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The Finite Element Method for problems in fluid mechanics

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List of acronyms

PDE: Partial Differential Equation.

EDP: Ecuación en Derivadas Parciales.

FEM: Finite Element Method.

FDM: Finite Difference Method.

FVM: Finite Volume Method.

CGM: Conjugate Gradient Method.

HPC: High Performance Computing.

CFD: Computational Fluid Dynamics.

CR: Crouzeix-Raviart.

LBB: Ladyzhenskaya–Babuška–Brezzi condition.

GMRES: Generalized Minimal Residual method.

RANS: Reynolds-Averaged Navier-Stokes.

Introduction

Applied Mathematics deals with the application of mathematical tools in science and engineering. Many mathematical methods have been used in physics, chemistry, biology, engineering, or economics. Especially after the recent development of computer science, these methods are an important part of the training of scientists and engineers. This field was born mainly with the study of differential equations, from analytical methods to modern simulation techniques and numerical analysis. The development of the theory of differential equations has always been closely related to newtonian physics and mechanics and quickly found applications, for example, in fluid mechanics, one of the most important fields of application nowadays.

The aim of this work is to study the Finite Element Method (FEM), a numerical method that allows solving all kinds of differential equations. This and other similar methods, such as the Finite Difference Method (FDM) and the Finite Volume Method (FVM) (see Appendix B), have a wide range of applications in engineering, from structural analysis to fluid mechanics and heat transfer; and in physics, for example, in acoustics, electrodynamics or plasma physics. In this work, we will focus on fluid dynamics. All these numerical methods in fluid physics constitute a broad field known as Computational Fluid Dynamics (CFD). The Navier-Stokes equations, which are the equivalent of Newton's laws in fluids, are of great importance in physics, engineering, and medicine, but are also particularly interesting from a mathematical point of view. So much so that the study of the existence and uniqueness of solutions is one of the 7 problems of the millennium. That is because, despite their importance in science and engineering, their theoretical properties are hardly known due to their complexity.

The first version of FEM was developed by Richard Courant in 1943 for a rod torsion problem, which was based on a 1908 Ritz paper in which he sought solutions to a problem involving linear combinations of simple functions. The first article on FEM was written in 1956 [1], already closely related to the branch of aerospace engineering. Finally, the widespread use of the method had to wait for the development of the first computers, where, for example, the first computer program called NASTRAN stands out. The FEM consists in discretizing the domain into small portions in which the solution is approximated by simple polynomials, reducing the problem to a system of linear algebraic equations. Current challenges of this method include the estimation of errors in non-linear or non-stationary problems and the increase in computation time as we make the discretization increasingly finer.

This work begins with an introduction to fluid physics. Then we introduce the mathematical foundations behind FEM, and finally we show how to apply it to Stokes' equations.

Resumen

La Matemática Aplicada es la rama que se encarga de la aplicación de herramientas matemáticas en otras ramas de la ciencia o la ingeniería. Especialmente tras el reciente desarrollo de la computación, estos métodos son una parte importante de la formación de científicos e ingenieros. Este campo nace fundamentalmente con el estudio de ecuaciones diferenciales, desde métodos analíticos hasta las modernas técnicas de simulación y análisis numérico.

El objetivo de este trabajo es estudiar el Método de Elementos Finitos (FEM), un método numérico que permite resolver todo tipo de ecuaciones diferenciales y que tiene un gran rango de aplicación en la ingeniería, desde el análisis estructural, hasta la mecánica de fluidos, pasando por la transferencia del calor. En este caso nos centraremos especialmente en la dinámica de fluidos. Las ecuaciones de Navier-Stokes, que son el equivalente a las leyes de Newton en fluidos, tienen una gran importancia en la física, la ingenieria y la medicina, pero también son especialmente interesantes desde el punto de vista matemático.

La primera versión del FEM fue desarrollada por Richard Courant en 1943 para un problema de torsión de una barra, que se basó en un artículo de Ritz en 1908 en el que buscaba soluciones a un problema en forma de combinaciones lineales de funciones simples. El primer artículo sobre el FEM fue escrito en 1956 [1], ya muy relacionado con la rama de la ingeniería aeroespacial. Finalmente, el uso extendido del método tuvo que esperar al desarrollo de los primeros ordenadores, donde destaca, por ejemplo, el primer programa informático llamado NASTRAN. Este método consiste en discretizar el dominio en pequeñas porciones en las que se aproxima la solución por polinomios sencillos, reduciendo el problema a un sistema de ecuaciones algebraicas.

Este trabajo comienza con una introducción a la física de fluidos, posteriormente veremos los fundamentos matemáticos detrás del FEM, y por último veremos como aplicarlo a las ecuaciones de Stokes.

Física de fluidos

Un fluido es una sustancia sin una forma fija y que se deforma continuamente cuando aplicamos una fuerza o presión. Las ecuaciones de Navier-Stokes son un sistema de dos ecuaciones, la conservación de la masa y la conservación del momento (segunda ley de Newton), que modelan el comportamiento de fluidos incompresibles. Consideremos $\Omega \subset \mathbb{R}^n$ abierto y acotado con frontera $\partial\Omega$. Utilizando el teorema de la divergencia y el hecho de que el cambio de masa dentro de Ω es igual al flujo de masa a través de la frontera, llegamos a:

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \mathbf{u} = 0, \tag{1}$$

donde ρ es la densidad del fluido, $\mathbf{u} = (u_1, ..., u_n)$ la velocidad y $\frac{D}{Dt} := \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla)$ la derivada material. En este trabajo consideramos solo fluidos incompresibles, que satisfacen $\frac{D\rho}{Dt} = 0$ y por tanto permiten expresar la conservación de la masa simplemente mediante $\nabla \cdot \mathbf{u} = 0$.

Un cálculo más complejo sobre la segunda ley de Newton permite obetener la conservación del momento en los fluidos:

$$\frac{D\mathbf{u}}{Dt} := \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{F} - \frac{1}{\rho}\nabla p + \nu \Delta \mathbf{u}, \tag{2}$$

donde $\mathbf{F}(x,t) \in \mathbb{R}^n$ es una fuerza externa, p la presión y ν la viscosidad del fluido. Cuando tenemos un fluido muy lento y viscoso, las ecuaciones de Navier-Stokes se pueden transformar en las más sencillas ecuaciones de Stokes, que son lineales y estacionarias, y que junto a las condiciones de contorno apropiadas, forman el problema que vamos a considerar en este trabajo:

$$\nabla \cdot \mathbf{u} = 0, \text{ en } \Omega, \tag{3}$$

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f}, \text{ en } \Omega, \tag{4}$$

$$\mathbf{u} = \mathbf{g}, \text{ en } \partial\Omega. \tag{5}$$

El Método de Elementos Finitos

Como hemos dicho, este método consiste, en primer lugar, en crear una malla en el dominio que lo subdivida en elementos más pequeños. Sea $\Omega \subset \mathbb{R}^2$ abierto y acotado, definimos una malla triangular en Ω como un conjunto $\mathcal{K} = \{K_i\}_{i=1}^n$ tal que

1. K_i es un triángulo, para todo $i = 1, ..., n$.
2. $\cup_{i=1}^n K_i = \bar{\Omega}$.
3. $K_i \cap K_j$ con $i \neq j$ es vacio, un vértice o un lado completo de un triángulo.

Entonces definimos sobre $\mathcal{K}$ un espacio finito de funciones V_h con una base $\{\varphi_j\}_{j=1}^N$ que aproxime el espacio funcional continuo al que pertenece la solución de la Ecuación en Derivadas Parciales (EDP). Normalmente consideramos el espacio de funciones continuas lineales a trozos

$$V_h = \{v \in C(\Omega) : v \mid_K \in P_1(K), \forall K \in \mathcal{K}\}, \tag{6}$$

donde $P_1(K) = \{v(x) = c_0 + c_1 x + c_2 y : (x,y) \in K, c_0, c_1, c_2 \in \mathbb{R}\}$ es el conjunto de funciones lineales en K y $h = \max\{h_K\}$ con h_K el lado mayor de cada K.

El siguiente paso es multiplicar nuestra ecuación por una función de test e integrar aplicando la fórmula de Green. De esta manera llegaremos a la formulación variacional o débil del problema.

$$(V) \begin{cases} \text{Encontrar } u \in V \text{ tal que} \\ a(u,v) = l(v), \ \forall v \in V, \end{cases} \tag{7}$$

donde $a(\cdot,\cdot)$ es una forma bilinial continua y coerciva, y $l(\cdot)$ una forma lineal continua. Ahora en el Método de Elementos Finitos aproximamos la función $u \in V$ con una función $u_h \in V_h$, dando lugar a:

$$\begin{cases} \text{Encontrar } u_h \in V_h \text{ tal que} \\ a(u_h, v_h) = l(v_h), \ \forall v_h \in V_h. \end{cases} \tag{8}$$

Considerando la base $\{\varphi_i\}_{i=1}^N$ de V_h el problema anterior es equivalente a

$$\begin{cases} \text{Encontrar } u_h \in V_h \text{ tal que} \\ a(u_h, \varphi_i) = l(\varphi_i), \ \forall i = 1, 2, ..., N. \end{cases} \tag{9}$$

Como $u_h \in V_h$, podemos escribirla como una combinación lineal $u_h = \sum_{j=1}^N \xi_j \varphi_j$. Sustituyendo:

$$b_i := l(\varphi_i) = \sum_{j=1}^N \xi_j a(\varphi_j, \varphi_i) := \sum_{j=1}^N A_{ij}\xi_j \Leftrightarrow A\xi = b. \tag{10}$$

Así hemos transformado el problema diferencial en un sistema de ecuaciones algebraicas. En este trabajo se va a estudiar teóricamente la existencia y unicidad de soluciones, así como la aplicación de distintos tipos de elementos finitos para la resolución numérica del problema.

Las ecuaciones de Stokes y el problema de la *lid-driven cavity*

En las ecuaciones de Stokes previamente mencionadas queremos encontrar la velocidad $\mathbf{u} = (u_1, u_2)$ y la presión p. Por ello, necesitamos introducir dos espacios de test distintos y trabajar con lo que se conocen como Elementos Finitos Mixtos.

$$V_g = \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v} \mid_{\partial\Omega} = \mathbf{g}\}, \qquad Q = \{q \in L^2(\Omega) : \int_\Omega q \; d\mathbf{x} = 0\}, \tag{11}$$

donde $H^1(\Omega)$ es el espacio de Sobolev de orden uno y en general $\mathbf{H}^1(\Omega) = H^1(\Omega) \times ..^n .. \times H^1(\Omega)$ (en nuestro caso $n = 2$), es decir, que cada componente del vector está en $H^1(\Omega)$. Multiplicando la segunda ecuación (4) por $v \in V_0$ e integrando usando la fórmula de Green, y multiplicando la primera ecuación (3) por $q \in Q$, llegamos a la formulación variacional del problema.

$$(V) \begin{cases} \text{Encontrar } \mathbf{u} \in V_g \text{ y } p \in Q \text{ tal que} \\ a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = l(\mathbf{v}), \; \forall \mathbf{v} \in V_0, \\ b(\mathbf{u}, q) = 0, \; \forall q \in Q, \end{cases} \tag{12}$$

donde

$$a(\mathbf{u}, \mathbf{v}) = (\nabla \mathbf{u} : \nabla \mathbf{v}) = \int_\Omega (\nabla u_1 \cdot \nabla v_1 + \nabla u_2 \cdot \nabla v_2) d\mathbf{x},$$

$$b(\mathbf{v}, p) = -(\nabla \cdot \mathbf{v}, p) = -\int_\Omega \nabla \cdot \mathbf{v} \; p \; d\mathbf{x},$$

$$l(\mathbf{v}) = (\mathbf{f}, \mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} \; d\mathbf{x}.$$

Como ejemplo vamos a calcular la solución del problema conocido como *lid-driven cavity*. Consideramos un fluido viscoso e incompresible en un dominio cuadrado $\Omega = (-1, 1)^2$ donde la pared superior se mueve horizontalmente, es decir, consideramos las siguientes condiciones de contorno

$$\begin{cases} \mathbf{u}(x, 1) = (1, 0), \forall x \in (-1, 1), \\ \mathbf{u}(x, -1) = (0, 0), \forall x \in (-1, 1), \\ \mathbf{u}(1, y) = (0, 0), \forall y \in (-1, 1), \\ \mathbf{u}(-1, y) = (0, 0), \forall y \in (-1, 1), \end{cases} \tag{13}$$

Para resolverlo utilizando el Método de Elementos Finitos, podemos usar, por ejemplo, el elemento $P_1 - P_0$ no conforme, que consiste en aproximar la presión por constantes y la velocidad por las funciones lineales de Crouzeix-Raviart S_i^{CR}, que solo son continuas en el punto medio de los lados de los triángulos. Al final llegamos al siguiente sistema de ecuaciones.

$$\begin{pmatrix} A & 0 & C_1 & 0 \\ 0 & A & C_2 & 0 \\ C_1^T & C_2^T & 0 & T \\ 0 & 0 & T^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{U}_x \\ \mathbf{U}_y \\ \mathbf{P} \\ \bar{p} \end{pmatrix} = \begin{pmatrix} \mathbf{l}_1 \\ \mathbf{l}_2 \\ \mathbf{l}_3 \\ 0 \end{pmatrix}, \tag{14}$$

donde $\mathbf{l}_i$ contiene información de las fuerzas actuando sobre el fluido (en nuestro caso ninguna) y las condiciones de contorno. Además

$$A_{ij} = \int_\Omega \nabla S_i^{CR} \cdot \nabla S_j^{CR}, \qquad T_i = \int_\Omega S_i^{CR}, \tag{15}$$

$$(C_1)_{ij} = \int_\Omega S_i^{CR} \frac{\partial S_j^{CR}}{\partial x}, \qquad (C_2)_{ij} = \int_\Omega S_i^{CR} \frac{\partial S_j^{CR}}{\partial y}, \tag{16}$$

y $\bar{p}$ es la presión media, de manera que la última condición implica que la presión promedio sea cero y por tanto estaremos considerando siempre las diferencias de presión.

Contents

1 Introduction to fluid mechanics

First of all, in order to understand the meaning of the equations that are going to be considered in this work, it is important to explain the physics used to model fluid mechanics.

1.1 Definition of fluid and continuum hypothesis

The main characteristic that defines a fluid, whether liquid or gas, is its ease of deformation. A fluid, unlike a solid, does not have a definite shape and its different elements may be rearranged freely without affecting the macroscopic properties. Thus, we can define a fluid as a substance that deforms continuously under the application of shear stress, regardless of its state of matter (liquid or gas). It is characterized by its ability to flow and has no fixed shape, allowing it to take the shape of its container.

On a small scale, mass is concentrated in the particles that make up the fluid separated by regions of vacuum. However, in fluid mechanics we are interested in studying large-scale properties and the molecular structure is not important. For this reason, we consider fluids to have a perfectly continuous structure. This is known as the continuum hypothesis.

Forces in a fluid and stress tensor

Acting on a fluid, two types of forces are distinguished:

- Volume forces: These are long-range forces that vary slowly with distance, in such a way that we can consider them constant at all points in the fluid. The most common are gravity, electromagnetic forces and fictitious forces due to accelerations. Since these forces are equal in all volume elements, we can assume that the total force is proportional to the volume. Then the total force will be

$$\int_V \mathbf{F}(\mathbf{x}, t)\rho dV \tag{17}$$

  where $\mathbf{F}$ is the acceleration ($\mathbf{g} = (0, 0, g)$ in the gravitational case), ρ the density of the fluid, dV a volume element, $\mathbf{x}$ its position and t the time.

- Surface forces: These short-range forces are of molecular origin and decrease rapidly as the distance between particles increase. They are negligible unless there is direct contact between the elements of the fluid. Therefore, they are determined by the area of the elements and not by their volume. If we consider a flat surface element (the faces of a small cube around a point, for example), the force is proportional to its area and is given by

$$\mathbf{\Sigma}(\mathbf{n}, \mathbf{x}, t)dA \tag{18}$$

  where $\mathbf{n}$ is the external normal unitary vector and $\mathbf{\Sigma}$ is the stress, which represents a force per unit area and is calculated using the strain tensor σ:

$$\Sigma_i(\vec{n}) = \sigma_{ij} n_j \tag{19}$$

where σ_{ij} represents the i component of the stress on a surface element with normal in the direction j. This tensor is symmetric ($\sigma_{ij} = \sigma_{ji}$), the three elements on the diagonal σ_{ii} are called normal stresses and those outside the diagonal are called shear stresses.

