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ORIGINAL ARTICLE

Linearization of holomorphic Lipschitz functions

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Abstract

Let X and Y be complex Banach spaces with B_X denoting the open unit ball of X. This paper studies various aspects of the *holomorphic Lipschitz space* $\mathcal{H}L_0(B_X,Y)$, endowed with the Lipschitz norm. This space consists of the functions in the intersection of the sets $\mathrm{Lip}_0(B_X,Y)$ of Lipschitz mappings and $\mathcal{H}^\infty(B_X,Y)$ of bounded holomorphic mappings, from B_X to Y. Thanks to the Dixmier–Ng theorem, $\mathcal{H}L_0(B_X,\mathbb{C})$ is indeed a dual space, whose predual $\mathcal{G}_0(B_X)$ shares linearization properties with both the Lipschitz-free space and Dineen–Mujica predual of $\mathcal{H}^\infty(B_X)$. We explore the similarities and differences between these spaces, and combine techniques to study the properties of the space of holomorphic Lipschitz functions. In particular, we get that $\mathcal{G}_0(B_X)$ contains a 1-complemented subspace isometric to X and that $\mathcal{G}_0(X)$ has the (metric) approximation property whenever X has it. We also analyze when $\mathcal{G}_0(B_X)$ is a subspace of $\mathcal{G}_0(B_Y)$, and we obtain an analog of Godefroy's characterization of functionals with a unique norm preserving extension in the holomorphic Lipschitz context.

KEYWORDS
Banach space, holomorphic function, linearization, Lipschitz function, symmetric regularity

1 | INTRODUCTION

Linearizing nonlinear functions is a typical procedure in infinite-dimensional analysis. Originating nearly 70 years ago with Grothendieck [32] (and his research about linearization of bilinear mappings through the projective tensor product), the practice of identifying spaces of continuous *nonlinear* functions with spaces of continuous *linear* mappings defined on Banach spaces has proved to be a useful technique. Accordingly, the study of geometric and topological properties of these *linearizations* has increasingly attracted interest.

Lipschitz functions (defined on pointed metric spaces) and holomorphic bounded functions (defined on the open unit ball of a Banach space) are really different both as sets and as function spaces. However, when looking at their linearization processes several similarities emerge. The purpose of this paper is to study, in light of these resemblances, the new set of functions consisting of the intersection of the previous sets. Lipschitz holomorphic functions defined on the open unit ball of a Banach space taking the value 0 at 0 will be our focus of attention. In the exploration of this set, we take advantage of a result of Ng [41] concerning the existence of preduals and all the background about related linearization processes.

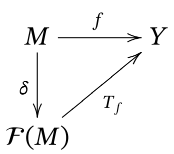
We begin with a brief review of important terms and concepts. General references for Lipschitz functions include [31, 46] and a standard reference for holomorphic functions on finite- or infinite-dimensional domains is [40]. The linearization process for bounded holomorphic functions was originally developed in [38]. A review of linearization procedures both for Lipschitz functions and for bounded holomorphic functions appeared in the recent survey [27], while a general approach to linearizing nonlinear sets of functions was settled in [18].

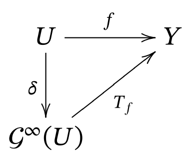
For a metric space (M, d) and a Banach space Y, let $\mathrm{Lip}(M, Y)$ be the vector space of all $f : M \to Y$ such that $\|f(x_1) - f(x_2)\| \leq Cd(x_1, x_2)$ for some $C > 0$ and for all $x_1 \neq x_2 \in M$. The smallest C in the above definition is the *Lipschitz constant* of f, $L(f)$. Let $0 \in M$ denote an arbitrary fixed point. In order to get a normed space, we will be particularly interested in the subspace $\mathrm{Lip}_0(M, Y)$ consisting of those $f \in \mathrm{Lip}(M, Y)$ such that $f(0) = 0$. In this way, $L(f) = 0$ if and only if $f = 0$, and so $\|\cdot\| = L(\cdot)$ defines a norm on $\mathrm{Lip}_0(M, Y)$.

For complex Banach spaces X and Y and open set $U \subset X$, denote by $\mathcal{H}^\infty(U, Y)$ the vector space of all $f : U \to Y$ such that f is holomorphic (i.e., complex Fréchet differentiable) and bounded on U, endowed with the supremum norm. In both the Lipschitz and $\mathcal{H}^\infty$ situations, if the range $Y = \mathbb{K}$, then the notation is shortened to $\mathrm{Lip}_0(M)$ and $\mathcal{H}^\infty(U)$.

It is known that $\mathrm{Lip}_0(M)$ and $\mathcal{H}^\infty(U)$ are dual spaces and that in some special situations, the predual is unique. The construction of a (or, in some cases, the) predual follows the same lines for both the Lipschitz and $\mathcal{H}^\infty$ situations: calling X one of Lip_0 or $\mathcal{H}^\infty$, we consider those functionals $\varphi \in X^*$ such that $\varphi|_{\overline{B}_X}$ is continuous when $\overline{B}_X$ is endowed with the compact-open topology. Among such functionals are the evaluations $f \rightsquigarrow \delta(x)(f) \equiv f(x)$ where x ranges over the domain of $f \in X$. In the case of $\mathrm{Lip}_0(M)$, the closed span of the set of such φ will be denoted as $\mathcal{F}(M)$, while the analogous closed span for $\mathcal{H}^\infty(U)$ is $\mathcal{G}^\infty(U)$. Each of these is a Banach space, being a closed subspace of $\mathrm{Lip}_0(M)^*$, and $\mathcal{H}^\infty(U)^*$, respectively. Using a standard technique developed by Ng [41], it follows that $\mathcal{F}(M)^* \equiv \mathrm{Lip}_0(M)$ and $\mathcal{G}^\infty(U)^* \equiv \mathcal{H}^\infty(U)$.

Among the most important common features of Lip_0 and $\mathcal{H}^\infty$ is *linearization*. In each of the two cases below, δ is the evaluation inclusion taking $x \rightsquigarrow \delta(x)$. Also, for f in either $\mathrm{Lip}_0(M, Y)$ or $\mathcal{H}^\infty(U, Y)$, T_f is the unique linear mapping making the diagram commute. Moreover, $\|f\| = \|T_f\|$.





1.1 | Notation

X, Y will stand for complex Banach spaces. We denote by B_X (respectively S_X) its open unit ball (respectively unit sphere). $\mathcal{L}(X, Y)$ denotes the space of continuous linear maps from X to Y, and $X^* = \mathcal{L}(X, \mathbb{C})$. $\mathcal{P}(^m X, Y)$ stands for the space of continuous m-homogeneous polynomials, that is, those $P : X \to Y$ so that there exists a continuous m-linear symmetric map $\check{P} : X \times \cdots \times X \to Y$ with $P(x) = \check{P}(x, \dots, x)$. We also write $\mathcal{P}(^m X) = \mathcal{P}(^m X, \mathbb{C})$. We say that $P \in \mathcal{P}(^m X, Y)$ is of *finite type* if $P(x) = \sum_{j=1}^n [x_j^*(x)]^m y_j$ for certain $x_j^* \in X^*$ and $y_j \in Y$. $\mathcal{P}_f(^m X, Y)$ stands for the space of finite type m-homogeneous polynomials. Moreover, we set $\mathcal{P}(X, Y)$ (resp. $\mathcal{P}_f(X, Y)$) to be the space of finite sums of continuous homogeneous polynomials (resp. homogeneous polynomials of finite type) from X to Y. Also, $\mathbb{D}(z,r)$ (resp. $C(z,r)$) denotes the open disc (resp. the circumference) in $\mathbb{C}$ centered at z with radius r, in particular $\mathbb{D} = \mathbb{D}(0,1)$.

Recall that X is said to have the bounded approximation property (BAP) if there is $\lambda > 0$ such that the identity $I : X \to X$ can be approximated by finite-rank operators in $\lambda B_{\mathcal{L}(X,X)}$ uniformly on compact sets (equivalently, pointwise). If $\lambda = 1$, then X is said to have the metric approximation property (MAP). If X has λ-BAP and Y is λ'-complemented in X, then Y has $\lambda\lambda'$-BAP. Recall, also, the version of this notion without control of the norms: X has the approximation property (AP) if the identity $I : X \to X$ can be approximated by finite-rank operators in $\mathcal{L}(X,X)$ uniformly on compact sets. We refer the reader to [19] for examples and applications.

1.2 | Organization of the paper

Section 2 introduces the main space of interest, $\mathcal{H}L_0(B_X,Y)$, consisting of those functions that are in both $\mathrm{Lip}_0(B_X,Y)$ and $\mathcal{H}^\infty(B_X,Y)$, endowed with the Lipschitz norm. A number of properties of $\mathcal{H}L_0(B_X,Y)$ are discussed and it is proved that this space really differs from $\mathrm{Lip}_0(B_X,Y)$ and $\mathcal{H}^\infty(B_X,Y)$ (in the sense that a nonseparable space can be injected between them). Then, we focus on the predual $\mathcal{G}_0(B_X)$ of $\mathcal{H}L_0(B_X)$ (where $Y = \mathbb{C}$). Specifically, we will see that $\mathcal{H}L_0(B_X)$ has a canonical predual whose properties echo those of $\mathcal{H}^\infty(B_X)$ and $\mathrm{Lip}_0(B_X)$. In Section 3, we deal with the (metric) approximation property (AP) for $\mathcal{G}_0(B_X)$, again inspired by the results for $\mathcal{G}^\infty(B_X)$. The next two sections involve a closer inspection of $\mathcal{G}_0(B_X)$ and its relationship with $\mathcal{G}_0(B_{X^{**}})$. Section 4 begins by considering the interaction between $\mathcal{G}_0(B_X)$ and $\mathcal{G}_0(B_Y)$ when $X \subset Y$ and then focuses on the case of $X \subset X^{**}$. Section 5 studies a natural connection between $\mathcal{G}_0(B_{X^{**}})$ and $\mathcal{G}_0(B_X)^{**}$ under the hypothesis of X^{**} having the MAP. Among other things, this enables us to characterize, under natural conditions on X and X^{**}, when a function $f \in \mathcal{H}L_0(B_X)$ has a unique norm preserving extension to $\mathcal{H}L_0(B_{X^{**}})$. Both Sections 4 and 5 make use of the concept of *(Arens) symmetric regularity*, which is reviewed in Section 4. The final section is the Appendix which provides an alternative argument for the main result in Section 2.

2 | THE SPACE OF HOLOMORPHIC LIPSCHITZ FUNCTIONS AND ITS PREDUAL

In the case that the metric space M is B_X, the open unit ball of a complex Banach space X, and Y is another complex Banach space, $\mathrm{Lip}_0(B_X,Y)$ is the space of Lipschitz functions $f : B_X \to Y$ with $f(0) = 0$ and:

$$L(f) = \sup\left\{ \frac{\|f(x) - f(y)\|}{\|x - y\|} \ : \ x \neq y \in B_X \right\}.$$

It is well known that $L(\cdot)$ defines a norm on $\mathrm{Lip}_0(B_X,Y)$ and $(\mathrm{Lip}_0(B_X,Y), L(\cdot))$ is a Banach space. Indeed, $\mathrm{Lip}_0(B_X,Y)$ is isometrically isomorphic to the space of operators $\mathcal{L}(\mathcal{F}(B_X),Y)$, where $\mathcal{F}(B_X)$ denotes the *Lipschitz-free space* over B_X (see, e.g., [29, 46], and [1] for the complex version).

Next, $\mathcal{H}^\infty(B_X,Y)$ stands for the space of bounded holomorphic functions from B_X to Y, which is a Banach space when endowed with the supremum norm. Analogous to the Lipschitz case above, we have that $\mathcal{H}^\infty(B_X,Y)$ is isometrically isomorphic to $\mathcal{L}(\mathcal{G}^\infty(B_X),Y)$, where $\mathcal{G}^\infty(B_X)$ is Mujica's canonical predual of $\mathcal{H}^\infty(B_X)$ [38] (we will review the space $\mathcal{G}^\infty(B_X)$ later in this section).

The parallel behavior of these Lipschitz and $\mathcal{H}^\infty$ spaces was the authors' motivation to introduce and study the following space and its canonical predual:

$$\mathcal{H}L_0(B_X,Y) = \{f \in \mathrm{Lip}_0(B_X,Y) \ : \ f \text{ is holomorphic on } B_X\}.$$

We will also denote $\mathcal{H}L_0(B_X) = \mathcal{H}L_0(B_X,\mathbb{C})$. Sometimes we will deal with holomorphic Lipschitz functions without assuming $f(0) = 0$, and then we use the notation $\mathcal{H}L(B_X,Y)$ and $\mathcal{H}L(B_X)$.

Since both normed spaces $\mathcal{H}^\infty(B_X,Y)$ and $\mathrm{Lip}_0(B_X,Y)$ are complete (with their respective norms) and each $f \in \mathcal{H}L_0(B_X,Y)$ satisfies $\|f\|_\infty \leq L(f)$ we easily derive that $\mathcal{H}L_0(B_X,Y)$ is a Banach space with norm $L(\cdot)$. Similar to Mujica's study we could define and study $\mathcal{H}L_0(U,Y)$ for any open set $U \ni 0$, but we have preferred to concentrate on, what is in our opinion, the most interesting case $U = B_X$.

If $f : B_X \to Y$ is a holomorphic function and $x \in B_X$ then $f(x+h) = \sum_{m=1}^{\infty} P_m(x)(h)$ for h in a suitable neighborhood of 0, where $P_m(x)$ is an m-homogeneous polynomial. Recall that the first differential df satisfies $df(x)(h) = P_1(x)(h)$ for every $h \in X$. Given $f \in \mathcal{H}^\infty(B_X, Y)$ such that $df \in \mathcal{H}^\infty(B_X, \mathcal{L}(X,Y))$ and $f(0) = 0$, by the mean value theorem, we have that $\|f(x) - f(y)\| \leq \|df\|\|x - y\|$ for any $x, y \in B_X$. Then, $f \in \mathrm{Lip}_0(B_X, Y)$ and $L(f) \leq \|df\|$. Conversely, if $f \in \mathcal{H}L_0(B_X, Y)$ we know that $df \in \mathcal{H}(B_X, \mathcal{L}(X,Y))$. Also, for $x, y \in B_X$,

$$\|df(x)(y)\| = \lim_{h\to 0} \left\| \frac{f(x+hy) - f(x)}{h} \right\| \leq L(f)\|y\| \leq L(f).$$

This means that df belongs to $\mathcal{H}^\infty(B_X, \mathcal{L}(X,Y))$ and $\|df\| \leq L(f)$.

This shows that there is another useful representation of our primary space of interest.

Proposition 2.1. $\mathcal{H}L_0(B_X, Y) = \{f \in \mathcal{H}^\infty(B_X, Y) : \; df \in \mathcal{H}^\infty(B_X, \mathcal{L}(X,Y)); \; f(0) = 0\}$. *Moreover, for every* $f \in \mathcal{H}L_0(B_X, Y)$, $L(f) = \|df\|$ *; that is,* $L(f) = \sup_{x\in B_X} \|df(x)\|$.

Note that $P|_{B_X} \in \mathcal{H}L_0(B_X, Y)$ for every $P \in \mathcal{P}(X,Y)$ such that $P(0) = 0$, a fact that will be useful later.

When $Y = \mathbb{C}$, we can define a mapping

$$\Phi : \mathcal{H}L_0(B_X) \to \mathcal{H}^\infty(B_X, X^*)$$

$$f \mapsto df$$

In general, Φ is an isometry *into* $\mathcal{H}^\infty(B_X, X^*)$, although if X also equals $\mathbb{C}$, then Φ is *onto*. Indeed, in the one-dimensional case, Φ is surjective since every holomorphic function f on $\mathbb{D}$ has a primitive that is Lipschitz whenever f is bounded. However, Φ is not surjective for $X \neq \mathbb{C}$. To see this, given $P \in \mathcal{P}(^2X)$, we have that $P|_{B_X} \in \mathcal{H}L_0(B_X)$ and $dP \in \mathcal{L}(X, X^*)$ is symmetric (i.e., $dP(x)(y) = dP(y)(x)$ for every $x, y \in X$). Note that df is linear only when f is a 2-homogeneous polynomial. Hence, a non-symmetric element of $\mathcal{L}(X, X^*)$ (which always exists whenever the dimension of X is strictly bigger than one) cannot be in the range of Φ.

In particular, we see that

$$\mathcal{H}L_0(\mathbb{D}) = \{f \in \mathcal{H}^\infty(\mathbb{D}) : \; f(0) = 0 \text{ and } f' \in \mathcal{H}^\infty(\mathbb{D})\}.$$

A lot of research has been done on $\mathcal{H}L_0(\mathbb{D})$ and on $\mathcal{H}L_0(U)$ for certain domains $U \subset \mathbb{C}^n$ such as the Euclidean ball. See, for example, [2, 11–13, 15, 26, 42, 44], where this topic is approached from different viewpoints than what is done here.

