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Optimal properties related to Total Positivity and Wronskian matrices

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B-bases are totally positive bases satisfying several optimal properties. In this paper we prove a new optimal property of B-bases related to the conditioning of their Wronskian matrices.

1. Introduction

Total Positivity is an interdisciplinary subject that includes the study of totally positive matrices. Totally positive matrices, which are also called totally nonnegative in the literature, arise in many fields, such as approximation theory, computer aided geometric

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design (CAGD), mechanics, differential or integral equations, statistics, combinatorics, economics and biology (see [1], [7], [8], [9], [11] or [16]). An important source of examples of totally positive matrices comes from the collocation matrices of systems of functions. A space of univariate functions with a basis whose collocation matrices are totally positive possesses a special totally positive basis called B-basis (see [3]), which generates all totally positive bases of the space by means of totally positive matrices (see Section 2).

In CAGD, normalized B-bases are the bases with optimal shape preserving properties. Other optimal properties of B-bases related to their collocation matrices can be seen in [4], [6] and [5]. This paper contributes to this field with a new optimal property of B-bases related in this case with the least conditioning of their Wronskian matrices. In fact, Wronskian matrices provide another source of totally positive matrices (cf. [12,13,15]). The layout of the paper is as follows. In Section 2 we introduce basic notations and auxiliary results related to B-bases, Wronskian matrices and Skeel condition numbers. In section 3, we present the main result of the paper.

2. Basic notations and auxiliary results

We start this section by introducing some notations and basic definitions. A matrix is *totally positive* (TP) if all its minors are nonnegative (see [1], [16]). A system of functions $(u_0, \dots, u_n)$ defined on the real subset I is TP if all its collocation matrices $(u_{j-1}(t_i))_{i,j=1,\dots,n+1}$, $t_1 < \cdots < t_{n+1}$ arbitrary in I are TP. A TP system of functions on I is *normalized* (NTP) if $\sum_{i=0}^n u_i(t) = 1$, for all $t \in I$. NTP bases are commonly used in computer-aided design due to their shape preserving properties (see [2]).

B-bases are TP bases that generate all TP bases of a space by means of TP matrices. The following characterization of a B-basis is a consequence of Corollary 3.10 of [3] and Proposition 3.11 of [3].

Theorem 1. *Let $(u_0, \dots, u_n)$ be a TP basis of a space U of univariate functions on I. Then $(u_0, \dots, u_n)$ is a B-basis if and only if for any other TP basis $(v_0, \dots, v_n)$ of U the matrix K of change of basis such that $(v_0, \dots, v_n) = (u_0, \dots, u_n)K$ is TP.*

The existence of a B-basis in a space with a TP basis was proved in [3] (see Remark 3.8 of [3]). Among all NTP bases of a space, we can find a unique normalized B-basis, which is the basis with optimal shape preserving properties (cf. [3]). For instance, the Bernstein bases and the B-spline bases are the normalized B-bases of their corresponding spaces. As an example of a B-basis that is not normalized, we can mention the monomial basis of the space of polynomials of degree at most n on $I = [0, \infty)$ (cf. [2]).

As usual, given an n-times continuously differentiable function f and x in its parameter domain, $f'(x)$ denotes the first derivative of f at x and, for any $i \leq n$, $f^{(i)}(x)$ denotes the i-th derivative of f at x. Let us recall that for a given basis $(u_0, \dots, u_n)$ of a space of n-times continuously differentiable functions, defined on a real interval I and $x \in I$, the *Wronskian matrix* at x is defined by

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$$W(u_0, \ldots, u_n)(x) := (u_{j-1}^{(i-1)}(x))_{i,j=1,\ldots,n+1}.$$