For a fluid at rest, $\sigma_{ij} = -p\delta_{ij}$, where p is the static pressure. For a general fluid we have two contributions to the stress:

1. Internal stress: Due to the movement that occurs in a fluid when the pressure changes. It has the same form $\sigma_1 = -pI$.
2. Viscous stress: The viscosity μ measures the resistance of a fluid to deform under stress and exists due to friction between neighboring fluid layers with different velocities. Thus, this stress is produced by the deformation of the fluid. A Newtonian fluid is defined as a fluid in which the viscous stress is proportional to the gradient of the velocity $\mathbf{u}$: $\sigma_2 = \mu\nabla\mathbf{u}$, where μ is the viscosity. However, it is better to define $\sigma_2 = \mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$ in order to get a symmetric matrix.

The total stress is $\sigma = -pI + \mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$.

Material derivative

Velocity is both position-dependent and time-dependent, and in fact, a fluid element may undergo an acceleration by moving to a new position where the velocity has a different value. For this reason, acceleration is not just $\frac{\partial \mathbf{u}}{\partial t}$, where $\mathbf{u}$ is the velocity of the fluid. Suppose that the velocity at time t is $\mathbf{u}$ and at time $t + dt$ is $\mathbf{u} + d\mathbf{u}$. Since $\mathbf{u}$ depends on $\mathbf{x}$ and t, we write explicitly:

$$\mathbf{u} = \mathbf{q}(\mathbf{x}, t), \qquad \mathbf{u} + d\mathbf{u} = \mathbf{q}(\mathbf{x} + d\mathbf{x}, t + dt). \tag{20}$$

Then

$$d\mathbf{u} = \mathbf{q}(\mathbf{x} + d\mathbf{x}, t + dt) - \mathbf{q}(\mathbf{x}, t + dt) + \mathbf{q}(\mathbf{x}, t + dt) - \mathbf{q}(\mathbf{x}, t), \tag{21}$$

and using Taylor series expansions

$$\begin{cases} \mathbf{q}(\mathbf{x} + d\mathbf{x}, t + dt) - \mathbf{q}(\mathbf{x}, t + dt) = (d\mathbf{x} \cdot \nabla)\mathbf{q}(\mathbf{x}, t + dt) + O(||d\mathbf{x}||^2), \\ \mathbf{q}(\mathbf{x}, t + dt) - \mathbf{q}(\mathbf{x}, t) = dt\frac{\partial}{\partial t}\mathbf{q}(\mathbf{x}, t) + O(dt^2). \end{cases} \tag{22}$$

Dividing by dt and taking the limit $dt \to 0$, we get

$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}. \tag{23}$$

Definition 1 *The material or convective derivative of a function in $\mathbb{R}^3 \times \mathbb{R}$ is defined as the operator*

$$\frac{D}{Dt} := \frac{\partial}{\partial t} + (\boldsymbol{u} \cdot \nabla), \tag{24}$$

and represents the change of a function in time following the fluid's current, where $\boldsymbol{u}$ is the flow velocity.

1.2 Conservation of mass

We now consider a surface $\partial\Omega$ enclosing a volume Ω, which is an open and connected set of $\mathbb{R}^3$. The total mass inside of the volume is

$$\int_\Omega \rho \, dV,$$

where $dV = dx \cdot dy \cdot dz$, and the amount of mass that flows outside Ω through $\partial\Omega$ is

$$\int_{\partial\Omega} \rho \mathbf{u} \cdot \mathbf{n} \, dS,$$

where $\mathbf{n}$ is the external normal unitary vector to $\partial\Omega$. What the conservation of mass tells us is that the change in time of the mass inside Ω is equal to the difference between the mass going in and going out through $\partial\Omega$, that is:

$$\frac{d}{dt}\int_\Omega \rho \, dV = -\int_{\partial\Omega} \rho \mathbf{u} \cdot \mathbf{n} \, dS. \tag{25}$$

Using the divergence theorem and the derivation under integral we reach

$$\int_\Omega \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right) dV = 0, \tag{26}$$

and since this is valid for any volume Ω, it must be

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \tag{27}$$

Using the material derivative, this equation becomes

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \mathbf{u} = 0. \tag{28}$$

Incompressible fluids

A fluid is said to be incompressible when the density of a fluid element does not change when the pressure changes. In this case, $\dfrac{D\rho}{Dt} = 0$ and due to conservation of mass, we get $\nabla \cdot \mathbf{u} = 0$.

Types of fluids, Reynolds number and vorticity

In addition to classifying fluids as viscous or non-viscous, and Newtonian or non-Newtonian, it is important to distinguish between laminar and turbulent flow. The flow is laminar when it is ordered and the stream lines (lines in which the velocity is tangent at each point) are parallel (they do not mix). Conversely, turbulent flow is chaotic and disordered, and the trajectories of the particles intersect forming aperiodic whirlpools. What differentiates these two types of flow is mainly the type of force that dominates in each case. In turbulent flow, inertial forces dominate over viscous forces, while in laminar flow, viscous forces dominate. To quantify this phenomenon numerically we define the Reynolds' number:

$$Re := \frac{\text{inertial forces}}{\text{viscous forces}} = \frac{Lu\rho}{\mu}, \tag{29}$$

where L is the size of the domain, $u = ||\mathbf{u}||$, ρ the density and μ the dynamic viscosity. It is also usually expressed as a function of the kinematic viscosity defined as $\nu := \mu/\rho$, $Re = LU/\nu$. When Re is low the flow is laminar, while when Re is high the flow is turbulent.

In this context, it is also important to define the vorticity of a fluid, which is given by the rotational of the velocity

$$\boldsymbol{\omega} := \nabla \times \mathbf{u}. \tag{30}$$

A fluid is said to be irrotational when $\nabla \times \mathbf{u} = 0$. In such case we can understand the velocity $\mathbf{u}$ as a conservative field, which implies the existence of a potential ϕ such that $\mathbf{u} = -\nabla\phi$. If the fluid is also incompressible ($\nabla \cdot \mathbf{u} = 0$) we can obtain the fluid dynamics simply by means of Laplace's equation

$$\nabla^2\phi = 0. \tag{31}$$

1.3 Conservation of momentum

The equation of motion is nothing more than Newton's second law, which states that the change of momentum over time is equal to the sum of the total forces acting on the fluid. The linear momentum in an open and connected volume $\Omega \subset \mathbb{R}^3$ is

$$\int_\Omega \rho\mathbf{u}\ dV, \tag{32}$$

and its change over time is given by

$$\frac{d}{dt}\int_\Omega \rho\mathbf{u}\ dV = \int_\Omega \frac{D\mathbf{u}}{Dt}\rho\ dV. \tag{33}$$

On the other hand, the sum of volume forces and surface forces will be

$$\int_\Omega \mathbf{F}\rho\ dV + \int_\Omega \nabla\cdot\sigma\ dV. \tag{34}$$

Newton's second law tells us that

$$\int_\Omega \frac{D\mathbf{u}}{Dt}\rho\ dV = \int_\Omega \mathbf{F}\rho\ dV + \int_\Omega \nabla\cdot\sigma\ dV, \tag{35}$$

and since it is fulfilled for any volume Ω we reach

$$\rho\frac{D\mathbf{u}}{Dt} = \rho\mathbf{F} + \nabla\cdot\sigma. \tag{36}$$

Now remembering that $\sigma = -pI + \mu(\nabla u + \nabla^T u)$ and substituting

$$\rho\frac{D\mathbf{u}}{Dt} = \rho\mathbf{F} - \nabla p + \mu(\nabla^2\mathbf{u} + \nabla(\nabla\cdot\mathbf{u})). \tag{37}$$

Therefore, for a Newtonian and incompressible fluid, the equations of conservation of mass and momentum are:

$$\nabla\cdot\mathbf{u} = 0, \tag{38}$$

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{F} - \frac{1}{\rho}\nabla p + \nu\Delta\mathbf{u}. \tag{39}$$

This system of partial differential equations is known as the Navier-Stokes equations. The term $\dfrac{\partial\mathbf{u}}{\partial t}$ represents the change in velocity, $(\mathbf{u}\cdot\nabla)\mathbf{u}$ is called the convective term, $-\frac{1}{\rho}\nabla p$ is the

acceleration produced by pressure changes, $\mathbf{F}$ is the acceleration produced by volume forces and $\nu\Delta\mathbf{u}$ is the diffusive term due to the fluids' viscosity. The Navier-Stokes equations are non-linear and therefore difficult to solve. However, in the case that we have a laminar flow (the viscous forces dominate over the inertial ones), the convective term is negligible. In that case and in stationary state the system of equations becomes linear and is called Stokes equations.

2 The Finite Element Method

2.1 Concepts of Functional Analysis

In this section we are going to see the mathematical foundations behind FEM. We need to introduce some concepts of Functional Analysis, starting with Hilbert spaces.

Definition 2 *A Hilbert space is a complete inner product vector space, which means a vector space with an inner product, such that every Cauchy sequence is convergent.*

Throughout this dissertation we will work especially with the space $L^2(\Omega) = \{f : \Omega \to \mathbb{R} : ||f||_{L^2(\Omega)} < \infty\}$ of square integrable functions, which is a Hilbert space under the norm:

$$||f||_{L^2(\Omega)} := \left(\int_\Omega |f(x)|^2 dx\right)^{1/2}, \tag{40}$$

inherited from the dot product

$$(f, g)_\Omega := \int_\Omega fg \; dx. \tag{41}$$

Definition 3 *Given two Hilbert spaces H_1 and H_2 and their induced norms $||\cdot||_{H_1}$ and $||\cdot||_{H_2}$, a linear map $L : H_1 \to H_2$ is said to be continuous if*

$$\exists c > 0 \text{ such that } ||Lu||_{H_2} \leq c||u||_{H_1}, \; \forall u \in H_1. \tag{42}$$

Existence and uniqueness

Theorem 1 (Riesz-Fréchet representation theorem) *Let H be a Hilbert space and H' its dual ($H' = \mathcal{L}(H, \mathbb{R}) = \{f : H \to \mathbb{R} : f \text{ linear}\}$), then*

$$\forall L \in H', \; \exists! u \in H \text{ such that } L(v) = (u, v)_H =: \mathcal{J}_u(v), \; \forall v \in H, \tag{43}$$

and the function

$$\mathcal{J} : \begin{array}{ccc} H & \longrightarrow & H' \\ u & \longrightarrow & \mathcal{J}_u = (u, \cdot)_H \end{array} \quad \text{with} \quad \mathcal{J}_u : \begin{array}{ccc} H & \longrightarrow & \mathbb{R} \\ v & \longrightarrow & \mathcal{J}_u(v) = (u, v)_H \end{array} \tag{44}$$

is an isometric isomorphism, which means $||L||_{H'} = ||u||_H$.

Proof: See the Appendix A.

Definition 4 *Let H be a Hilbert space. A bilinear form*

$$\begin{array}{rccc} a: & H \times H & \longrightarrow & \mathbb{R} \\ & (u,v) & \longrightarrow & a(u,v) \end{array} \tag{45}$$

is continuous if $\exists M > 0$ such that $|a(u,v)| \leq M||u||_H||v||_H$, $\forall u, v \in H$, where M is called continuity constant.

Definition 5 *Let H be a Hilbert space. A bilinear form*

$$\begin{array}{rccc} a: & H \times H & \longrightarrow & \mathbb{R} \\ & (u,v) & \longrightarrow & a(u,v) \end{array} \tag{46}$$

is coercive if $\exists \alpha > 0$ such that $|a(v,v)| \geq \alpha||v||_H^2$, $\forall v \in H$, where α is called coercivity constant.

Definition 6 *Let $a : H \times H \to \mathbb{R}$ be bilinear and continuous and $l \in H'$ acting on the Hilbert space H, we define an abstract variational problem as*

$$(V) \begin{cases} \text{Find } u \in H \text{ such that} \\ a(u,v) = l(v), \ \forall v \in H \end{cases} \tag{47}$$

If the bilinear form a is also symmetric, then it defines a scalar product. In the context of FEM this product is usually called energy product. Besides, if a is coercive, then a is also positive definite.

Lemma 1 (Lax-Milgram lemma) *If $a(\cdot,\cdot)$ is a bilinear continuous and coercive form, and l is a linear continuous form, then the abstract variational problem (V) that we have just defined has a solution $u \in H$ and is unique.*

Proof: See the Appendix A.

Distributions

Definition 7 *Let Ω be an open set in $\mathbb{R}^n$. Given $f : \Omega \to \mathbb{R}$, we define the support of f as the closure of the set of points where f takes non-zero values, that is, $suppf = \overline{\{x \in \Omega : f(x) \neq 0\}}$.*

Definition 8 *A function $f : \Omega \to \mathbb{R}$ is said to have a compact closure in Ω if $\exists K \subset \Omega$ compact such that $suppf \subseteq K$. We call $D(\Omega) = \{\varphi \in C^\infty(\Omega) : \exists K \subseteq \Omega$ compact such that $supp\varphi \subseteq K\}$ to the vector space of infinitely differentiable functions with compact support in Ω.*

Definition 9 *A linear form $T : D(\Omega) \to \mathbb{R}$ is said to be continuous if $\forall (\varphi_n)_{n\in\mathbb{N}} \subset D(\Omega)$ such that $\varphi_n \to \varphi$ in $D(\Omega)$, we get $T(\varphi_n) \to T(\varphi)$.*

Definition 10 *We define a distribution on Ω as a linear and continuous function*

$$\begin{array}{rccc} T: & D(\Omega) & \longrightarrow & \mathbb{R} \\ & \varphi & \longrightarrow & T(\varphi) \end{array} \tag{48}$$

Example 1 *Let $f \in L^2(\Omega)$. Then*

$$\begin{array}{rccc} T_f: & D(\Omega) & \longrightarrow & \mathbb{R} \\ & \varphi & \longrightarrow & T_f(\varphi) = \displaystyle\int_\Omega f\varphi = (T_f, \varphi)_\Omega \end{array}$$

is a distribution, since it is linear and continuous.

Weak derivatives

Example 2 *Let $\Omega = (a,b) \subset \mathbb{R}$ be bounded and $f \in C^1((a,b))$. Let T_f be the distribution in the previous example. Consider now its derivative $f' \in C((a,b))$, which has a distribution associated to it $T_{f'} : D(a,b) \to \mathbb{R}$ given by*

$$T_{f'}(\varphi) = \int_a^b f'(x)\varphi(x)dx \underbrace{=}_{part.int.} -\int_a^b f(x)\varphi'(x)dx + \underbrace{f(b)\varphi(b)}_{=0} - \underbrace{f(a)\varphi(a)}_{=0} = -(T_f, \varphi')_\Omega.$$

Definition 11 *Let $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + ... + \alpha_n$ and $T \in D'(\Omega)$. We define the weak derivative or derivative in the sense of distributions as $D^\alpha T \in D'(\Omega)$ given by $(D^\alpha T, \varphi) = (-1)^{|\alpha|}(T, D^\alpha \varphi)$.*

Definition 12 *We call Sobolev space of order m, $H^m(\Omega)$, to the space of functions in $L^2(\Omega)$ such that their weak derivatives up to order m are in $L^2(\Omega)$*

$$H^m(\Omega) = \{u \in L^2(\Omega) : u^{(k)} \in L^2(\Omega), 1 \le k \le m\}. \tag{49}$$

In $H^1(\Omega)$ we define the dot product $(u,v)_{H^1} := (u,v)_{L^2} + (u',v')_{L^2}$. $H^1(\Omega)$ is a Hilbert space with this dot product. We can generalise this to $H^m(\Omega)$ by defining $(u,v)_{H^m} := \sum_{|\alpha| \le m} (u^{(\alpha)}, v^{(\alpha)})_{L_2}$.