Note that there are plenty of examples of non-Lipschitz functions in $\mathcal{H}^\infty(\mathbb{D})$. For instance, given a sequence $(b_n) \subset \mathbb{C} \setminus \{1\}$ with $|b_n| = 1$ and $b_n \to 1$, define $f : \{b_n\} \cup \{1\} \to \mathbb{C}$ by $f(1) = 0$ and $f(b_n) = \sqrt{|b_n - 1|}$. Then, the Rudin–Carleson theorem provides an extension of f which lies in the disc algebra $\mathcal{A}(\mathbb{D})$ (i.e., the space of uniformly continuous functions in $\mathcal{H}^\infty(\mathbb{D})$) and has the same supremum norm, but it is not Lipschitz.

Our next goal is to show that $\mathcal{H}L_0(B_X)$ is indeed much smaller than both $\mathcal{H}^\infty(B_X)$ and $\mathrm{Lip}_0(B_X)$. More precisely, we will prove the following result, where we denote $\mathcal{H}_0^\infty(B_X) = \{f \in \mathcal{H}^\infty(B_X) : \; f(0) = 0\}$.

Theorem 2.2. *Let X be a non-null complex Banach space. Then*

(a) ℓ_∞ *is isomorphic to a subspace of* $\mathcal{H}_0^\infty(B_X) \setminus \mathcal{H}L_0(B_X) \cup \{0\}$.
(b) ℓ_∞ *is isomorphic to a subspace of* $\mathrm{Lip}_0(B_X) \setminus \mathcal{H}L_0(B_X) \cup \{0\}$.

We will provide a different proof of Theorem 2.2 *(a)* in the Appendix. There, we build an isomorphism into its image $F : \ell_\infty \longrightarrow \mathcal{H}^\infty(B_X)$ such that, additionally, its restriction to c_0 satisfies that $F|_{c_0} : c_0 \longrightarrow \mathcal{A}_u(B_X)$, the Banach algebra of *uniformly continuous* holomorphic functions $B_X \to \mathbb{C}$.

Proof.

(a) For the case $X = \mathbb{C}$, by a classical result, given (z_j) an interpolating sequence in $\mathbb{D}$ there exists a topological into isomorphism $S : \ell_\infty \longrightarrow \mathcal{H}^\infty(\mathbb{D})$ such that $S(c)(z_j) = c_j$ for every j and every $c = (c_n) \in \ell_\infty$ We can also get that $S(c)(0) = 0$.

For this, see, for example, [28, Theorem VII.2.1 and applications, p. 285], where the assertion is made for the upper half-plane $\mathbb{H}$. This can be translated to our case by considering the Cayley transform $\Phi : \mathbb{H} \to \mathbb{D}$, $\Phi(z) = \frac{z-i}{z+i}$. As Φ is a biholomorphic mapping, its associated composition operator $C_\Phi : \mathcal{H}^\infty(\mathbb{H}) \to \mathcal{H}^\infty(\mathbb{D})$ is an isometric isomorphism onto. Moreover, a sequence (z_j) in $\mathbb{D}$ is interpolating if and only if the corresponding sequence $(\Phi^{-1}(z_j))$ in $\mathbb{H}$ is interpolating. Thus, if $T : \ell_\infty \longrightarrow \mathcal{H}^\infty(\mathbb{H})$ is the isomorphism into such that for every $c = (c_n) \in \ell_\infty$, $T(c)(\Phi^{-1}(z_j)) = c_j$ for every j, then $S = T \circ C_\Phi$ satisfies our claim.

Now, let $z_j \to 1$ and partition $\mathbb{N}$ into infinitely many disjoint infinite sequences $\mathbb{N}_i = (n_{i,k})_k$. For $c \in \ell_\infty$ define $x_c \in \ell_\infty$ by $x_c(n_{i,k}) = (-1)^i c_k$, and let $Y = \{S(x_c) : c \in \ell_\infty\}$. Then, Y is a subspace of $\mathcal{H}^\infty(\mathbb{D})$ isomorphic to ℓ_∞. Given $c \neq 0$, we have $c_k \neq 0$ for some k, so

$$|S(x_c)(z_{n_{2i,k}}) - S(x_c)(z_{n_{2i+1,k}})| = |c_k - (-c_k)| = 2|c_k|$$

for every i, while $z_j \to 1$. Thus, $S(x_c)$ cannot be uniformly continuous, and hence it is not Lipschitz.

For the general case, we fix $x_0 \in S_X$ and consider $x^* \in X^*$ such that $x^*(x_0) = 1 = \|x^*\|$. We define

$$\Psi : \mathcal{H}^\infty(\mathbb{D}) \longrightarrow \mathcal{H}^\infty(B_X)$$

by $\Psi(f) = f \circ x^*$. Clearly, Ψ is a well-defined linear mapping and since $x^*(B_X) = \mathbb{D}$ we have that Ψ is an isometry onto its image. Moreover, considering its restriction we have that

$$\Psi : \mathcal{H}L(\mathbb{D}) \longrightarrow \mathcal{H}L(B_X)$$

is again an isometry, now with the Lipschitz norms. Indeed, if $f \in \mathcal{H}L(\mathbb{D})$ then

$$L(\Psi(f)) = L(f \circ x^*) \leq L(f)L(x^*) = L(f).$$

But if $\lambda, \mu \in \mathbb{D}$, then

$$\begin{aligned}|f(\lambda) - f(\mu)| =& |f \circ x^*(\lambda x_0) - f \circ x^*(\mu x_0)| = |\Psi(f)(\lambda x_0) - \Psi(f)(\mu x_0)| \\ \leq& L(\Psi(f))\|\lambda x_0 - \mu x_0\| = L(\Psi(f))|\lambda - \mu|,\end{aligned}$$

and we get $L(f) \leq L(\Psi(f))$. Finally, due to the injectivity of Ψ we have that

$$\Psi(\mathcal{H}_0^\infty(\mathbb{D}) \setminus \mathcal{H}L(\mathbb{D})) \subset \mathcal{H}_0^\infty(B_X) \setminus \mathcal{H}L(B_X).$$

Now, the claim follows.

(b) First, we consider the one-dimensional case $X = \mathbb{C}$. Let $l : \mathbb{R} \to [0,1]$ be a C^1 function such that $l(x) = 0$ for $x \leq 1/2$, l is strictly increasing on $(1/2, 1)$, and $l(x) = 1$ for $x \geq 1$. Define $f : \mathbb{D} \to [0,1]$ as $f(z) = l(|z|)$. Note that $L(f) \leq L(l)$ so $f \in \mathrm{Lip}_0(\mathbb{D})$. Now, we define $T : \mathcal{H}L_0(\mathbb{D}) \to \mathrm{Lip}_0(\mathbb{D})$ as $T(g) = f \cdot g$. We claim that T is an isomorphism onto its image. Indeed, given $g \in \mathcal{H}L_0(\mathbb{D})$ and $z, u \in \mathbb{D}$,

$$|f(z)g(z) - f(u)g(u)| \leq |f(z) - f(u)||g(z)| + |f(u)||g(z) - g(u)| \leq 2L(f)L(g)|z - u|.$$

Therefore, T is a continuous linear mapping with $\|T\| \leq 2L(f)$.

We now check that T is bounded below. By continuity of l and l', given $0 < \varepsilon < 1$, there exists $0 < r < 1$ such that $L(f|_{\mathbb{D}\setminus r\mathbb{D}}) \leq \varepsilon$ and if $|z| \geq r$ then $f(z) > 1 - \varepsilon$. Thus, for $g \in \mathcal{H}L_0(\mathbb{D})$, we have $L(g) = \sup_{\mathbb{D}} |g'| = \sup_{\mathbb{D}\setminus r\mathbb{D}} |g'|$ by the maximum modulus theorem. So, we may find $z, u \in \mathbb{D}\setminus r\mathbb{D}$ with

$$|g(z) - g(u)| \geq (1 - \epsilon)L(g)|z - u|.$$

Then,

$$\begin{aligned}|f(z)g(z)-f(u)g(u)| &\geq |f(z)|\cdot|g(z)-g(u)|-|g(u)|\cdot|f(z)-f(u)|\\ &\geq (1-\varepsilon)^2L(g)|z-u|-L(g)|u|\cdot\varepsilon|z-u|\\ &\geq ((1-\varepsilon)^2-\varepsilon)L(g)|z-u|\end{aligned}$$

and we get $L(f\cdot g)\geq((1-\varepsilon)^2-\varepsilon)L(g)$, for every $0<\varepsilon<1$. As a consequence,

$$L(Tg)=L(f\cdot g)\geq L(g),$$

and T is bounded below. Moreover, $T(g)=f\cdot g$ is never holomorphic on $\mathbb{D}$ for any $g\in\mathcal{H}L_0(\mathbb{D})\setminus\{0\}$, since f vanishes on $\mathbb{D}(0,1/2)$, and $T(\mathcal{H}L_0(\mathbb{D}))$ is isomorphic to $\mathcal{H}L_0(\mathbb{D})$ which in turn is isometric to $\mathcal{H}^\infty(\mathbb{D})$ that has a subspace isomorphic to ℓ_∞.

The general case is a straightforward consequence of the above argument in the following natural way. Let X be a non-null complex Banach space and take $x^*\in S_{X^*}$. Defining $R:\mathrm{Lip}_0(\mathbb{D})\to\mathrm{Lip}_0(B_X)$ by $R(h)=h\circ x^*$, we have that R is an isometry into. Hence, $R\circ T:\mathcal{H}L_0(\mathbb{D})\to\mathrm{Lip}_0(B_X)$ is an isomorphism into its image and we get that ℓ_∞ is isomorphic to a subspace of $\mathrm{Lip}_0(B_X)$. But if $g\in\mathcal{H}L_0(\mathbb{D})\setminus\{0\}$, then $R\circ T(g)=(f\cdot g)x^*$ is not a Gateaux holomorphic function since its restriction to $\{zx : z\in\mathbb{D}\}$ is not holomorphic. We conclude that $\ell_\infty\setminus\{0\}\subset\mathrm{Lip}_0(B_X)\setminus\mathcal{H}L_0(B_X)$. □

In the rest of this section, we will focus our attention on the canonical predual of the space $\mathcal{H}L_0(B_X)$ and show that it shares many properties with the canonical preduals of $\mathcal{H}^\infty(B_X)$ and $\mathrm{Lip}_0(B_X)$.

Let us denote by τ_0 the compact-open topology on $\mathcal{H}L_0(B_X)$. An easy argument using Montel's theorem [22, Theorem 15.50] shows that $\overline{B}_{\mathcal{H}L_0(B_X)}$ is τ_0-compact. In fact, on this ball, convergence in the topology τ_0 coincides with pointwise convergence. Thus, the Dixmier–Ng theorem [41] says that $\mathcal{H}L_0(B_X)$ is a dual space with predual given by

$$\mathcal{G}_0(B_X):=\{\varphi\in\mathcal{H}L_0(B_X)^* : \varphi|_{\overline{B}_{\mathcal{H}L_0(B_X)}}\text{ is }\tau_0\text{-continuous}\}.$$

For $x\in B_X$ and $f\in\mathcal{H}L_0(B_X)$, denote $\delta(x)(f)=f(x)$. Clearly $\delta(x):\mathcal{H}L_0(B_X)\to\mathbb{C}$ is linear and continuous, meaning that $\delta(x)\in\mathcal{H}L_0(B_X)^*$. Also, $\delta(x)|_{\overline{B}_{\mathcal{H}L_0(B)}}$ is τ_0-continuous so $\delta(x)\in\mathcal{G}_0(B_X)$.

Proposition 2.3. *Let X be a complex Banach space.*

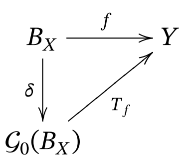
(a) The mapping

$$\begin{aligned}\delta : B_X&\to\mathcal{G}_0(B_X)\\ x&\mapsto\delta(x)\end{aligned}$$

is holomorphic and $\|\delta(x)-\delta(y)\|=\|x-y\|$ for every $x,y\in B_X$. In particular, $\delta\in\mathcal{H}L_0(B_X,\mathcal{G}_0(B_X))$ with $L(\delta)=1$.

(b) $\mathcal{G}_0(B_X)=\overline{\mathrm{span}}\{\delta(x) : x\in B_X\}$.

(c) For any complex Banach space Y and any $f\in\mathcal{H}L_0(B_X,Y)$, there is a unique operator $T_f\in\mathcal{L}(\mathcal{G}_0(B_X),Y)$ such that the following diagram commutes:



The map $f\mapsto T_f$ defines an isometric isomorphism from $\mathcal{H}L_0(B_X,Y)$ onto $\mathcal{L}(\mathcal{G}_0(B_X),Y)$. These properties characterize $\mathcal{G}_0(B_X)$ uniquely up to an isometric isomorphism.

(d) *A bounded net* $(f_\alpha) \subset \mathcal{H}L_0(B_X)$ *is weak-star convergent to a function* $f \in \mathcal{H}L_0(B_X)$ *if and only if* $f_\alpha(x) \to f(x)$ *for every* $x \in B_X$.

Proof.

(a) The map δ is weakly holomorphic since for any $f \in \mathcal{G}_0(B_X)^* = \mathcal{H}L_0(B_X)$ we have that $f \circ \delta = f$ is holomorphic. Thus, δ is holomorphic (see [40, Theorem 8.12]). Also, given $x, y \in B_X$, we have

$$\|\delta(x) - \delta(y)\| = \sup_{f \in B_{\mathcal{H}L_0(B_X)}} |\langle f, \delta(x) - \delta(y)\rangle| = \sup_{f \in B_{\mathcal{H}L_0(B_X)}} |f(x) - f(y)| \le \|x - y\|,$$

and equality holds since we may take $f = x^*|_{B_X}$ where $\|x^*\| = 1$ and $x^*(x - y) = \|x - y\|$.

(b) Just observe that for every $f \in \mathcal{H}L_0(B_X) = \mathcal{G}_0(B_X)^*$ we have that $f = 0$ whenever $f|_{\{\delta(x):x \in B_X\}} = 0$.

(c) First, note that an interpolation argument shows that the set $\{\delta(x) : x \in B_X \setminus \{0\}\}$ is linearly independent in $\mathcal{G}_0(B_X)$. Indeed, assume that $\sum_{j=1}^n \lambda_j \delta(x_j) = 0$ for different points $x_j \in B_X \setminus \{0\}$ and $\lambda_j \in \mathbb{C}$. Let $x_0 = 0$ and $\lambda_0 = 0$. Take $x_{ij}^* \in S_{X^*}$ with $x_{ij}^*(x_i - x_j) = \left\|x_i - x_j\right\|$ and define $f(x) = \sum_{j=0}^n \overline{\lambda_j} \prod_{i \ne j} \frac{x_{ij}^*(x_i - x)}{\|x_i - x_j\|}$. Then, $f \in \mathcal{H}L_0(B_X)$ and $0 = \langle f, \sum_{j=1}^n \lambda_j \delta(x_j)\rangle = \sum_{j=1}^n |\lambda_j|^2$.

Now, given $f \in \mathcal{H}L_0(B_X, Y)$, we define $T_f(\delta(x)) := f(x)$ for every $x \in B_X$ (this is the only possibility to get a commutative diagram) and extend it linearly to $\operatorname{span}\{\delta(x) : x \in B_X\}$. Note that, given $u = \sum_{j=1}^n \lambda_j \delta(x_j)$,

$$\left\|T_f u\right\| = \left\|\sum_{j=1}^n \lambda_j f(x_j)\right\| = \sup_{y^* \in B_{Y^*}} \left|\sum_{j=1}^n \lambda_j (y^* \circ f)(x_j)\right| = \sup_{y^* \in B_{Y^*}} |\langle u, y^* \circ f\rangle|$$
$$\le \sup\{L(y^* \circ f) : y^* \in B_Y\}\|u\| = L(f)\|u\|.$$

Thus, T_f extends uniquely to an operator $T_f \in \mathcal{L}(\mathcal{G}_0(B), Y)$ with $\left\|T_f\right\| \le L(f)$. Since $L(\delta) = 1$ and $f = T_f \circ \delta$, we get that $\left\|T_f\right\| = L(f)$.

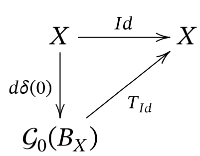
Moreover, the map $f \mapsto T_f$ is onto since given any $T \in L(\mathcal{G}_0(B_X), Y)$, we have that $f := T \circ \delta$ is a holomorphic Lipschitz map with $f(0) = 0$ and $T = T_f$.

The uniqueness of $\mathcal{G}_0(B_X)$ follows from the diagram property and the fact that $\left\|T_f\right\| = L(f)$.

