAC, ABB, AAB, AAA, BBB, CCC\} = \{0, 1, \ldots, 9\}$. The notation of the states is quite intuitive; they represent the opinions of the players in the following order: individual with initial opinion A, B, and C. Let us call them person number 1, 2 and 3, respectively. For instance, in state AAB the person number 1, whose opinion was initially A is still A; the person number 2, who initially held opinion B now holds opinion A; and person number 3, whose opinion was initially C, now it is B.
The states 7, 8 and 9, the ones that correspond to each opinion being the prevalent, are absorbing: once the three individuals share the same opinion it is impossible for any of them to change to another one. The rest of states are transient and from all of them there is at least an absorbing state that can be reached. This makes it an absorbing Markov chain.

Proposition 2.1. *An absorbing chain with a transition probability matrix of the form* (A.7) *when taking the strategy* (1.4) *has the following absorption probabilities at states 7,8 and 9, respectively:* [1]

$$b_{07} = \frac{a^2(2c+a)}{(c+a)^2(c+b+a)}, \quad b_{08} = \frac{b^2(3c+b)}{(c+b)^2(c+b+a)}, \quad b_{09} = \frac{c^2(c^3+3bc^2+3ac^2+8abc+a^2c+ab^2+3a^2b)}{(c+a)^2(c+b)^2(c+b+a)}.$$

Proof. The proof can be found in the appendix, as the proof of Proposition A.2. □

[1] It is noteworthy that the value of b_{07} is incorrect, its correct value is presented in Proposition 2.1. It is most possible that this is a transcription mistake, as the calculations involve quite complex expressions.

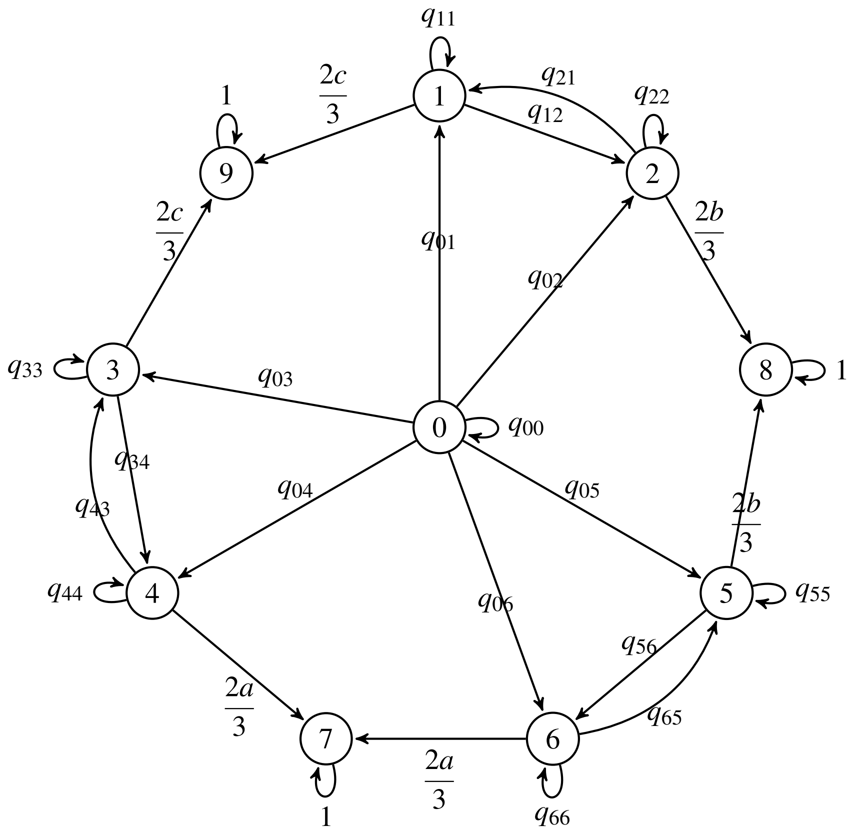

Figure 2.1: diagram of the Markov chain representing the opinion spread model. Own elaboration.

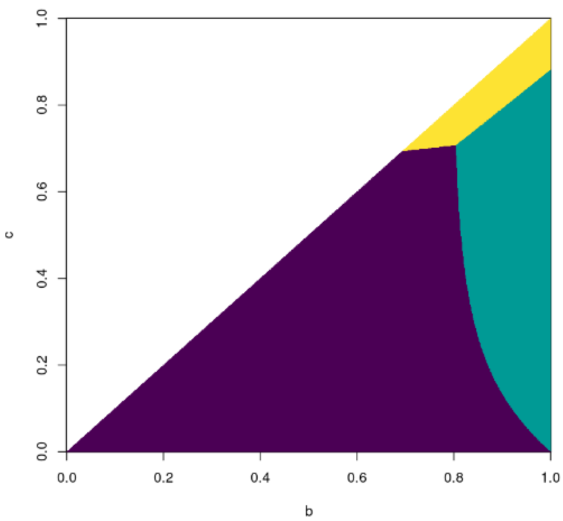

Figure 2.2: The most probable opinion to prevail whem $a = 1$ and $0 < c < b < 1$. Purple corresponds to opinion A, green to opinion B and yellow to opinion C.

In this graph the three different regions correspond to where each opinion is the most likely to prevail among the three of them. That is, the probability that finally all the individuals share either opinion A, B or C, respectively.

The fact that participants do not get eliminated but rather transform their opinions prevents us from encountering as paradoxical a result as we could observe previously, thus the region where opinion C prevails is very small. This can be intuitively understood by noting that once one of the individuals persuades another, the remaining one will have to contend with both of them to convince them both of their viewpoint. In essence, we find it to be a "two against one" struggle, which is considerably more challenging to win. Therefore, this makes opinion A the most likely to be shared in the end by all of the individuals.

2.1 Spread of opinions in a population of N individuals

As a continuation of the concept of using the truel model to study how the transmission of opinions can happen in a population of three individuals, we can consider how this transmission would unfold if we had a population of N individuals, with each of them having initially one of the three distinct possible viewpoints A, B and C, and a certain convincing ability of a, b and c, respectively, where $1 \geq a > b > c > 0$.

Toral and Amengual suggest in [3] a model along these lines to explain the development of a *collective random firing truel* in a population of N people, where each of the individuals has a marksmanship of either a, b or c. There is particularly one simulation in this article concerning the propagation of opinion. As the detailed explanation is given for the collective random truel, we provide below an adapted explanation of the steps taken to specifically carry out the simulation of interest to us.