Definition 13 *We denote $H_0^1(\Omega) = \{u \in H^1(\Omega) : u(\omega) = 0, \forall \omega \in \partial\Omega\}$.*

Proposition 1 (Poincaré inequality) *Let $\Omega \subset \mathbb{R}$ be bounded. Then there exists a constant $c \in \mathbb{R}$ such that $\forall u \in H_0^1(\Omega)$, $||u||_{L^2} \le c||u'||_{L^2}$ (this can also be generalised to $\mathbb{R}^n$).*

Proof: First, in the case $\Omega = (0, \pi)$, we consider the sine Fourier series of the even extension of $u \in H_0^1(\Omega)$ (which means $u(0) = u(\pi) = 0$), that is,

$$u(x) = \sum_{k=1}^\infty b_k \sin(kx), \qquad u'(x) = \sum_{k=1}^\infty k b_k \cos(kx). \tag{50}$$

Using Parseval inequality $\displaystyle\int_{-\pi}^\pi |u(x)|^2 dx = \pi \sum_{k=1}^\infty b_k^2$, twice we get:

$$||u'||_{L^2}^2 = \int_0^\pi |u'(x)|^2 dx = \frac{1}{2}\int_{-\pi}^\pi |u'(x)|^2 dx = \frac{\pi}{2}\sum_{k=1}^\infty k^2 b_k^2 \tag{51}$$

$$\ge \frac{\pi}{2}\sum_{k=1}^\infty b_k^2 = \frac{1}{2}\int_{-\pi}^\pi |u(x)|^2 dx = \int_0^\pi |u(x)|^2 dx = ||u||_{L^2}^2. \tag{52}$$

Using the transformation $y = a + \frac{b-a}{\pi}x$ and the chain rule one can prove that $\forall u \in H_0^1((a,b))$:

$$||u||_{L^2((a,b))} \le \frac{b-a}{\pi}||u'||_{L^2((a,b))}. \tag{53}$$

■

2.2 Approximation with Finite Elements

Example in 1D

First of all, let us see what this method consists of in 1D and then we will how to generalize it to 2D. Let us consider the following problem

$$\begin{cases} \text{Find } u \text{ such that} \\ -u''(x) = f(x),\ x \in I = (0, L), \\ u(0) = u(L) = 0. \end{cases} \tag{54}$$

The first step is to find what is known as the variational formulation of the problem. To do this we multiply the differential equation by a test function v and integrate by parts:

$$\int_0^L fv\ dx = -\int_0^L u''v\ dx = \int_0^L u'v'\ dx - u'(L)v(L) + u'(0)v(0) = \int_0^L u'v'\ dx, \tag{55}$$

where the last equality follows from assuming $v(0) = v(L) = 0$. Specifically, what we are doing is imposing that the solution u and the test function v are in the set $V_0 = H_0^1(I)$. Thus, we have the variational problem

$$\begin{cases} \text{Find } u \in V_0 \text{ such that} \\ \int_0^L u'v'dx = \int_0^L fv\ dx,\ \forall v \in V_0. \end{cases} \tag{56}$$

The second step is discretizing the variational problem using the Galerkin method, which approximates u by a continuous piecewise linear function. That is, we consider a partition $\mathcal{I} : 0 = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = L$ of I into n subintervals $I_i = [x_{i-1}, x_i]$ of lengths $h_i = x_i - x_{i-1}$, $i = 1, ..., n$; and the set $V_h = \{v \in C(I) : v\ |_{I_i} \in P_1(I_i)\}$, where $C(I)$ is the set of continuous functions on I and $P_1(I_i) = \{v(x) = c_0 + c_1 x : x \in I_i, c_0, c_1 \in \mathbb{R}\}$ is the set of linear functions on I_i and $h = \max\{h_i\}$. Now, the variational problem becomes the so-called Galerkin's problem:

$$\begin{cases} \text{Find } u_h \in V_{0,h} \text{ such that} \\ \int_0^L u_h'v_h'dx = \int_0^L fv_h\ dx,\ \forall v_h \in V_{0,h}, \end{cases} \tag{57}$$

where $V_{0,h} = \{v \in V_h : v(0) = v(L) = 0\}$.

Finally, we transform this problem into a system of algebraic equations. To do this, we realise that solving the Galerkin's problem is equivalent to solving

$$\begin{cases} \text{Find } u_h \in V_{0,h} \text{ such that} \\ \int_0^L u_h'\varphi_i'dx = \int_0^L f\varphi_i dx,\ i = 1, 2, ..., n-1, \end{cases} \tag{58}$$

where $\{\varphi_i\}_{i=1}^{n-1}$ is a basis of the space $V_{0,h}$. This can be done because V_h is a finite-dimensional vector space and we can introduce a basis $\{\varphi_i\}_{i=1}^{n-1}$, the so-called hat functions, associated to the nodes of the partition such that

$$\varphi_i(x_j) = \begin{cases} 1, \text{ if } j = i \\ 0, \text{ if } j \neq i \end{cases}, \qquad \text{which implies} \qquad \varphi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h_i}, \text{ if } x \in I_i \\ \dfrac{x_{i+1} - x}{h_{i+1}}, \text{ if } x \in I_{i+1} \\ 0, \text{otherwise} \end{cases}. \tag{59}$$

Since $u_h \in V_{0,h}$, u_h can be written as a linear combination of the base elements

$$u_h = \sum_{j=1}^{n-1} \xi_j \varphi_j, \text{ with } \xi_j \in \mathbb{R} \text{ for all } j = 1, ..., n-1. \tag{60}$$

Then

$$\int_0^L f\varphi_i \, dx = \int_I \left(\sum_{j=1}^{n-1} \xi_j \varphi_j' \right) \varphi_i' dx = \sum_{j=1}^{n-1} \xi_j \int_I \varphi_j' \varphi_i' dx \tag{61}$$

Introducing the matrix notation $A_{ij} = \int_I \varphi_j' \varphi_i' dx$, $i, j = 1, ..., n-1$ known as the stiffness matrix, and $b_i = \int_0^L f\varphi_i dx$, $i = 1, ..., n-1$ the load vector, we are left to solve the system $A\boldsymbol{\xi} = \mathbf{b}$. Quadrature methods such as the trapezoidal rule or Simpson's formula are used to calculate the integrals of the elements of A and $\mathbf{b}$. To solve the system, as in general we will have very large matrices, the matrix is not inverted, but numerical methods such as Cholesky or the Conjugent Gradient Method are used.

Generalization to 2D

Partial differential equations have two or more variables, so it is necessary to extend the method above to spaces with larger dimensions. Let us look at the 2-dimensional case. Now, instead of partitioning an interval, we have to discretise a two-dimensional domain.

Definition 14 *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded portion of the plane with a polygonal boundary, we call mesh or triangulation of Ω to a set of elements $\mathcal{K} = \{K_i\}_{i=1}^n$ so that:*

1. *K_i is a triangle, $i = 1, ..., n$.*
2. *$\cup_{i=1}^n K_i = \bar{\Omega}$.*
3. *$K_i \cap K_j$ with $i \neq j$ is empty, a vertex or a complete side of a triangle.*

Definition 15 *Let $\mathcal{K}$ be a mesh or triangulation of Ω, we define the space V_h of continuous piecewise linear functions as*

$$V_h = \{v \in C(\Omega) : v \mid_K \in P_1(K), \forall K \in \mathcal{K}\}, \tag{62}$$

where now $P_1(K) = \{v(x, y) = c_0 + c_1 x + c_2 y : (x, y) \in K, c_0, c_1, c_2 \in \mathbb{R}\}$ is the set of linear functions in K and $h = \max\{h_K\}$ with h_K the longest side of each K.

In the same way, we can define a basis of hat functions $\{\varphi_i\}_{i=0}^{n_p}$ of V_h, see Figure 1, and the linear interpolant $\pi f(x) = \sum_{i=1}^{n_p} f(N_i)\varphi_i(x)$ from the nodal values $f(N_i)$ in each of the n_p nodes.

The process of getting a variational or weak problem out of the classical problem can be generalized to problems in more dimensions. We simply have to use Green's formula instead of applying integration by parts. We recall that Green's formula tells us that for an open and connected set $\Omega \subset \mathbb{R}^2$, with boundary $\partial\Omega$ and exterior unit normal vector $\mathbf{n}$, the following holds:

$$\int_\Omega -(\Delta u)v \, dx = \int_\Omega \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} \mathbf{n} \cdot (\nabla u) v \, ds. \tag{63}$$

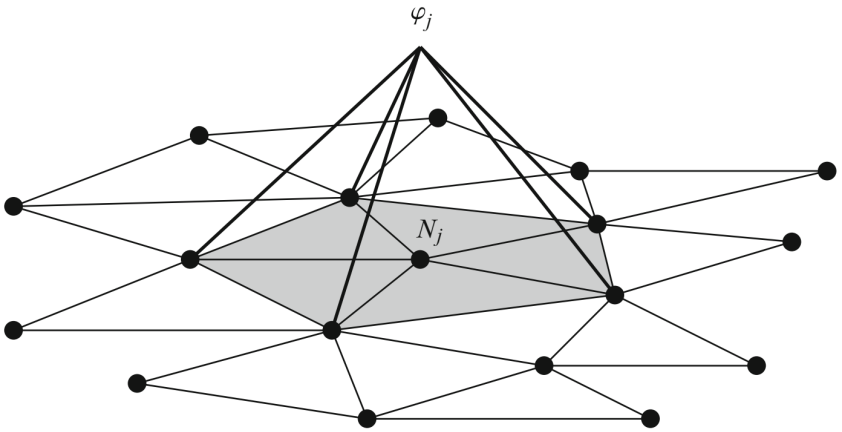


Figure 1: *Example of a triangular mesh with the hat function in one of its nodes (Source: [2]).*

Example 3 *We can see an example of application to Poisson's equation.*

$$\begin{cases} -\Delta u = f, \ in \ \Omega, \\ u = 0, \ in \ \partial\Omega. \end{cases} \tag{64}$$

Multiplying by a test function $v \in H_0^1(\Omega) = \{u \in L^2(\Omega) : u' \in L^2(\Omega), u \mid_{\partial\Omega} = 0\}$ *and using Green's formula*

$$\int_\Omega fv \ dx = -\int_\Omega (\Delta u)v \ dx = \int_\Omega \nabla u \cdot \nabla v \ dx - \int_{\partial\Omega} \boldsymbol{n} \cdot \nabla u \ v \ dx = \int_\Omega \nabla u \cdot \nabla v \ dx. \tag{65}$$

So the variational problem is

$$\begin{cases} Find \ u \in H_0^1(\Omega) \ such \ that \\ \int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx, \ \forall v \in H_0^1(\Omega). \end{cases} \tag{66}$$

Taking $a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx$ *and* $l(v) = \int_\Omega f v dx$*, we can apply the Lax-Milgram lemma and obtain that there exists solution and is unique. The hypotheses that we have to verify are:*

1. *a is coercive:*
$$a(u,u) = \int_\Omega \nabla u \cdot \nabla u \ dx = ||\nabla u||_{L_2}^2 \geq \alpha ||u||_{H^1}^2,$$
where in the last inequality the Poincaré inequality has been applied in $\mathbb{R}^n$*.*

2. *a is continuous: using the Cauchy-Schwarz inequality,*
$$a(u,v) = (\nabla u, \nabla v)_{L^2} \leq ||\nabla u||_{L^2} ||\nabla v||_{L^2} \leq ||u||_{H^1} ||v||_{H^1}.$$

3. *l is continuous:*
$$l(v) = (f,v)_{L^2} \leq ||f||_{L^2} ||v||_{L^2} \leq ||f||_{L^2} ||v||_{H^1}.$$

In section 3.1 we will discuss how to solve the dynamics of an irrotational fluid using Laplace's equation, which is the case when $f = 0$. In addition, the energy product a for the Poisson equation with homogeneous Dirichlet boundary conditions is positive definite:

$$a(u,u) = \int_\Omega (\nabla u(x))^2 \ dx \geq 0, \text{ for all } u \in H_0^1(\Omega) \tag{67}$$

$$a(u,u) = 0 \Leftrightarrow u = 0 \tag{68}$$

Sumary of Finite Elements

Starting with an abstract variational problem

$$(V)\begin{cases} \text{Find } u \in V \text{ such that} \\ a(u,v) = l(v),\ \forall v \in V, \end{cases} \tag{69}$$

the Galerkin approximation takes in general the form

$$\begin{cases} \text{Find } u_h \in V_h \text{ such that} \\ a(u_h, v_h) = l(v_h),\ \forall v_h \in V_h. \end{cases} \tag{70}$$

Let $\{\varphi_i\}_{i=1}^n$ be a basis of V_h. Then the Finite Element Method is

$$\begin{cases} \text{Find } u_h \in V_h \text{ such that} \\ a(u_h, \varphi_i) = l(\varphi_i),\ \forall i = 1, 2, ..., n. \end{cases} \tag{71}$$

Since $u_h \in V_h$, we can write it as $u_h = \sum_{j=1}^n \xi_j \varphi_j$. Substituting:

$$b_i := l(\varphi_i) = \sum_{j=1}^n \xi_j a(\varphi_j, \varphi_i) := \sum_{j=1}^n A_{ij}\xi_j \Leftrightarrow A\boldsymbol{\xi} = \mathbf{b}. \tag{72}$$

Conjugate Gradient Method

Proposition 2 *For the Poisson equation:*

1. *The stiffness matrix $A \in \mathbb{R}^{n\times n}$ is symmetric.*
2. *The stiffness matrix A is positive definite, that is, $\boldsymbol{x}^T A\boldsymbol{x} > 0$, for all $0 \neq \boldsymbol{x} \in \mathbb{R}^n$.*

Proof: The first statement is obvious since $A_{ij} = a(\varphi_j, \varphi_i)$ and a is symmetric. For the second:

$$\mathbf{x}^T A\mathbf{x} = \sum_{i,j=1}^n x_i a(\varphi_i, \varphi_j) x_j = a\left(\sum_{i=1}^n x_i\varphi_i, \sum_{j=1}^n x_j\varphi_j\right) = a(x,x) > 0,\ \forall x \neq 0 \tag{73}$$

■

Taking into account these properties, we can create fast linear system solvers, such as the Conjugate Gradient Method (CGM), which, together with the properties of the matrix A, reduces substantially the number of operations. To run large simulations in computers one can also use some High Performance Computing (HPC) techniques, like parallel programming.

Definition 16 *Two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are conjugate (with respect to A) if $\boldsymbol{u}^T A\boldsymbol{v} = 0$. Since A is symmetric and positive definite this defines an inner product $(\boldsymbol{u}, \boldsymbol{v})_A = (A\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}, A\boldsymbol{v})$. Thus, two vectors are conjugate if and only if they are orthogonal with respect to this inner product.*

Let $B = \{\mathbf{p}_1, ..., \mathbf{p}_n\}$ be a basis of $\mathbb{R}^n$ with n mutually conjugate vectors. We can express the solution $\mathbf{x}^*$ of $A\mathbf{x} = \mathbf{b}$ in this basis as $\mathbf{x}^* = \sum_{i=1}^n \alpha_i \mathbf{p}_i$. Then $A\mathbf{x}^* = \sum_{i=1}^n \alpha_i A\mathbf{p}_i$. Multiplying $A\mathbf{x}^* = \mathbf{b}$ on the left by $\mathbf{p}_k^T$

$$\mathbf{p}_k^T \mathbf{b} = \mathbf{p}_k^T A\mathbf{x}^* = \sum_{i=1}^{n} \alpha_i \mathbf{p}_k^T A\mathbf{p}_i = \alpha_k (\mathbf{p}_k, \mathbf{p}_k)_A. \tag{74}$$

Thus, the coefficients of $\mathbf{x}^*$ are $\alpha_k = \frac{(\mathbf{p}_k, \mathbf{b})}{(\mathbf{p}_k, \mathbf{p}_k)_A}$. Now, to finish the method we need to find n different conjugate vectors. But, if what we need is an approximation of $\mathbf{x}^*$, we do not need to calculate all the $\mathbf{p}_k$. Thus, we build an iterative process starting at $\mathbf{x}_0$, which can be assumed to be $\mathbf{x}_0 = \mathbf{0}$. To build the next iteration we need a way to decide whether we are closer to $\mathbf{x}^*$. The solution $\mathbf{x}^*$ is also the unique minimum of the quadratic function $f(\mathbf{x}) := \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{x}^T\mathbf{b}$. The existence of unique minimum is guaranteed by the fact that the Hessian $H(f(\mathbf{x})) = A$ is symmetric positive definite and $\mathbf{x}^*$ solves $\nabla f(\mathbf{x}) = A\mathbf{x} - \mathbf{b} = 0$, our initial problem. We take $\mathbf{p}_0$ as the negative of the gradient of f at $\mathbf{x}_0$, that is $\mathbf{p}_0 = \mathbf{b} - A\mathbf{x}_0$. Let $\mathbf{r}_k = \mathbf{b} - A\mathbf{x}_k$ be the residual at the k-th step. As above, $\mathbf{r}_k$ is the negative gradient of f at $\mathbf{x}_k$, so the gradient descent method would indicate to move in the direction $\mathbf{r}_k$. However, the directions $\mathbf{p}_k$ must be conjugate to each other. Hence we can use the Gram-Schmidt method to create the next $\mathbf{p}_k$

$$\mathbf{p}_k = \mathbf{r}_k - \sum_{i<k} \frac{\mathbf{p}_i^T A\mathbf{r}_k}{\mathbf{p}_i^T A\mathbf{p}_i} \mathbf{p}_i. \tag{75}$$

Following the direction $\mathbf{p}_k$, the next iterative step for the solution is

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{p}_k \text{ , where } \alpha_k = \frac{\mathbf{p}_k^T \mathbf{b}}{\mathbf{p}_k^T A\mathbf{p}_k}. \tag{76}$$

For more details on the CGM and other fast solvers we refer to [3].