(d) The ball $\overline{B}_{\mathcal{H}L_0(B_X)}$ is τ_0-compact and the weak-star topology is coarser than τ_0, so they coincide on $\overline{B}_{\mathcal{H}L_0(B_X)}$. □

Proposition 2.4. *For every complex Banach space* X*, we have that* X *is isometric to a 1-complemented subspace of* $\mathcal{G}_0(B_X)$.

Proof. In the particular case of $f = \mathrm{Id} : B_X \to X$, differentiating the diagram in Proposition 2.3 and using that $d(\mathrm{Id})(x) = \mathrm{Id}$ for all $x \in B_X$, we obtain another commutative diagram where all the arrows are linear:



Moreover, $d\delta(0)$ is an isometry. Indeed, given $x \in X$ and $f \in \mathcal{H}L_0(B_X)$ we have

$$\langle f, d\delta(0)(x)\rangle = \lim_{t \to 0} \left\langle f, \frac{\delta(tx) - \delta(0)}{t}\right\rangle = \lim_{t \to 0} \frac{f(tx) - f(0)}{t} = df(0)(x)$$

and so

$$\|d\delta(0)(x)\| = \sup\{|df(0)(x)| : f \in B_{\mathcal{H}L_0(B_X)}\} \le \|x\|.$$

The other inequality is clear due to the commutative diagram:

$$\|x\| = \|T_{\mathrm{Id}} \circ d\delta(0)(x)\| \leq \|d\delta(0)(x)\|.$$

Finally, let $P = d\delta(0) \circ T_{\mathrm{Id}}$. Then, using that $T_{\mathrm{Id}} \circ d\delta(0) = \mathrm{Id}$, we have

$$P^2 = d\delta(0) \circ T_{\mathrm{Id}} \circ d\delta(0) \circ T_{\mathrm{Id}} = d\delta(0) \circ T_{\mathrm{Id}} = P,$$

so P is a norm-one projection from $\mathcal{G}_0(B_X)$ onto $d\delta(0)(X)$. □

Note that this result also holds for $\mathcal{G}^\infty(B_X)$ [38] but not in general for $\mathcal{F}(B_X)$. In [31], it is proved that this is true for X separable although for nonseparable X it could even occur that $\mathcal{F}(B_X)$ does not contain a subspace isomorphic to X. Another useful property of Lipschitz-free spaces is the fact that they contain a complemented copy of ℓ_1 [20]; the same holds for $\mathcal{G}_0(B_X)$.

Proposition 2.5. *Let X be a complex Banach space. Then, there is a complemented subspace of $\mathcal{G}_0(B_X)$ isomorphic to ℓ_1.*

Proof. ℓ_∞ is isomorphic to a subspace of $\mathcal{H}^\infty(\mathbb{D})$, and the latter is isometric to $\mathcal{H}L_0(\mathbb{D})$. Also, one can easily prove that $\mathcal{H}L_0(\mathbb{D})$ is complemented in $\mathcal{H}L_0(B_X)$ (anyway, we will show a stronger fact in Proposition 4.1) so we get that $\mathcal{H}L_0(B_X)$ contains a copy of ℓ_∞. It is a classical result (see [14, Theorem 4]) that this implies its predual $\mathcal{G}_0(B_X)$ contains a complemented copy of ℓ_1. □

Next, we want to describe the closed unit ball of $\mathcal{G}_0(B_X)$. For that, we introduce some more notation. We denote by *conv* (resp., Γ) the (resp., absolute) convex hull of a set. As usual in the Lipschitz world, for every $x, y \in B_X$ with $x \neq y$, $m_{x,y}$ stands for the *elementary molecule* $\frac{\delta(x)-\delta(y)}{\|x-y\|}$. Also, for every $x \in B_X$, $y \in X$, and $f \in \mathcal{H}L_0(B_X)$, we denote $e_{x,y}(f) := df(x)(y)$. Then, $e_{x,y} \in \mathcal{G}_0(B_X)$ with $\left\|e_{x,y}\right\| = \|y\|$. Indeed, it is clear that

$$\left\|e_{x,y}\right\| = \sup\{|df(x)(y)| : f \in B_{\mathcal{H}L_0(B_X)}\} \leq \sup\{\|df(x)\| : f \in B_{\mathcal{H}L_0(B_X)}\}\|y\| \leq \|y\|.$$

Conversely, take $x^* \in X^*$ with $x^*(y) = \|y\|$ and $\|x^*\| = 1$. Then, $x^*|_{B_X} \in \mathcal{H}L_0(B_X)$ and $e_{x,y}(x^*|_{B_X}) = x^*(y) = \|y\|$. This shows that $e_{x,y}$ belongs to $\mathcal{H}L_0(B_X)^*$ and the equality of norms. Finally, by a simple application of Cauchy's integral formula we derive that the restriction of $e_{x,y}$ to $\overline{B}_{\mathcal{H}L_0(B_X)}$ is τ_0-continuous and so it belongs to $\mathcal{G}_0(B_X)$.

Proposition 2.6. *Let X be a complex Banach space. Then,*

$$\overline{B}_{\mathcal{G}_0(B_X)} = \overline{\Gamma}\{m_{x,y} : x, y \in B_X,\ x \neq y\} = \overline{\mathrm{conv}}\{e_{x,y} : x \in B_X, y \in S_X\}$$

Proof. By Proposition 2.3, we have that $\left\|m_{x,y}\right\| = 1$ for every $x, y \in B_X$ with $x \neq y$. Also,

$$L(f) = \sup\{|\langle f, m_{x,y}\rangle| : x, y \in B_X, x \neq y\} \text{ for all } f \in \mathcal{H}L_0(B_X).$$

Thus, $\{m_{x,y} : x, y \in B_X, x \neq y\}$ is 1-norming for $\mathcal{H}L_0(B_X)$. Equivalently, $\overline{B}_{\mathcal{G}_0(B_X)} = \overline{\Gamma}\{m_{x,y} : x, y \in B_X,\ x \neq y\}$. Analogously, we have that

$$L(f) = \|df\| = \sup\{\|df(x)\| : x \in B_X\} = \sup\{|\langle f, e_{x,y}\rangle| : x \in B_X, y \in S_X\}$$

and so $\overline{B}_{\mathcal{G}_0(B_X)} = \overline{\Gamma\{e_{x,y} : x \in B_X, y \in S_X\}}$. But $e_{x,\lambda y_1+\eta y_2} = \lambda e_{x,y_1} + \eta e_{x,y_2}$ for every $\lambda, \eta \in \mathbb{C}$ so actually $\overline{B}_{\mathcal{G}_0(B_X)} = \overline{\mathrm{conv}}\{e_{x,y} : x \in B_X, y \in S_X\}$. □

As a consequence, the density characters of X and $\mathcal{G}_0(B_X)$ coincide. In particular X is separable if and only if $\mathcal{G}_0(B_X)$ is separable.

We will now relate $\mathcal{G}_0(B_X)$ with the Lipschitz-free space $\mathcal{F}(B_X)$ and Mujica's predual $\mathcal{G}^\infty(B_X)$ of $\mathcal{H}^\infty(B_X)$. Note that each element of $\mathcal{F}(B_X)$ can also be seen as an element of $\mathcal{G}_0(B_X)$, but maybe with a different behavior. For instance, consider $z \in B_X \setminus \{0\}$ and μ given by $\langle \mu, f\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it}z)dt$ for $f \in \mathrm{Lip}_0(B_X)$. Then $\mu \neq 0$ in $\mathcal{F}(B_X)$ but $\langle \mu, f\rangle = 0$ for all $f \in \mathcal{H}L_0(B_X)$, so $\mu = 0$ when considered as an element of $\mathcal{G}_0(B_X)$. The next proposition formalizes this situation. We say that an operator $T : X \to Y$ is a *quotient operator* if T is surjective and $\|y\| = \inf\{\|x\| : Tx = y\}$ for every $y \in Y$; this implies that $X/\ker T$ is isometrically isomorphic to Y.

Proposition 2.7. *Let X be a complex Banach space.*

(a) The operator

$$\pi : \mathcal{F}(B_X) \to \mathcal{G}_0(B_X)$$
$$\delta(x) \mapsto \delta(x)$$

is a quotient operator with kernel $\mathcal{H}L_0(B_X)_\perp = \{\mu \in \mathcal{F}(B_X) : \langle f, \mu\rangle = 0$ for every $f \in \mathcal{H}L_0(B_X)\}$. Thus, $\mathcal{G}_0(B_X) \equiv \mathcal{F}(B_X)/\mathcal{H}L_0(B_X)_\perp$ isometrically.

(b) The operator

$$\Psi : \mathcal{G}^\infty(B_X)\widehat{\otimes}_\pi X \to \mathcal{G}_0(B_X)$$
$$\delta(x) \otimes y \mapsto e_{x,y}$$

is a quotient map with $\|\Psi\| = 1$. In addition, the operator Ψ is injective if and only if $X = \mathbb{C}$.

Proof.

(a) First note that the existence of such an operator π follows from the linearization property of Lipschitz-free spaces applied to the 1-Lipschitz map $B_X \to \mathcal{G}_0(B_X)$ given by $x \mapsto \delta(x)$. Also, $\pi^* : \mathcal{H}L_0(B_X) \to \mathrm{Lip}_0(B_X)$ is just the inclusion map since, for every $f \in \mathcal{H}L_0(B_X)$ and every $x \in B_X$:

$$\pi^* f(x) = \langle \pi^* f, \delta(x)\rangle = \langle f, \pi(\delta(x))\rangle = \langle f, \delta(x)\rangle = f(x).$$

Thus, π^* is an isometry into. It is a standard fact that this implies that π is a quotient operator. Moreover, $\ker \pi = \pi^*(\mathcal{H}L_0(B_X))_\perp = \mathcal{H}L_0(B_X)_\perp$.

(b) Consider the isometry into

$$\Phi : \mathcal{H}L_0(B_X) \to \mathcal{H}^\infty(B_X, X^*)$$
$$f \mapsto df$$

defined after Proposition 2.1. Recall that $\mathcal{G}^\infty(B_X)\widehat{\otimes}_\pi X$ is a predual of $\mathcal{L}(\mathcal{G}^\infty(B_X), X^*) \simeq \mathcal{H}^\infty(B_X, X^*)$ (see, e.g., [45]). Thus, if we restrict Φ^* to this predual we obtain $\Psi = \Phi^*|_{\mathcal{G}^\infty(B_X)\widehat{\otimes}_\pi X}$, note that $\Psi(\delta(x) \otimes y) = e_{x,y} \in \mathcal{G}_0(B_X)$ for all x, y and so $\Psi(\mathcal{G}^\infty(B_X)\widehat{\otimes}_\pi X) \subset \mathcal{G}_0(B_X)$. Then, $\|\Psi\| = 1$ and Ψ is a quotient operator since $\Psi^* = \Phi$ is an isometry into. In the case $X = \mathbb{C}$, we have indeed that $\Phi : \mathcal{H}L_0(\mathbb{D}) \to \mathcal{H}^\infty(\mathbb{D})$ is an onto isometry, and thus Ψ is also an isometry from $\mathcal{G}^\infty(\mathbb{D})$ onto $\mathcal{G}_0(\mathbb{D})$. However, Ψ is not injective for $X \neq \mathbb{C}$ since Φ is not surjective. □

Remark 2.8. We suspect, but cannot prove, that in general $\mathcal{H}L_0(B_X)$ is not complemented in $\mathrm{Lip}_0(B_X)$. The authors are grateful to the referee for doubting an argument in an earlier version of this paper, and to Tommaso Russo for confirming that the question may be more complicated than it first appears. Indeed, one can prove that $\mathcal{P}(^2X)$ is complemented in $\mathcal{H}L_0(B_X)$. Also, by a result of Hajek and Russo [33], $\mathcal{P}(^2X)$ is not complemented in $\mathrm{Lip}_0(B_X)$ *in the real case*. However, it is not at all clear that their argument carries over to the complex case.

It follows from Proposition 2.7 that $\mathcal{G}_0(\mathbb{D})$ is isometric to $\mathcal{G}^\infty(\mathbb{D})$ (which is the unique predual of $\mathcal{H}^\infty(\mathbb{D})$ [4]). We have some immediate consequences.

Corollary 2.9. *A function f is an extreme point of $\overline{B}_{\mathcal{H}L_0(\mathbb{D})}$ if and only if f' is an extreme point of $\overline{B}_{\mathcal{H}^\infty(\mathbb{D})}$.*

Corollary 2.10. *A function $f \in \mathcal{H}L_0(\mathbb{D})$ attains its norm as a functional on $\mathcal{G}_0(\mathbb{D})$ if and only if $f' \in \mathcal{H}^\infty(\mathbb{D})$ attains its norm as a functional on $\mathcal{G}^\infty(\mathbb{D})$.*

Let us state one more consequence of Proposition 2.7.

Corollary 2.11. *Let X be a complex Banach space and $\varphi \in \mathcal{G}_0(B_X)$.*

(a) There are sequences $(x_n), (y_n) \subset B_X$ with $x_n \neq y_n$ and $(a_n) \subset \ell_1$ such that

$$\varphi = \sum_{n=1}^{\infty} a_n m_{x_n, y_n}.$$

Moreover, $\|\varphi\| = \inf \sum_{n=1}^{\infty} |a_n|$ where the infimum is taken over all such representations of φ.

(b) There are sequences $(x_n) \subset B_X$, $(y_n) \subset S_X$, and $(a_n) \subset \ell_1$ such that

$$\varphi = \sum_{n=1}^{\infty} a_n e_{x_n, y_n}.$$

Moreover, $\|\varphi\| = \inf \sum_{n=1}^{\infty} |a_n|$ where the infimum is taken over all such representations of φ.

Proof. Given $\varepsilon > 0$, Proposition 2.7 *(a)* provides an element $\mu \in \mathcal{F}(B_X)$ with $\pi(\mu) = \varphi$ and $\|\mu\| \leq \|\varphi\| + \varepsilon$. It is known (see, e.g., [3, Lemma 3.3]) that there are points $x_n, y_n \in B_X$ and $(a_n) \subset \ell_1$ with $\mu = \sum_{n=1}^{\infty} a_n \frac{\delta(x_n) - \delta(y_n)}{\|x_n - y_n\|}$ and $\sum_{n=1}^{\infty} |a_n| \leq \|\mu\| + \varepsilon \leq \|\varphi\| + 2\varepsilon$ (here δ denotes the canonical embedding $\delta : B_X \to \mathcal{F}(B_X)$). Then, $\varphi = \sum_{n=1}^{\infty} a_n \pi(\frac{\delta(x_n) - \delta(y_n)}{\|x_n - y_n\|}) = \sum_{n=1}^{\infty} a_n m_{x_n, y_n}$.

Item *(b)* follows similarly using the corresponding property for projective tensor products (see, e.g., [45, Proposition 2.8]) and $\mathcal{G}^\infty(B_X)$ [39, Theorem 5.1]. □

Another consequence of the linearization process shows that functions in $\mathcal{H}L_0$ behave similarly to functions in $\mathrm{Lip}_0(B_X, B_Y)$ that can be isometrically factored through the free-Lipschitz spaces $\mathcal{F}(B_X)$ and $\mathcal{F}(B_Y)$. Given $f \in \mathcal{H}L_0(B_X, Y)$ with $f(B_X) \subset B_Y$ we can easily obtain a commutative diagram:

$$\begin{array}{ccc} B_X & \xrightarrow{\ f\ } & B_Y \\ {\scriptstyle \delta_X}\downarrow & & \downarrow{\scriptstyle \delta_Y} \\ \mathcal{G}_0(B_X) & \xrightarrow[T_{\delta_Y \circ f}]{} & \mathcal{G}_0(B_Y), \end{array} \tag{2.1}$$

where $T_{\delta_Y \circ f}$ is linear and $\|T_{\delta_Y \circ f}\| = L(f)$.

3 | APPROXIMATION PROPERTIES ON $\mathcal{G}_0(B_X)$

Following Mujica's ideas [38], we devote this section to study the MAP and the AP for $\mathcal{G}_0(B_X)$ whenever X has the same property. Beginning with the MAP, we prove the following result about approximation of elements in the closed unit ball of the dual space. We first introduce the notation:

- $\mathcal{P}_0(X, Y)$: The vector space of polynomials $P : X \to Y$ such that $P(0) = 0$ endowed with the norm $\|dP\| = L(P|_{B_X})$.
- $\mathcal{P}_{f,0}(X, Y)$: The subspace of $\mathcal{P}_0(X, Y)$ consisting of finite-type polynomials.

Proposition 3.1. *Let X and Y be complex Banach spaces. Then*

(a) $\overline{B}_{\mathcal{H}L_0(B_X,Y)} = \overline{B_{\mathcal{P}_0(X,Y)}}^{\tau_0}$.
(b) If X has the MAP then $\overline{B}_{\mathcal{H}L_0(B_X,Y)} = \overline{B_{\mathcal{P}_{f,0}(X,Y)}}^{\tau_0}$.

Proof.

