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Tesis Doctoral

PERIPHERAL STRUCTURES AND TOPOLOGICAL INVARIANTS OF KNOTTED SUBMANIFOLDS

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TESIS DOCTORAL

Peripheral structures and topological invariants of knotted submanifolds

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Moi, j'ai trahi la musique respectable pour la musique concrète, et le violoncelle de mon enfance pour le magnétophone et pour le potentiomètre. [. . .]
Nous autres dans nos studios avec nos armes automatiques, multipliant d'un coup de pouce le nombre des exécutants, gonflant le volume des orchestres, nous trichons.

Me, I have betrayed respectable music for practical music, and the cello of my childhood for the tape recorder and for the potentiometer. [. . .]
The rest of us in our studios, with our automatic weapons, multiplying the number of performers with a flick, swelling up entire orchestras, we cheat. *(translated)*

Yo, he traicionado a la música respetable para la música concreta, y al violonchelo de mi infancia para el magnetófono y para el potenciómetro. [. . .]
Nosotros en nuestros estudios, con nuestras armas automáticas, multiplicando los intérpretes con una toba, hinchando las orquestas, hacemos trampas. *(traducido)*

— Pierre Schaeffer
La leçon de musique
De la musique concrète à la musique même, 1979

ABSTRACTS

Abstract (in English)

We study knotted codimension-two objects in manifolds of dimension 3 and 4: complex line arrangements in $\mathbb{CP}^2$ and links in S^3. We introduce new topological invariants of their embedding, derived from the interaction between their complement and their peripheral structure.

The motivation for line arrangements is to identify *Zariski pairs* which have the same combinatorics but different embeddings. Building on ideas developed by B. Guerville-Ballé and W. Cadiegan-Schlieper, we consider the inclusion map of boundary manifold to the exterior and its effect on homology classes. A careful study of the graph Waldhausen structure of the boundary manifold allows to identify specific generators of the homology. Their potential images are encoded in a group, the *graph stabiliser*, with a nice combinatorial presentation. The invariant related to the inclusion map is an element of this group. Using a computer implementation in `Sage` and the braid monodromy, we compute the invariant for some examples and exhibit new ordered Zariski pairs.

The second part concerns knot theory and a generalisation of a slope invariant developed by A. Degtyarev, V. Florens and A.G. Lecuona. Similarly to the context of line arrangements, we consider the inclusion map of the boundary components of a neighbourhood of a link in its exterior. On twisted homology, the kernel of this map is a Lagrangian subspace – for the intersection form – and its slopes provide a topological invariant of the link. We present two applications of this idea. In the first, developed in collaboration with L. Bénard, we consider knots and $\mathrm{SL}_2(\mathbb{C})$ representations. This new slope invariant appears to be closely related to a higher-level invariant called the A-polynomial. The second application uses a Lagrangian characterisation method due to V. Arnol'd. It provides a concordance invariant with several relations to Sato-Levine invariant and Milnor linking numbers.

Resumen (en castellano)

Estudiamos objetos anudados de codimensión dos en variedades de dimensión 3 y 4: configuraciones de rectas complejas en $\mathbb{CP}^2$ y enlaces en S^3. Introducimos nuevos invariantes topológicos de su encaje, que provienen de la interacción entre el complementario y su estructura periférica.

La motivación para las configuraciones de rectas es identificar *pares de Zariski* que tienen la misma combinatoria pero diferentes encajes. Basándonos en las ideas desarrolladas por B. Guerville-Ballé y W. Cadiegan-Schlieper, consideramos el mapa de inclusión de la variedad límite hacia el exterior y su efecto sobre las clases de homología. Un estudio cuidadoso de la estructura de grafo de Waldhausen de la variedad del borde permite identificar generadores específicos de la homología. Sus imágenes potenciales están codificadas en un grupo, el *estabilizador del grafo*, con una elegante presentación combinatoria. El invariante relacionado con la inclusión es un elemento de este grupo. Utilizando una implementación informática en `Sagemath` y la monodromía de trenzas, calculamos el invariante para algunos ejemplos y encontramos nuevos pares ordenados de Zariski.