A priori errors

To get information about the error $e := u - u_h$ between the exact and the Galerkin solution, we subtract the equations (69) and (70) and get $a(e, v_h) = 0$, $\forall v_h \in V_h$, which means that the error e is orthogonal to V_h with respect to the inner product a. This is called Galerkin orthogonality.

The a priori error estimates involve the exact unknown solution u, as formulated in Céa's lemma.

Theorem 2 (Céa's lemma) *The error $e = u - u_h$ between the exact and the Galerkin solution satisfies*

$$||e||_V \leq \frac{M}{\alpha} ||u - v_h||_V, \ \forall v_h \in V_h. \tag{77}$$

Proof: Since a is coercive, we have $\forall v \in V_h$

$\alpha||e||_V^2 \leq a(e,e) = a(e, u - v_h + v_h - u_h) = a(e, u - v_h) + a(e, v_h - u_h) = a(e, u - v_h) \leq M||e||_V||u - v_h||_V$

where $a(e, v_h - u_h) = 0$ by Galerkin's orthogonality, and in the last inequality the continuity of a has been used. ■

Theorem 3 *The error $e = u - u_h$ between the exact and the Galerkin solution satisfies*

$$||e||_V = ||u - u_h||_V \leq Ch||u||_{H^2}. \tag{78}$$

A posteriori errors

These error estimates do not use the unknown solution u, but the calculated one, so they are more useful. Knowing the following bounds of a posteriori errors allows to perform refinements in the mesh, which is a process known as Adaptive Finite Element Methods. It consists in creating a finer mesh (smaller h) in the areas where $||e||_V$ is bigger.

Theorem 4 *The error $e = u - u_h$ between the exact and the Galerkin solution satisfies*

$$||e||_V \leq C \left(\sum_{K \in \mathcal{K}} h_K^2 ||l + \Delta u_h||_K^2 + \frac{1}{4} h_K ||[\boldsymbol{n} \cdot \nabla u_h]||_{\partial K}^2 \right) \tag{79}$$

where $[\boldsymbol{n} \cdot \nabla u_h]$ denotes the jump in the directional derivative of u_h on the sides of element K.

Proof: The proofs of the two previous theorems can be found in [2].

2.3 Types of Finite Elements

Up to now, we have only used approximations with piecewise linear functions, but in general we can use other kind of polynomials. In this section, we see how to do this.

Definition 17 *Formally, a finite element is a triplet consisting of*

1. *A polygon $K \subset \mathbb{R}^d$: lines, triangles, quadrilaterals, tetrahedrons, cubes...*
2. *A polynomial functional space P over K with base $\{S_j\}_{j=1}^N$, where each S_j is called shape function.*
3. *A set of $N = dim(P)$ linear functions $L_i(\cdot)$ defining the degrees of freedom. These define the shape functions by imposing the conditions $L_i(S_j) = \delta_{ij}$, $\forall i, j = 1, 2, ..., N$.*

Definition 18 *The Lagrange family is the family of finite elements that has $L_i(v) = v(N_i)$, $\forall i = 1, 2, ..., N$, which consists of the evaluation of the function v in the N nodes.*

Shape functions in the reference triangle

For the two-dimensional case, if K is a triangle and $P = P_1(K)$ the space of linear polynomials on the triangle K, the nodes are the 3 vertices of the triangle and we have 3 shape functions S_1, S_2, S_3. We consider the so-called reference triangle $\overline{K} = \{(r, s) \in \mathbb{R}^2 : 0 < r, s < 1; r + s < 1\}$. Then we take $L_1(v) = v(0, 0)$, $L_2(v) = v(1, 0)$ and $L_3(v) = v(0, 1)$. The simplest base of P is $\{1, r, s\}$ and the first form function, for example, will take the form $S_1 = c_1 + c_2 r + c_3 s$. To determine the coefficients we impose $L_i(S_1) = \delta_{i1}$, which results in the system

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} L_1(1) & L_1(r) & L_1(s) \\ L_2(1) & L_2(r) & L_2(s) \\ L_3(1) & L_3(r) & L_3(s) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \tag{80}$$

whose solution gives rise to $S_1 = 1 - r - s$. Doing the same, $S_2 = r$ and $S_3 = s$. We can use this process to define other more complex finite elements, such as quadratic elements, or elements with other geometries, such as quadrilaterals or tetrahedrons.

Isoparametric mapping

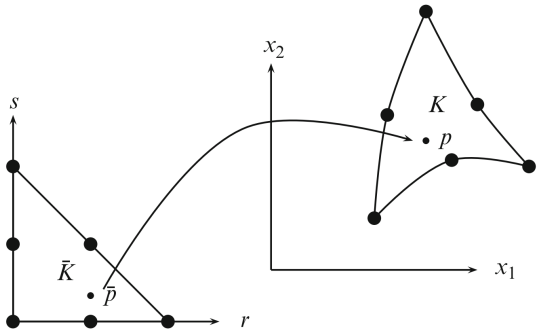


Figure 2: *Scheme of the isoparametric mapping (Source: [2]).*

In order to calculate the elements of the stiffness matrix in any element or in curved domains we do an affine transformation that takes us to the reference element $\overline{K}$. Suppose we have a triangle K with nodes $N_i = (x_1^i, x_2^i)$, $i = 1, 2, ..., n$. We build a map that takes us from a given point $(r, s) \in \overline{K}$ to a point $(x_1, x_2) \in K$:

$$x_1(r,s) = \sum_{i=1}^{n} x_1^i S_i(r,s), \tag{81}$$

$$x_2(r,s) = \sum_{i=1}^{n} x_2^i S_i(r,s). \tag{82}$$

Any function on a finite element K can be written as $v(r,s) = \sum_{i=1}^{n} v_i S_i(r,s)$, where $v_i = v(N_i)$ are the nodal values of v. In the stiffness matrix A of the element K appear the partial derivatives of v with respect to x_1 and x_2. In order to express them as a function of r and s we use the chain rule:

$$\frac{\partial v}{\partial x_1} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x_1} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial x_1}, \tag{83}$$

$$\frac{\partial v}{\partial x_2} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x_2} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial x_2}, \tag{84}$$

or with matrix notation

$$\begin{pmatrix} \frac{\partial v}{\partial x_1} \\ \frac{\partial v}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial r}{\partial x_1} & \frac{\partial s}{\partial x_1} \\ \frac{\partial r}{\partial x_2} & \frac{\partial s}{\partial x_2} \end{pmatrix} \begin{pmatrix} \frac{\partial v}{\partial r} \\ \frac{\partial v}{\partial s} \end{pmatrix} = J_K \begin{pmatrix} \frac{\partial v}{\partial r} \\ \frac{\partial v}{\partial s} \end{pmatrix}, \tag{85}$$

where the elements of the change matrix can be calculated by doing

$$(J_K^{-1})_{11} = \frac{\partial x_1}{\partial r} = \sum_{i=1}^{n} \frac{\partial S_i}{\partial r} x_1^i, \qquad (J_K^{-1})_{21} = \frac{\partial x_2}{\partial r} = \sum_{i=1}^{n} \frac{\partial S_i}{\partial r} x_2^i, \tag{86}$$

$$(J_K^{-1})_{12} = \frac{\partial x_1}{\partial s} = \sum_{i=1}^{n} \frac{\partial S_i}{\partial s} x_1^i, \qquad (J_K^{-1})_{22} = \frac{\partial x_2}{\partial s} = \sum_{i=1}^{n} \frac{\partial S_i}{\partial s} x_2^i. \tag{87}$$

In this way, we can calculate the partial derivatives of any function v at any point (x_1, x_2) of a generic element by solving a system of equations given by the Jacobian matrix of the change of coordinates. In particular, we can calculate the stiffness matrix and the load vector on any element from the calculation on the reference element using the Jacobi transformation formula. Going back to Poisson's equation, remember that we had

$$a_{ij} = a(\varphi_j, \varphi_i) = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j = \sum_{K\in\mathcal{K}} \int_K \nabla\varphi_i \cdot \nabla\varphi_j \qquad \text{and} \qquad b_i = \sum_{K\in\mathcal{K}} \int_K f\varphi_i$$

Introducing the change of coordinates to the reference element $(x_1, x_2) \to (r, s)$ in the integrals, we get

$$a_{ij} = \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i = \sum_{K\in\mathcal{K}} \int_K \nabla\varphi_j \cdot \nabla\varphi_i \tag{88}$$

$$= \sum_{K\in\mathcal{K}} |\det J_K|^{-1} \int_{\bar{K}} \left(\frac{\partial\varphi_j(r,s)}{\partial x}, \frac{\partial\varphi_j(r,s)}{\partial y}\right) \cdot \left(\frac{\partial\varphi_i(r,s)}{\partial x}, \frac{\partial\varphi_i(r,s)}{\partial y}\right) \tag{89}$$

$$= \sum_{K\in\mathcal{K}} |\det J_K|^{-1} \int_{\bar{K}} \left(\frac{\partial S_j(r,s)}{\partial r}, \frac{\partial S_j(r,s)}{\partial s}\right) J_K^T J_K \left(\frac{\partial S_i(r,s)}{\partial r}, \frac{\partial S_i(r,s)}{\partial s}\right) \tag{90}$$

$$:= \sum_{K\in\mathcal{K}} |\det J_K|^{-1} \int_{\bar{K}} \left((C_K)_{11}\frac{\partial S_i}{\partial r}\frac{\partial S_j}{\partial r} + (C_K)_{12}\frac{\partial S_i}{\partial r}\frac{\partial S_j}{\partial s} + (C_K)_{21}\frac{\partial S_i}{\partial s}\frac{\partial S_j}{\partial r} + (C_K)_{22}\frac{\partial S_i}{\partial s}\frac{\partial S_j}{\partial s}\right) \tag{91}$$

$$:= \sum_{K\in\mathcal{K}} |\det J_K|^{-1} \left((C_K)_{11} m_{ij}^{rr} + (C_K)_{12}(m_{ij}^{rs} + m_{ji}^{rs}) + (C_K)_{22} m_{ij}^{ss}\right) \tag{92}$$

$$b_i = \sum_{K\in\mathcal{K}} |\text{det}J_K|^{-1} \int_{\bar{K}} \bar{f} S_i \tag{93}$$

where $\bar{f}(r,s) = f(x_1(r,s), x_2(r,s))$, $M^{rr} = (m_{ij}^{rr}) = \int_{\bar{K}} \frac{\partial S_j}{\partial r}\frac{\partial S_i}{\partial r}$, $M^{rs} = (m_{ij}^{rs}) = \int_{\bar{K}} \frac{\partial S_j}{\partial r}\frac{\partial S_i}{\partial s}$, $M^{ss} = (m_{ij}^{ss}) = \int_{\bar{K}} \frac{\partial S_j}{\partial s}\frac{\partial S_i}{\partial s}$ and $C_K = J_K^T J_K$.

3 Application to CFD problems

3.1 Irrotacional fluids and Laplace equation

As explained in the first section, an irrotational fluid can be described by defining a potential ϕ, which is a scalar function such that $\mathbf{u} = -\nabla\phi$. This potential satisfies the Laplace's equation

$$-\Delta\phi = 0, \text{ in } \Omega. \tag{94}$$

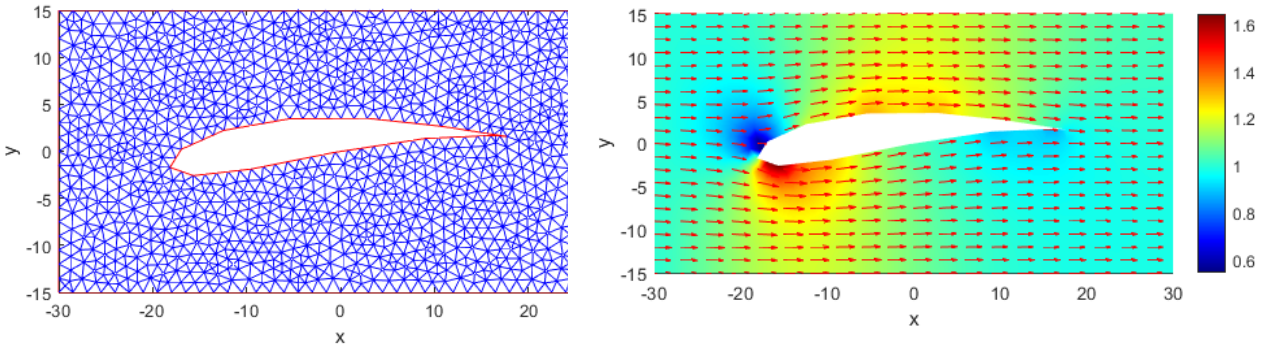
The simplest example of airfoil design can be done assuming that we have such a flow around a wing. Although air does not really behave in this way, but tends to have a turbulent flow due to its low viscosity (the Navier-Stokes or Euler equations would have to be solved), this is a first example of application of the Finite Element Method. Furthermore, we assume that the wing is much longer than it is wide, in order to reduce the problem to two dimensions. Vortices will be produced at the wingtips that cannot be explained in this way. The domain $\Omega = (-30, 30) \times (-15, 15)$ is a rectangle with the wing inside and $\Gamma = \partial\Omega$, see Figure 3. Let's assume an airflow from left to right. Finally, it remains to impose the boundary conditions:

$$\mathbf{n} \cdot \nabla\phi = 1, \text{ in } \Gamma_{in}, \qquad \phi = 0, \text{ in } \Gamma_{out}, \tag{95}$$

$$\mathbf{n} \cdot \nabla\phi = 0, \text{ otherwise}, \tag{96}$$

where $\mathbf{n}$ is the exterior normal unit vector, so the last condition tells us that the velocity is horizontal on the top and bottom walls. The second condition establishes the source of potential, and the first represents the incoming flow.

Programming the code that creates the stiffness matrix for the Laplacian using P_1 elements (continuous piecewise linear) and solving the system of equations, we arrive at Figure 3. On the left we see the discretization of the domain. On the right we see the direction of the velocity and, in colours, the values of the pressure. A pressure difference creates a force equal to the pressure gradient, so the wing will produce an upward lift force and a drag force.