(a) If $f \in \overline{B}_{\mathcal{H}L_0(B_X,Y)}$ then $f \in \mathcal{H}^\infty(B_X, Y)$ and $f(0) = 0$. Consider the Taylor series expansion of f at 0: $f(x) = \sum_{k=0}^{\infty} P^k f(0)(x)$. As in [38], for each $m \in \mathbb{N} \cup \{0\}$, we denote

$$S_m f(x) = \sum_{k=0}^{m} P^k f(0)(x) \qquad \text{and} \qquad \sigma_m f(x) = \frac{1}{m+1} \sum_{k=0}^{m} S_k f(x).$$

Since $df = \sum_{k=0}^{\infty} dP^k f(0) \in \mathcal{H}^\infty(B_X, \mathcal{L}(X, Y))$ it follows from [38, Proposition 5.2] that $\sigma_m f(x) \to f(x)$ for all $x \in B_X$ and

$$\|d\sigma_m f\| = \|\sigma_m(df)\| \le \|df\| \le 1.$$

This implies that $f \in \overline{B_{\mathcal{P}_0(X,Y)}}^{\tau_0}$.

For the reverse inclusion, let $f \in \mathcal{H}L_0(B_X, Y)$ and $(P_\alpha) \subset B_{\mathcal{P}_0(X,Y)}$ such that $P_\alpha(x) \to f(x)$ for all $x \in B_X$. Then, $L(f) \le 1$ and so $f \in \overline{B}_{\mathcal{H}L_0(B_X,Y)}$.

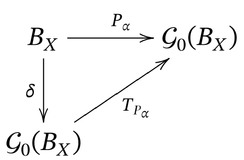
(b) If X has the MAP there is a net of finite rank operators $(T_\alpha) \subset \mathcal{L}(X, X)$ such that $T_\alpha(x) \to x$ for all $x \in X$ and $\|T_\alpha\| \le 1$ for every α. Given $P \in B_{\mathcal{P}_0(X,Y)}$ we have that $P \circ T_\alpha$ belongs to $B_{\mathcal{P}_{f,0}(X,Y)}$ (since $L(P \circ T_\alpha|_{B_X}) < 1$) and $P(T_\alpha x) \to P(x)$ for every x. This means that $P \in \overline{B_{\mathcal{P}_{f,0}(X,Y)}}^{\tau_0}$. Finally, an appeal to *(a)* yields the result. □

Recall that, by definition, the image of each $P \in \mathcal{P}_{f,0}(X, Y)$ is contained in a finite-dimensional space. We will use this fact repeatedly in the following.

Theorem 3.2. *X has the MAP if and only if* $\mathcal{G}_0(B_X)$ *has the MAP.*

Proof. X being isometric to a 1-complemented subspace of $\mathcal{G}_0(B_X)$ it is clear that X has the MAP when $\mathcal{G}_0(B_X)$ has it.

Now, suppose that X has the MAP and consider the mapping $\delta \in \overline{B}_{\mathcal{H}L_0(B_X,\mathcal{G}_0(B_X))}$. By Proposition 3.1, there exist a net $(P_\alpha) \subset B_{\mathcal{P}_{f,0}(X,\mathcal{G}_0(B_X))}$ such that $P_\alpha(x) \to \delta(x)$ for all $x \in B_X$. Applying a linearization as in Proposition 2.3 we obtain finite rank linear mappings T_{P_α} with norm bounded by 1, such that the following diagram commutes:



Note that $T_{P_\alpha}(\delta(x)) = P_\alpha(x) \to \delta(x) = \mathrm{Id}(\delta(x))$. Then, we have that $T_{P_\alpha} \to \mathrm{Id}$ on $\mathrm{span}\{\delta(x) : x \in B_X\}$. Since the net (T_{P_α}) is bounded the same holds for the closure. Hence, $\mathcal{G}_0(B_X)$ has the MAP. □

Note that our arguments cannot be adapted to the case in which X has the BAP since the approximations of the identity could send the unit ball B_X to a bigger ball (and, hence, we cannot control the Lipschitz norm of $P \circ T_\alpha|_{B_X}$ as in Proposition 3.1 *(b)*).

Question 1. Does $\mathcal{G}_0(B_X)$ have the BAP whenever X has the BAP?

The same question for $\mathcal{G}^\infty(B_X)$ was posed by Mujica in [38]. As far as we know, this question is still open.

In contrast to this unknown case about the BAP, the analogous statement for the AP (approximation property—without bounds) was successfully solved by Mujica [38] for $\mathcal{G}^\infty(B_X)$. We now turn to this goal for our space $\mathcal{G}_0(B_X)$, following Mujica's scheme but somewhat simplifying the arguments.

Note that in the results about the MAP we used several times that a bounded net of linear operators converges uniformly on compact sets if and only if it converges pointwise on a dense set. For the AP we cannot make use of this kind of argument, so our first step will be to describe a locally convex topology τ_γ such that the following topological isomorphism holds:

$$(\mathcal{H}L_0(B_X, Y), \tau_\gamma) \cong (\mathcal{L}(\mathcal{G}_0(B_X), Y), \tau_0). \tag{3.1}$$

Remark 3.3. Note that for a topology τ_γ satisfying Equation (3.1), if (f_α) is a bounded net in $\mathcal{H}L_0(B_X, Y)$ which converges pointwise to $f \in \mathcal{H}L_0(B_X, Y)$ then $f_\alpha \stackrel{\tau_\gamma}{\to} f$. Indeed, linearizing we obtain a bounded net $(T_{f_\alpha}) \subset \mathcal{L}(\mathcal{G}_0(B_X), Y)$ which converges pointwise to T_f. Then, $T_{f_\alpha} \stackrel{\tau_0}{\to} T_f$ implying that $f_\alpha \stackrel{\tau_\gamma}{\to} f$.

As a consequence, we derive from Proposition 3.1 *(a)* the following identity:

$$\overline{B}_{\mathcal{H}L_0(B_X,Y)} = \overline{B_{\mathcal{P}_0(X,Y)}}^{\tau_\gamma}. \tag{3.2}$$

In order to work with the τ_0-topology in $\mathcal{L}(\mathcal{G}_0(B_X), Y)$ it would be good to have a useful description of the compact sets of the space $\mathcal{G}_0(B_X)$. For that, we appeal to the following variation of the classical Grothendieck description of compact sets (which can be proved, for instance, by slightly modifying the proof of [43, Proposition 9, p. 134]):

Lemma 3.4. *Let X be a Banach space and $V \subset S_X$ such that $\overline{B}_X = \overline{\Gamma}(V)$. For each compact set $K \subset X$, there exist sequences $(\alpha_j) \in c_0$ (with $\alpha_j > 0$ for all j) and $(v_j) \subset V$ such that $K \subset \overline{\Gamma}(\{\alpha_j v_j\})$.*

A direct consequence of this lemma, along with Proposition 2.6 is the following:

Corollary 3.5. *Let $K \subset \mathcal{G}_0(B_X)$ be a compact set. Then, there exist sequences $(\alpha_j) \in c_0$ and $(x_j, y_j) \subset B_X \times B_X$ (with $\alpha_j > 0$ and $x_j \neq y_j$ for all j) such that $K \subset \overline{\Gamma}(\{\alpha_j m_{x_j y_j}\})$.*

Now, we can introduce, as in [38, Theorem 4.8], a topology τ_γ satisfying Equation (3.2).

Theorem 3.6. *Let τ_γ be the locally convex topology on $\mathcal{H}L_0(B_X, Y)$ generated by the family of seminorms*

$$p(f) = \sup_j \alpha_j \frac{\|f(x_j) - f(y_j)\|}{\|x_j - y_j\|}$$

for varying $(\alpha_j) \in c_0$, $0 < \alpha_j < 1$, and $(x_j, y_j) \subset B_X \times B_X$ with $x_j \neq y_j$ for all j. Then, the mapping

$$(\mathcal{H}L_0(B_X, Y), \tau_\gamma) \to (\mathcal{L}(\mathcal{G}_0(B_X), Y), \tau_0)$$

$$f \mapsto T_f$$

is a topological isomorphism.

Proof. If $K \subset \mathcal{G}_0(B_X)$ is a compact set, by the previous corollary there are sequences $(\alpha_j) \in c_0$, $(x_j, y_j) \subset B_X \times B_X$ with $\alpha_j > 0$, $x_j \neq y_j$ for all j, such that $K \subset \overline{\Gamma}(\{\alpha_j m_{x_j y_j}\})$. Then, for all $f \in \mathcal{H}L_0(B_X, Y)$,

$$\sup_{u \in K} \|T_f u\| \leq \sup_j \|T_f(\alpha_j m_{x_j y_j})\| = \sup_j \alpha_j \frac{\|f(x_j) - f(y_j)\|}{\|x_j - y_j\|},$$

showing that the mapping $f \mapsto T_f$ is $\tau_\gamma - \tau_0$ continuous.

To prove the continuity of the inverse mapping note that for a seminorm p of τ_γ, the associated sequence $(\alpha_j m_{x_j y_j})$ converges to 0 in $\mathcal{G}_0(B_X)$. Thus, the set $K = \{\alpha_j m_{x_j y_j}\} \cup \{0\}$ is a compact set in $\mathcal{G}_0(B_X)$ and $p(f) = \sup_j \|T_f(\alpha_j m_{x_j y_j})\| = \sup_{u\in K} \|T_f u\|$. □

Although the corresponding result for Lipschitz-free spaces will not be used in this work, we include it here since it may be of independent interest.

Theorem 3.7. *Let M be a complete pointed metric space. Then*

(i) For each compact subset K of $\mathcal{F}(M)$, there exist sequences $(\alpha_j) \in c_0$ and $(x_j, y_j) \subset M \times M$ (with $\alpha_j > 0$ and $x_j \neq y_j$ for all j) such that $K \subset \overline{\Gamma}(\{\alpha_j m_{x_j y_j}\})$.

(ii) Given a Banach space Y, let τ_γ be the locally convex topology on $\mathrm{Lip}_0(M, Y)$ generated by the seminorms

$$p(f) = \sup_j \alpha_j \frac{\|f(x_j) - f(y_j)\|}{d(x_j, y_j)}$$

where $(\alpha_j) \in c_0$, $(x_j, y_j) \subset M \times M$ and $\alpha_j > 0$, $x_j \neq y_j$ for all j. Then, the mapping

$$(\mathrm{Lip}_0(M, Y), \tau_\gamma) \to (\mathcal{L}(\mathcal{F}(M), Y), \tau_0)$$
$$f \mapsto T_f$$

is a topological isomorphism.

Now, we examine the relationship between the topologies τ_γ and τ_0 on $\mathcal{H}L_0(B_X, Y)$.

Proposition 3.8. *Let X and Y be complex Banach spaces. Then, τ_γ is finer than τ_0 on $\mathcal{H}L_0(B_X, Y)$, and these topologies are equivalent on $\mathcal{P}(^m X, Y)$ for each $m \in \mathbb{N}$.*

Proof. If $K \subset B_X$ is a compact set, then $\delta(K) \subset \mathcal{G}_0(B_X)$ is compact. By Corollary 3.5, there exist sequences $(\alpha_j) \in c_0$ and $(x_j, y_j) \subset B_X \times B_X$ (with $\alpha_j > 0$ and $x_j \neq y_j$ for all j) such that $\delta(K) \subset \overline{\Gamma}(\{\alpha_j m_{x_j y_j}\})$. Hence, for all $f \in \mathcal{H}L_0(B_X, Y)$,

$$\sup_{x\in K} \|f(x)\| \leq \sup_j \alpha_j \frac{\|f(x_j) - f(y_j)\|}{\|x_j - y_j\|},$$

proving the first assertion.

For the second statement, take a seminorm p that generates τ_γ: $p(f) = \sup_j \alpha_j \frac{\|f(x_j)-f(y_j)\|}{\|x_j-y_j\|}$, with $(\alpha_j) \in c_0$, $(x_j, y_j) \subset B_X \times B_X$, $\alpha_j > 0$ and $x_j \neq y_j$ for all j. For a homogeneous polynomial $P \in \mathcal{P}(^m X, Y)$, we have:

$$p(P) = \sup_j \alpha_j \frac{\|P(x_j) - P(y_j)\|}{\|x_j - y_j\|} = \sup_j \frac{\|P(\alpha_j^{1/m} x_j) - P(\alpha_j^{1/m} y_j)\|}{\|x_j - y_j\|}$$
$$= \sup_j \frac{\left\| \sum_{k=1}^m \binom{m}{k} \check{P}\left((\alpha_j^{1/m}(x_j - y_j))^k, (\alpha_j^{1/m} y_j)^{m-k}\right) \right\|}{\|x_j - y_j\|}$$
$$= \sup_j \left\| \sum_{k=1}^m \binom{m}{k} \check{P}\left(\left(\frac{\alpha_j^{1/m}(x_j - y_j)}{\|x_j - y_j\|^{1/k}} \right)^k, (\alpha_j^{1/m} y_j)^{m-k} \right) \right\|.$$

Note that there exist compact sets K_1 and K_2 in X such that $\left\{\alpha_j^{1/m}\frac{(x_j-y_j)}{\|x_j-y_j\|^{1/k}}\right\}\subset K_1$ and $\left\{\alpha_j^{1/m}y_j\right\}\subset K_2$ (since both sequences go to 0). Then,

$$p(P)\le\sum_{k=1}^{m}\binom{m}{k}\sup_{a\in K_1,b\in K_2}\|\check{P}(a^k,b^{m-k})\|.$$

Using the polarization formula, for each $k\in\{1,\dots,m\}$,

$$\check{P}(a^k,b^{m-k})=\frac{1}{2^m m!}\sum_{\varepsilon_i=\pm1}\varepsilon_1\cdots\varepsilon_m P\left(\left(\sum_{i=1}^{k}\varepsilon_i\right)a+\left(\sum_{i=k+1}^{m}\varepsilon_i\right)b\right).$$

Taking into account that the following set is compact

$$C(K_1,K_2)=\left\{\left(\sum_{i=1}^{k}\varepsilon_i\right)a+\left(\sum_{i=k+1}^{m}\varepsilon_i\right)b\ :\ a\in K_1,b\in K_2,k\in\{1,\dots,m\},\varepsilon_i=\pm1\right\}$$

and that

$$\sup_{a\in K_1,b\in K_2}\|\check{P}(a^k,b^{m-k})\|\le\frac{1}{m!}\sup_{u\in C(K_1,K_2)}\|P(u)\|,$$

we derive the intended inequality:

$$p(P)\le\frac{2^m-1}{m!}\sup_{u\in C(K_1,K_2)}\|P(u)\|.$$

□

We can now combine all the pieces of our study of the topology τ_γ to obtain the following.

Proposition 3.9. *If X has the AP, for a given $f\in\mathcal{H}L_0(B_X,Y)$ there exists a net $(P_\alpha)\subset\mathcal{P}_{f,0}(X,Y)$ such that $P_\alpha\overset{\tau_\gamma}{\to}f$.*

Proof. It is enough to consider $f\in\overline{B}_{\mathcal{H}L_0(B_X,Y)}$. Moreover, taking into account the equality (3.2) we just need to prove the result for each homogeneous polynomial $P\in\mathcal{P}(^mX,Y)$ (for any m). Applying [38, Lemma 5.3] (or composing the polynomial with the approximations of the identity supplied by the AP of X) we obtain a net $(P_\alpha)\subset\mathcal{P}_{f,0}(X,Y)$ such that $P_\alpha\overset{\tau_0}{\to}P$. Now, Proposition 3.8 implies that $P_\alpha\overset{\tau_\gamma}{\to}P$, which completes the proof. □

Finally, we are in position to prove the announced result:

Theorem 3.10. *X has the AP if and only if $\mathcal{G}_0(B_X)$ has the AP.*

Proof. One implication is clear because X is isometric to a complemented subspace of $\mathcal{G}_0(B_X)$.

For the other, take $\delta\in\mathcal{H}L_0(B_X,\mathcal{G}_0(B_X))$. By Proposition 3.9 there exists a net $(P_\alpha)\subset\mathcal{P}_{f,0}(X,\mathcal{G}_0(B_X))$ such that $P_\alpha\overset{\tau_\gamma}{\to}\delta$. By the linearization process, appealing to the isomorphism (3.1), we obtain that $(T_{P_\alpha})\subset\mathcal{L}(\mathcal{G}_0(B_X),\mathcal{G}_0(B_X))$ is a net of finite rank linear mappings satisfying $T_{P_\alpha}\overset{\tau_0}{\to}Id$. □

We finish this section with some comments about $\mathcal{H}L(B_X,Y)$ (i.e., the space of all holomorphic Lipschitz functions). It is easy to check that this is a Banach space with the norm $\|f\|_{\mathcal{H}L}=\max\{\|f(0)\|,L(f)\}$.