La segunda parte se refiere a la teoría de nudos y una generalización de un invariante llamado *pendiente* («slope») desarrollado por A. Degtyarev, V. Florens y A.G. Lecuona. De manera similar al contexto de configuraciones de rectas, consideramos la inclusión de los componentes del borde de un entorno de un enlace en su exterior. En homología torcida, el núcleo de esta aplicación es un subespacio lagrangiano –para la forma de intersección– y sus pendientes proporcionan un invariante topológico del enlace. Presentamos dos aplicaciones de esta idea. En el primero, desarrollado en colaboración con L. Bénard, consideramos nudos y representaciones $\mathrm{SL}_2(\mathbb{C})$. Este nuevo invariante de pendiente parece estar estrechamente relacionado con un invariante de nivel superior llamada polinomio A. La segunda aplicación utiliza un método de caracterización lagrangiano debido a V. Arnol'd. Proporciona un invariante de concordancia con varias relaciones con el invariante de Sato-Levine y los números de enlace de Milnor.

Résumé (en français)

On étudie des objets noués de codimension 2 dans des variétés de dimension 3 et 4 : des arrangements de droites complexes dans $\mathbb{CP}^2$ et des entrelacs dans S^3. On introduit de nouveaux invariants topologiques de leurs plongements, dérivés des interactions entre leur complémentaire et leur structure périphérique.

La motivation concernant les arrangements de droites est d'identifier des *paires de Zariski* qui ont la même combinatoire mais des plongements différents. En utilisant des idées développées par B. Guerville-Ballé et W. Cadiegan-Schlieper, on considère l'application inclusion de la variété bord dans l'extérieur et son effet sur les classes d'homologie. Une étude approfondie de la structure graphée de Waldhausen de la variété bord permet d'identifier des générateurs spécifiques de son homologie. L'information de leurs images potentielles est collectée dans un groupe, le *stabilisateur du graphe*, qui a une présentation combinatoire simple. On utilise une implémentation en `Sage` et la monodromie de tresses pour calculer l'invariant dans certains exemples et produire de nouvelles paires de Zariski ordonnées.

La seconde partie est consacrée à la théorie des nœuds et à une généralisation d'un invariant de pente (*« slope »*) développé par A. Degtyarev, V. Florens et A.G. Lecuona. Similairement aux arrangements de droites, on considère l'application inclusion des composantes de bord d'un voisinage de l'entrelacs dans l'extérieur. Au niveau de l'homologie tordue, le noyau de cette application est un sous-espace Lagrangien – pour la forme d'intersection – et sa pente est un invariant topologique de l'entrelacs. On présente deux applications de cette construction. Dans la première, en collaboration avec L. Bénard, on considère le cas des nœuds et des representations dans $\mathrm{SL}_2(\mathbb{C})$. L'invariant *slope* obtenu est étroitement relié à un invariant de haut niveau, le A-polynôme. La seconde application utilise une caractérisation des sous-espaces lagrangiens due à V. Arnol'd. On construit un invariant de concordance qui a de nombreux liens avec l'invariant de Sato-Levine et les enlacements de Milnor.

CONTENTS

INTRODUCTION

Peripheral structures in codimension 2

The subject of this thesis is to study certain families of curves of co-dimension 2. The works of Smale [Sma62] and Milnor [Mil65] made it possible to determine an algebraic classification of smooth manifolds in dimension ≥ 5. However, the study of smooth manifolds in dimension 3 and 4 is still a very active domain which involves a wide variety of tools from algebraic topology and geometry. In our work we notably make use of the fundamental group, several types of (co-)homology, intersections and mapping class groups. Graphs and related combinatorial methods also make a significant contribution.

Consider a manifold M of dimension 3 or 4 with boundary. In the cases we study this manifold will be the complement of a regular neighbourhood of certain curves of co-dimension 2. Our approach is to study the inclusion

$$i : \partial M \hookrightarrow M$$

In several situations the boundary has a much simpler structure than the manifold itself. Yet this inclusion and its induced morphisms still contain significant topological information. We consider several types of algebraic topology structures for the induced morphisms, mainly the fundamental group $\pi_1(M)$, the homology group $H_1(M, \mathbb{Z})$ and the twisted homology group $H_1(M; \rho)$ with respect to a representation $\rho : \pi_1(M) \to \mathrm{GL}(V)$ of the fundamental group over a vector space V. This last structure, also called *homology with local coefficients*, computes homological groups which have the structure of vector spaces. General construction of twisted homology is detailed for example in [Hat02], and more specialised results used in our work are presented in Appendix A.

We apply this approach to the complements of two families of curves: singular plane algebraic curves embedded in $\mathbb{CP}^2$, in dimension 4, and knots and links embedded in S^3, in dimension 3.