First of all, the individuals only interact with others close to them in the sense that, considering a simple two-dimensional grid with N places, a person only interacts with the ones in the four direct links from their location. Initially, on the grid, we find the N individuals distributed arbitrarily; the population consists of individuals with opinions A, B, and C in proportions x_A, x_B, and x_C respectively (satisfying $x_A + x_B + x_C = 1$).

After establishing the initial conditions, this is how the propagation of opinions unfolds:

1. A person is selected randomly from the remaining group.

2. The selected person arbitrarily picks two players from the neighboring sites with different viewpoints from him. Then, the three of them undergo a process as described above, so that in a random order, each person will attempt to persuade the other individual between his two opponents who possess the highest probability of conviction.
   In the case that the chosen player has only one of his four neighbors with a different opinion from him, the two engage in an "opinion duel" and finally only one opinion will be shared between the two of them.

3. The steps above are repeated until there is only one opinion shared by the whole population.

Figure 2.3 shows the evolution of a simulation illustrating the propagation of opinions within a population under the following characteristics: there are $N = 2500$ individuals distributed in a proportion of $x_A = 0.3$ (black), $x_B = 0.3$ (red) and $x_C = 0.4$ (green). The respective convincing abilities for the people holding opinion A, B or C are, respectively, $a = 1$, $b = 0.8$ and $c = 0.5$.

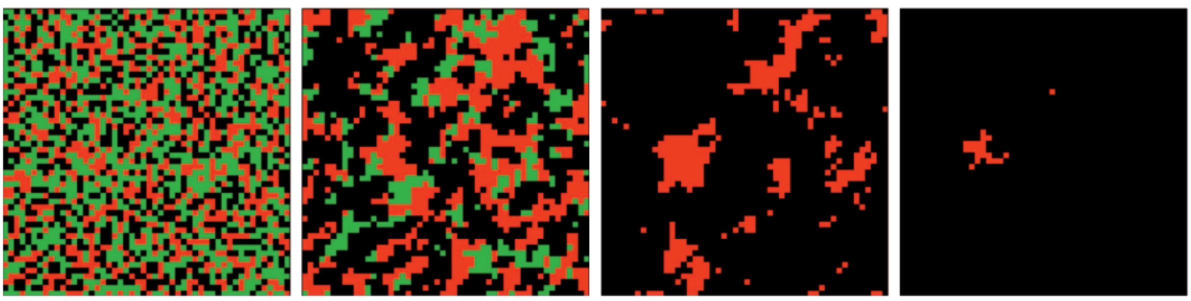

Figure 2.3: Pictures capturing various phases of a simulation of the *collective opinion spread*. (Amengual and Toral, [3])

In the fourth image, we can see how the space is mostly occupied by people with opinion A. This is not surprising, since for the opinion model, although there are exceptions, the probability for an opinion to prevail over the others is directly proportional to the persuasive capacity associated with the viewpoint. That being said, the effect of the proportion in which each opinion initially appears should not be disregarded.

Chapter 3

A practical example

In this chapter, the focus is the examination of the random truel by assigning actual values to variables a, b and c. Then, the survival probabilities of each player are analyzed for this designated values, depending on the pure strategy chosen by the players. Also the expected duration of the game is studied.

The code that was used to calculate general formulas for the survival probability of each player, taking the strategy given by the Nash equilibrium, can be reused to find these same probabilities when other strategies are considered, both pure and mixed. In this case, we are focusing on pure strategies. We are going to present all the pure strategies that can occur for the three players and, using a specific example, we arrive at the conclusion that indeed the set of pure strategies mentioned in (1.4) is the Nash equilibrium for the random truel.

For this purpose, the values chosen for the marksmanships of the players are $a = 1, b = 0.85, c = 0.6$. They have been chosen with the objective of finding a combination of the three that would result in the largest possible difference between the winning probability of A and that of C, in order to make the paradox noticeable.

Notation	Set of strategies	$P(A\text{wins})$	$P(B\text{wins})$	$P(C\text{wins})$
BAA	$P_A^B = P_B^A = P_C^A = 1$	0.255102	0.346938	0.397959
BAB	$P_A^B = P_B^A = P_C^B = 1$	0.408163	0.203377	0.388458
BCB	$P_A^B = P_B^C = P_C^B = 1$	0.595697	0.159404	0.244897
BCA	$P_A^B = P_B^C = P_C^A = 1$	0.442636	0.302965	0.254398
CCB	$P_A^C = P_B^C = P_C^B = 1$	0.561224	0.346938	0.091836
CCA	$P_A^C = P_B^C = P_C^A = 1$	0.408163	0.490499	0.101337
CAB	$P_A^C = P_B^A = P_C^B = 1$	0.373690	0.390912	0.235397
CAA	$P_A^C = P_B^A = P_C^A = 1$	0.220628	0.534473	0.244897

Table 3.1: Survival probabilities of the players in the random firing truel depending on the chosen set of strategies when $a = 1$, $b = 0.85$, $c = 0.6$.

Notice that in the set of strategies column only the elements $P_i^j, i, j \in \{A, B, C\}$ that are equal to 1 are shown. It is assumed that the rest of the elements P_i^j that do not appear in that column take a value of 0. In the table, a simple notation for the pure strategies is also presented, in which the player targeted by A is shown first, followed by the one targeted by B and finally the one targeted by C. Furthermore, the scenario in which a player loses his turn is not considered because, as we have already explained, this does not benefit him in any case.

Here, it can be clearly noted what it means that the players are assumed to be rational beings. Let us see how, by choosing the strategy that best responds to the strategies of the other players, we arrive at the conclusion that the equilibrium is reached when all players follow the *strongest opponent strategy*. In other words, no one will gain anything by deciding to change their strategy under the assumption that the other individuals do not change theirs.

We are going to round the probability values to make the explanation more comprehensible. Now, let us consider that the players adopt the set of strategies *BCB*. While player A has the highest survival probability at 60 percent, followed by player C with a probability of 24 percent, we have player B with a probability of only 16 percent. This is why B is interested in changing his strategy, so we move to the set of strategies *BAB*. In this case, we observe how B's survival probability increases and becomes 20 percent, while A and C now have respective probabilities of 41 and 39 percent. Now, let us consider that C wants to improve their chances, so they make a change, and we now have the set of strategies *BAA*. This way, C's survival probability becomes 40 percent, and their counterparts A and B have probabilities of 25 and 35 percent, respectively.