(a) *Triangular mesh on the domain.* (b) *Solution of Laplace equation.*

Figure 3: *Solution of Laplace's equation to model an irrotational flow around an airfoil. The arrows indicate the direction of the speed and the colour the values of the pressure.*

3.2 Incompresible fluids and Stokes' equations

When we have laminar flow, that is, the fluid is viscous, it has low velocity or the flow lines are parallel and we assume a steady state, this can be modelled with the Stokes equations:

$$\nabla \cdot \mathbf{u} = 0, \text{ in } \Omega, \tag{97}$$

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f}, \text{ in } \Omega, \tag{98}$$

$$\mathbf{u} = \mathbf{g}, \text{ in } \partial\Omega, \tag{99}$$

where we have assumed that the viscosity is $\nu = 1$ and no slip condition at the boundary $\mathbf{u} = \mathbf{g}$. The pressure only appears within the gradient, so when solving the equation it will be determined up to a certain constant. We could set this constant as a hydrostatic pressure level or,as we do in this case, we can force the mean pressure to be 0, so that the solution will actually give pressure changes with respect to the mean:

$$\frac{1}{|\Omega|}(p, 1)_\Omega = \frac{1}{|\Omega|}\int_\Omega p \, dx = 0. \tag{100}$$

3.2.1 Weak formulation

In this case, we want to find the velocity $\mathbf{u} = (u_1, u_2)$ and the pressure p, so it will be necessary to introduce two test function spaces and work with the so-called mixed finite elements.

$$V_g = \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \mathbf{u} \mid_{\partial\Omega} = \mathbf{g}\}, \tag{101}$$

$$Q = \{q \in L^2(\Omega) : \int_\Omega q \, dx = 0\}. \tag{102}$$

Now, we multiply the second equation by a test function $\mathbf{v} \in V_0$ and integrate using Green's formula.

$$\int_\Omega \mathbf{v} \cdot \mathbf{f} \, d\Omega = \int_\Omega -(\Delta \mathbf{u})\mathbf{v} \, d\Omega + \int_\Omega \nabla p \, \mathbf{v} \, d\Omega \tag{103}$$

$$= -\int_{\partial\Omega} (\mathbf{n} \cdot \nabla \mathbf{u})\mathbf{v} \, dS + \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} \, d\Omega + \int_{\partial\Omega} (\mathbf{n} \cdot \mathbf{v}) p \, dS - \int_\Omega p \nabla \cdot \mathbf{v} \, d\Omega \tag{104}$$

$$= \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} \, d\Omega - \int_\Omega p \nabla \cdot \mathbf{v} \, d\Omega, \tag{105}$$

where $\nabla \mathbf{u} : \nabla \mathbf{v} = \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j} \frac{\partial v_j}{\partial x_i}$. For the first equation we multiply by $q \in Q$ and get $\int_\Omega q \nabla \cdot \mathbf{u} \, d\Omega = 0$. Finally, the variational problem is as follows

$$(V) \begin{cases} \text{Find } \mathbf{u} \in V_g \text{ and } p \in Q \text{ such that} \\ a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = l(\mathbf{v}), \ \forall \mathbf{v} \in V_0, \\ b(\mathbf{u}, q) = 0, \ \forall q \in Q, \end{cases} \tag{106}$$

where

$$a(\mathbf{u}, \mathbf{v}) = (\nabla \mathbf{u} : \nabla \mathbf{v}) = \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} \, d\Omega,$$

$$b(\mathbf{v}, p) = -(\nabla \cdot \mathbf{v}, p) = -\int_\Omega \nabla \cdot \mathbf{v} \ p \ dx, \qquad l(\mathbf{v}) = (\mathbf{f}, \mathbf{v}) = \int_\Omega \mathbf{f} \ \mathbf{v} \ dx.$$

To avoid working with two test spaces V_0 and V_g, we can expand $\mathbf{g}$ to all of Ω and write $\mathbf{u} = \mathbf{g} + \mathbf{u}_0$, where $\mathbf{u}_0 \in V_0$ is the new unknown that satisfies

$$\begin{cases} a(\mathbf{u}_0, \mathbf{v}) + b(\mathbf{v}, p) = l(\mathbf{v}) - a(\mathbf{g}, \mathbf{v}), \ \forall \mathbf{v} \in V_0, \\ b(\mathbf{u}_0, q) = 0, \ \forall q \in Q. \end{cases} \tag{107}$$

3.2.2 Existence, uniqueness and Inf-Sup/LBB condition

The solution existence condition is not given by the Lax-Milgram Lemma, since the coercivity of $B((\mathbf{u}, p), (\mathbf{v}, q)) = a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + b(\mathbf{u}, q)$ cannot be defined.

Proposition 3 *Let us consider the variational problem (106) with $a : V_g \times V_g \to \mathbb{R}$ bilinear, coercive and continuous, $l : V_g \to \mathbb{R}$ linear and continuous and $b : V_g \times Q \to \mathbb{R}$ bilinear and continuous. Let $F(\boldsymbol{v}) = \frac{1}{2} a(\boldsymbol{v}, \boldsymbol{v}) - l(\boldsymbol{v})$. Then $\boldsymbol{u} \in V_g$ is a solution of (106) if and only if $F(\boldsymbol{u}) = \min_{\boldsymbol{v} \in V_g} F(\boldsymbol{v})$ under the condition $b(\boldsymbol{v}, q) = 0, \forall q \in Q$.*

Proof: By defining the Lagrangian $L(\mathbf{v}, q) := \frac{1}{2} a(\mathbf{v}, \mathbf{v}) - l(\mathbf{v}) + b(\mathbf{v}, q)$, finding the minimum u of F under the condition $b(\mathbf{v}, q) = 0, \forall q \in Q$ is equivalent to solving

$$\min_{q \in Q} \max_{\mathbf{v} \in V_g} L(\mathbf{v}, q). \tag{108}$$

Assuming that the first derivatives of L with respect to $\mathbf{v}$ and q are null, we arrive at the two equations in (106). ■

Definition 19 *Let $b : V_g \times Q \longrightarrow \mathbb{R}$ be a bilinear form. b is said to satisfy the Inf-Sup condition or LBB condition if*

$$\exists \beta > 0 \text{ such that } \beta \leq \inf_{q \in Q} \sup_{\boldsymbol{v} \in V_g} \frac{|b(\boldsymbol{v}, q)|}{||\boldsymbol{v}||_V ||q||_Q}. \tag{109}$$

Theorem 5 *Let $b : V_g \times Q \to \mathbb{R}$ be bilinear and continuous and as defined in (106). Then there exists a constant $\beta > 0$ such that*

$$\beta ||q||_Q \leq \sup_{\boldsymbol{v} \in V_g} \frac{b(\boldsymbol{v}, q)}{||\boldsymbol{v}||_{V_g}}, \forall q \in Q, \tag{110}$$

which is equivalent to the LBB condition.

Proof: To prove that the b term in the Stokes' equations satisfy the LBB condition, you can consult page 160 of [4].

Theorem 6 (Brezzi) *Let V_g and Q be two Hilbert spaces, let $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ be two continuous bilinear forms in $V_g \times V_g$ and $V_g \times Q$ respectively and let $Z = Ker\{b\} = \{\boldsymbol{v} \in V_g : b(\boldsymbol{v}, q) = 0, \forall q \in Q\}$. If $a(\cdot,\cdot)$ is coercive in Z and $b(\cdot,\cdot)$ satisfies the LBB condition, then there exists a unique solution $(\boldsymbol{u}, p) \in V_g \times Q$ of the problem (106).*

Proof: First we prove the uniqueness of p. Suppose that both p and $\tilde{p}$ satisfy

$$b(\mathbf{v}, p) = (r, \mathbf{v}), \forall \mathbf{v} \in V \tag{111}$$

where r is the residual defined by $(r, \mathbf{v}) = l(\mathbf{v}) - a(\mathbf{u}, \mathbf{v}), \forall \mathbf{v} \in V$. By subtraction we get

$$b(\mathbf{v}, p - \tilde{p}) = 0, \forall \mathbf{v} \in V \tag{112}$$

Combining this with the inf-sup condition, we have

$$\beta ||p - \tilde{p}||_Q \leq \sup_{\mathbf{v} \in V} \frac{b(\mathbf{v}, p - \tilde{p})}{||\mathbf{v}||_V} = 0 \tag{113}$$

from which it follows $||p - \tilde{p}||_Q = 0$ and thus $p = \tilde{p}$. For the velocity we can use the same idea using the coercivity of a instead of the LBB condition. ■

3.2.3 Finite Elements approximation

Let V_h and Q_h be two piecewise polynomial finite spaces approximating V_g and Q, the finite element approximation takes the form:

$$\begin{cases} \text{Find } \mathbf{u}_h \in V_h \text{ and } p_h \in Q_h \text{ such that} \\ a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = l(\mathbf{v}_h), \ \forall \mathbf{v}_h \in V_h, \\ b(\mathbf{u}_h, q_h) = 0, \ \forall q_h \in Q_h, \end{cases} \tag{114}$$

where $h = \max\{h_K\}$ and h_K the longest side of each element K, giving a sense of the size of the triangularization. As explained in the previous section, if $\{\boldsymbol{\varphi}_i\}_{i=1}^n$ is a basis of V_h and $\{\chi_i\}_{i=1}^m$ is a basis of Q_h, then $\mathbf{u}_h = \sum_{i=1}^n \xi_i \boldsymbol{\varphi}_i$ and $p_h = \sum_{j=1}^m \psi_j \chi_j$. The problem is reduced to a system of equations:

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \psi \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}, \tag{115}$$

where $A_{ij} = a(\varphi_j, \varphi_i)$, $B_{ij} = b(\varphi_i, \chi_j)$ and $b_i = l(\varphi_i)$.

Next, we have to choose a finite element type that satisfies the discrete LBB condition. We could think of the element $P_1 - P_1$, but this is not LBB stable.

Brezzi-Pitkäranta stabilization There are modifications of the $P_1 - P_1$ element that make it stable. An example is the Brezzi-Pitkäranta stabilization, which consists of adding an additional term in the second equation.

$$(V) \begin{cases} \text{Find } \mathbf{u}_h \in V_h \text{ y } p_h \in Q_h \text{ such that} \\ a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = l(\mathbf{v}_h), \ \forall \mathbf{v}_h \in V_h, \\ b(\mathbf{u}_h, q_h) + \dfrac{1}{12} h^2 \displaystyle\int_\Omega \nabla p_h \cdot \nabla p_h = 0, \ \forall q_h \in Q_h. \end{cases} \tag{116}$$

Taylor-Hood element It is one of the most used elements in the simulation of incompressible fluids, since it is not numerically very complicated and it provides a good approximation of both the velocity and the pressure. It consists of approximating the velocity by piecewise quadratic polynomials and the pressure by piecewise linear polynomials. In Figure 5 we can see the pressure and velocity nodes of different elements.

$$V_h = \{\mathbf{v} \in \left(C(\Omega)\right)^2 : \mathbf{v}|_K \in \left(P_2(K)\right)^2\}, \tag{117}$$

$$Q_h = \{q \in C(\Omega) : q|_K \in P_1(K)\}. \tag{118}$$

MINI element It is probably the simplest element that satisfies the inf-sup condition. Velocity and pressure are approximated by piecewise linear polynomials, but in addition, to each velocity element a cubic bubble function $\varphi_{\text{bubble}} = S_1S_2S_3$ is added. The finite element spaces are then

$$V_h = \{\mathbf{v} \in \left(C(\Omega)\right)^2 : \mathbf{v}|_K \in \left(P_1(K)\right)^2 \oplus \left(B_3(K)\right)^2\}, \text{ where } B_3(K) = \text{span}\{\varphi_{\text{bubble}}\}, \tag{119}$$

$$Q_h = \{q \in C(\Omega) : q|_K \in P_1(K)\}. \tag{120}$$

Although it is easier to implement than the previous one, the pressure errors are much larger.

Theorem 7 (Fortin's trick) *If there exists an interpolation operator $\Pi : V_g \longrightarrow V_h$ such that*

$$b(\boldsymbol{v} - \Pi\boldsymbol{v}, q_h) = 0,\ \forall \boldsymbol{v} \in V_g, q_h \in Q_h \qquad \textit{and} \qquad ||\Pi\boldsymbol{v}||_{V_h} \leq C||\boldsymbol{v}||_{V_g},\ \forall \boldsymbol{v} \in V_g, \tag{121}$$

then the discrete inf-sup condition is satisfied.

Proof:

$$\beta||q_h||_Q \leq \sup_{\mathbf{v}\in V_g} \frac{b(\mathbf{v}, q_h)}{||\mathbf{v}||_{V_g}} = \sup_{\mathbf{v}\in V_g} \frac{b(\Pi\mathbf{v}, q_h)}{||\mathbf{v}||_{V_g}} \leq C \sup_{\mathbf{v}\in V_g} \frac{b(\Pi\mathbf{v}, q_h)}{||\Pi\mathbf{v}||_{V_h}}, \tag{122}$$

and thus $\forall q_h \in Q_h$

$$\beta||q_h||_Q \leq C \sup_{\mathbf{v}\in V_h} \frac{b(\mathbf{v}, q_h)}{||\mathbf{v}||_{V_h}}. \tag{123}$$

■

Theorem 8 *Assume that Ω is convex or has smooth boundary. Then the MINI element satisfies the inf-sup condition.*

Proof: Let $\pi_h^0 : H_0^1(\Omega) \to V_h$ be the L^2-projector, which for any $\mathbf{v} \in H_0^1(\Omega)$ returns $\pi_h^0\mathbf{v} \in V_h$ such that

$$\int_\Omega (\mathbf{v} - \pi_h^0\mathbf{v})\mathbf{w} \cdot \mathbf{x} = 0,\ \forall \mathbf{w} \in V_h.$$

A proposition in [4] tells us that $||\pi_h^0\mathbf{v}||_{H^1} \leq c_1||\mathbf{v}||_{H^1}$ and $||\mathbf{v} - \pi_h^0\mathbf{v}||_{L^2} \leq c_2 h||\mathbf{v}||_{H^1}$. We fix a linear mapping $\pi_h^1 : L^2(\Omega) \to B_3$ such that

$$\int_K (\pi_h^1\mathbf{v} - \mathbf{v})dx = 0,\ \forall K \in \mathcal{K}.$$

We can interpret this map as a two-step process. First we apply the L^2-projection onto the space of piecewise constant functions. Then, in each triangle the constant is replaced by a

bubble function with the same integral. In this way we get $||\pi_h^1 \mathbf{v}||_{L^2} \leq c_3 ||\mathbf{v}||_{L^2}$. Now, we define $\Pi_h \mathbf{v} := \pi_h^0 \mathbf{v} + \pi_h^1(\mathbf{v} - \pi_h^0 \mathbf{v})$. Since p is continuous, we can apply Green's formula and the fact that the gradient of the pressure is piecewise constant:

$$b(\mathbf{v}-\Pi_h\mathbf{v}, q_h) = \int_\Omega \nabla\cdot(\mathbf{v}-\Pi_h\mathbf{v}) q_h \, dx = \int_{\partial\Omega} (\mathbf{v}-\Pi_h\mathbf{v})\cdot\mathbf{n} \, q_h \, ds - \int_\Omega (\mathbf{v}-\Pi_h\mathbf{v})\cdot\nabla q_h \, dx = 0, \quad (124)$$

and the boundedness of Π_h is given by:

$$\begin{aligned} ||\Pi_h \mathbf{v}||_{H^1} &\leq ||\pi_h^0 \mathbf{v}||_{H^1} + ||\pi_h^1(\mathbf{v} - \pi_h^0 \mathbf{v})||_{H^1} && (125)\\ &\leq c_1 ||\mathbf{v}||_{H^1} + \frac{c_4}{h} ||\pi_h^1(\mathbf{v} - \pi_h^0 \mathbf{v})||_{L^2} && (126)\\ &\leq c_1 ||\mathbf{v}||_{H^1} + \frac{c_4}{h} c_3 ||\mathbf{v} - \pi_h^0 \mathbf{v}||_{L^2} && (127)\\ &\leq c_1 ||\mathbf{v}||_{H^1} + c_4 c_3 c_2 ||\mathbf{v}||_{H^1}. && (128) \end{aligned}$$

where c_4 comes from the Poincare inequality. In addition, we have the following error estimate

$$||\mathbf{u} - \mathbf{u}_h||_V + ||p - p_h||_Q \leq ch(||\mathbf{u}||_{H^2} + ||p||_{H^1}). \quad (129)$$

■

$\mathbf{P}_1$-$\mathbf{P}_0$ non-conforming element

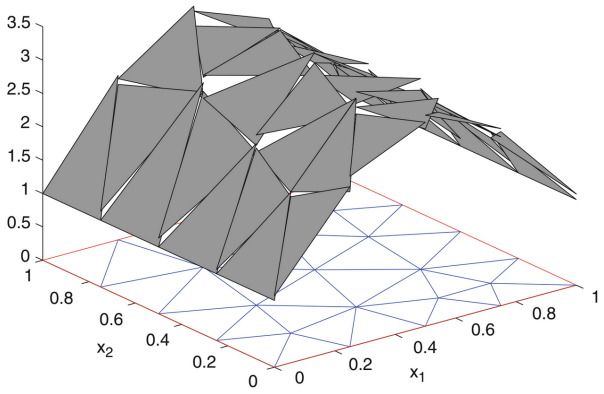


Figure 4: *Scheme of Crouzeix-Raviart elements (Source: [2]).*

It consists of approximating the velocity by Crouzeix-Raviart functions and the pressure by piecewise constants. The Crouzeix-Raviart element is a linear element that is only continuous at the midpoint of the edges of the triangles. The corresponding basis functions have the form $S_i^{CR} = -S_i + S_j + S_k$, where the S_i, S_j and S_k are the classical hat functions and with a cyclic permutation of the indices $\{i, j, k\}$.