Topology of line arrangements

Generalities

A plane algebraic curve $\mathcal{C}$ is the zero locus of a complex homogenous polynomial. The topological study of these curves was initiated by O. Zariski which considered them as branch curves of algebraic functions. He showed that all such smooth curves of a given degree are isotopic [Zar29], which enticed to focus on curves with singularities. The *combinatorics* of the curve is the topological type of the pair $(\mathcal{C}, N_{\mathcal{C}})$ where $N_{\mathcal{C}}$ is a tubular neighbourhood of $\mathcal{C}$. These data is determined by the topological type of the singularities and the incidence relations between the components. Following the works of O. Zariski [Zar31; Zar37] and E. van Kampen [Kam33] it is known that there exist pairs of curves $\mathcal{C}_1, \mathcal{C}_2$ with the same combinatorics but different embeddings in $\mathbb{CP}^2$, which were dubbed *Zariski pairs* by E. Artal in [Art94].

Line arrangements are finite collections of complex lines in $\mathbb{CP}^2$ that is, curves whose defining polynomial has irreducible factors of degree 1. They form a subclass of plane algebraic curves whose combinatorics depends only on the incidence relations, which can be encoded in a graph called the *incidence graph*. The components of a line arrangement $\mathcal{A}$ are non-singular and the singularities are all pairwise transverse intersections. For all these reasons the study of line arrangements provides a favourable setting to create topological methods and invariants that could then be extended to algebraic curves in general. It also offers some interesting questions in itself. The first potential Zariski pair of line arrangements was discovered by G. Rybnikov [Ryb11]. It was definitively confirmed as a Zariski pair by E. Artal, J. Carmona, J.I. Cogolludo and M.Á. Marco in [Art+06]. They considered the complement of $\mathcal{A}$ and proved that the two complements of the pair have non-homeomorphic fundamental groups. This showed that the combinatorics does not determine the topological type of a curve even in the simplest case of line arrangements. The same team also found a Zariski pair of arithmetic complexified real arrangements in [Art+05]. The search for more Zariski pairs and a finer comprehension of the relationship between combinatorics and topology of curves and line arrangements has been a very active topic since the 2000s. S. Nazir, M. Yoshinaga [NY12] and F. Ye [Ye13] have shown that no Zariski pairs exist for line arrangements with less than 10 lines. Many new Zariski pairs have been found since thanks to the works of B. Guerville-Ballé [Gue16] and J. Viu Sos [GV19]. Readers can refer to [ACT08; Gue22] for a more detailed review of the subject.

Review of common invariants

A wide variety of common invariants from algebraic topology have been applied to the study of line arrangements and Zariski pairs. E. van Kampen [Kam33] gave a method to compute a presentation of the fundamental group of the exterior of an algebraic curve, now called the *Zariski-van Kampen method*. Direct comparison of fundamental groups can sometimes give a Zariski pair, as for the original example of G. Rybnikov. However, there are known examples of Zariski pairs where the fundamental groups of the complements are isomorphic (see [Shi09; Shi19; ACM19a; Gue20]) and it has since then be shown that not even the characteristic varieties determine the topology of line arrangements in general, including for complexified arrangement drawn in $\mathbb{RP}^2$.

The Zariski-van Kampen method makes use of a construction called *braid monodromy*, introduced by O. Chisini in [Chi33], which encodes as a braid the relative position of each component of the curve in the vicinity of each singularity. This approach was refined by B. Moishezon in [Moi81] and for the specific case of real line arrangements by M. Salvetti [Sal88]. A. Libgober [Lib86] has shown that the braid monodromy does determine the homotopy type of a line arrangement, and J. Carmona [Car03] has extended it to the topological type (see also [ACC03]). The braid monodromy has appeared to be a slightly more effective invariant than the fundamental group and more Zariski pairs have been obtained using it, see [ACT08] for a survey.

Homology inclusion invariants

As we mentioned earlier, another approach to build invariants of line arrangements is to focus on the *boundary manifold* $B_{\mathcal{A}} := \partial E_{\mathcal{A}}$ of the exterior $E_{\mathcal{A}}$ of the arrangement, defined as the complement of a tubular neighbourhood of $\mathcal{A}$. T. Jiang, S. S.-T. Yau [JY93] and E. Westlund [Wes97] have shown that $B_{\mathcal{A}}$ has the structure of a *graph manifold* as defined by F. Waldhausen [Wal67a; Wal67b] and W. Neumann [Neu81]. The topology of $B_{\mathcal{A}}$ is directly determined by the combinatorics $C_{\mathcal{A}}$ of the line arrangement $\mathcal{A}$, encoded in the form of a graph $\Gamma_{\mathcal{A}}$. This graph provides a 'blueprint' to reconstruct $B_{\mathcal{A}}$ by gluing together circle bundles (Seifert pieces) corresponding to the boundary of local neighbourhoods around each line component and singularity of $\mathcal{A}$. Our own interest lies in the study of the inclusion