What is most interesting about this reasoning is that, regardless of the initial set of strategies chosen and the order in which we focus on one player or another, we eventually arrive at the same set of strategies as the optimal one. That is why it constitutes the unique Nash equilibrium point. As observed, for the values we have assigned to the players' marksmanships, at the Nash equilibrium point the survival probabilities of the three players are in reverse order compared to the values of a, b, and c.

Furthermore, we can incorporate an additional reasoning which might be more straightforward in some circumstances. First of all, it must be noted that, regardless of the strategy employed by B an C, the most favorable strategy for A is to aim at B:

Set of strategies	$P(A\text{wins})$			Set of strategies
BAA	0.255102	>	0.220628	CAA
BAB	0.408163	>	0.373690	CAB
BCB	0.595697	>	0.561224	CCB
BCA	0.442636	>	0.408163	CCA

Secondly, despite of the strategy used by A and C, it is always more convenient for B to shoot at A:

Set of strategies	$P(B\text{wins})$			Set of strategies
BAA	0.346938	>	0.302965	BCA
BAB	0.203377	>	0.159404	BCB
CAB	0.390912	>	0.346938	CCB
CAA	0.534473	>	0.490499	CCA

Lastly, player C is left with the options *BAA* and *BAB*. The respective probabilities of survival in these cases are 0.397959 and 0.388458. This makes it evident that the best option simultaneously for all players is to adopt *BAA*.

Notice that right now, if any of the three players were to decide to change his strategy and therefore target a different player unilaterally, none of them would achieve a higher survival probability than what they have in the case of the *BAA* set of strategies.

Let us see now how the survival probabilities of the three players in the sequential truel would look again for $a = 1$, $b = 0.85$ and $c = 0.6$. These variable values lead to $g(a,b,c) < 0$, so the equilibrium point is the set of strategies *BA*0.

The notation in tables 3.2 and 3.3 is used analogously to the case of the random truel, with the only difference being that if C chooses to shoot into the air, it is denoted by 0. It is only considered that C would want to shoot into the air to benefit from it, as he has the lowest marksmanship.

Pure strategy	$P(A\text{wins})$	$P(B\text{wins})$	$P(C\text{wins})$
BAA	0.024000	0.665531	0.310468
BAB	0.624000	0.122978	0.253021
BA0	0.060000	0.307446	0.632553
BCA	0.364000	0.542553	0.093446
BCB	0.964000	0.000000	0.036000
BC0	0.910000	0.000000	0.090000
CAA	0.008999	0.716531	0.274468
CAB	0.609000	0.173978	0.217021
CA0	0.022499	0.434946	0.542553
CCA	0.349000	0.593553	0.057446
CCB	0.949000	0.051000	0.000000
CC0	0.872499	0.127500	0.000000

Table 3.2: Survival probabilities of the players in the sequential firing truel depending on the chosen set of strategies when $a = 1$, $b = 0.85$, $c = 0.6$; $g(a,b,c) < 0$.

The survival probability of C for the equilibrium point has substantially increased in the sequential case compared to the random case: in the initial scenario, C has a survival probability of 40 percent at the equilibrium point (*BAA*), whereas in the second scenario, the likelihood of C winning the game at the equilibrium point (*BA*0) is 63 percent. And, just as in the previous case, the players' survival probabilities at the equilibrium point are higher as their marksmanships decrease.

To confirm that in fact the optimal set of strategies here is not *BAA*, let us observe that it does not satisfy the key characteristic of the Nash equilibrium, which is that none of the players benefit by being the only one to change their strategy. In the course of our analysis, we will also arrive at the conclusion that the equilibrium in this context is *BA*0.

If the players adopt the set of strategies *BAA*, B is the most likely to win the game, with a probability of 67 percent, whereas A and C have respective winning probabilities of 2 and 31 percent. Let us assume now that player C changes his strategy and instead of aiming at player A he shoots into the air, then the players would be in the case of *BA*0. While that might seem unreasonable for C to do, the reality is that by doing so he improves his chances of surviving and now they are of 63 percent. This change in the set of strategies leads players A and B to have as their new respective probabilities of surviving 6 and 31 percent. Examining the remaining sets of strategies, it becomes evident that neither A, B, nor C can upgrade their survival probability by being the only ones to change their strategy.

The second reasoning that was used in the random firing truel can be used here analogously. It can be verified that the most favorable strategy for player A is to aim at B, in spite of the strategies chosen by B and C. Also, the most convenient strategy for B is to fire at A, regardless of what A and C do. That leaves the following possibilities for player C: *BAA* and *BA*0. Given that in those cases the survival probability of C is, respectively, 0.310468 and 0.632553, *BA*0 is chosen.

Finally, we choose the values $a = 1$, $b = 0.32$ and $c = 0.28$, which lead to $g(a,b,c) > 0$ and therefore now the Nash equilibrium is the set of strategies *BAA*. A procedure analogous to that of the random truel can be followed, where starting from any set of strategies, it can be shown that *BAA* is indeed the Nash equilibrium. Again, we encounter the paradoxical situation where in the optimal strategy, the least skilled player, C, has the highest probability of survival among the three, at 36 percent. And again, if we consider what benefits A and B the most, separately, without taking into account the strategies adopted by the other players, we see that they should shoot each other, leaving C again with the options *BAA* and *BA*0. From these two options, the survival probability of C is the highest with the first one.

Pure strategy	$P(A\text{wins})$	$P(B\text{wins})$	$P(C\text{wins})$
BAA	0.352511	0.279553	0.367934
BAB	0.632512	0.104005	0.263482
BA0	0.489599	0.144451	0.365948
BCA	0.582911	0.175548	0.241539
BCB	0.862911	0.000000	0.137088
BC0	0.809599	0.000000	0.190400
CAA	0.332928	0.436225	0.230846
CAB	0.612928	0.260677	0.126394
CA0	0.462400	0.362051	0.175548
CCA	0.563327	0.332220	0.104451
CCB	0.843328	0.156672	0.000000
CC0	0.782400	0.217600	0.000000

Table 3.3: Survival probabilities of the players in the sequential firing truel depending on the chosen set of strategies when $a = 1$, $b = 0.32$, $c = 0.28$; $g(a,b,c) > 0$.