Theorem 9 *The non-conforming P_1-P_0 element satisfies the inf-sup condition.*

Proof: In this case

$$V_h = \{\mathbf{v} \in \left(L^2(\Omega)\right)^2 : \mathbf{v} \in \left(P_1(K)\right)^2, \forall K \in \mathcal{K}, ([v_i], 1)_{\partial\Omega} = 0, i = 1, 2\}, \quad (130)$$

$$Q_h = \{q \in L_0^2(\Omega) : q|_K \in P_0(K), \forall K \in \mathcal{K}\}. \quad (131)$$

We define the linear interpolant $\Pi\mathbf{v} = (\Pi v_i)_{i=1}^2$ in each triangle K as:

$$\Pi v_i = \sum_{j=1}^{3} \frac{(v_i, 1)_{E_j}}{|E_j|} S_j^{CR}, \quad (132)$$

where the three E_j's are the edges of the triangle. Let m_j be each edge mid-point. Using the divergence theorem:

$$(\nabla \cdot \mathbf{v}, 1)_K = (\mathbf{v}, \mathbf{n})_{\partial K} = \sum_{j=1}^{3} (\mathbf{v}, \mathbf{n})_{E_j} = \sum_{j=1}^{3} \Pi\mathbf{v}(m_j) \cdot \mathbf{n}|E_j| = \sum_{j=1}^{3} (\Pi\mathbf{v}, \mathbf{n})_{E_j} =$$

$$= (\Pi\mathbf{v}, \mathbf{n})_{\partial K} = (\nabla \cdot \Pi\mathbf{v}, 1)_K \Rightarrow b(\Pi\mathbf{v}, q_h) = b(\mathbf{v}, q_h). \tag{133}$$

Finally, since the norm of all $\mathbf{v} \in V$ is invariant under the addition of a constant, we take $v_i = \tilde{v}_i + \bar{v}_i$ with $\bar{v}_i = \Pi v_i(m_0)$ where m_0 is the mid-point of the edge. As a consequence, $\tilde{v}_i(m_0) = 0$ and $\Pi\mathbf{v} = \Pi(\tilde{\mathbf{v}} + \bar{\mathbf{v}}) = \Pi\tilde{\mathbf{v}} + \bar{\mathbf{v}}$. Also, $\Pi\tilde{\mathbf{v}}$ is a Crouzeix-Raviart function, so $\Pi\tilde{\mathbf{v}} = \sum_{j=1}^{3} \frac{(\tilde{v}_i,1)_{E_j}}{|E_j|} S_j^{CR}$. Putting all this together we get

$$||\nabla\Pi v_i||_K = ||\nabla\Pi\tilde{v}_i||_K \leq C \max_j |(\tilde{v}_j, 1)_{E_j}| \leq C||\tilde{v}_i||_{\partial K} \leq C(||\tilde{v}_i||_K + ||\nabla\tilde{v}_i||_K) \leq C||\nabla v_i||_K$$

Adding to all elements K and all spatial dimensions, we obtain

$$||\Pi\mathbf{v}||_V \leq C||\mathbf{v}||_V. \tag{134}$$

■

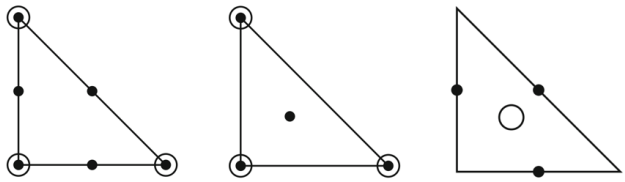


Figure 5: *Pressure nodes (circles) and velocity nodes (points) for the elements: Taylor-Hood, MINI, P1-P0 non-conforming. The number of nodes coincides with the number of degrees of freedom.*

3.2.4 Example: lid-driven cavity

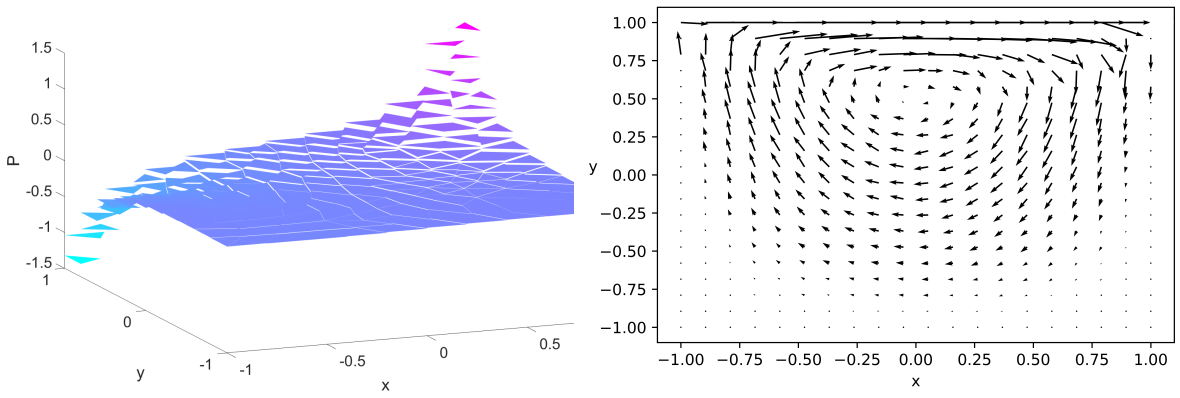
Finally, to put what has been learned into practice, a typical example of fluid dynamics, known as *lid-driven cavity*, will be solved.

A viscous and incompressible fluid is considered in a square domain $\Omega = (-1, 1)^2$ in which the upper wall moves with horizontal velocity, that is, the boundary conditions are:

$$\begin{cases} \mathbf{u}(x, 1) = (1, 0), \forall x \in (-1, 1), \\ \mathbf{u}(x, -1) = (0, 0), \forall x \in (-1, 1), \\ \mathbf{u}(1, y) = (0, 0), \forall y \in (-1, 1), \\ \mathbf{u}(-1, y) = (0, 0), \forall y \in (-1, 1), \end{cases} \tag{135}$$

and the finite element approximation in the Stokes equations is given by

$$\begin{cases} \text{Find } (u_{hx}, u_{hy}, p_h) \in V_h \times V_h \times Q_h \text{ such that} \\ \int_\Omega \nabla u_{hx} \cdot \nabla v_{hx} \, d\Omega + \int_\Omega \nabla u_{hy} \cdot \nabla v_{hy} \, d\Omega - \int_\Omega \frac{\partial v_{hx}}{\partial x} p_h \, d\Omega - \int_\Omega \frac{\partial v_{hy}}{\partial y} p_h \, d\Omega = \int_\Omega f v_h \, d\Omega, \ \forall v_{hx}, v_{hy} \in V_{h0}, \\ \int_\Omega \frac{\partial u_{hx}}{\partial x} q_h \, d\Omega + \int_\Omega \frac{\partial u_{hy}}{\partial y} q_h \, d\Omega, \ \forall q_h \in Q_h, \end{cases} \tag{136}$$



(a) *Pressure graph.* (b) *Velocity graph.*

Figure 6: *Solution of the Stokes equations with the boundary conditions at (135) and the $P_1 - P_0$ non-conforming element.*

where $V_{h0} = \{v_h \in V_h : v|_{\partial\Omega} = 0\}$. Using the non-conformimg $P_1 - P_0$ element with the Crouzeix-Raviart functions, we are left with the following system of equations to be solved:

$$\begin{pmatrix} A & 0 & C_1 & 0 \\ 0 & A & C_2 & 0 \\ C_1^T & C_2^T & 0 & T \\ 0 & 0 & T^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{U}_x \\ \mathbf{U}_y \\ \mathbf{P} \\ \bar{p} \end{pmatrix} = \begin{pmatrix} \mathbf{l}_1 \\ \mathbf{l}_2 \\ \mathbf{l}_3 \\ 0 \end{pmatrix}, \tag{137}$$

where $\mathbf{l}_i$ contains information about the forces acting on the fluid (none in this case) and the boundary conditions. Besides

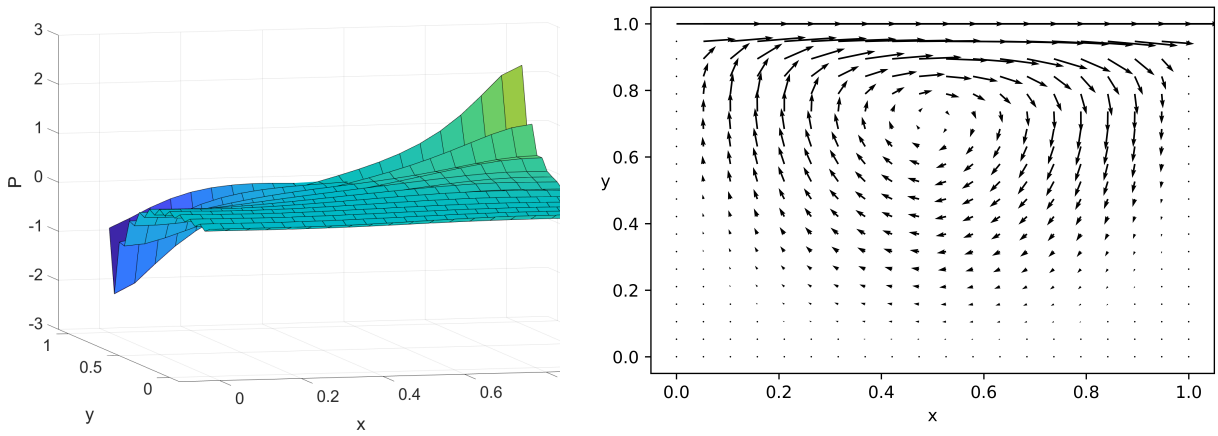
$$A_{ij} = \int_\Omega \nabla S_i^{CR} \cdot \nabla S_j^{CR}, \qquad T_i = \int_\Omega S_i^{CR}, \tag{138}$$

$$(C_1)_{ij} = \int_\Omega S_i^{CR} \frac{\partial S_j^{CR}}{\partial x}, \qquad (C_2)_{ij} = \int_\Omega S_i^{CR} \frac{\partial S_j^{CR}}{\partial y}, \tag{139}$$

and $\bar{p}$ is the mean pressure, so the last condition dictates that the mean pressure must be zero. Thus, the pressure obtained is actually the pressure difference with respect to the average. The obtained results are shown in Figure 6. As we can see, the movement of the upper wall drags the viscous fluid and it begins to rotate around an interior point. The pressure shows a maximum in the upper right corner, where the fluid hits the wall, and a minimum in the upper left corner.

Another option may be to use the MINI element, which is a bit more difficult to program but is much more widely used. Remember that in this type of element the pressure is approximated by functions $Q_h = P_1$ and the velocity by $V_h = P_1 \oplus B_3$. We now call $\{S_i^l\}$ the base functions of P_1 and $\{S_i^b\}$ those of B. Then, we have to solve the following system,

$$\begin{pmatrix} A & 0 & 0 & 0 & C_1 \\ 0 & A & 0 & 0 & C_2 \\ 0 & 0 & D & 0 & C_3 \\ 0 & 0 & 0 & D & C_4 \\ C_1^T & C_2^T & C_3^T & C_4^T & 0 \\ 0 & 0 & 0 & 0 & T \end{pmatrix} \begin{pmatrix} \mathbf{U}_x^l \\ \mathbf{U}_y^l \\ \mathbf{U}_x^b \\ \mathbf{U}_y^b \\ \mathbf{p} \\ \bar{p} \end{pmatrix} = \begin{pmatrix} \mathbf{l}_1 \\ \mathbf{l}_2 \\ \mathbf{l}_3 \\ \mathbf{l}_4 \\ \mathbf{l}_5 \\ 0 \end{pmatrix}, \tag{140}$$



(a) *Pressure graph.* (b) *Velocity graph*

Figure 7: *Solution of the Stokes equations with the MINI element.*

where the matrices A, C_1, C_2 and T are as in the previous case but with the new basis linear functions of P_1 and also:

$$D_{ij} = \delta_{ij} \int_\Omega \nabla S_i^b \cdot \nabla S_j^b \qquad (C_3)_{ij} = \int_\Omega S_i^l \frac{\partial S_j^b}{\partial x} \qquad (C_4)_{ij} = \int_\Omega S_i^l \frac{\partial S_j^b}{\partial y} \tag{141}$$

As we can see in Figure 7, similar results to the previous case are obtained. This time $\Omega = (0,1)^2$ has been considered instead to simplify the code for the boundary conditions.

3.2.5 Solving the linear systems

In the Stokes equations, the obtained stiffness matrices are not positive definite. Therefore we cannot use the CGM. Other techniques, similar to the CGM and based on Krylov subspaces, have been developed for non definite matrices. One example is the Generalized Minimal Residual Method (GMRES), which iteratively approximates the solution by the minimal residual vector in a Krylov subspace.

Definition 20 *The n-th Krylov subspace for a system* $A\boldsymbol{x} = \boldsymbol{b}$ *is defined as* $K_n = K_n(A, \boldsymbol{r}_0) := \{\boldsymbol{r}_0, A\boldsymbol{r}_0, A^2\boldsymbol{r}_0, ..., A^{n-1}\boldsymbol{r}_0\}$*, where* $\boldsymbol{r}_0 = \boldsymbol{b} - A\boldsymbol{r}_0$ *and* x_0 *is an initial guess.*

The GMRES method aims at finding an approximate solution $\mathbf{x}_n \in K_n$ that minimizes the euclidean norm of the residual $\mathbf{r}_n = \mathbf{b} - A\mathbf{x}_n$. Instead of the linearly dependent vectors $\mathbf{r}_0$, $A\mathbf{r}_0$, ..., $A^{n-1}\mathbf{r}_0$, the Arnoldi iteration is used to obtain orthonormal vectors $\mathbf{q}_1, ..., \mathbf{q}_n$, which serve as a basis for K_n. Specifically, $\mathbf{q}_1 = ||\mathbf{r}_0||_2^{-1}\mathbf{r}_0$. Consequently, the vector $\mathbf{x}_n \in K_n$ can be expressed as $\mathbf{x}_n = \mathbf{x}_0 + Q_n\mathbf{y}_n$, where $\mathbf{y}_n \in \mathbb{R}^n$ and Q_n is the m-by-n matrix composed of $\mathbf{q}_1, ..., \mathbf{q}_n$. The Arnoldi process also produces an upper Hessenberg matrix $H_n \in \mathbb{R}^{(n+1)\times n}$ with $AQ_n = Q_{n+1}H_n$. Since the columns of Q_n are orthonormal

$$||\mathbf{r}_n|| = ||\mathbf{b} - A\mathbf{x}_n|| = ||\mathbf{b} - A(\mathbf{x}_0 + Q_n\mathbf{y}_n)|| = ||\mathbf{r}_0 - AQ_n\mathbf{y}_n|| = ||\alpha\mathbf{q}_1 - Q_{n+1}H_n\mathbf{y}_n|| \tag{142}$$

$$= ||Q_{n+1}(\alpha\mathbf{e}_1 - H_n\mathbf{y}_n)|| = ||\alpha\mathbf{e}_1 - H_n\mathbf{y}_n||, \tag{143}$$

$$\tag{144}$$

where $\alpha = ||\mathbf{r}_0||$ and $\mathbf{e}_1 = (1, 0, ..., 0)$. Thus, $\mathbf{x}_n$ can be found by $\mathbf{x}_n = \mathbf{x}_0 + Q_n\mathbf{y}_n$ and $\mathbf{y}_n$ minimizes the norm of the residual $\mathbf{r}_n = H_n\mathbf{y}_n - \alpha\mathbf{e}_1$, which is a linear least squares problem.

In practice, the CGM and the GMRES need preconditioning to really beat other methods. The convergence speed is given by the condition number $\kappa_2(A) = \dfrac{\lambda_{max}}{\lambda_{min}}$. With a preconditioner we can solve $A\mathbf{u} = \mathbf{b}$ with $A \in \mathbb{R}^{n\times n}$ by defining another matrix $B \in \mathbb{R}^{n\times n}$ and $\bar{A} = AB^{-1}$. Then, we solve $(\bar{A})\bar{\mathbf{u}} = \mathbf{b}$ and finally $B\mathbf{u} = \bar{\mathbf{u}}$. The advantage is that B is choosen so that $\kappa_2(\bar{A}) < \kappa_2(A)$ and $B\mathbf{u} = \bar{\mathbf{u}}$ is easy to solve. An example is the Gauss-Seidel preconditioner, where, given $A = L + D + U$ a separation of A in its diagonal D, lower L and upper U triangular parts, we take $B = A + LD^{-1}U$.