Note that $\|f\|_\infty\le2\|f\|_{\mathcal{H}L}$ for any $f\in\mathcal{H}L(B_X,Y)$. Also, it is plain to see that $\mathcal{H}L_0(B_X,Y)$ is a 1-complemented subspace of $\mathcal{H}L(B_X,Y)$. Moreover, motivated by a similar result for Lip_0-spaces (see [46, Theorem 1.7.2]) we get:

Proposition 3.11. *Let X, Y be complex Banach spaces. Then, $\mathcal{H}L(B_X, Y)$ is isometric to a 1-complemented subspace of $\mathcal{H}L_0(B_{X\oplus_1\mathbb{C}}, Y)$.*

Proof. Consider $\Phi : \mathcal{H}L(B_X, Y) \to \mathcal{H}L_0(B_{X\oplus_1\mathbb{C}}, Y)$ given by $\Phi f(x,\lambda) = f(x) + (\lambda - 1)f(0)$. It is easy to check that Φf is Lipschitz with $L(\Phi f) \leq \|f\|_{\mathcal{H}L}$ for every $f \in \mathcal{H}L(B_X, Y)$. Note that

$$L(\Phi f) \geq \sup\left\{ \frac{\|\Phi f(x,0) - \Phi f(y,0)\|}{\|x-y\|} : x \neq y \in B_X \right\} = L(f)$$

and also

$$L(\Phi f) \geq \frac{\|\Phi f(0,1) - \Phi f(0,0)\|}{\|(0,1)-(0,0)\|_1} = \|f(0)\|,$$

so we actually have $L(\Phi f) = \|f\|_{\mathcal{H}L}$. Thus, Φ is an isometry into.

Now, consider $T : \mathcal{H}L_0(B_{X\oplus_1\mathbb{C}}, Y) \to \mathcal{H}L(B_X, Y)$ given by $Tg(x) = g(x,0) + g(0,1)$. One can easily check that $\|T\| \leq 1$ and $T\circ\Phi = I_{\mathcal{H}L(B_X,Y)}$. Therefore, $P = \Phi\circ T$ is a norm-one projection from $\mathcal{H}L_0(B_{X\oplus_1\mathbb{C}}, Y)$ onto $\Phi(\mathcal{H}L(B_X, Y))$. □

With the same procedure as at the beginning of the previous section, we can produce a canonical predual $\mathcal{G}(B_X)$ of $\mathcal{H}L(B_X)$ made up of elements of $\mathcal{H}L(B_X)^*$ which are τ_0-continuous when restricted to the closed unit ball. The facts that $\mathcal{H}L_0(B_X)$ is a 1-complemented subspace of $\mathcal{H}L(B_X)$ and that the projection from $\mathcal{H}L(B_X)$ onto $\mathcal{H}L_0(B_X)$ is $\tau_0 - \tau_0$ continuous allow us to derive that $\mathcal{G}_0(B_X)$ is isometric to a 1-complemented subspace of $\mathcal{G}(B_X)$. Moreover, Proposition 3.11 actually shows that $\mathcal{G}(B_X)$ is 1-complemented in $\mathcal{G}_0(B_{X\oplus_1\mathbb{C}})$ since, for $Y = \mathbb{C}$, Φ is the adjoint of the map given by $\delta(x,\lambda) \mapsto \delta(x) + (\lambda - 1)\delta(0)$ and T is the adjoint of the one given by $\delta(x) \mapsto \delta(x,0) + \delta(0,1)$.

With standard adaptations most of the results of this and the previous sections can be stated for $\mathcal{G}(B_X)$ instead of $\mathcal{G}_0(B_X)$. That is the case of Propositions 2.3, 2.4, 3.1, and Theorem 3.2. The version of Proposition 2.6 for $\mathcal{G}(B_X)$ requires the addition of $\delta(0)$ to both considered sets. This addition has impact in Corollary 3.5 and Theorem 3.6, which in turn affects the proofs of Propositions 3.8 and 3.9 and Theorem 3.10. All these results are valid for $\mathcal{G}(B_X)$ after the mentioned modifications. Alternatively, this also follows from the fact that $\mathcal{G}(B_X)$ is isometric to a 1-complemented subspace of $\mathcal{G}_0(B_{X\oplus_1\mathbb{C}})$ (just note that the map Φ in Proposition 3.11 is the adjoint of the linearization T_F of the map $F(x,\lambda) = \delta(x) + (\lambda - 1)\delta(0)$). Also, note that the square diagram (2.1) can be made for $\mathcal{G}(B_X)$, but there is no equality between the norms of $T_{\delta_Y\circ f}$ and f.

4 | RELATION BETWEEN $\mathcal{G}_0(B_X)$ AND $\mathcal{G}_0(B_Y)$ WHEN $X \subset Y$

Recall that, given metric spaces M, N with $0 \in M \subset N$, the (real) Lipschitz-free space $\mathcal{F}(M)$ canonically identifies with a subspace of $\mathcal{F}(N)$. This follows from the McShane extension theorem asserting that for every $f \in \mathrm{Lip}_0(M,\mathbb{R})$ there is $\tilde{f} \in \mathrm{Lip}_0(N,\mathbb{R})$ with $\tilde{f}|_M = f$ and $L(f) = L(\tilde{f})$, see, for example [46, Theorem 1.33]. Note in passing that in the complex-valued case all extensions can have a larger Lipschitz constant. This is why our next goal is to analyze the corresponding relation between $\mathcal{G}_0(B_X)$ and $\mathcal{G}_0(B_Y)$ when $X \subset Y$. Then, $B_X \subset B_Y$ and the restriction mapping has norm one:

$$\mathcal{H}L_0(B_Y) \to \mathcal{H}L_0(B_X)$$
$$f \mapsto f|_{B_X}.$$

Then, the following mapping also has norm one:

$$\rho : \mathcal{G}_0(B_X) \to \mathcal{G}_0(B_Y)$$
$$\varphi \mapsto \widehat{\varphi},$$

where $\widehat{\varphi}(f) = \varphi(f|_{B_X})$.

Whenever ρ is an isometry, we write $\mathcal{G}_0(B_X) \subset \mathcal{G}_0(B_Y)$. Then, by the Hahn–Banach theorem, every element of $\mathcal{H}L_0(B_X)$ would have a norm preserving extension to $\mathcal{H}L_0(B_Y)$. Since there exist polynomials which cannot be extended to a larger space it is not always true that $\mathcal{G}_0(B_X) \subset \mathcal{G}_0(B_Y)$. Moreover, the previous argument can be clearly reversed, so: $\mathcal{G}_0(B_X) \subset \mathcal{G}_0(B_Y)$ if and only if every $f \in \mathcal{H}L_0(B_X)$ has a norm preserving extension to $\mathcal{H}L_0(B_Y)$.

We study some situations where this norm preserving extension occurs. All are cases where we have an extension morphism. The simplest occurs when X is 1-complemented in Y. Here, the complementation also spreads to $\mathcal{G}_0(B_X)$.

Proposition 4.1. *If X is 1-complemented in Y then ρ is an isometry and $\mathcal{G}_0(B_X)$ is a 1-complemented subspace of $\mathcal{G}_0(B_Y)$.*

Proof. Let $\pi : Y \to X$ be a norm-one projection. Given $f \in \mathcal{H}L_0(B_X)$ the mapping $f \circ \pi$ belongs to $\mathcal{H}L_0(B_Y)$ with $L(f \circ \pi) \leq L(f)$ and $(f \circ \pi)|_{B_X} = f$. Now, for each $\varphi \in \mathcal{G}_0(B_X)$,

$$\|\varphi\| = \sup_{f \in B_{\mathcal{H}L_0(B_X)}} |\varphi(f)| = \sup_{f \in B_{\mathcal{H}L_0(B_X)}} |\widehat{\varphi}(f \circ \pi)| \leq \|\widehat{\varphi}\|.$$

Thus, $\|\varphi\| = \|\widehat{\varphi}\|$, meaning that ρ is an isometry. Finally, we derive that $\mathcal{G}_0(B_X)$ is 1-complemented in $\mathcal{G}_0(B_Y)$ through the following projection:

$$\mathcal{G}_0(B_Y) \to \mathcal{G}_0(B_X)$$
$$\psi \mapsto [f \mapsto \psi(f \circ \pi)].$$

□

Jung has proved recently that $\mathcal{G}^\infty(B_X)$ does not have the Radon–Nikodym property (RNP) for any X [35]. Here we obtain the same result for $\mathcal{G}_0(B_X)$.

Corollary 4.2. *The space $\mathcal{G}_0(B_X)$ fails to have the Radon–Nikodym property for every complex Banach space X.*

Proof. The space $\mathcal{G}^\infty(\mathbb{D})$ fails to have the RNP since its unit ball does not have extreme points [4]. Thus, by the isometry presented in Proposition 2.7, the same holds for $\mathcal{G}_0(\mathbb{D})$. Since $\mathbb{C}$ is 1-complemented in X, Proposition 4.1 yields that $\mathcal{G}_0(\mathbb{D})$ is a subspace of $\mathcal{G}_0(B_X)$ and we are done. □

Another situation in which we have an extension morphism is when $Y = X^{**}$. Recall that, given $f \in \mathcal{H}^\infty(B_X)$, we can consider its standard, canonical extension $\tilde{f} \in \mathcal{H}^\infty(B_{X^{**}})$ [7]. This extension, which defines an isometry from $\mathcal{H}^\infty(B_X)$ to $\mathcal{H}^\infty(B_{X^{**}})$ [21], is a topic widely developed in the literature. For instance, it is essential in the description of the spectrum (or maximal ideal space) of the Banach algebra $\mathcal{H}^\infty(B_X)$. Another ingredient that usually appears associated with this extension and its properties is the notion of *symmetrically regular space*. Both these concepts have their origin in the study initiated by Arens [5, 6] about extending the product of a Banach algebra to its bidual, which we now review.

For an n-linear mapping $A : X \times \cdots \times X \to Y$ the canonical extension $\widetilde{A} : X^{**} \times \cdots \times X^{**} \to Y^{**}$ is given by consecutive weak-star convergence in the following way:

$$\widetilde{A}(x_1^{**}, \dots, x_n^{**})(y^*) = \lim_{\alpha_1} \dots \lim_{\alpha_n} y^*(A(x_{\alpha_1}, \dots, x_{\alpha_n}))$$

where each $(x_{\alpha_i}) \subset X$ is a net which is weak-star convergent to x_i^{**} and $y^* \in Y^*$. Now, the *canonical extension* of a homogeneous polynomial $P \in \mathcal{P}(^m X, Y)$ is given by $\widetilde{P} \in \mathcal{P}(^m X^{**}, Y^{**})$ which is defined, for $x^{**} \in X^{**}$, in the expected way:

$$\widetilde{P}(x^{**}) = \widetilde{\check{P}}(x^{**}, \dots, x^{**}).$$

This provides a method to canonically extend bounded holomorphic functions $f \in \mathcal{H}^\infty(B_X, Y) \rightsquigarrow \widetilde{f} \in \mathcal{H}^\infty(B_{X^{**}}, Y^{**})$ and we know from [21] that this extension is an isometry: $\|f\| = \|\widetilde{f}\|$.

Recall that X is said to be *regular* if every continuous bilinear mapping $A : X \times X \to \mathbb{C}$ is Arens regular. That is, the following two extensions of A to $X^{**} \times X^{**} \to \mathbb{C}$ coincide:

$$\lim_{\alpha} \lim_{\beta} A(x_\alpha, y_\beta) \quad \text{and} \quad \lim_{\beta} \lim_{\alpha} A(x_\alpha, y_\beta),$$

where (x_α) and (y_β) are nets in X converging weak-star to points x_0^{**} and y_0^{**} in X^{**}. The space X is *symmetrically regular* if the above holds for every continuous symmetric bilinear form. Equivalently, X is (symmetrically) regular if any continuous (symmetric) linear mapping $T : X \to X^*$ is weakly compact. Several equivalent characterizations of this notion can be seen in [9, Theorem 8.3] and some interesting properties appeared in [10, Section 1]. As examples of non-reflexive regular (and hence, symmetrically regular) Banach spaces we have, for instance, those that satisfy property (V) of Pełczyński, like c_0, $C(K)$ or $\mathcal{H}^\infty(\mathbb{D})$, while typical non symmetrically regular spaces are ℓ_1 and $X \oplus X^*$, for any non-reflexive space X. Also, Leung [37, Theorem 12] provided an example of a symmetrically regular space that is not regular and in [10] it is shown that $c_0(\ell_1^n)$ is regular but its bidual $\ell_\infty(\ell_1^n)$ is not symmetrically regular.

We now want to work with the canonical extension of elements in $\mathcal{H}L_0(B_X)$. For $f \in \mathcal{H}L_0(B_X)$, in order to compute the Lipschitz constant of $\tilde{f}$ we need to deal with the differential of the extension, $d\tilde{f}$, which belongs to $\mathcal{H}(B_{X^{**}}, X^{***})$. Now, we do know the norm of the extension of the differential $\widetilde{df} \in \mathcal{H}^\infty(B_{X^{**}}, X^{***})$. Fortunately, on symmetrically regular spaces they coincide:

Proposition 4.3. *If X is symmetrically regular and $f \in \mathcal{H}L_0(B_X)$, then $d\tilde{f} = \widetilde{df}$.*

Proof. If $f = \sum_{k=0}^\infty P^k f(0)$ then the series expansion of df at 0 is given by $df = \sum_{k=0}^\infty dP^k f(0)$. Thus, $\widetilde{df} = \sum_{k=0}^\infty \widetilde{(dP^k f(0))}$. On the other hand, $\tilde{f} = \sum_{k=0}^\infty \widetilde{P^k f(0)}$ and so $d\tilde{f} = \sum_{k=0}^\infty d(\widetilde{P^k f(0)})$.

Therefore, the result is proved once we show that for any given $m \in \mathbb{N}$ and any $P \in \mathcal{P}(^m X)$, $\widetilde{dP} = d\widetilde{P}$. Note that in this case $\widetilde{P} \in \mathcal{P}(^m X^{**})$, $dP \in \mathcal{P}(^{m-1}X, X^*)$ while both $\widetilde{dP}$ and $d\widetilde{P}$ belong to $\mathcal{P}(^{m-1}X^{**}, X^{***})$.

When X is symmetrically regular, it follows from [9, Theorem 8.3] that $\widetilde{\overset{\vee}{P}} = \overset{\vee}{\widetilde{P}}$. The argument is now complete because, for each $x^{**}, y^{**} \in X^{**}$ we have $\widetilde{dP}(x^{**})(y^{**}) = m\widetilde{\overset{\vee}{P}}(x^{**}, \dots, x^{**}, y^{**})$ and $d\widetilde{P}(x^{**})(y^{**}) = m\overset{\vee}{\widetilde{P}}(x^{**}, \dots, x^{**}, y^{**})$. □

A generalization of this procedure (which, however, uses the canonical extension in its definition) is when there exists an isometric extension morphism $s : X^* \to Y^*$. This happens, for instance, when X is an M-ideal in Y. More generally, if $X \subset Y$ then the existence of an isometric extension morphism $s : X^* \to Y^*$ is equivalent to X^{**} being 1-complemented in Y^{**}. Actually, the existence of an isometric extension morphism $s : X^* \to Y^*$ is equivalent to X being 1-locally complemented in Y (see the definition in the next section and the comment before Corollary 5.5).

Note that $s(x^*)(x) = x^*(x)$ for all $x \in X$, $x^* \in X^*$ and that $\|s(x^*)\| = \|x^*\|$. This extension transfers to $\mathcal{H}^\infty(B_X)$ in the following way:

$$\bar{s} : \mathcal{H}^\infty(B_X) \to \mathcal{H}^\infty(B_Y)$$
$$f \mapsto \tilde{f} \circ s^* \circ i_Y,$$

where $i_Y : Y \to Y^{**}$ is the canonical inclusion.

The mapping $\bar{s}$ is an isometric extension from $\mathcal{H}^\infty(B_X)$ to $\mathcal{H}^\infty(B_Y)$. Now, we can show that the canonical extension and the mapping $\bar{s}$ are isometric on $\mathcal{H}L_0(B_X)$ when X is a symmetrically regular Banach space.

Proposition 4.4. *If X is symmetrically regular, then the extension mapping*

$$E : \mathcal{H}L_0(B_X) \to \mathcal{H}L_0(B_{X^{**}})$$
$$f \mapsto \tilde{f}$$

is an isometry. If, in addition, $X \subset Y$ and there is an isometric extension morphism $s : X^* \to Y^*$ then

$$\bar{s} : \mathcal{H}L_0(B_X) \to \mathcal{H}L_0(B_Y)$$
$$f \mapsto \tilde{f} \circ s^* \circ i_Y$$

is an isometric extension.

Proof. If $f \in \mathcal{H}L_0(B_X)$ then its norm is given by $\|df\|$. By [21], $\|d\tilde{f}\| = \|\widetilde{df}\|$. Also, by the previous proposition we know that $d\tilde{f} = \widetilde{df}$. So, we obtain that $\|df\| = \|d\tilde{f}\|$, meaning that $\tilde{f}$ does indeed belong to $\mathcal{H}L_0(B_{X^{**}})$ and that the mapping $f \mapsto \tilde{f}$ is an isometry.