As a side note, the code in *Maxima* was also used to compute the expected number of rounds in order for the game to have a winner in the random truel depending on the set of strategies chosen by the participants. That is, the number of steps until the Markov chain representing the random truel is absorbed considering that it starts in state 0, when A, B and C all remain alive. To do this, we use the result provided by Corollary 1.1. For all strategies, the expected number of rounds before the game ends is around 2.4. In particular, for the equilibrium point *BAA* it is 2.55 rounds.

Set of strategies	Expected number of rounds
BAA	2.55
BAB	2.51
BCB	2.41
BCA	2.44
CCB	2.34
CCA	2.37
CAB	2.45
CAA	2.48

Table 3.4: Expected duration of the random truel when $a = 1$, $b = 0.85$, $c = 0.6$.

Finally, the expected number of rounds for the sequential truel at the equilibrium point when $g(a,b,c) < 0$ and $a = 1$, $b = 0.85$, $c = 0.6$ is 2.36 and, when $g(a,b,c) > 0$ and $a = 1$, $b = 0.32$, $c = 0.28$ is 3.05, the highest between the three cases at the Nash equilibrium point.

Bibliography

[1] P. AMENGUAL AND R. TORAL, A Markov chain analysis of truels, *Proceedings of the 8th Granada Seminar on Computational Physics* (2005).

[2] P. AMENGUAL AND R. TORAL, Distribution of winners in truel games, *AIP Conference Proceedings*, **779** (2005), 128-141.

[3] P. AMENGUAL AND R. TORAL, Truels, or survival of the weakest, *Computing in Science and Engineering*, **8** (2006), 88-95.

[4] S.J. BRAMS AND D.M. KILGOUR, The Truel, *Mathematics Magazine*, **70** (5) (1997), 315-326.

[5] E. CERDÁ, J.L. JIMENO AND J. PÉREZ, *Teoría de Juegos*, Pearson-Prentice Hall, Madrid, 2004.

[6] M. GARDNER, *Mathematical Puzzles and Diversions*, Simon & Schuster, New York, 1959.

[7] C. GRINSTEAD AND J. SNELL, *Introduction to Probability*, American Mathematical Society, 1997.

[8] F.S. HILLIER AND G.J. LIEBERMAN, *Introducción a la Investigación de Operaciones*, 8ª ed., McGrawHill, Mexico, 2006.

[9] D.M. KILGOUR, The Sequential Truel, *International Journal of Game Theory*, **4** (3) (1975), 151-174.

[10] D.M. KILGOUR, Equilibrium points of infinite sequential truels, *International Journal of Game Theory*, **6** (1977), 167-180.

[11] C. KINNAIRD, *Encyclopedia of puzzles and pastimes*, The Citadel Press, New York, 1946.

[12] H.D. LARSON AND L. MOSER, A dart game, *American Mathematical Monthly*, **55** (10) (1948), 640-641.

[13] E. MUTLU AND M. WEGENER, The good, the bad, the well-connected, *International Journal of Game Theory*, **50** (2021), 759-771.

[14] M. SHUBIK, *Game Theory in the Social Sciences*, MIT Press, Cambridge, 1985.

[15] J. VON NEUMANN ET AL., *Contributions to the theory of games (AM-40), Volume IV*, Princeton University Press, Princeton, 1959.

Appendices

Appendix A

Transition probability matrices of the models

A.1 Random firing truel

This is the transition probability matrix obtained for the **random firing truel** before contemplating each player's strategy:

$$P = \left(\begin{array}{cccc|ccc} q_{00} & q_{01} & q_{02} & q_{03} & 0 & 0 & 0 \\ 0 & q_{11} & 0 & 0 & \frac{a}{2} & \frac{b}{2} & 0 \\ 0 & 0 & q_{22} & 0 & \frac{a}{2} & 0 & \frac{c}{2} \\ 0 & 0 & 0 & q_{33} & 0 & \frac{b}{2} & \frac{c}{2} \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right). \tag{A.1}$$

Where:

- $q_{00} = 1 - \frac{(1-P_C^0)c + (1-P_B^0)b + (1-P_A^0)a}{3}$
- $q_{01} = \frac{bP_B^C + aP_A^C}{3}$
- $q_{02} = \frac{cP_C^B + aP_A^B}{3}$
- $q_{11} = 1 - \frac{a+b}{2}$
- $q_{22} = 1 - \frac{a+c}{2}$
- $q_{33} = 1 - \frac{b+c}{2}$

A.2 Sequential firing truel

The transition probability matrix for the **sequential firing truel** without considering each individual's specific strategy is the following:

$$P = \left(\begin{array}{ccccccccc|ccc}
0 & 0 & q_{02} & 0 & q_{04} & 0 & q_{06} & 0 & 0 & 0 & 0 & 0 \\
q_{10} & 0 & 0 & q_{13} & 0 & 0 & 0 & 0 & q_{18} & 0 & 0 & 0 \\
0 & q_{21} & 0 & 0 & 0 & q_{25} & 0 & q_{27} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q_{34} & 0 & 0 & 0 & 0 & a & 0 & 0 \\
0 & 0 & 0 & q_{43} & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & q_{56} & 0 & 0 & a & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q_{65} & 0 & 0 & 0 & 0 & 0 & c \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_{78} & 0 & b & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q_{87} & 0 & 0 & 0 & c \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right). \tag{A.2}$$

Where:

- $q_{02} = (1-a) + aP_A^0$
- $q_{04} = aP_A^C$
- $q_{06} = aP_A^B$
- $q_{10} = (1-b) + bP_B^0$
- $q_{13} = bP_B^C$
- $q_{18} = bP_B^A$
- $q_{21} = (1-c) + cP_C^0$
- $q_{25} = cP_B^C$
- $q_{27} = cP_C^A$
- $q_{34} = 1-a$
- $q_{43} = 1-b$
- $q_{56} = 1-a$
- $q_{65} = 1-c$
- $q_{78} = 1-b$
- $q_{87} = 1-c$

Proposition A.1. *An absorbing chain with a transition probability matrix of the form* (A.2) *has the following absorption probabilities at states 9, 10 and 11, respectively:*