3.3 Turbulent flow and the Navier-Stokes equations

After studying the basic properties of laminar and incompressible flows through the Stokes equations, the next step would be to study more general flows as the Navier-Stokes and the Euler equations, which depend on time and can be dominated by the convective term, making them nonlinear and much more difficult to solve. Some important properties such as the existence and uniqueness of solutions have not yet been proven. Nevertheless, due to its wide range of application, numerous techniques have been developed to solve them numerically under the name of Computational Fluid Dynamics or CFD. The main problem in solving the Navier-Stokes equations is turbulence, which is a chaotic behaviour exhibited by fluids with low viscosity moving at high speed (high Reynolds number). Turbulence is caused by the dissipation of energy in the form of heat on a microscopic scale with large momentum transport on a macroscopic scale. Due to the microscopic nature of turbulence it is very difficult to simulate it with finite elements, since we cannot make the elements infinitely small without increasing the computation time. Some models, such as RANS (Reynolds-averaged Navier-Stokes), attempt to model turbulence by taking an average, so that each magnitude is expressed as the sum of its average over time plus a fluctuating component.

4 Conclusions

This work has analyzed the Finite Element Method as a powerful numerical technique for solving a wide range of PDEs, particularly those appearing in fluid mechanics. We have examined the fundamental principles and key steps involved in the FEM and shown an example from very basic principles, including the variational formulation, the shape functions, the Galerkin method, the assembly of the global system of equations and its iterative solvers.

The FEM provides a systematic framework for dividing a domain into smallerand more manageable elements, enabling the formulation of local approximations and the construction of a global solution. By discretizing the problem into a finite set of unknowns, the FEM transforms the original differential problem into a system of algebraic equations, that can be solved using well-established numerical methods. One of the main advantages of the FEM is its ability to handle irregular geometries and complex boundary conditions. The FEM has proven to be a valuable tool in the field of computational mathematics and engineering, providing insights into complex physical phenomena and aiding in the design and optimization of various systems.

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A Proofs of Riesz-Fréchet and Lax-Milgram theorems

Riesz-Fréchet representation theorem: *Let H be a Hilbert space and H' its dual ($H' = \mathcal{L}(H, \mathbb{R})$), then*

$$\forall L \in H',\ \exists! u \in H \text{ such that } L(v) = (u, v) =: \mathcal{J}_u(v),\ \forall v \in H \tag{145}$$

and the function

$$\begin{array}{rccc} \mathcal{J}: & H & \longrightarrow & H' \\ & u & \longrightarrow & \mathcal{J}_u = (u, \cdot) \end{array} \quad with \quad \begin{array}{rccc} \mathcal{J}_u: & H & \longrightarrow & \mathbb{R} \\ & v & \longrightarrow & \mathcal{J}_u(v) = (u, v) \end{array} \tag{146}$$

is an isometric isomorphism, which means $||L||_{H'} = ||u||_H$.

Proof: First, let us see that $\mathcal{J}$ is well defined, that is, $\mathcal{J}_u \in H'$. By the linearity of the dot product $(u, v + w) = (u, v) + (u, w)$ we get that $\mathcal{J}_u(v + w) = \mathcal{J}_u(v) + \mathcal{J}_u(w)$, $\forall v, w \in H$. In addition, $(u, \lambda v) = \lambda(u, v)$ implies that $\mathcal{J}_u(\lambda v) = \lambda \mathcal{J}_u(v)$, $\forall v \in H$. Last but not least, $|\mathcal{J}_u(v)| \leq ||u||_H ||v||_H$, which in total means $\mathcal{J}_u \in H'$.

Now, we see that $\mathcal{J}$ is isometric, that is we have to prove that $\forall u \in H$, $||L||_{H'} = ||u||_H$. By the Cauchy-Schwarz inequality

$$\forall 0 \neq v \in H,\ |\mathcal{J}_u(v)| = |(u, v)| \leq ||u||_H ||v||_H \Longrightarrow \frac{|\mathcal{J}_u(v)|}{||v||_H} \leq ||u||_H.$$

On the other hand, by the definition of norm of a linear map

$$||\mathcal{J}_u||_{H'} \leq \sup_{0 \neq v \in H} \frac{|\mathcal{J}_u(v)|}{||v||_H}.$$

Besides that, for $v = u$

$$|\mathcal{J}_u(u)| = |(u, u)| = ||u||_H^2 \Longrightarrow \frac{|\mathcal{J}_u(u)|}{||u||_H} = ||u||_H.$$

Thus, on the one hand, we have $||\mathcal{J}_u||_{H'} \leq ||u||_H$, and on the other $||u||_H \leq ||\mathcal{J}_u||_{H'}$.

Now, we see that $\mathcal{J}_u$ is inyective. We have alredy seen that $||u||_H = ||\mathcal{J}_u||_{H'} = ||\mathcal{J}(u)||_{H'}$. Then, if $\mathcal{J}_u = 0$, we get $||\mathcal{J}_u||_{H'} = 0 \Longrightarrow ||u||_H = 0 \Longrightarrow u = 0_H$.

Finally, we have to show that $\mathcal{J}$ is surjective, which means that $\forall L \in H'$, $\exists u \in H$ such that $\mathcal{J}(u) = \mathcal{J}_u = L$, and therefore $\mathcal{J}_u(v) \in L(v)$, $\forall v \in H$ and $(u, v) = L(v)$, $\forall v \in H$. Let $w \in (\text{Ker } L)^{\perp}$, then

$$\forall v \in H,\ L\left(v - \frac{L(v)}{L(w)} w\right) = L(v) - \frac{L(v)}{L(w)} L(w) = 0 \Longrightarrow$$

$$\Longrightarrow v - \frac{L(v)}{L(w)} w \in \text{Ker } L \Longrightarrow \left(v - \frac{L(v)}{L(w)} w, w\right) = 0 \Longrightarrow (v, w) - \frac{L(v)}{L(w)} = 0 \Longrightarrow$$

$$\Longrightarrow L(v) = (v, L(w)w).$$

So, just take $u = L(w)w$. ■

Lax-Milgram's lemma: *If $a(\cdot,\cdot)$ is a bilinear continuous and coercive form, and l is a linear continuous form, then the following abstract variational problem has a solution $u \in H$ and is unique.*

$$(V)\begin{cases} \text{Find } u \in H \text{ such that} \\ a(u,v) = l(v),\ \forall v \in H \end{cases} \tag{147}$$

Proof: Let $u \in H$ be fixed. Consider the map

$$\begin{array}{rccc} A_u: & H & \longrightarrow & \mathbb{R} \\ & v & \longrightarrow & a(u,v) \end{array}$$

A_u is linear by the bilinearity of a and $|A_u(v)| = |a(u,v)| \leq M||u||_H||v||_H \leq c||v||_H$, $\forall v \in H$, which means that A_u is continuous. Therefore $A_u \in H'$ and

$$||A_u||_{H'} = \sup_{0 \neq v \in H} \frac{|A_u(v)|}{||v||_H}.$$

Now, we define

$$\begin{array}{rccc} A: & H & \longrightarrow & H' \\ & u & \longrightarrow & A(u) = A_u \end{array}$$

Then the abstract variational problem is equivalent to

$$(V)\begin{cases} \text{Find } u \in H \text{ such that} \\ a(u,v) = l(v),\ \forall v \in H \end{cases} \Leftrightarrow \begin{cases} \text{Find } u \in H \text{ such that} \\ A_u = l \text{ in H'} \end{cases} \Leftrightarrow \begin{cases} \text{Find } u \in H \text{ such that} \\ A(u) = l \text{ in H'} \end{cases} \tag{148}$$

By the Riesz-Frechét representation theorem, for $l \in H'$, $\exists! f \in H$ such that $l(v) = (f,v)$, $\forall v \in H$ and $||l||_{H'} = ||f||_H$. Likewise, for $A_u \in H'$, $\exists! \mathcal{A}_u \in H$ such that $A_u(v) = (\mathcal{A}_u, v)$, $\forall v \in H$ y $||A_u||_{H'} = ||\mathcal{A}_u||_H$. Now, we define

$$\begin{array}{rccc} \mathcal{A}: & H & \longrightarrow & H \\ & u & \longrightarrow & \mathcal{A}(u) = \mathcal{A}_u \end{array} \qquad ||\mathcal{A}(u)|| = ||\mathcal{A}_u||_H \leq M||u||_H.$$

Uniqueness: Let us see that $\mathcal{A}$ is inyective. $\forall v \in H$, $\alpha||v||_H^2 \leq a(v,v) = (\mathcal{A}_v, v) \leq ||\mathcal{A}_v||_H||v||_H$. If $v \in H$ verifies that $\mathcal{A}(v) = 0$, then $v = 0_H$.

Existence: Let us see that $\mathcal{A}$ is surjective. We have to see that $\mathcal{A}H = H$. If $\mathcal{A}H$ is closed, then $H = \mathcal{A}H \oplus (\mathcal{A}H)^\perp$. Then we can just prove that $(\mathcal{A}H)^\perp = \{0_H\}$: Let $v_0 \in (\mathcal{A}H)^\perp$, $\alpha||v_0||_H^2 \leq a(v_0,v_0) = A_{v_0}(v_0) = (\mathcal{A}_{v_0}, v_0) = (\mathcal{A}(v_0), v_0) = 0 \Rightarrow v_0 = 0_H$. Now we see that $\mathcal{A}H$ is closed. To do that we consider $(\mathcal{A}v_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{A}H$ converging to $\omega \in H$. In the injectivity we saw that $||\mathcal{A}v_n - \mathcal{A}v_m||_H \geq \alpha||v_n - v_m||_H$. Then

$$(\mathcal{A}v_n) \text{ converges } \Rightarrow (\mathcal{A}v_n) \text{ is Cauchy } \Rightarrow (v_n) \text{ is Cauchy } \Rightarrow v_n \to v \text{ in } H$$

On the other hand,

$$||\mathcal{A}(v_n - v)||_H \leq M||v_n - v||_H \Rightarrow \mathcal{A}v_n \to \mathcal{A}v,$$

but also $\mathcal{A}v_n \to \omega$. Thus $\omega = \mathcal{A}(v) \in H$. ■

B Appendix: Why FEM? Comparison with other methods

In this appendix we are going to compare the FEM with the FDM and the FVM. To do that we consider a test problem, consisting in the Poisson equation with homogenuous Dirichlet conditions:

$$\begin{cases} -\Delta u = f, \text{ in } \Omega = (0,1)^2 \subset \mathbb{R}^2 \\ u(x) = 0, \ \forall x \in \partial\Omega \end{cases} \tag{149}$$

We know this problem is well-posed in the sense of Hadamard, which means that exists a unique solution $u \in C^2(\Omega)$ and its dependence on the data f is continuous.

Finite Difference Method

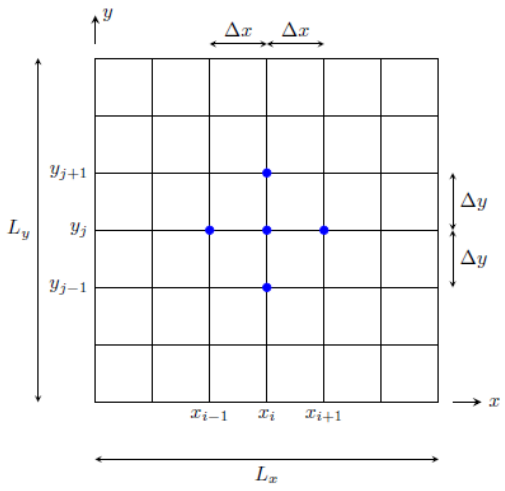

Figure 8: *Point grid in the FDM.*

In the FDM we create a uniform point grid Ω_h of step size $h = \Delta x = \Delta y$. In the interior points we approximate the derivatives using the quotients:

$$\frac{\partial u}{\partial x}(x_i, y_j) = \frac{u(x_i + h, y_j) - u(x_i - h, y_j)}{2h} + O(h^2),$$

$$\frac{\partial^2 u}{\partial x^2}(x_i, y_j) = \frac{u(x_i + h, y_j) - 2u(x_i, y_j) + u(x_i - h, y_j)}{h^2} + O(h^2).$$

The discrete laplacian takes the form

$$-\Delta u(x_i, y_j) = \frac{4u(x_i, y_j) - u(x_i + h, y_j) - u(x_i - h, y_j) - u(x_i, y_j + h) - u(x_i, y_j - h)}{h^2}. \tag{150}$$

It is useful to represent this with stencils. Looking into one individual point, we have the following representation.

$$-\Delta_h = \frac{1}{h^2} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{151}$$

If N is the number of divisions on each direction and $h = 1/N$, then we have a total of $(N-1)^2$ interior points and the whole laplacian can be represented by a matrix $L_h \in \mathbb{R}^{(N-1)^2 \times (N-1)^2}$.

$$L_h = \begin{pmatrix} 4 & -1 & ... & 0 & ... & -1 & ... & 0 & ... \\ -1 & 4 & -1 & ... & 0 & ... & -1 & ... & 0 \\ ... & -1 & 4 & -1 & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ -1 & 0 & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & -1 & 4 & -1 \\ ... & 0 & -1 & 0 & ... & ... & 0 & -1 & 4 \end{pmatrix}. \tag{152}$$

At the end we arrive at a system of linear equations $L_h u_h = f_h$, where u_h is the solution vector at the points of the grid.

Finite Volume Method

We consider the same grid Ω_h on the domain Ω. Instead of considering the vertices/points we consider the little squares E_j's. In the problem will appear some integrals over these E_j. Using the Green's formula, we transform these integrals in integrals over the boundary ∂E_j.

$$-\int_{\partial E_j} \frac{\partial u}{\partial n} ds = \int_{E_j} (-\Delta u) dx - \int_{E_j} f dx, \tag{153}$$

and then we separate the integral over the boundary in 4 integrals over the 4 sides of each square:

$$\int_{\partial E_j} = \int_{right} + \int_{up} + \int_{left} + \int_{down}, \tag{154}$$

$$\begin{aligned} \int_{right} \frac{\partial u}{\partial n} ds &\approx h \frac{u_h(x_j + h, y_j) - u_h(x_j, y_j)}{h}, \\ \int_{up} \frac{\partial u}{\partial n} ds &\approx h \frac{u_h(x_j, y_j + h) - u_h(x_j, y_j)}{h}, \\ \int_{left} \frac{\partial u}{\partial n} ds &\approx h \frac{u_h(x_j - h, y_j) - u_h(x_j, y_j)}{h}, \\ \int_{down} \frac{\partial u}{\partial n} ds &\approx h \frac{u_h(x_j, y_j - h) - u_h(x_j, y_j)}{h}. \end{aligned}$$

At the end

$$\int_{E_j} f dx = 4u(x_j, y_j) - u(x_j + h, y_j) - u(x_j - h, y_j) - u(x_j, y_j + h) - u(x_j, y_j - h) \approx h^2 \frac{1}{|E_j|} \int_{E_j} f dx = h^2 f(x_j, y_j). \tag{155}$$

Finite Element Method

As we saw during this work, the FEM consists in reducing the general function space V of the solution into a finite dimensional function space V_h with a basis $\{\varphi\}_{i=1}^n$. Then, we approximate the solution u with $U_h = \sum_{i=1}^n u_h(x_j, y_j)\varphi_j(x, y) \in V_h$ and solve a linear system of equations.

Comparison between FDM, FVM and FEM

In the next table we study the main advantages and disadvantages of each method. The main advantages of the FEM are the possibility of using higher orders (not only linear), the fact that it can be used for complex geometries and the preservation of the structure in the matrices. However, the coding is not that simple and sometimes we need to use external packages, and the fact that in the first place we have to transform the differential equation into a variational problem.

	PDEs	high orders	geometry	structure preservation	coding
FD	universal	possible	unflexible	difficult	simple
FV	conservative/divergence	difficult	flexible	good	need packages
FE	variational problems	possible	flexible	good	need packages

Table 1: *Comparison between FDM, FVM and FEM.*

Although there are pros and cons in all three methods, for CFD the FVM has been traditionally used for many reasons. The FEM is very well suited for second order PDEs, and consequently is very adecuate to problems such as elasticity and more generally structures, but also for the Stokes equations in fluids. Other fluid problems are described by first order differential equations (for example, the Euler equations of gas dynamics) or equations in which the first order terms are dominant (the compressible Navier-Stokes equations). For these cases, solutions are often discontinuous (for example in the form of shocks) and the Galerkin method is not useful. One can fix this through stabilization techniques developed in the past years, but still the preferred approach is to base methods on the conservation properties, thus the use of finite volume methods, where, for example, mass conservation is directly inherited.