A matrix $A = (a_{ij})_{0 \le i,j \le n}$ is called *row-stochastic* if all its entries are nonnegative and has row sums 1: $a_{ij} \ge 0$ for all i, j and, for all $i = 0, \ldots, n$, $\sum_{j=0}^n a_{ij} = 1$, that is, $Ae = e$ for $e := (1, \ldots, 1)^T$. If $A = (a_{ij})_{0 \leqslant i,j \leqslant n}$, we denote by $|A|$ the matrix whose (i, j)-entry is $|a_{ij}|$. Given a nonsingular matrix A, the condition number of A, denoted by $\kappa_\infty(A)$, is given by

$$\kappa_\infty(A) := \|A\|_\infty \, \|A^{-1}\|_\infty,$$

where

$$\|A\|_\infty := \max_i \sum_{j=1}^n |a_{ij}|.$$

We shall also use the following condition numbers for a nonsingular matrix A and a *vector* f:

$$\mathrm{Cond}(A, f) := \frac{\| \, |A^{-1}| \, |A| \, |f| \|_\infty}{||f||_\infty},$$

$$\mathrm{Cond}(A) := \sup_{f \neq 0} \mathrm{Cond}(A, f) = \mathrm{Cond}(A, e) = \| \, |A^{-1}| \, |A| \, \|_\infty. \tag{1}$$

The numbers $\mathrm{Cond}(A, f)$ and $\mathrm{Cond}(A)$ were introduced by Skeel in 1979, see [17], and measure effects of perturbations of the data in linear systems $Af = c$.

We can observe the following two properties:

- $\mathrm{Cond}(A) \leqslant \kappa_\infty(A)$ and it can be much smaller, and,
- in contrast to $\kappa_\infty(A)$, $\mathrm{Cond}(A)$ is invariant under row scaling: if D is a nonsingular diagonal matrix then

$$\mathrm{Cond}(DA) = \mathrm{Cond}(A).$$

These two properties provide some of the reasons that explain why the Skeel condition number $\mathrm{Cond}(A)$ is more adequate than the traditional condition number $\kappa_\infty(A)$ (cf. also Section 7.2 of [10]) because it provides sharper bounds and because the geometry of the linear system is preserved.

3. Main result

In previous sections, we have recalled some optimal properties of B-bases. This section will provide new optimal properties. In particular, a property about better conditioning

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of their Wronskian matrices with respect to Wronskian matrices of other TP bases of their space.

There are B-bases such that all their Wronskian matrices are TP. An example is provided by the monomial basis of degree not greater than n on $[0, \infty]$ (cf. [12]). In fact, in Theorem 6.3 of [2] it was proved that, for a basis of the space of polynomials of degree at most n, a basis is TP on $[a, \infty]$ if and only if its Wronskian matrices are TP for all $t \geq a$. The fact that all Wronskian matrices are TP does not hold for other TP bases (see Example 6.4 of [2]) or even for B-bases, such as the Bernstein basis (cf. [14]).

The following result proves that, if a nonsingular Wronskian matrix of a B-basis is nonnegative or even TP, these properties are inherited by the corresponding Wronskian matrices of the remaining TP bases of the space. The final part of the following result shows that, when they are nonnegative, the transposes of the Wronskian matrices of a B-basis are the best conditioned for the Skeel condition number (1) among all the transposes of the corresponding Wronskian matrices of TP bases.

Theorem 2. *Let $(b_0, \ldots, b_n)$ be a B-basis of a space of functions U on a real subset I and $(u_0, \ldots, u_n)$ be any TP basis of U. If a Wronskian matrix $W(b_0, \ldots, b_n)(x)$ is nonsingular and nonnegative (respectively, TP) for $x \in I$, then the corresponding Wronskian matrix $W(u_0, \ldots, u_n)(x)$ is also nonsingular and nonnegative (respectively, TP) and*

$$\mathrm{Cond}((W(b_0, \ldots, b_n)(x))^T) \leqslant \mathrm{Cond}((W(u_0, \ldots, u_n)(x))^T). \tag{2}$$