To prove the second statement, note that for any $P \in \mathcal{P}(^mX)$ we have that $\bar{s}(P) \in \mathcal{P}(^mY)$ and $d(\bar{s}(P)) \in \mathcal{P}(^{m-1}Y, Y^*)$. Now, for $y, z \in B_Y$,

$$d(\bar{s}(P))(y)(z) = m\widetilde{(\bar{s}(P))}(y, \dots, y, z) = m\check{\tilde{P}}(s^*(i_Y(y)), \dots, s^*(i_Y(y)), s^*(i_Y(z)))$$
$$= d\tilde{P}(s^*(i_Y(y)))(s^*(i_Y(z))) = (i_Y^* \circ s^{**} \circ d\tilde{P} \circ s^* \circ i_Y)(y)(z).$$

This says that $d(\bar{s}(P)) = i_Y^* \circ s^{**} \circ d\tilde{P} \circ s^* \circ i_Y$ for every polynomial $P \in \mathcal{P}(^mX)$. Then, the same equality holds for every $f \in \mathcal{H}L_0(B_X)$:

$$d(\bar{s}(f)) = i_Y^* \circ s^{**} \circ d\tilde{f} \circ s^* \circ i_Y.$$

Since X is symmetrically regular, the first part of the proof shows that $\|d(\bar{s}(f))\| \le \|d\tilde{f}\| = \|df\|$. Also, note that for $x \in B_X$, we have $s^* \circ i_Y(x) = i_X(x)$. This implies that $d\tilde{f}(s^*(i_Y(x)) = i_{X^*}(df(x))$. Therefore,

$$d(\bar{s}(f))(x) = i_Y^* \circ s^{**}(i_{X^*}(df(x))) = s(df(x)).$$

This equality and the fact that s is an isometry allow us to derive the other inequality:

$$\|d(\bar{s}(f))\| \ge \sup_{x \in B_X} \|d(\bar{s}(f))(x)\| = \sup_{x \in B_X} \|s(df(x))\|$$
$$= \sup_{x \in B_X} \|df(x)\| = \|df\|,$$

which concludes the proof. □

In the previous result, symmetric regularity is used to obtain that $d\tilde{f} = \widetilde{df}$. Actually we only need the identity of their norms: $\|d\tilde{f}\| = \|\widetilde{df}\|$. We do not know if this equality holds in general.

Corollary 4.5. *If X is symmetrically regular, then $\mathcal{G}_0(B_X) \subset \mathcal{G}_0(B_{X^{**}})$. If, in addition, $X \subset Y$ and there is an isometric extension morphism $s : X^* \to Y^*$, then $\mathcal{G}_0(B_X) \subset \mathcal{G}_0(B_Y)$.*

Note that in the above corollary the hypothesis of symmetric regularity is not a necessary condition: $X = \ell_1$ is not symmetrically regular and by Proposition 4.1, $\mathcal{G}_0(B_{\ell_1}) \subset \mathcal{G}_0(B_{\ell_1^{**}})$.

4.1 | Dual isometric spaces

It is known that there exist non-isomorphic Banach spaces with isomorphic duals. Attending to that, Díaz and Dineen [23] posed the following question: if X and Y are Banach spaces such that X^* and Y^* are isomorphic, under which conditions is it true that $\mathcal{P}(^mX)$ and $\mathcal{P}(^mY)$ are isomorphic for every $m \ge 1$? That is, if X^* and Y^* are isomorphic (i.e., the spaces of 1-homogeneous polynomials are isomorphic) does it imply that the spaces of m-homogeneous polynomials are isomorphic for every m? They also gave a partial answer to this question. Later, a relaxation of the conditions was independently obtained by Cabello-Sánchez et al. [16, Theorem 1] and Lassalle and Zalduendo [36, Theorem 4]], who proved that the answer is affirmative whenever X and Y are symmetrically regular. We present here a version of this result for holomorphic Lipschitz functions on the ball. Since we need to remain inside the ball when changing the space we have to restrict ourselves to the case of isometric isomorphisms.

Proposition 4.6. *If X and Y are symmetrically regular Banach spaces such that X^* and Y^* are isometrically isomorphic then $\mathcal{H}L_0(B_X)$ and $\mathcal{H}L_0(B_Y)$ are isometrically isomorphic as well.*

Proof. Let us denote by $s : X^* \to Y^*$ the isometric isomorphism and consider the mapping $\overline{s} : \mathcal{H}L_0(B_X) \to \mathcal{H}L_0(B_Y)$ as in Proposition 4.4. By the proof of that proposition we derive that $\overline{s}$ is continuous and $\|\overline{s}\| \le 1$. Since Y is symmetrically regular, we can use the same procedure for the mapping $\overline{s^{-1}} : \mathcal{H}L_0(B_Y) \to \mathcal{H}L_0(B_X)$ leading to $\|\overline{s^{-1}}\| \le 1$. Finally, appealing to [36, Corollary 3] we obtain that $\overline{s^{-1}}\circ\overline{s}(P) = P$ for every homogeneous polynomial P on X and, hence, $\overline{s^{-1}}\circ\overline{s}(f) = f$ for every $f \in \mathcal{H}L_0(B_X)$. Indeed, if $\sum_{k=0}^{\infty} P^k$ is the Taylor series expansion of a given $f \in \mathcal{H}L_0(B_X)$, then $\tilde{f}(z) = \sum_{k=0}^{\infty} \tilde{P}^k(z)$ for every $z \in B_{X^{**}}$. Thus,

$$\overline{s}(f)(y) = \tilde{f}(s^*(i_Y(y))) = \sum_{k=0}^{\infty} \tilde{P}^k(s^*(i_Y(y))) = \sum_{k=0}^{\infty} \overline{s}(P^k)(y),$$

for every $y \in Y$. From here

$$\overline{s^{-1}}(\overline{s}(f))(x) = \widetilde{\overline{s}(f)}\big((s^{-1})^*(i_X(x))\big) = \sum_{k=0}^{\infty} \widetilde{\overline{s}(P^k)}\big((s^{-1})^*(i_X(x))\big)$$
$$= \sum_{k=0}^{\infty} \overline{s^{-1}}(\overline{s}(P^k))(x) = \sum_{k=0}^{\infty} P^k(x) = f(x),$$

for every $x \in X$. Analogously, one can check that $\overline{s}\circ\overline{s^{-1}}(f) = f$ for every $f \in \mathcal{H}L(B_Y)$. □

4.2 | Mapping between $\mathcal{G}_0(B_X)$ and $\mathcal{G}_0(B_Y)$

Any linear mapping between X and Y produces a mapping between $\mathcal{G}_0(B_X)$ and $\mathcal{G}_0(B_Y)$ by a canonical procedure (actually, two canonical procedures depending on the norm of the mapping).

(i) Let $\psi : X \to Y$ be a linear mapping with $\|\psi\| \le 1$. Note that $L(\psi) = \|\psi\|$ in this case. Since $\psi(B_X) \subset B_Y$ we can define the canonical mapping with norm ≤ 1:

$$\mathcal{H}L_0(B_Y) \to \mathcal{H}L_0(B_X)$$
$$f \mapsto f\circ\psi.$$

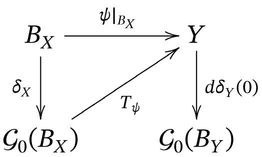
Thus, the following also has norm ≤ 1:

$$T_{\delta_Y\circ\psi} : \mathcal{G}_0(B_X) \to \mathcal{G}_0(B_Y)$$
$$\varphi \mapsto \widehat{\varphi},$$

where $\widehat{\varphi}(f) = \varphi(f\circ\psi)$.

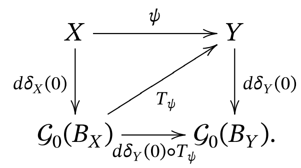
(ii) When $\|\psi\| > 1$ the previous construction does not work but we can appeal to a linearization plus differentiation process (as we used to show that X is a 1-complemented subspace of $\mathcal{G}_0(B_X)$).

Let $\psi \in \mathcal{L}(X,Y)$ so that $\psi|_{B_X} \in \mathcal{H}L_0(B_X, Y)$. We have the usual commutative diagram:



where $T_\psi \in \mathcal{L}(\mathcal{G}_0(B_X), Y)$.

Applying the differential at 0 to the equality $\psi|_{B_X} = T_\psi \circ \delta_X$ we get the commutative diagram:



Note that the linear mapping $d\delta_Y(0)\circ T_\psi : \mathcal{G}_0(B_X) \to \mathcal{G}_0(B_Y)$ has norm less than or equal to $\|\psi\|$.

5 | LOCAL COMPLEMENTATION IN THE BIDUAL

In this section, we are interested in the relationship between $\mathcal{G}_0(B_{X^{**}})$ and $\mathcal{G}_0(B_X)^{**}$ under the hypothesis of X^{**} having the MAP, in the spirit of what is done in [17].

We begin with a result about a special approximation behavior in the case that the bidual space has the MAP.

Proposition 5.1. *Let X, Y be Banach spaces such that X^{**} has the MAP. For each $f \in \mathcal{H}L_0(B_{X^{**}}, Y)$ with $L(f) = 1$, there exists a net $(Q_\alpha) \subset \mathcal{P}_{f,0}(X, Y)$ with $L(Q_\alpha|_{B_X}) \leq 1$ satisfying $\widetilde{Q}_\alpha(x^{**}) \to f(x^{**})$ for all $x^{**} \in B_{X^{**}}$.*

Proof. By Proposition 3.1, it is enough to consider $f = P \in \mathcal{P}_0(X^{**}, Y)$ with $L(P|_{B_{X^{**}}}) \leq 1$. If X^{**} has the MAP we can appeal to [17, Corollary 1] to obtain a net of finite rank mappings $(t_\alpha) \subset \mathcal{L}(X, X^{**})$ with $\|t_\alpha\| \leq 1$ and $t_\alpha^{**}(x^{**}) \to x^{**}$ for all $x^{**} \in X^{**}$. Now, we define $Q_\alpha = P\circ t_\alpha$, which clearly belongs to $\mathcal{P}_{f,0}(X, Y)$. Note that, for any $x, y \in B_X$,

$$\|Q_\alpha(x) - Q_\alpha(y)\| = \|P(t_\alpha(x)) - P(t_\alpha(y))\| \leq L(P|_{B_{X^{**}}})\|t_\alpha\|\|x - y\| \leq \|x - y\|.$$

Then, $L(Q_\alpha|_{B_X}) \leq 1$. Since t_α is a finite rank mapping, we have that $t_\alpha^{**} \in \mathcal{L}(X^{**}, X^{**})$. Hence, $\widetilde{Q}_\alpha = \widetilde{P}\circ t_\alpha^{**} = P\circ t_\alpha^{**}$. As a consequence, $\widetilde{Q}_\alpha(x^{**}) = P(t_\alpha^{**}(x^{**})) \to P(x^{**})$ for all $x^{**} \in B_{X^{**}}$. □

For a symmetrically regular space X, we consider the following mapping:

$$\Theta : B_{X^{**}} \to \mathcal{G}_0(B_X)^{**} = \mathcal{H}L_0(B_X)^*$$
$$x^{**} \mapsto [f \in \mathcal{H}L_0(B_X) \mapsto \widetilde{f}(x^{**})].$$

Proposition 5.2. *If X is symmetrically regular then Θ belongs to $\mathcal{H}L_0(B_{X^{**}}, \mathcal{G}_0(B_X)^{**})$ with $L(\Theta) = 1$.*

Proof. If X is symmetrically regular, by Proposition 4.4, the canonical extension is an isometry from $\mathcal{H}L_0(B_X)$ into $\mathcal{H}L_0(B_{X^{**}})$, so Θ is well defined. For any $f \in \mathcal{H}L_0(B_X)$, we have $\Theta(\cdot)(f) = \widetilde{f}$, meaning that Θ is weak-star holomorphic, and thus it is holomorphic. Also, $\Theta(0) = 0$ and for any $x^{**}, y^{**} \in B_{X^{**}}$, once again by the symmetric regularity of X we have

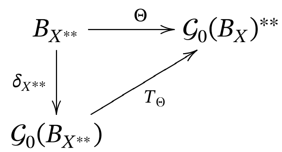
$$\|\Theta(x^{**}) - \Theta(y^{**})\| = \sup_{f\in B_{\mathcal{H}L_0(B_X)}} \|\widetilde{f}(x^{**}) - \widetilde{f}(y^{**})\| \leq \|x^{**} - y^{**}\|.$$

This means that $\Theta \in \mathcal{H}L_0(B_{X^{**}}, \mathcal{G}_0(B_X)^{**})$ with $L(\Theta) \leq 1$. On the other hand,

$$\|\Theta(x^{**}) - \Theta(y^{**})\| \geq \sup_{x^*\in B_{X^*}} |x^{**}(x^*) - y^{**}(x^*)| = \|x^{**} - y^{**}\|.$$

Therefore, $L(\Theta) = 1$. □

As a consequence of the previous proposition, if X is symmetrically regular we can linearize the mapping Θ:



This produces a linear mapping $T_\Theta \in \mathcal{L}(\mathcal{G}_0(B_{X^{**}}), \mathcal{G}_0(B_X)^{**})$ with $\|T_\Theta\| = L(\Theta) = 1$.

Motivated by the principle of local reflexivity, Kalton [34] introduced the following definition.

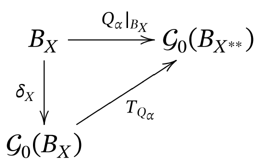
Definition 5.3. Given Banach spaces $X \subset Y$ we say that X is 1-locally complemented in Y if for every $\varepsilon > 0$ and every finite-dimensional subspace F of Y there exist a linear mapping $T : F \to X$ such that $\|T\| \leq 1 + \varepsilon$ and $T(x) = x$ for all $x \in F \cap X$.

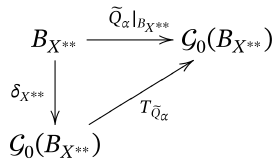
Note that the principle of local reflexivity says that X is 1-locally complemented in X^{**}, for any Banach space X.

Theorem 5.4. *If X is symmetrically regular and X^{**} has the MAP then T_Θ embeds $\mathcal{G}_0(B_{X^{**}})$ as a 1-locally complemented subspace of $\mathcal{G}_0(B_X)^{**}$. In particular, T_Θ is an isometry.*

Proof. We know that the mapping $\delta_{X^{**}}$ belongs to $\mathcal{H}L_0(B_{X^{**}}, \mathcal{G}_0(B_{X^{**}}))$ with $L(\delta_{X^{**}}) = 1$. Thus, we can apply Proposition 5.1 to get a net $(Q_\alpha) \subset \mathcal{P}_{f,0}(X, \mathcal{G}_0(B_{X^{**}}))$ with $L(Q_\alpha|_{B_X}) \leq 1$ such that $\widetilde{Q}_\alpha(x^{**}) \to \delta_{X^{**}}(x^{**})$ for all $x^{**} \in B_{X^{**}}$.

Consider the following two commutative diagrams:

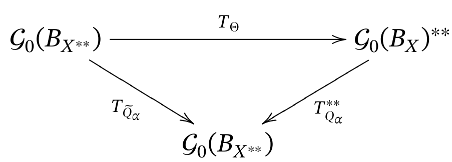




Note that, since X is symmetrically regular we have

$$\|T_{Q_\alpha}\| = L(Q_\alpha|_{B_X}) = L(\widetilde{Q}_\alpha|_{B_X}) = \|T_{\widetilde{Q}_\alpha}\| \leq 1.$$

For each α, since T_{Q_α} is a finite rank operator we have that $T^{**}_{Q_\alpha}$ belongs to $\mathcal{L}(\mathcal{G}_0(B_X)^{**}, \mathcal{G}_0(B_{X^{**}}))$. Thus, we have the following diagram:



The space $\mathcal{G}_0(B_{X^{**}})$ has the MAP witnessed by the net $(T_{\widetilde{Q}_\alpha})$ thanks to (the proof of) Theorem 3.2. Appealing to [17, Lemma 4], the proof will be completed once we check that the previous diagram is commutative. For this, it is enough to prove that $T_{\widetilde{Q}_\alpha}(\delta_{X^{**}}(x^{**})) = T^{**}_{Q_\alpha} \circ T_\Theta(\delta_{X^{**}}(x^{**}))$ for every $x^{**} \in B_{X^{**}}$.