There are also many other methods not so well known but widely used for depending on what type of applications, such as spectral methods, boundary element methods, particle methods or the lattice Boltzmann methods.

C Appendix: Computer programs

MINI element:

```
[mxx, myy, mxy, mx, my] = matriz2; %Matrices con las integrales en el elemento de referencia
ns1 = 20; %Numero de divisiones en x
ns2 = 20; %Numero de divisiones en y
d = 3; %Grados de libertad
nt = (ns1 + 1)*(ns2 + 1); %Numero de vertices

[globales, x, y, nel] = gen2(ns1, ns2); %Generar malla, nel=numero de elementos
a = zeros(nt,nt);
d1 = zeros(nt,nt);
d2 = zeros(nt,nt);
d3 = zeros(nel,nt);
d4 = zeros(nel,nt);
D = zeros(nel,nel);
A = zeros(3*nt + 2*nel +1, 3*nt + 2*nel);
b = zeros(3*nt + 2*nel +1, 1);
nodos = zeros(1,3);
mk = zeros(d,d);
nk1 = zeros(d,d);
nk2 = zeros(1,d);
rk = zeros(1,1);
lk = zeros(d,1);
t = zeros(1, nt);
aux1 = [-1/120, 1/120, 0;]; %mx, my para las funciones burbuja
sol = zeros(3*nt + 2*nel, 1);
coord = [x;y]';

%Ensamblado
for k=1:nel
    nodos = globales(k,:); %Vertices de cada elemento
    [c,J,deter] = InvDet(coord(nodos,:)); %Transformaci n al elemento de referencia
    mk = deter*(c(1,1)*mxx + c(1,2)*(mxy+mxy') + c(2,2)*myy);
    a(nodos,nodos) = a(nodos,nodos) + mk;
    rk = deter*(c(1,1) + c(1,2) + c(2,2))/180;
    D(k,k) = D(k,k) + rk;
    nk1 = deter*(J(1,1)*mx + J(1,2)*my);
    d1(nodos,nodos) = d1(nodos,nodos) + nk1;
    nk1 = deter*(J(2,1)*mx + J(2,2)*my);
    d2(nodos,nodos) = d2(nodos,nodos) + nk1;
    nk2 = deter*(J(1,1)*aux1 + J(1,2)*aux1);
    d3(k,nodos) = d3(k,nodos) + nk2;
    nk2 = deter*(J(2,1)*aux1 + J(2,2)*aux1);
    d4(k,nodos) = d4(k,nodos) + nk2;
    t(nodos) = t(nodos) + deter/6;
end

%Construccion de la matriz por bloques
a = a*50;
A(1:nt,1:nt) = a;
A(nt+1:2*nt, nt+1:2*nt) = a;
```

```
A(2*nt+1:2*nt+nel, 2*nt+1:2*nt+nel) = D;
A(2*nt+nel+1:2*nt+2*nel, 2*nt+nel+1:2*nt+2*nel) = D;
A(1:nt, 2*nt+1+2*nel:3*nt+2*nel) = d1;
A(nt+1:2*nt, 2*nt+2*nel+1:3*nt+2*nel) = d2;
A(2*nt+1:2*nt+nel, 2*nt+2*nel+1:3*nt+2*nel) = d3;
A(2*nt+nel+1:2*nt+2*nel, 2*nt+2*nel+1:3*nt+2*nel) = d4;
A(2*nt+2*nel+1:3*nt+2*nel, 1:nt) = d1';
A(2*nt+2*nel+1:3*nt+2*nel, nt+1:2*nt) = d2';
A(2*nt+2*nel+1:3*nt+2*nel, 2*nt+1:2*nt+nel) = d3';
A(2*nt+2*nel+1:3*nt+2*nel, 2*nt+nel+1:2*nt+2*nel) = d4';
A(3*nt+2*nel+1, 2*nt+2*nel+1:3*nt+2*nel) = t;

%Condicion Dirichlet en u_x
%0 en y=0
for k = 1:(ns1 + 1)
    for i = 1:3*nt
        A(k,i) = 0;
    end
    b(k) = 0;
    A(k,k) = 1;
end
%0 en x=0
for k = 1:ns2+1
    for i = 1:3*nt
        A(ns1*(k-1)+k,i) = 0;
    end
    b(ns1*(k-1)+k) = 0;
    A(ns1*(k-1)+k,ns1*(k-1)+k) = 1;
end
%0 en x=1
for k = 1:ns2+1
    for i = 1:3*nt
        A(ns1*k+k,i) = 0;
    end
    b(ns1*k+k) = 0;
    A(ns1*k+k,ns1*k+k) = 1;
end
%1 en y=1
for k = 1:ns1+1
    for i = 1:3*nt
        A(ns1*ns2+ns2+k,i) = 0;
    end
    b(ns1*ns2+ns2+k) = 1;
    A(ns1*ns2+ns2+k,ns1*ns2+ns2+k) = 1;
end

%Condicion Dirichlet en u_y
%0 en y=0
for k = 1:(ns1 + 1)
    for i = 1:3*nt
        A(k+nt,i) = 0;
    end
    b(k+nt) = 0;
    A(k+nt,k+nt) = 1;
end
```

```
%0 en x=0
for k = 1:ns2+1
    for i = 1:3*nt
        A(ns1*(k-1)+k+nt,i) = 0;
    end
    b(ns1*(k-1)+k+nt) = 0;
    A(ns1*(k-1)+k+nt,ns1*(k-1)+k+nt) = 1;
end
%0 en x=1
for k = 1:ns2+1
    for i = 1:3*nt
        A(ns1*k+k+nt,i) = 0;
    end
    b(ns1*k+k+nt) = 0;
    A(ns1*k+k+nt,ns1*k+k+nt) = 1;
end
%0 en y=1
for k = 1:ns1+1
    for i = 1:3*nt
        A(ns1*ns2+ns2+k+nt,i) = 0;
    end
    b(ns1*ns2+ns2+k+nt) = 0;
    A(ns1*ns2+ns2+k+nt,ns1*ns2+ns2+k+nt) = 1;
end

%Resolver el sistema de ecuaciones
disp('Solving the linear system...')
sol = A\b;

%Separar la soluci n
X = reshape(coord(:,1), ns1+1, ns2+1);
ux = reshape(sol(1:nt), ns1+1, ns2+1);
Y = reshape(coord(:,2), ns1+1, ns2+1);
uy = reshape(sol(nt+1:2*nt), ns1+1, ns2+1);
Z = reshape(sol(2*nt+2*nel+1:3*nt+2*nel), ns1+1, ns2+1);

%Grafica de la presion
surf(X,Y,-Z/500)
xlim([-0.1 1.1])
ylim([-0.1 1.1])
xlabel('x', 'FontSize', 30)
ylabel('y', 'FontSize', 30)
zlabel('P', 'FontSize', 30)
ax=gca;
ax.FontSize = 30;
pause

%Grafica de la velocidad
h =quiver(X,Y,ux,uy);
xlim([-0.1 1.1])
ylim([-0.1 1.1])
xlabel('x', 'FontSize', 30)
ylabel('y', 'FontSize', 30)
ax=gca;
ax.FontSize = 30;
```

```
axis square
shading interp
```

$P^1 - P^0$ non-conforming element:

```
function NCStokesSolver()
[p,e,t]=initmesh('squareg'); %Dominio cuadrado [-1,1]^2
figure(1)
set(gcf,'DefaultLineLineWidth',0.2);
pdemesh(p,e,t,'EdgeColor','blue')
xlim([-1 1])
ylim([-1 1])
xlabel('x')
ylabel('y')

t2e=Tri2Edge(p,t); %Lados de los triangulos
nt=size(t,2); %Numero de triangulos
ne=max(t2e(:)); %Numero de lados
[A11,B1,B2,areas]=NCAssembler(p,t2e,t); %Valor de las integrales
nu=0.1; %Viscosidad
LHS=[nu*A11 sparse(ne,ne) B1';
    sparse(ne,ne) nu*A11 B2';
    B1 B2 sparse(nt,nt)]; %Matriz de rigidez
rhs=zeros(2*ne+nt,1); %Vector de carga
last=[zeros(2*ne,1); areas]; %Ultima fila y ultima columna
LHS=[LHS last; last' 0];
rhs=[rhs; 0];

[xmid,ymid,edges] = EdgeMidPoints(p,t2e,t);
fixed=[]; %Nodos internos
gvals=[]; %Condiciones de contorno
for i=1:length(edges) %Bucle en los lados
    n=edges(i); %indice del lado
    x=xmid(i); %coordenada x del punto medio del lado
    y=ymid(i); %coordenada y del punto medio del lado
    if (x<-0.99 | x>0.99 | y<-0.99 | y>0.99) % boundary
        fixed=[fixed; n; n+ne];
        u=0; v=0; %valores en la frontera
        if (y>0.99), u=1; end % u=1,v=0 en [-1,1]x{1}
        gvals=[gvals; u; v];
    end
end

neq=2*ne+nt+1; %Numero de ecuaciones
free=setdiff([1:neq],fixed);
rhs=rhs(free)-LHS(free,fixed)*gvals; %Vector de nodos internos
LHS=LHS(free,free); %Matriz de nodos internos
sol=zeros(neq,1);
sol(fixed)=gvals; %Insertar condiciones de contorno
sol(free)=LHS\rhs; %Resolver el sistema
U=sol(1:ne); V=sol(1+ne:2*ne); P=sol(2*ne+1:2*ne+nt);
figure(2), pdesurf(p,t,P') %Grafica de la presi n
xlim([-1 1])
ylim([-1 1])
xlabel('x', 'FontSize', 30)
ylabel('y', 'FontSize', 30)
```

```
zlabel('P', 'FontSize', 30)
ax=gca;
ax.FontSize = 30;
figure(3), quiver(xmid,ymid,U',V') %Grafica de la velocidad
xlim([-1.1 1.1])
ylim([-1.1 1.1])
xlabel('x', 'FontSize', 30)
ylabel('y', 'FontSize', 30)
ax=gca;
ax.FontSize = 30;

function [A11,B1,B2,areas] = NCAssembler(p,t2e,t)
nt=size(t,2);
ne=max(t2e(:));
A11=sparse(ne,ne);
B1=sparse(nt,ne);
B2=sparse(nt,ne);
areas=zeros(nt,1);
for i=1:nt
    vertex=t(1:3,i);
    x=p(1,vertex);
    y=p(2,vertex);
    [area,Sx,Sy]=CRGradients(x,y);
    edges=t2e(i,:);
    %Integrales numericas
    A11(edges,edges)=A11(edges,edges)+(Sx*Sx'+Sy*Sy')*area;
    B1(i,edges)=-Sx'*area;
    B2(i,edges)=-Sy'*area;
    areas(i)=area;
end

%Gradiente y area de las funciones Crouzeix-Raviart
function [area,Sx,Sy] = CRGradients(x,y)
[area,b,c]=HatGradients(x,y);
Sx=[-b(1)+b(2)+b(3); b(1)-b(2)+b(3); b(1)+b(2)-b(3)];
Sy=[-c(1)+c(2)+c(3); c(1)-c(2)+c(3); c(1)+c(2)-c(3)];

%Gradiente y area de las funciones sombrero
function [area,b,c] = HatGradients(x,y)
area = polyarea(x,y);
b = [y(2)-y(3); y(3)-y(1); y(1)-y(2)]/2/area;
c = [x(3)-x(2); x(1)-x(3); x(2)-x(1)]/2/area;

function [xmid,ymid,e] = EdgeMidPoints(p,t2e,t)
i=t(1,:); j=t(2,:); k=t(3,:); %Vertices del triangulo
t2e=t2e(:);
start=[j i i]; %Vertice inicial del lado
stop =[k k j]; %Vertice final del lado
xmid=(p(1,start)+p(1,stop))/2; %coordenada x del punto medio del lado
ymid=(p(2,start)+p(2,stop))/2; %coordenada y del punto medio del lado
[e,idx]=unique(t2e);
xmid=xmid(idx);
ymid=ymid(idx);

function edges = Tri2Edge(p,t)
```

```
np=size(p,2); %Numero de vertices
nt=size(t,2); %Numero de triangulos
i=t(1,:); % i=1st vertice de todos los elementos
j=t(2,:); % j=2nd
k=t(3,:); % k=3rd
A=sparse(j,k,-1,np,np); % 1st lado entre (j,k)
A=A+sparse(i,k,-1,np,np); % 2nd lado entre (i,k)
A=A+sparse(i,j,-1,np,np); % 3rd lado entre (i,j)
A=-((A+A.')<0);
A=triu(A); %Submatriz triangular superior de A
[r,c,v]=find(A); %fila, columna y valor (=-1)
v=[1:length(v)]; %Renombrar lados
A=sparse(r,c,v,np,np); %Reconstruir A
A=A+A'; %Hacer A simetrica
edges=zeros(nt,3);
for k=1:nt
    edges(k,:)=[A(t(2,k),t(3,k))
                A(t(1,k),t(3,k))
                A(t(1,k),t(2,k))]';

end
```

Matrices:

```
function [mxx,myy,mxy,mx,my] = matriz2
mxx = [0.5, -0.5, 0.;
       -0.5, 0.5, 0.;
       0., 0., 0.];
myy = [0.5, 0., -0.5;
       0., 0., 0.;
       -0.5, 0., 0.5];
mxy = [0.5, 0., -0.5;
      -0.5, 0., 0.5;
      0., 0., 0.];
mx = [-1/6, -1/6, -1/6;
      1/6, 1/6, 1/6;
      0, 0, 0];
my = [-1/6, -1/6, -1/6;
      0, 0, 0;
      1/6, 1/6, 1/6];
return
```

Isoparametric mapping:

```
function [c, J, deter]=InvDet(v)
%Calcula c=B_k^{-1}(B_k^{-1})^T y deter=abs(detB_k)
M=[v(2,1)-v(1,1),v(3,1)-v(1,1);v(2,2)-v(1,2),v(3,2)-v(1,2)]; %Matriz de la
    transformaci n afin

deter=det(M);
M=inv(M);
J=M';
deter=abs(deter);
c=M*M';
return
```

Mesh generation:

```
%Programa para generar mallas 2D.

function [globales,x,y,nel]=gen2(ns1,ns2)

nel  = ns1*ns2*2; %Numero total de elementos.
nt   = (ns1+1)*(ns2+1);
aux  = zeros(nt,1);

%En el siguiente bucle creamos la matriz de conectividad.
for j = 1:ns2
    ind = 1+(j-1)*(ns1+1);
    for i = 1:2:2*ns1-1
        elem = i + (j-1)*2*ns1;
        globales(elem,1:3)= [ind,ind+1,ind+ns1+1];
        ind = ind+1;
    end
    ind = ns1+(j-1)*(ns1+1)+3;
    for i = 2:2:2*ns1
        elem = i + (j-1)*2*ns1;
        globales(elem,1:3)= [ind,ind-1,ind-ns1-1];
        ind = ind+1;
    end
end

%En el siguiente bucle definimos las coordenadas de los nodos de la malla.
x1 = 0;
x2 = 1;
y1 = 0;
y2 = 1;
h1 = (x2-x1)/ns1;
h2 = (y2-y1)/ns2;

for j = 1:ns2+1
    for i = 1:ns1+1
        nodo = (j-1)*(ns1+1) + i;
        x(nodo) = h1*(i-1) + x1;
        y(nodo) = h2*(j-1) + y1;
    end
end

%Dibujamos la malla de triangulos.
for i=1:2:nel
    nodos = globales(i,:);
    hold on
    fill(x(nodos),y(nodos),'w')
    x1 = int2str(i);
    x2 = int2str(nodos(1));
    text(x(nodos(1))+ 0.35*h1,y(nodos(1))+0.25*h2,x1,'Color',[1 0 0]);
    text(x(nodos(1))+0.1*h1,y(nodos(1))+0.1*h2,x2,'Color',[0 0 0]);
end
for i=2:2:nel
    nodos = globales(i,:);
    hold on
    fill(x(nodos),y(nodos),'w')
    x1 = int2str(i);
```

```
    text(x(nodos(1))- 0.25*h1,y(nodos(1))-0.25*h2,x1,'Color',[1 0 0]);
end
pause
hold off

return
```