- *If* $g(a,b,c) > 0$*, when the values* P_i^j *are evaluated as in* (1.4)*:*

$$b_{2;9} = \frac{a^2(b-1)(c-1)^2}{(ac-c-a)(abc-bc-ac+c-ab+b+a)}, \quad b_{2;10} = -\frac{b(bc^2-2bc+c+b)}{(bc-c-b)(abc-bc-ac+c-ab+b+a)},$$

$$b_{2;11} = -\frac{c(ab^2c^2-4abc^2+2bc^2+2ac^2-c^2-2ab^2c+6abc-bc-2ac+ab^2-2ab)}{(ac-c-a)(bc-c-b)(abc-bc-ac+c-ab+b+a)}.$$

- *If* $g(a,b,c) < 0$*, when the values* P_i^j *are evaluated as in* (1.8)*:*

$$b_{2;9} = \frac{a^2(b-1)(c-1)}{(ab-b-a)(ac-c-a)}, \quad b_{2;10} = -\frac{b^2(c-1)}{(ab-b-a)(bc-c-b)},$$

$$b_{2;11} = -\frac{c(ab^2c-3abc+bc+ac-ab^2+2ab)}{(ab-b-a)(ac-c-a)(bc-c-b)}.$$

Proof. The divisions of the matrix (A.2) correspond to its canonical form's elements, Q, R and U.

If $g(a,b,c) > 0$, as the players are rational, they take the strategy (1.4). Then, the fundamental matrix is (A.2). And, by Theorem 1.3, the matrix $B = NR$ obtained is (A.2). The absorption probabilities of interest are in the third row of the matrix B.

If $g(a,b,c) < 0$ the players take the strategy (1.8). Then, (A.5) is the fundamental matrix $N = (I-Q)^{-1}$. And, by Theorem 1.3, (A.6) is the matrix B, of which we are interested in its third row. □

$$N=(I-Q)^{-1}=$$

$$\begin{pmatrix}
\frac{1}{abc-bc-ac+c-ab+b+a} & \frac{(a-1)(c-1)}{abc-bc-ac+c-ab+b+a} & -\frac{a-1}{abc-bc-ac+c-ab+b+a} & 0 & 0 & \frac{a(c-1)}{(ac-c-a)(abc-bc-ac+c-ab+b+a)} & -\frac{a}{(ac-c-a)(abc-bc-ac+c-ab+b+a)} & \frac{(a-1)\left(bc^2-2bc+c+b\right)}{(bc-c-b)(abc-bc-ac+c-ab+b+a)} & -\frac{(a-1)(2bc-c-b)}{(bc-c-b)(abc-bc-ac+c-ab+b+a)} \\
-\frac{b-1}{abc-bc-ac+c-ab+b+a} & \frac{1}{abc-bc-ac+c-ab+b+a} & \frac{(a-1)(b-1)}{abc-bc-ac+c-ab+b+a} & 0 & 0 & -\frac{a(b-1)(c-1)}{(ac-c-a)(abc-bc-ac+c-ab+b+a)} & \frac{a(b-1)}{(ac-c-a)(abc-bc-ac+c-ab+b+a)} & -\frac{abc-2bc-ac+c+b}{(bc-c-b)(abc-bc-ac+c-ab+b+a)} & \frac{ab^2c-b^2c-2abc+2bc+ac-c-b}{(bc-c-b)(abc-bc-ac+c-ab+b+a)} \\
\frac{(b-1)(c-1)}{abc-bc-ac+c-ab+b+a} & -\frac{c-1}{abc-bc-ac+c-ab+b+a} & \frac{1}{abc-bc-ac+c-ab+b+a} & 0 & 0 & \frac{a(b-1)(c-1)^2}{(ac-c-a)(abc-bc-ac+c-ab+b+a)} & -\frac{a(b-1)(c-1)}{(ac-c-a)(abc-bc-ac+c-ab+b+a)} & -\frac{bc^2-2bc+c+b}{(bc-c-b)(abc-bc-ac+c-ab+b+a)} & \frac{2bc-c-b}{(bc-c-b)(abc-bc-ac+c-ab+b+a)} \\
0 & 0 & 0 & -\frac{1}{ab-b-a} & \frac{a-1}{ab-b-a} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{b-1}{ab-b-a} & -\frac{1}{ab-b-a} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{ac-c-a} & \frac{a-1}{ac-c-a} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{c-1}{ac-c-a} & -\frac{1}{ac-c-a} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{bc-c-b} & \frac{b-1}{bc-c-b} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{c-1}{bc-c-b} & -\frac{1}{bc-c-b}
\end{pmatrix} \quad (A.3)$$

$$B=NR=$$

$$\begin{pmatrix}
\frac{a^2(c-1)}{(ac-c-a)(abc-bc-ac+c-ab+b+a)} & \frac{(a-1)b\left(bc^2-2bc+c+b\right)}{(bc-c-b)(abc-bc-ac+c-ab+b+a)} & -\frac{c\left(2a^2bc^2-4abc^2+2bc^2-a^2c^2+2ac^2-c^2-3a^2bc+5abc-bc+a^2c-2ac+a^2b-2ab\right)}{(ac-c-a)(bc-c-b)(abc-bc-ac+c-ab+b+a)} \\
-\frac{a^2(b-1)(c-1)}{(ac-c-a)(abc-bc-ac+c-ab+b+a)} & -\frac{b(abc-2bc-ac+c+b)}{(bc-c-b)(abc-bc-ac+c-ab+b+a)} & \frac{c\left(a^2b^2c^2-2ab^2c^2+b^2c^2-2a^2bc^2+4abc^2-2bc^2+a^2c^2-2ac^2+c^2-a^2b^2c+2ab^2c+2a^2bc-5abc+bc-a^2c+2ac-ab^2+2ab\right)}{(ac-c-a)(bc-c-b)(abc-bc-ac+c-ab+b+a)} \\
\frac{a^2(b-1)(c-1)^2}{(ac-c-a)(abc-bc-ac+c-ab+b+a)} & -\frac{b\left(bc^2-2bc+c+b\right)}{(bc-c-b)(abc-bc-ac+c-ab+b+a)} & -\frac{c\left(ab^2c^2-4abc^2+2bc^2+2ac^2-c^2-2ab^2c+6abc-bc-2ac+ab^2-2ab\right)}{(ac-c-a)(bc-c-b)(abc-bc-ac+c-ab+b+a)} \\
-\frac{a}{ab-b-a} & \frac{(a-1)b}{ab-b-a} & 0 \\
\frac{a(b-1)}{ab-b-a} & -\frac{b}{ab-b-a} & 0 \\
-\frac{a}{ac-c-a} & 0 & \frac{(a-1)c}{ac-c-a} \\
\frac{a(c-1)}{ac-c-a} & 0 & -\frac{c}{ac-c-a} \\
0 & -\frac{b}{bc-c-b} & \frac{(b-1)c}{bc-c-b} \\
0 & \frac{b(c-1)}{bc-c-b} & -\frac{c}{bc-c-b}
\end{pmatrix} \quad (A.4)$$