On the one hand, we know that $T_{\widetilde{Q}_\alpha}(\delta_{X^{**}}(x^{**})) = \widetilde{Q}_\alpha(x^{**})$. On the other hand, $T^{**}_{Q_\alpha} \circ T_\Theta(\delta_{X^{**}}(x^{**})) = T^{**}_{Q_\alpha}(\Theta(x^{**}))$. In order to understand this element of $\mathcal{G}_0(B_{X^{**}})$, let us see how it acts on any $f \in \mathcal{H}L_0(B_{X^{**}})$:

$$\langle T^{**}_{Q_\alpha}(\Theta(x^{**})), f\rangle = \langle \Theta(x^{**}), T^*_{Q_\alpha}(f)\rangle. \tag{5.1}$$

Now, $T^*_{Q_\alpha}(f)$ belongs to $\mathcal{H}L_0(B_X)$ and, for any $x \in B_X$, $T^*_{Q_\alpha}(f)$ satisfies

$$T^*_{Q_\alpha}(f)(x) = \langle T^*_{Q_\alpha}(f), \delta_X(x)\rangle = \langle f, T_{Q_\alpha}(\delta_X(x))\rangle = \langle f, Q_\alpha(x)\rangle = (T_f \circ Q_\alpha)(x).$$

Then, $T^*_{Q_\alpha}(f) = T_f \circ Q_\alpha$. Replacing this equality in Equation (5.1) and using the definition of Θ and the fact that the range of $\widetilde{Q}_\alpha$ is contained in $\mathcal{G}_0(B_{X^{**}})$ we derive

$$\begin{aligned}\langle T^{**}_{Q_\alpha}(\Theta(x^{**})), f\rangle &= \langle \Theta(x^{**}), T_f \circ Q_\alpha\rangle = \widetilde{T_f \circ Q}_\alpha(x^{**}) = T^{**}_f \circ \widetilde{Q}_\alpha(x^{**}) \\ &= T_f(\widetilde{Q}_\alpha(x^{**})) = \langle \widetilde{Q}_\alpha(x^{**}), f\rangle, \quad \text{for all } f \in \mathcal{H}L_0(B_{X^{**}}).\end{aligned}$$

Therefore, $T^{**}_{Q_\alpha}(\Theta(x^{**})) = \widetilde{Q}_\alpha(x^{**})$ and thus $T^{**}_{Q_\alpha} \circ T_\Theta(\delta_{X^{**}}(x^{**})) = T_{\widetilde{Q}_\alpha}(\delta_{X^{**}}(x^{**}))$ for every $x^{**} \in B_{X^{**}}$, which completes the proof. □

It is known (see, for instance, [17, Lemma 3] or [34, Theorem 3.5]) that X is 1-locally complemented in Y if and only if X^* is 1-complemented in Y^* (with projection being the restriction mapping). This is also equivalent to X^{**} being 1-complemented in Y^{**} (under the natural embedding).

Corollary 5.5. *If X is symmetrically regular and X^{**} has the MAP then $\mathcal{H}L_0(B_{X^{**}})$ is isometric to a 1-complemented subspace of $\mathcal{H}L_0(B_X)^{**}$.*

Under the same conditions as the previous results, we can also obtain a version for holomorphic Lipschitz functions of the following characterization of unique norm preserving extensions to the bidual, proved by Godefroy in [30].

Lemma 5.6. *Let X be a Banach space and $x^* \in X^*$ with $\|x^*\| = 1$. The following are equivalent:*

(i) x^* has a unique norm preserving extension to a functional on X^{**}.
(ii) The function $Id_{\overline{B}_{X^*}} : (\overline{B}_{X^*}, w^*) \longrightarrow (\overline{B}_{X^*}, w)$ is continuous at x^*.

Aron et al. [8] gave a version of this result for homogeneous polynomials. Later, other extensions appeared (for instance, in [25] for ideals of homogeneous polynomials and in [24] for bilinear mappings in operator spaces).

Now, the statement of the theorem in our setting is the following:

Theorem 5.7. *Suppose X is symmetrically regular and X^{**} has the MAP. Consider a function $f \in \mathcal{H}L_0(B_X)$ with $L(f) = 1$. Then, the following are equivalent:*

(i) f has a unique norm preserving extension to $\mathcal{H}L_0(B_{X^{**}})$.
(ii) The canonical extension from $(\overline{B}_{\mathcal{H}L_0(B_X)}, w^*)$ to $(\overline{B}_{\mathcal{H}L_0(B_{X^{**}})}, w^*)$ is continuous at f.
(iii) If the net $(f_\alpha) \subset \overline{B}_{\mathcal{H}L_0(B_X)}$ converges pointwise to f, then $(\widetilde{f}_\alpha) \subset \overline{B}_{\mathcal{H}L_0(B_{X^{**}})}$ converges pointwise to $\widetilde{f}$.

Proof. $(i) \Rightarrow (ii)$ Let $(f_\alpha) \subset \overline{B}_{\mathcal{H}L_0(B_X)}$ be a net which weak-star converges to a function $f \in \overline{B}_{\mathcal{H}L_0(B_X)}$. By the weak-star compactness of the ball $\overline{B}_{\mathcal{H}L_0(B_{X^{**}})}$ there is a subnet $(\widetilde{f}_\beta)$ which is weak-star convergent to a function $g \in \overline{B}_{\mathcal{H}L_0(B_{X^{**}})}$. Since for each $x \in B_X$, $\widetilde{f}_\alpha(x) = f_\alpha(x) \to f(x)$ we derive that $g|_{B_X} = f$. Also, since $L(g) \le 1 = L(f)$, it follows that $L(g) = L(f)$, which means that g is a norm preserving extension of f. By (i) and Proposition 4.4 we obtain that $g = \widetilde{f}$. Now, a standard subnet argument shows that the whole net $(\widetilde{f}_\alpha)$ must converge weak-star to $\widetilde{f}$.

$(ii) \Rightarrow (iii)$ It is clear due to Proposition 2.3 (d).

$(iii) \Rightarrow (i)$ Let $g \in \overline{B}_{\mathcal{H}L_0(B_{X^{**}})}$ be a norm preserving extension of f. By Proposition 5.1, there is a net $(Q_\alpha) \subset \mathcal{P}_{f,0}(X, Y)$ with $L(Q_\alpha|_{B_X}) \le 1$ satisfying $\widetilde{Q}_\alpha(x^{**}) \to g(x^{**})$ for all $x^{**} \in B_{X^{**}}$. But for any $x \in B_X$ we have $\widetilde{Q}_\alpha(x) = Q_\alpha(x) \to g(x) = f(x)$. Now, (iii) clearly implies that $g = \widetilde{f}$. □

All the numbered results of Sections 4 and 5 have easily adapted analogous versions for $\mathcal{G}$ and $\mathcal{H}L$ instead of $\mathcal{G}_0$ and $\mathcal{H}L_0$.

5.1 | Extensions of $\mathcal{H}^\infty(B_X)$ and $\mathcal{G}^\infty(B_X)$

The arguments of this section can be canonically translated to prove analogous results for the case of $\mathcal{G}^\infty$ instead of $\mathcal{G}_0$ (and $\mathcal{H}^\infty$ instead of $\mathcal{H}L_0$). Moreover, for this case the hypothesis of symmetrical regularity is unnecessary. Let us state the results without proofs, since they are similar to the previous arguments.

Theorem 5.8. *If X^{**} has the MAP then $\mathcal{G}^\infty(B_{X^{**}})$ is isometric to a 1-locally complemented subspace of $\mathcal{G}^\infty(B_X)^{**}$ and $\mathcal{H}^\infty(B_{X^{**}})$ is isometric to a 1-complemented subspace of $\mathcal{H}^\infty(B_X)^{**}$.*

The following question is posed in [17]: When X^{**} has the BAP, is it true that $\mathcal{H}^\infty(B_{X^{**}})$ is isomorphic to a complemented subspace of $\mathcal{H}^\infty(B_X)^{**}$? Note that the previous theorem affirmatively answers this open question for the case X^{**} having MAP.

Theorem 5.9. *Suppose X^{**} has the MAP. Consider a function $f \in \mathcal{H}^\infty(B_X)$ with $\|f\| = 1$. Then, the following are equivalent:*

(i) f has a unique norm preserving extension to $\mathcal{H}^\infty(B_{X^{**}})$.
(ii) The canonical extension from $(\overline{B}_{\mathcal{H}^\infty(B_X)}, w^*)$ to $(\overline{B}_{\mathcal{H}^\infty(B_{X^{**}})}, w^*)$ is continuous at f.
(iii) If the net $(f_\alpha) \subset \overline{B}_{\mathcal{H}^\infty(B_X)}$ converges pointwise to f, then $(\widetilde{f}_\alpha) \subset \overline{B}_{\mathcal{H}^\infty(B_{X^{**}})}$ converges pointwise to $\widetilde{f}$.

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REFERENCES

[1] A. Abbar, C. Coine, and C. Petitjean, *A pre-adjoint approach on weighted composition operators between spaces of Lipschitz functions*, Results Math. **79** (2024), no. 2, 85.
[2] P. Ahern and R. Schneider, *Holomorphic Lipschitz functions in pseudoconvex domains*, Amer. J. Math. **101** (1979), no. 3, 543–565.
[3] R. J. Aliaga, E. Pernecká, C. Petitjean, and A. Procházka, *Supports in Lipschitz-free spaces and applications to extremal structure*, J. Math. Anal. Appl. **489** (2020), no. 1, 124128.
[4] T. Ando, *On the predual of $\mathcal{H}^\infty$*, Comment. Math. **1** (1978), 33–40.
[5] R. Arens, *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc. **2** (1951), 839–848.
[6] R. Arens, *Operations induced in function classes*, Monatsh. Math. **55** (1951), 1–19.
[7] R. Aron and P. Berner, *A Hahn–Banach extension theorem for analytic mappings*, Bull. Soc. Math. France **106** (1978), 3–24.
[8] R. Aron, C. Boyd, and Y. S. Choi, *Unique Hahn–Banach theorems for spaces homogeneous polynomials*, J. Aust. Math. Soc. **70** (2001), 387–400.
[9] R. Aron, B. J. Cole, and T. W. Gamelin, *Spectra of algebras of analytic functions on a Banach space*, J. Reine Angew. Math. **415** (1991), 51–93.
[10] R. Aron, P. Galindo, D. García, and M. Maestre, *Regularity and algebras of analytic functions in infinite dimensions*, Trans. Amer. Math. Soc. **348** (1996), no. 2, 543–559.
[11] L. Bernal-González, J. López-Salazar, and J. B. Seoane-Sepúlveda, *On Weierstrass' monsters in the disc algebra*, Bull. Belg. Math. Soc. Simon Stevin **25** (2018), no. 2, 241–262.
[12] L. Bernal-González, A. Bonilla, J. López-Salazar, and J. B. Seoane-Sepúlveda, *Boundary-nonregular functions in the disc algebra and in holomorphic Lipschitz spaces*, Mediterr. J. Math. **15** (2018), 1–20.
[13] L. Bernal-González, A. Bonilla, J. López-Salazar, and J. B. Seoane-Sepúlveda, *Nowhere hölderian functions and Pringsheim singular functions in the disc algebra*, Monatsh. Math. **188** (2019), no. 4, 591–609.

[14] C. Bessaga and A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. **17** (1958), 151–164.
[15] J. Bruna and F. Tugores, *Free interpolation for holomorphic functions regular to the boundary*, Pac. J. Math. **108** (1983), no. 1, 31–49.
[16] F. Cabello Sánchez, J. Castillo, and R. García, *Polynomials on dual-isomorphic spaces*, Ark. Mat. **38** (2000), no. 1, 37–44.
[17] F. Cabello Sánchez and R. García, *The bidual of tensor product of Banach spaces*, Rev. Mat. Iberoam. **21** (2005), 843–861.
[18] D. Carando and I. Zalduendo., *Linearization of functions*, Math. Ann. **328** (2004), 683–700.
[19] P. G. Casazza, *Approximation properties*, Handbook of the Geometry of Banach Spaces, vol. I, North-Holland, Amsterdam, 2001, pp. 271–316.
[20] M. Cúth, M. Doucha, and P. Wojtaszczyk, *On the structure of Lipschitz-free spaces*, Proc. Amer. Math. Soc. **144** (2016), 3833–3846.
[21] A. M. Davie and T. W. Gamelin, *A theorem on polynomial-star approximation*, Proc. Amer. Math. Soc. **106** (1989), no. 2, 351–356.
[22] A. Defant, D. García, M. Maestre, and P. Sevilla-Peris, *Dirichlet series and holomorphic functions in high dimensions*, New Mathematical Monographs, vol. 37, Cambridge University Press, 2019.
[23] J. C. Díaz and S. Dineen, *Polynomials on stable spaces*, Ark. Mat. **36** (1998), 87–96.
[24] V. Dimant and M. Fernández-Unzueta, *Biduals of tensor products in operator spaces*, Studia Math. **230** (2015), no. 2, 165–185.
[25] V. Dimant, D. Galicer, and R. García, *Geometry of integral polynomials, M-ideals and unique norm preserving extensions*, J. Funct. Anal. **262** (2012), 1987–2012.
[26] P. L. Duren, *Theory of H_p spaces*, Dover, Mineola, NY, 2000.
[27] E. García-Sánchez, D. de Hevia, and P. Tradacete, Free objects in Banach space theory, https://arxiv.org/abs/2302.10807.
[28] J. B. Garnett, *Bounded analytic functions*, Graduate Texts in Mathematics, vol. 236, Springer, 2007.
[29] G. Godefroy, *A survey on Lipschitz-free Banach spaces*, Comment. Math. **55** (2015), 89–118.
[30] G. Godefroy, *Points de Namioka, espaces normante, applications á la Théorie isométrique de la dualité*, Israel J. Math. **38** (1981) 209–220.
[31] G. Godefroy and N. Kalton, *Lipschitz-free Banach spaces*, Studia Math. **159** (2003), 121–141.
[32] A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Soc. de Matemática de São Paulo, 1956.
[33] P. Hájek and T. Russo, *Projecting Lipschitz functions onto spaces of polynomials*, Mediterr. J. Math. **19** (2022), no. 4, 190.
[34] N. Kalton, *Locally complemented subspaces and $\mathcal{L}_p$-spaces for* $0 < p < 1$, Math. Nachr. **115** (1984), 71–97.
[35] M. Jung, *Daugavet property of Banach algebras of holomorphic functions and norm-attaining holomorphic functions*, Adv. Math. **421** (2023), 109005.
[36] S. Lassalle and I. Zalduendo, *To what extent does the dual Banach space E' determine the polynomials over E?*, Ark. Mat. **38** (2000), no. 2, 343–354.
[37] D. Leung, *Some remarks on regular Banach spaces*, Glasgow Math. J. **38** (1996), no. 2, 243–248.
[38] J. Mujica, *Linearization of bounded holomorphic mappings on Banach spaces*, Trans. Amer. Math. Soc. **324** (1991), 867–887.
[39] J. Mujica, *Linearization of holomorphic mappings of bounded type*, Progress in Functional Analysis, North-Holl. Math. Stud., vol. 170, North-Holland, Amsterdam, 1992, pp. 149–162.
[40] J. Mujica, *Complex analysis in Banach spaces*, Dover, Mineola, NY, 2010.
[41] K. F. Ng, *On a theorem of Dixmier*, Math. Scand. **29** (1971), 279–280.
[42] M. Pavlovic, *Function classes on the unit disc. An introduction*, Walter de Gruyter, Berlin, 2014.
[43] W. Robertson, *Topological vector spaces*, vol. 53, Cambridge University Press, 1980.
[44] W. Rudin, *Holomorphic Lipschitz functions in balls*, Comment. Math. Helv. **53** (1978), no. 1, 143–147.
[45] R. A. Ryan, *Introduction to tensor products of Banach spaces*, Springer Monogr. Math., Springer, London, 2002.
[46] N. Weaver, *Lipschitz Algebras*, 2nd ed., World Sci., Hackensack, NJ, 2018.

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APPENDIX A

Finally, we will prove the following result as promised in Section 2.

Theorem A.1. *There exists an isomorphism into* $F : \ell_\infty \to \mathcal{H}_0^\infty(\mathbb{D})$ *such that* $F(\ell_\infty \setminus \{0\}) \subset H_0^\infty(\mathbb{D}) \setminus \mathcal{H}L_0(\mathbb{D})$ *and* $F(c_0 \setminus \{0\}) \subset \mathcal{A}(\mathbb{D}) \setminus \mathcal{H}L_0(\mathbb{D})$.

Note that one can easily prove a version for holomorphic functions on B_X for any X using the same ideas as in the proof of the second part of Theorem 2.2(a).

In what follows, we will use the function $\varphi_\lambda : \mathbb{C} \to \mathbb{C}$ given by

$$\varphi_\lambda(z) = \frac{\overline{\lambda} z + 1}{2}.$$

It is standard that

$$\varphi_\lambda(\lambda) = 1, \quad |\varphi_\lambda(z)| < 1 \text{ for all } z \in \overline{\mathbb{D}} \setminus \{\lambda\}. \tag{A.1}$$

We also need the following technical lemma, which in particular provides another example of a non-Lipschitz function in the disc algebra $\mathcal{A}(\mathbb{D})$.