$$i : B_{\mathcal{A}} \hookrightarrow E_{\mathcal{A}}$$

which still contains topological information on the exterior $E_{\mathcal{A}}$. This approach was first taken by E. Hironaka [Hir01] on the fundamental groups of complexified real line arrangements, and then continued by E. Artal, V. Florens, B. Guerville-Ballé and M.Á. Marco in [FGM15; AFG17]. They initiated the study of the induced morphism

$$i_* : H_1(B_{\mathcal{A}}, \mathbb{C}) \longrightarrow H_1(E_{\mathcal{A}}, \mathbb{C})$$

The group $H_1(B_{\mathcal{A}}, \mathbb{C})$ has two types of generators: the meridians of the line components, and generators arising from the cycles of the graph $\Gamma_{\mathcal{A}}$ (see Theorem 1.5.16). The main difficulty lies in the ambiguity of defining the set of cycle generators. In other words the map

$$H_1(\Gamma_{\mathcal{A}}) \xrightarrow{\gamma_*} H_1(B_{\mathcal{A}}, \mathbb{C}) \xrightarrow{i_*} H_1(E_{\mathcal{A}}, \mathbb{C}) \tag{G}$$

depends on the choice of the embedding

$$\gamma : \Gamma_{\mathcal{A}} \longrightarrow B_{\mathcal{A}}$$

The cycle generators form a freely generated subgroup of $H_1(B_{\mathcal{A}}, \mathbb{C})$, yet their images are sums of the meridians in $H_1(E_{\mathcal{A}}, \mathbb{C})$. The composition $i_* \circ \gamma_*$ contains information from both the combinatorics, through the graph itself, and from the topology of the line arrangement $\mathcal{A}$, through the closure of the cycles by homological discs in the exterior. A proper definition of γ where the combinatorial dependence is precisely determined is the key to access only the topological part of the information.

The approach to this problem in [FGM15; AFG17] is to consider characters of the fundamental group $\pi_1(E_{\mathcal{A}})$. They introduce *inner cyclic triplets* (C, ω, c) where C is a combinatorics, ω is a character and $c \in H_1(\Gamma_{\mathcal{A}})$ is a cycle of the graph such that ω is trivial on all meridian generators of $H_1(B_{\mathcal{A}})$ along c. This allows to define a local embedding γ such that $\omega \circ i_* \circ \gamma_*$ is a topological invariant on all neighbour cycles of c. The restriction of $\omega \circ i_*$ on these cycles constitutes the *$\mathcal{I}$-invariant*. It successfully detects known Zariski pairs [AFG17] as well as a new Zariski quadruplet [Gue16].

The main downside of the $\mathcal{I}$-invariant is that it can only be applied to certain combinatorics because it does not provide a complete description of $i_* \circ \gamma_*$. A more algebraic approach of the problem is taken by W. Cadegan-Schlieper in his Ph.D. thesis [Cad18] to generalise the $\mathcal{I}$-invariant. The *loop-linking number* gives the images by i_* of all cycle generators of $H_1(B_{\mathcal{A}}, \mathbb{C})$ using an embedding γ that respects a list of combinatorial conditions on the graph $\Gamma_{\mathcal{A}}$, based on a construction due to P. Orlik and L. Solomon [OS80]. These conditions depend only on the ordered combinatorics. B. Guerville-Ballé [Gue22] has successfully used the loop-linking number to obtain more Zariski pairs with non-isomorphic fundamental groups.

Our work presents a new generalisation of these ideas. Our new approach is to completely determine the combinatorial dependence of the values of the map $i_* \circ \gamma_*$ in Eq. (G). We use a new construction of graphed embeddings which fully exploits the ordered graph manifold structure of the boundary $B_{\mathcal{A}}$. We deduce from this a new set of relations that completely accounts for the differences in values caused by a combinatorial change of generators. Inside the quotient, i_* becomes a topological invariant on any ordered line arrangement, which we call the *homology inclusion*.

Summary of Part I

Chapter 1 is dedicated to the combinatorial study of the boundary manifold $B_{\mathcal{A}}$ and our new class of graphed embeddings.