Proof. Since $(b_0, \ldots, b_n)$ is a B-basis of U and $(u_0, \ldots, u_n)$ is a TP basis of U, we can write by Theorem 1

$$(u_0, \ldots, u_n) = (b_0, \ldots, b_n)K,$$

where K is a nonsingular TP matrix. Then we can derive the following relationship for the Wronskians at x:

$$W(u_0, \ldots, u_n)(x) = W(b_0, \ldots, b_n)(x)K, \tag{3}$$

or, equivalently,

$$(W(u_0, \ldots, u_n)(x))^T = K^T(W(b_0, \ldots, b_n)(x))^T. \tag{4}$$

If $W(b_0, \ldots, b_n)(x)$ is nonsingular and nonnegative, then by (3) and the fact that K is also nonsingular and nonnegative, we conclude that the Wronskian matrix $W(u_0, \ldots, u_n)(x)$ is again nonsingular and nonnegative. If $W(b_0, \ldots, b_n)(x)$ is TP, since K is also TP and the product of TP matrices is also as TP matrix (cf. Theorem 3.1 of [1]), we deduce that $W(u_0, \ldots, u_n)(x)$ is also TP. Thus, it remains to prove (2) when $W(b_0, \ldots, b_n)(x)$ is nonsingular and nonnegative.

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Since $(W(b_0,\dots,b_n)(x))^T =: (b_{ij})_{1\le i,j\le n+1}$ is nonsingular and nonnegative, then by the first part of the proof $(W(u_0,\dots,u_n)(x))^T =: (u_{ij})_{0\le i,j\le n}$ is also nonsingular and nonnegative and so both matrices are nonnegative matrices with nonzero row sums. Then let us define the diagonal matrices with positive diagonal entries given by:

$$D_1 := \operatorname{diag}\left(\sum_{j=0}^n u_{0j}, \sum_{j=0}^n u_{2j}, \dots, \sum_{j=0}^n u_{nj}\right)$$

and

$$D_2 := \operatorname{diag}\left(\sum_{j=0}^n b_{0j}, \sum_{j=0}^n b_{1j}, \dots, \sum_{j=0}^n b_{nj}\right)$$

and so the following matrices

$$W_u := D_1^{-1}(W(u_0,\dots,u_n)(x))^T, \quad W_b := D_2^{-1}(W(b_0,\dots,b_n)(x))^T \tag{5}$$

are row-stochastic matrices. Then, by (4) and (5), we can derive

$$W_u = D_1^{-1}K^T(W(b_0,\dots,b_n)(x))^T = (D_1^{-1}K^TD_2)W_b. \tag{6}$$

Now we define the matrix

$$M := D_1^{-1}K^TD_2, \tag{7}$$

which is nonnegative and satisfies by (5)

$$W_u = MW_b. \tag{8}$$

Since the matrices W_b and W_u are row-stochastic, we deduce from (8) and (7) that

$$Me = MW_be = W_ue = e.$$

Therefore, the matrix M is also row-stochastic.

In conclusion, matrices M, W_b and W_u are row-stochastic, and so

$$\|W_b\|_\infty = \|W_u\|_\infty = \|M\|_\infty = 1. \tag{9}$$

Since the Skeel condition number is invariant under row scaling and taking into account that $|W_b|e = W_be = e$ because W_b is a row-stochastic matrix, we deduce from (5) and (9) that

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$$\mathrm{Cond}(W(b_0,\ldots,b_n)(x))^T) = \mathrm{Cond}(W_b) = \||W_b^{-1}|\,|W_b|\|_\infty \tag{10}$$

$$= \||W_b^{-1}|\,|W_b|e\|_\infty = \|W_b^{-1}\|_\infty = \kappa_\infty(W_b). \tag{11}$$

In a similar way, we can deduce that:

$$\mathrm{Cond}((W(u_0,\ldots,u_n)(x))^T) = \kappa_\infty(W_u). \tag{12}$$

By (10), (11), (9), (8) and (12), we can deduce that

$$\mathrm{Cond}((W(b_0,\ldots,b_n)(x))^T) = \kappa_\infty(W_b) = \|W_b^{-1}\|_\infty = \|\bar{W}_u^{-1}M\|_\infty$$
$$\leqslant \|W_u^{-1}\|_\infty = \kappa_\infty(W_u) = \mathrm{Cond}((W(u_0,\ldots,u_n)(x))^T),$$

and the proof is finished. □

Declaration of competing interest

There is no competing interest.

Data availability

No data was used for the research described in the article.

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