Lemma A.2. *Fix $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and define $f_\lambda : \mathbb{C} \to \mathbb{C}$ by*

$$f_\lambda(z) = \begin{cases} 1 + (\overline{\lambda} z - 1)e^{1/(\overline{\lambda} z - 1)} & \text{if } z \neq \lambda \\ 1 & \text{if } z = \lambda. \end{cases}$$

Then

(a) *f_λ is holomorphic in $\mathbb{C} \setminus \{\lambda\}$.*
(b) *The restriction of f_λ to $\overline{\mathbb{D}}$ belongs to $\mathcal{A}(\mathbb{D}) \setminus \mathcal{H}L(\mathbb{D})$.*
(c) *$|f_\lambda(z)| \leq 3$ for all $z \in \overline{\mathbb{D}}$.*
(d) *If $0 < s < 1$, then $|f'_\lambda(z)| \leq \frac{s+1}{s}$ for all $z \in \mathbb{D}$ such that $|z - \lambda| \geq s$.*
(e) *Given $k \in \mathbb{N}$ and $0 < \delta < 1$, we have that*

$$\sup_{z \in \mathbb{D}(\lambda, \delta) \cap \mathbb{D}} |(f_\lambda \cdot \varphi_\lambda^k)'(z)| = +\infty.$$

Proof. A standard computation shows that (a) holds. Now, to prove the rest of the claims it is enough to consider the case $\lambda = 1$. Denote $f = f_1$ and take $z = a + ib \in \overline{\mathbb{D}} \setminus \{1\}$, with $a, b \in \mathbb{R}$. We have that

$$\left| \mathrm{e}^{\frac{1}{z-1}} \right| = \mathrm{e}^{\mathrm{Re}\,\frac{1}{z-1}} = \mathrm{e}^{\frac{a-1}{(a-1)^2 + b^2}} \leq \mathrm{e}^0 = 1.$$

Hence f, defined as $f(z) = 1 + (z-1)\mathrm{e}^{\frac{1}{z-1}}$ is holomorphic on $\mathbb{C} \setminus \{1\}$ and continuously extends to $\overline{\mathbb{D}}$. Further $|f(z)| \leq 3$ for every $z \in \overline{\mathbb{D}}$. Let us show that f is not a Lipschitz function. For that, it is enough to check that f' is not bounded on $\mathbb{D}$. Taking a null sequence (θ_n), $0 < \theta_n < 1$, and setting $z_n := \cos\theta_n(\cos\theta_n + \mathrm{i}\sin\theta_n)$, we obtain that the sequence $(z_n) \subset \mathbb{D}$ converges to 1 and

$$|f'(z_n)| = \left| \frac{z_n - 2}{z_n - 1} \right| \mathrm{e}^{\mathrm{Re}\left(\frac{1}{z_n - 1}\right)} = \left| \frac{z_n - 2}{z_n - 1} \right| \mathrm{e}^{-1}.$$

Consequently, $\lim_{n \to +\infty} |f'(z_n)| = +\infty$. Thus far we have proved (a), (b), and (c). Let us check (d). We have

$$|f'(z)| = \left| \frac{z-2}{z-1} \right| \cdot \left| \mathrm{e}^{\frac{1}{z-1}} \right| \leq 1 + \frac{1}{|z-1|},$$

for all $z \in \mathbb{D}$. Hence, if $0 < s < 1$ and $z \in \mathbb{D}$ with $|z - 1| \geq s$ we have that $|f'(z)| \leq \frac{s+1}{s}$.

Finally, (e) is a consequence of $(f\varphi^k)'(z) = f'(z)\varphi^k(z) + f(z)(\varphi^k)'(z)$ for all $z \in \mathbb{C} \setminus \{1\}$. □

Proof of Theorem A.1. To begin with, we choose a sequence $(\lambda_n) \subset \mathbb{C} \setminus \{1\}$ convergent to 1 with $|\lambda_n| = 1$ and $\lambda_n \neq \lambda_m$ for every $n \neq m$. Consider the functions $\Phi : \mathbb{C}^2 \to \mathbb{C}$ and $\varphi_n : \mathbb{C} \to \mathbb{C}$, $\varphi_n(z) := \Phi(z, \lambda_n)$ defined as

$$\Phi(z, \lambda) = \frac{\overline{\lambda} z + 1}{2}$$

and, for each $p \in \mathbb{N}$, the compact subset of $\mathbb{C}^2$

$$K_p = \{(\lambda_p, \lambda_n) : n \in \mathbb{N}, n \neq p\} \cup \{(\lambda_p, 1)\}.$$

We have $|\Phi(z,\lambda)| < 1$ for every $(z,\lambda) \in K_p$ by Equation (A.1), and Φ is continuous on $\mathbb{C}^2$. Hence, there exists $0 < s_p < 1$ such that $|\Phi(z,\lambda)| < 1$ for every $(z,\lambda) \in K_p + \overline{\mathbb{D}}((0,0), s_p)$. In particular,

$$|\varphi_n(z)| = |\Phi(z,\lambda_n)| < 1, \tag{A.2}$$

for all $z \in \overline{\mathbb{D}}(\lambda_p, s_p)$ and all $n \neq p$.

Now, since the sequence (λ_n) is convergent to 1 we can find a sequence of positive numbers (r_n) that tends to 0 such that $0 < 2r_n < s_n$ for all $n \in \mathbb{N}$ and such that $\overline{\mathbb{D}}(\lambda_n, 2r_n) \cap \overline{\mathbb{D}}(\lambda_p, 2r_p) = \emptyset$, for all $n \neq p$. Moreover, as (r_n) converges to 0, for each $n \in \mathbb{N}$ the set

$$L_n := \bigcup_{p \neq n} \overline{\mathbb{D}}(\lambda_p, 2r_p) \cup \{1\},$$

is also a compact subset of $\mathbb{C}$, (although it is not a subset of $\overline{\mathbb{D}}$) and $|\varphi_n(z)| < 1$ for all $z \in L_n$. Since $|\varphi_n|$ is continuous on $\mathbb{C}$ we obtain that

$$\max\{|\varphi_n(z)| \,:\, z \in C_n \cup L_n\} < 1,$$

for all n, where $C_n = \overline{\mathbb{D}} \setminus \mathbb{D}(\lambda_n, r_n)$. As a consequence, for each n the sequence $\left(\varphi_n^k\right)_{k=1}^{\infty}$ converges uniformly to 0 on $C_n \cup L_n$ and we can find a $k_n \in \mathbb{N}$ such that

$$|\varphi_n^{k_n}(z)| < \frac{r_n}{3^{n+1}}, \tag{A.3}$$

for every $z \in C_n \cup L_n$.

We denote $f_n := f_{\lambda_n}$, for $n \in \mathbb{N}$ and we define $F : \ell_\infty \longrightarrow \mathcal{H}^\infty(\mathbb{D})$ by

$$F(a_n) := \sum_{n=1}^{\infty} a_n f_n \varphi_n^{k_n},$$

For each $(a_n) \in \ell_\infty$, the series $F(a_n)(z)$ is convergent for each $z \in \overline{\mathbb{D}}$. To see this, we first suppose that

(a) $z \in \overline{\mathbb{D}} \setminus \left(\bigcup_{n=1}^{\infty} \overline{\mathbb{D}}(\lambda_n, r_n)\right)$. In that case, by Equation (A.3) and Lemma A.2. (c),

$$\sum_{n=1}^{\infty} |a_n f_n(z) \varphi_n^{k_n}(z)| \leq \sum_{n=1}^{\infty} 3|a_n| \frac{r_n}{3^{n+1}} \leq \frac{1}{2} \|(a_n)\|_\infty. \tag{A.4}$$

Hence, $F(a_n)(z)$ converges. Moreover, the series $F(a_n)$ converges absolutely and uniformly on the open set $\mathbb{D} \setminus \left(\bigcup_{n=1}^{\infty} \overline{\mathbb{D}}(\lambda_n, r_n)\right)$. Thus, $F(a_n)$ is holomorphic in that open set.

If (a) does not occur, then it must be that we have:

(b) There exists a unique $n_0 \in \mathbb{N}$ such that $z \in \mathbb{D}(\lambda_{n_0}, 2r_{n_0})$. By Equation (A.3), for every $u \in \mathbb{D}(\lambda_{n_0}, 2r_{n_0})$ we have that

$$|a_n f_n(u) \varphi_n^{k_n}(u)| \leq 3|a_n| \frac{r_n}{3^{n+1}} < \frac{|a_n|}{3^n},$$

for all $n \neq n_0$ and

$$|a_{n_0} f_{n_0}(u) \varphi_{n_0}^{k_{n_0}}(u)| \leq 3|a_{n_0}|.$$

Hence,

$$\sum_{n=1}^{\infty} |a_n f_n(z) \varphi_n^{k_n}(z)| \leq 4\|(a_n)\|_\infty, \tag{A.5}$$

and we have obtained that for every $z \in \mathbb{D}(\lambda_{n_0}, 2r_{n_0})$, $F(a_n)(z)$ exists and in fact $|F(a_n)(z)| \leq 4\|(a_n)\|_\infty$. But our argument shows that the series $F(a_n)$ is absolutely and uniformly convergent in the open disc $\mathbb{D}(\lambda_{n_0}, 2r_{n_0})$. Hence, $F(a_n)$ is holomorphic on $\mathbb{D} \cup \bigcup_{n=1}^{\infty} \mathbb{D}(\lambda_n, 2r_n)$ and $F : \ell_\infty \to \mathcal{H}^\infty(\mathbb{D})$ is a continuous linear mapping since $\|F(a_n)\| \leq 4\|(a_n)\|_\infty$ for all $(a_n) \in \ell_\infty$.

Now, we check that F is bounded below. We already know that for each $(a_n) \in \ell_\infty$, the function $F(a_n)$ is holomorphic on $\mathbb{D} \cup \bigcup_{n=1}^{\infty} \mathbb{D}(\lambda_n, 2r_n)$ and bounded on $\mathbb{D}$. Thus, using Equation (A.3) and the fact that $\lambda_p \in \overline{\mathbb{D}}$, we get

$$\|F(a_n)\| = \sup_{z\in\mathbb{D}} |F(a_n)(z)| \geq \sup_{p\in\mathbb{N}} |F(a_n)(\lambda_p)| \geq \sup_{p\in\mathbb{N}} \left\{ |a_p| - \sum_{n\neq p} 3|a_n| \frac{r_n}{3^{n+1}} \right\}$$

$$\geq \sup_{p\in\mathbb{N}} \left\{ |a_p| - \frac{\|(a_n)\|_\infty}{2} \right\} = \frac{\|(a_n)\|_\infty}{2}$$

for every $(a_n) \in \ell_\infty$.

Let us check that if $(b_n) \in c_0$, then $F(b_n)$ belongs to $\mathcal{A}(\mathbb{D})$. Given $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that $|b_n| < \frac{\varepsilon}{3}$, for every $n \geq n_1$. Thus, if $z \in \overline{\mathbb{D}}$.

$$\sum_{n=n_1}^{\infty} |b_n f_n(z)\varphi_n^{k_n}(z)| \leq 3\varepsilon \sum_{n=n_1}^{\infty} |\varphi_n^{k_n}(z)|. \tag{A.6}$$

Now if, $z \in \overline{\mathbb{D}} \setminus \left(\bigcup_{n=1}^{\infty} \overline{\mathbb{D}}(\lambda_n, r_n)\right)$, then by Equation (A.3), $|\varphi_n^{k_n}(z)| \leq \frac{r_n}{3^{n+1}}$. Hence, by Equation (A.6),

$$\sum_{n=n_1}^{\infty} |b_n f_n(u)\varphi_n^{k_n}(z)| < \varepsilon.$$

Otherwise, if $z \in \overline{\mathbb{D}} \cap \left(\bigcup_{n=1}^{\infty} \overline{\mathbb{D}}(\lambda_n, r_n)\right)$, there is a unique $n_0 \in \mathbb{N}$ such that $z \in \overline{\mathbb{D}}(\lambda_{n_0}, r_{n_0})$ and

$$\sum_{n=n_1}^{\infty} |b_n f_n(z)\varphi_n^{k_n}(z)| \leq \varepsilon + \sum_{\substack{n=n_1\\ n\neq n_0}}^{\infty} \varepsilon \frac{r_n}{3^{n+1}} < 2\varepsilon.$$

Consequently, the series $\sum_{n=1}^{\infty} b_n f_n(z)\varphi_n^{k_n}(z)$ converges absolutely and uniformly on $\overline{\mathbb{D}}$ and $F_{|c_0} : c_0 \to \mathcal{A}(\mathbb{D})$ is a well-defined continuous linear mapping.

Consider $(a_n) \in \ell_\infty \setminus \{0\}$. There exists n_0 such that $a_{n_0} \neq 0$. We are going to show that $F(a_n)'(z)$ is not bounded on $\mathbb{D}(\lambda_{n_0}, \frac{r_{n_0}}{3}) \cap \mathbb{D}$.

By the Weierstrass theorem,

$$F(a_n)'(z) = \sum_{n=1}^{+\infty} a_n \left(f_n \varphi_n^{k_n}\right)'(z),$$

for every $z \in \mathbb{D} \cup \bigcup_{n=1}^{\infty} \mathbb{D}(\lambda_n, 2r_n)$. If $n \neq n_0$, then by the Cauchy integral formula

$$\left(\varphi_n^{k_n}\right)'(z) = \frac{1}{2\pi \mathrm{i}} \int_{C(\lambda_{n_0}, r_{n_0})} \frac{\varphi_n^{k_n}(u)}{(u-z)^2} du,$$

for every $z \in \mathbb{D}(\lambda_{n_0}, \frac{r_{n_0}}{3})$. Thus, by Equations (A.2) and (A.3), we obtain

$$\sup_{z\in\mathbb{D}(\lambda_{n_0}, \frac{r_{n_0}}{3})} \left|\left(\varphi_n^{k_n}\right)'(z)\right| \leq \frac{r_{n_0}}{(\frac{2}{3} r_{n_0})^2} \sup_{|u-\lambda_{n_0}|=r_{n_0}} |\varphi_n^{k_n}(u)| < \frac{9}{4r_{n_0}} \frac{r_n}{3^{n+1}} < \frac{1}{r_{n_0}} \frac{1}{3^n},$$

and we get

$$|\left(f_n\varphi_n^{k_n}\right)'(z)| \le |f_n'(z)||\varphi_n^{k_n}(z)| + |f_n(z)||(\varphi_n^{k_n})'(z)| < \frac{1}{3^n} + \frac{1}{r_{n_0}}\frac{1}{3^{n-1}},$$

where in the second inequality we have applied, Equations (A.2), (A.3), and the properties of f_n and f_n' given in Lemma A.2. Hence,

$$|F(a_n)'(z)| \ \ge |a_{n_0}||\left(f_{n_0}\varphi_{n_0}^{k_{n_0}}\right)'(z)| - \|(a_n)\|_\infty\left(\frac{1}{2} + \frac{3}{2r_{n_0}}\right),$$

for every $z \in \mathbb{D}(\lambda_{n_0}, \frac{r_{n_0}}{3})$. Finally, by Lemma A.2.(e), we have that $F(a_n)'$ is unbounded on $\mathbb{D}(\lambda_{n_0}, \frac{r_{n_0}}{3}) \cap \mathbb{D}$ and hence, $F(a_n)$ does not belong to $\mathcal{H}L(\mathbb{D})$.

To conclude, if we define $F_1 : \ell_\infty \to \mathcal{H}_0^\infty(\mathbb{D})$ by $F_1(a_n)(z) := zF(a_n)(z)$ for $(a_n) \in \ell_\infty$ and $z \in \mathbb{D}$, it is clear that F_1 is an isomorphism onto its image and that $F_1(\ell_\infty \setminus \{0\}) \subset \mathcal{H}_0^\infty(\mathbb{D}) \setminus \mathcal{H}L(\mathbb{D})$. □

Finally, we note that if we are only interested in $\mathcal{A}(\mathbb{D})$, then there are known results related to Theorem A.1. Indeed, in three relevant papers [11–13], Bernal et al. have obtained many results on the existence of large subspaces of functions that belong to $\mathcal{A}(\mathbb{D}) \setminus \mathcal{H}L(\mathbb{D}) \cup \{0\}$. In particular, in [11, Theorem 4.1.c] the authors show that there exists an infinite-dimensional Banach space X contained in $\mathcal{A}(\mathbb{D})$ such that any non-null function in X is not differentiable on any point of a fixed dense subset of $\mathbb{T}$. Also, in [13, Theorem 3.4], the authors prove that there exists an infinite-dimensional Banach space X, contained in $\mathcal{A}(\mathbb{D})$, (which, however, is endowed with a stronger norm than the one inherited from $\mathcal{A}(\mathbb{D})$) such that if $f \in X$, then the restriction of f to $\mathbb{T}$ is nowhere Hölder on $\mathbb{T}$.