$$N=(I-Q)^{-1}=\begin{pmatrix}
-\frac{1}{ab-b-a} & \frac{a-1}{ab-b-a} & \frac{a-1}{ab-b-a} & 0 & 0 & -\frac{a(c-1)}{(ab-b-a)(ac-c-a)} & \frac{a}{(ab-b-a)(ac-c-a)} & \frac{(a-1)b(c-1)}{(ab-b-a)(bc-c-b)} & -\frac{(a-1)b}{(ab-b-a)(bc-c-b)} \\
\frac{b-1}{ab-b-a} & -\frac{1}{ab-b-a} & -\frac{(a-1)(b-1)}{ab-b-a} & 0 & 0 & \frac{a(b-1)(c-1)}{(ab-b-a)(ac-c-a)} & -\frac{a(b-1)}{(ab-b-a)(ac-c-a)} & -\frac{b(c-1)}{(ab-b-a)(bc-c-b)} & \frac{b}{(ab-b-a)(bc-c-b)} \\
\frac{b-1}{ab-b-a} & -\frac{1}{ab-b-a} & -\frac{1}{ab-b-a} & 0 & 0 & \frac{a(b-1)(c-1)}{(ab-b-a)(ac-c-a)} & -\frac{a(b-1)}{(ab-b-a)(ac-c-a)} & -\frac{b(c-1)}{(ab-b-a)(bc-c-b)} & \frac{b}{(ab-b-a)(bc-c-b)} \\
0 & 0 & 0 & -\frac{1}{ab-b-a} & \frac{a-1}{ab-b-a} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{b-1}{ab-b-a} & -\frac{1}{ab-b-a} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{ac-c-a} & \frac{a-1}{ac-c-a} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{c-1}{ac-c-a} & -\frac{1}{ac-c-a} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{bc-c-b} & \frac{b-1}{bc-c-b} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{c-1}{bc-c-b} & -\frac{1}{bc-c-b}
\end{pmatrix}. \quad (A.5)$$

$$B = NR = \begin{pmatrix} -\frac{a^2(c-1)}{(ab-b-a)(ac-c-a)} & \frac{(a-1)b^2(c-1)}{(ab-b-a)(bc-c-b)} & -\frac{c\left(a^2bc-3abc+bc+ac-a^2b+2ab\right)}{(ab-b-a)(ac-c-a)(bc-c-b)} \\ \frac{a^2(b-1)(c-1)}{(ab-b-a)(ac-c-a)} & -\frac{b^2(c-1)}{(ab-b-a)(bc-c-b)} & -\frac{c\left(ab^2c-3abc+bc+ac-ab^2+2ab\right)}{(ab-b-a)(ac-c-a)(bc-c-b)} \\ \frac{a^2(b-1)(c-1)}{(ab-b-a)(ac-c-a)} & -\frac{b^2(c-1)}{(ab-b-a)(bc-c-b)} & -\frac{c\left(ab^2c-3abc+bc+ac-ab^2+2ab\right)}{(ab-b-a)(ac-c-a)(bc-c-b)} \\ -\frac{a}{ab-b-a} & \frac{(a-1)b}{ab-b-a} & 0 \\ \frac{a(b-1)}{ab-b-a} & -\frac{b}{ab-b-a} & 0 \\ -\frac{a}{ac-c-a} & 0 & \frac{(a-1)c}{ac-c-a} \\ \frac{a(c-1)}{ac-c-a} & 0 & -\frac{c}{ac-c-a} \\ 0 & -\frac{b}{bc-c-b} & \frac{(b-1)c}{bc-c-b} \\ 0 & \frac{b(c-1)}{bc-c-b} & -\frac{c}{bc-c-b} \end{pmatrix} \quad \text{(A.6)}$$

A.3 Opinion spread model

The transition probability matrix for the **opinion spread model** without taking into account each individual's strategy is the following:

$$P = \left(\begin{array}{ccccccc|ccc} q_{00} & q_{01} & q_{02} & q_{03} & q_{04} & q_{05} & q_{06} & 0 & 0 & 0 \\ 0 & q_{11} & q_{12} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2c}{3} \\ 0 & q_{21} & q_{22} & 0 & 0 & 0 & 0 & 0 & \frac{2b}{3} & 0 \\ 0 & 0 & 0 & q_{33} & q_{34} & 0 & 0 & 0 & 0 & \frac{2c}{3} \\ 0 & 0 & 0 & q_{43} & q_{44} & 0 & 0 & \frac{2a}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q_{55} & q_{56} & 0 & \frac{2b}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & q_{65} & q_{66} & \frac{2a}{3} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right). \tag{A.7}$$

Where:

$$q_{00} = \frac{3-a-b-c}{3} \qquad q_{06} = \frac{P_A^C a}{3} \qquad q_{43} = \frac{c}{3}$$

$$q_{01} = \frac{P_C^A c}{3} \qquad q_{11} = \frac{2(1-c)}{3} + \frac{1-b}{3} \qquad q_{44} = \frac{1-c}{3} + \frac{2(1-a)}{3}$$

$$q_{02} = \frac{P_B^A b}{3} \qquad q_{12} = \frac{b}{3} \qquad q_{55} = \frac{2(1-b)}{3} + \frac{1-a}{3}$$

$$q_{21} = \frac{c}{3}$$

$$q_{03} = \frac{P_C^B c}{3} \qquad q_{22} = \frac{1-c}{3} + \frac{2(1-b)}{3} \qquad q_{56} = \frac{a}{3}$$

$$q_{04} = \frac{P_A^B a}{3} \qquad q_{33} = \frac{2(1-c)}{3} + \frac{1-a}{3} \qquad q_{65} = \frac{b}{3}$$

$$q_{05} = \frac{P_B^C b}{3} \qquad q_{34} = \frac{a}{3} \qquad q_{66} = \frac{1-b}{3} + \frac{2(1-a)}{3}$$

Proposition A.2. *An absorbing chain with a transition probability matrix of the form* (A.7) *has the following absorption probabilities at states 7,8 and 9, respectively:*

$$b_{07} = \frac{a^2(2c+a)}{(c+a)^2(c+b+a)}, \quad b_{08} = \frac{b^2(3c+b)}{(c+b)^2(c+b+a)}, \quad b_{09} = \frac{c^2(c^3+3bc^2+3ac^2+8abc+a^2c+ab^2+3a^2b)}{(c+a)^2(c+b)^2(c+b+a)}.$$

Proof. The submatrices of P that correspond to its canonical form's elements, Q, R and U, can be observed through the divisions of the matrix (A.7). Taking into account that in the opinion spread model the participants take the strategy (1.4), the fundamental matrix $N = (I-Q)^{-1}$ is (A.8). And, by Theorem 1.3, B is the matrix (A.9). □

$$N=(I-Q)^{-1}=\begin{pmatrix} \frac{3}{c+b+a} & \frac{3c(c+3b)}{2(c+b^2)(c+b+a)} & \frac{3b(3c+b)}{2(c+b)^2(c+b+a)} & \frac{3ac}{2(c+a)^2(c+b+a)} & \frac{3a(2c+a)}{2(c+a)^2(c+b+a)} & 0 & 0 \\ 0 & \frac{3(c+2b)}{2(c+b)^2} & \frac{3b}{2(c+b)^2} & 0 & 0 & 0 & 0 \\ 0 & \frac{3c}{2(c+b)^2} & \frac{3(2c+b)}{2(c+b)^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3(c+2a)}{2(c+a)^2} & \frac{3a}{2(c+a)^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{3c}{2(c+a)^2} & \frac{3(2c+a)}{2(c+a)^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3(b+2a)}{2(b+a)^2} & \frac{3a}{2(b+a)^2} \\ 0 & 0 & 0 & 0 & 0 & \frac{3b}{2(b+a)^2} & \frac{3(2b+a)}{2(b+a)^2} \end{pmatrix}. \tag{A.8}$$

$$B=NR=\begin{pmatrix} \frac{a^2(2c+a)}{(c+a)^2(c+b+a)} & \frac{b^2(3c+b)}{(c+b)^2(c+b+a)} & \frac{c^2(c^3+3bc^2+3ac^2+8abc+a^2c+ab^2+3a^2b)}{(c+a)^2(c+b)^2(c+b+a)} \\ 0 & \frac{b^2}{(c+b)^2} & \frac{c(c+2b)}{(c+b)^2} \\ 0 & \frac{b(2c+b)}{(c+b)^2} & \frac{c^2}{(c+b)^2} \\ \frac{a^2}{(c+a)^2} & 0 & \frac{c(c+2a)}{(c+a)^2} \\ \frac{a(2c+a)}{(c+a)^2} & 0 & \frac{c^2}{(c+a)^2} \\ \frac{a^2}{(b+a)^2} & \frac{b(b+2a)}{(b+a)^2} & 0 \\ \frac{a(2b+a)}{(b+a)^2} & \frac{b^2}{(b+a)^2} & 0 \end{pmatrix}. \tag{A.9}$$

Appendix B

Maxima code for the calculation of absorption probabilities

The subsequent are the codes used to compute the absorption probabilities of the Markov chains representing the different models.

B.1 Random firing truel

```
/*Transition probabilities*/
p00: 1- (1/3)*(a*(1-PA0)+b*(1-PB0)+c*(1-PC0));
p02: (1/3)*(a*PAB + c*PCB);
p03: (1/3)*(b*PBA + c*PCA);
p24: (1/2)*a;
p35: (1/2)*b;
p26: (c/2);
p36: p26;
p22: 1- (a+c)/2;
p33: 1- (b+c)/2;

/*Elements of the canonical form*/
Q: matrix([p00,p02,p03],[0,p22,0],[0,0,p33]);
R: matrix([0,0,0],[p24,0,p26],[0,p35,p36]);
I3: diagmatrix(3,1);

/*Fundamental matrix*/
N:invert(I3-Q);

/*Absorption probability matrix*/
B: N.R;

/*Absorption probabilities starting from 0 when taking the BAA set of strategies*/
ev(row(B,1), PCB=0, PBC=0, PAC=0, PA0=0, PB0=0, PC0=0, PAB=1,PBA=1, PCA=1);
factor(%);
```

B.2 Sequential firing truel

```
/*Transition probabilities*/
p02:(1-a)+a*PA0;
p04:a*PAC;
```

```
p06:a*PAB;
p10:(1-b)+b*PB0;
p13:b*PBC;
p18:b*PBA;
p21:(1-c)+c*PC0;
p25:c*PCB;
p27:c*PCA;
p34:1-a;
p39:a;
p43:1-b;
p410:b;
p56:1-a;
p59:a;
p65:1-c;
p611:c;
p78:1-b;
p710:b;
p87:1-c;
p811:c;

/*Elements of the canonical form*/
Q: matrix(
[0,0,p02,0,p04,0,p06,0,0],
[p10,0,0,p13,0,0,0,0,p18],
[0,p21,0,0,0,p25,0,p27,0],
[0,0,0,0,p34,0,0,0,0],
[0,0,0,p43,0,0,0,0,0],
[0,0,0,0,0,0,p56,0,0],
[0,0,0,0,0,p65,0,0,0],
[0,0,0,0,0,0,0,0,p78],
[0,0,0,0,0,0,0,p87,0]);
R: matrix(
[0,0,0],
[0,0,0],
[0,0,0],
[p39,0,0],
[0,p410,0],
[p59,0,0],
[0,0,p611],
[0,p710,0],
[0,0,p811]);
I9: diagmatrix(9,1);

/*Fundamental matrix*/
N:invert(I9-Q);

/*Absorption probability matrix*/
B: N.R;

/*Absorption probabilities starting from 2 when taking the BAA set of strategies*/
ev(row(B,3),PCB=0, PBC=0, PAC=0, PA0=0, PB0=0, PC0=0, PAB=1, PBA=1, PCA=1);
```

```
factor(%);

/*Absorption probabilities starting from 2 when taking the BAO set of strategies*/
ev(row(B,3),PCB=0, PBC=0, PAC=0, PAO=0, PBO=0, PCO=1, PAB=1, PBA=1, PCA=0);
factor(%);
```

B.3 Opinion spread model

```
/*Transition probabilities*/
p00:(1/3)*(3-a-b-c);
p04:(1/3)*c*PCA;
p05:(1/3)*b*PBA;
p06:(1/3)*c*PCB;
p07:(1/3)*a*PAB;
p08:(1/3)*b*PBC;
p09:(1/3)*a*PAC;
p41:(2/3)*c;
p44:(2/3)*(1-c)+(1/3)*(1-b);
p45:(1/3)*b;
p52:(2/3)*b;
p54:(1/3)*c;
p55:(1/3)*(1-c)+(2/3)*(1-b);
p66:(2/3)*(1-c)+(1/3)*(1-a);
p67:(1/3)*a;
p73:(2/3)*a;
p77:(1/3)*(1-c)+(2/3)*(1-a);
p88:(2/3)*(1-b)+(1/3)*(1-a);
p99:(1/3)*(1-b)+(2/3)*(1-a);
p61:p41;
p76:p54;
p82:p52;
p89:p67;
p93:p73;
p98:p45;

/*Elements of the canonical form*/
Q:matrix([r0,p04,p05,p06,p07,p08,p09],
[0,p44,p45,0,0,0,0],
[0,p54,p55,0,0,0,0],
[0,0,0,p66,p67,0,0],
[0,0,0,p76,p77,0,0],
[0,0,0,0,0,p88,p89],
[0,0,0,0,0,p98,p99]);
R:matrix([0,0,0],
[p41,0,0],
[0,p52,0],
[p61,0,0],
[0,0,p73],
[0,p82,0],
[0,0,p93]);
I7:diagmatrix(7,1);
```

```
/*Fundamental matrix*/
N:invert(I7-Q);

/*Absorption probability matrix*/
B: N.R;

/*Absorption probabilities starting from 0 when taking the BAA set of strategies*/
ev(row(B,1),PCB=0, PBC=0, PAC=0, PA0=0, PB0=0, PC0=0, PAB=1, PBA=1, PCA=1);
factor(%);
```

Appendix C

R code for the survival probability graphs

The following is the code used in R to draw the graphs for the following cases:

C.1 Random firing truel

```
mifunc1<-function(c,b,a=1){

  if(c > b) return(NA)
  else {
  pc<-(2*a+c)*c/((a+c)*(a+b+c))
  pb<-b/(a+b+c)
  pa<- a^2/((a+c)*(a+b+c))
  }
  p<-c(pa,pb,pc)
  col<-as.numeric(which.max(p))
  return(col+2)
}

b<-seq(0,1,0.001)
c<-seq(0,1,0.001)
z<-matrix(0,nrow=length(b),ncol=length(c))
for (i in 1:length(b)){
  for(j in 1:length(c)){
    z[i,j]<-mifunc1(c[j],b[i])
  }
}

image(b,c,z, col=hcl.colors(100))
```

C.2 Sequential firing truel

```
    g<-function(a,b,c) {
     (a^2)*((1-b)^2)*(1-c)-(b^2)*c-a*b*(1-b*c)
     }
mifunc1<-function(c,b,a=1){
  if(c > b) return(NA)
  if (g(a,b,c)> 0 ) {
  pc<-((-c)*(a*(b^2)*(c^2)-4*a*b*(c^2)+2*b*(c^2)+2*a*(c^2)-(c^2)-2*a*(b^2)*c+6*a*b*c-b*c
      -2*a*c+a*(b^2)-2*a*b))/((a*c-a-c)*(b*c-c-b)*(a*b*c-b*c-a*c+c-a*b+b+a))
```

```
  pb<-((-b)*(b*(c^2)-2*b*c+c+b))/((b*c-c-b)*(a*b*c-b*c-a*c+c-a*b+b+a))
  pa<-(a^2*(b-1)*((c-1)^2))/((a*c-a-c)*(a*b*c-b*c-a*c+c-a*b+b+a))
  }
    else
  {
  pc<- ((-c)*(a*(b^2)*c-3*a*b*c+b*c+a*c-a*(b^2)+2*a*b))/((a*b-b-a)*(a*c-c-a)*(b*c-b-c))
  pb<- ((-b^2)*(c-1))/((a*b-b-a)*(b*c-b-c))
  pa<-((a^2)*(b-1)*(c-1))/((a*c-a-c)*(a*b-b-a))
  }
  p<-c(pa,pb,pc)
  col<-as.numeric(which.max(p))
  return(col+2)
}

b<-seq(0,1,0.001)
c<-seq(0,1,0.001)
z<-matrix(0,nrow=length(b),ncol=length(c))
for (i in 1:length(b)){
  for(j in 1:length(c)){
    z[i,j]<-mifunc1(c[j],b[i])
  }
}

image(b,c,z, col=hcl.colors(100))
g_values <- matrix(0, nrow=length(b), ncol=length(c))
for (i in 1:length(b)) {
  for (j in 1:length(c)) {
    g_values[i, j] <- g(1, b[i], c[j])
  }
}

image(b, c, z, col=hcl.colors(100))

contour(b, c, g_values, levels=0, add=TRUE, col="red")
```

C.3 Opinion spread model

```
    mifunc1<-function(c,b,a=1){

  if(c > b) return(NA)
  else {
  pc<- ((c^2)*(3*b*(c^2)+3*a*(c^2)+8*a*b*c+(a^2)*c+a*(b^2)+3(a^2)*b))/
       (((c+b)^2)*(c+b+a)*((c+a)^2))
  pb<- ((b^2)*(3*c+b))/(((c+b)^2)*(c+b+a))
  pa<- ((a^2)*(2*c+a))/(((c+a)^2)*(c+b+a))
  }
  p<-c(pa,pb,pc)
  col<-as.numeric(which.max(p))
  return(col+2)
}
```

```
b<-seq(0,1,0.001)
c<-seq(0,1,0.001)
z<-matrix(0,nrow=length(b),ncol=length(c))
for (i in 1:length(b)){
  for(j in 1:length(c)){
    z[i,j]<-mifunc1(c[j],b[i])
  }
}

image(b,c,z, col=hcl.colors(100))
```