After some preliminaries in Section 1.1, we present in Section 1.2 the *ordered stars* which are the 'elementary bricks' used to build the embeddings. In Section 1.3 we recall the structure of a graph manifold M as a union of circle bundles S_i joined by a set of tori Θ along a graph Γ. The fundamental property of graph manifolds is the unicity of the minimal graph structure:

Theorem A ([Wal67b, Satz 8.1])**.** *Let M and N be two graph manifolds with respective minimal graph structures Θ_M and Θ_N. Let $\Phi : M \to N$ be a homeomorphism. Then, if M and N do not belong to one of the exceptional cases, Φ is isotopic to a homeomorphism $\Psi : M \to N$ such that $\Psi(\Theta_M) = \Psi(\Theta_N)$.* ◁

As a consequence we consider *strongly positive graphed homeomorphisms* (see Definitions 1.3.22 and 1.3.23) which respect the graph structure and the local orientation of each Seifert piece. Definition 1.3.20 introduce the crucial concept of the *graph ordering* Ω which is the data of *local orders* ω_i around each vertex, used for the ordered stars. We can then define *ordered graphed embeddings* in Section 1.4 as the reunion of a set of ordered stars, one for each Seifert piece S_i, respecting the local order.

Theorem B. *Let M be a graphed manifold ordered whose graph Γ is ordered by Ω. There is a well-defined action of the group of strongly positive homeomorphism on the set of ordered graphed embeddings. This action is transitive.* ◁

In Section 1.5 we explain how the choice of an ordered graphed embedding determines the cycle generators of $\pi_1(M)$ and $H_1(M)$ in Theorems 1.5.9 and 1.5.16.

We then present in Section 1.6 the main construction of Chapter 1, the graph stabiliser. The topological map $i_* \circ \gamma_*$ of Eq. (G) lies in the group

$$\mathcal{H} := \mathrm{Hom}(H_1(\Gamma_{\mathcal{A}}), H_1(E_{\mathcal{A}}))$$

which is combinatorially determined, see Proposition 2.4.22. The *graph stabiliser* $\mathcal{G}_\Gamma$ is the quotient of $\mathcal{H}$ by the differences on the cycle generators between every two ordered graphed embeddings, see Definition 1.6.1. Using Theorems A and B, we can reduce the computation of this difference down to the ordered stars, and thus give in Theorem 1.6.11 a combinatorial presentation of the graph stabiliser.

Chapter 2 uses the graph stabiliser to define the *homology inclusion* as an invariant of ordered line arrangements.

We start by recalling the basic definitions of line arrangements in Section 2.1, and combinatorics based on incidence data in Section 2.2, where we also list the exceptional cases that we do not consider. In Section 2.3 we give the description of the minimal graph structure of the boundary manifold $B_{\mathcal{A}}$ and of the circle bundles that compose it (see Theorems 2.3.15 and 2.3.17), using the blow-up operation to resolve the singularities. Section 2.4 presents the main tools used to characterise the topological type the exterior manifold and its fundamental group, namely the *wiring diagram* and the *braid monodromy*. It is this last tool that we use as the primary invariant to get the topological information of the arrangements.

Section 2.5 collects all previous results to establish the invariance of the homology inclusion. By construction of the graph stabiliser, the class of the map $i^*_{\mathcal{A}'} \circ \gamma_*$ does not depend on the choice of the ordered graphed embedding. Every combinatorial contribution has been accounted for by the quotient, so we get:

Theorem C. *Let $\mathcal{A}, \mathcal{A}' \subset \mathbb{CP}^2$ be two non-exceptional line arrangements with the same combinatorics. Endow $\mathcal{A}$ and $\mathcal{A}'$ with the same ordering on their set of lines, which induces a graph ordering on the minimal graph structure. If the ordered arrangements $\mathcal{A}$ and $\mathcal{A}'$ are topologically equivalent then for every ordered graphed embedding γ we have $|i^*_{\mathcal{A}} \circ \gamma_*| = |i^*_{\mathcal{A}'} \circ \gamma_*|$ inside the graph stabiliser $\mathcal{G}_\Gamma$.* $\triangleleft$

Chapter 3 explains the method to compute the homology inclusion. As mentioned before, in Section 3.1 we use the braid monodromy to construct a geometrical *standard ordered graphed embedding* $\gamma_{\mathbf{B}}$. The values of the map $|i^*_{\mathcal{A}} \circ \gamma^*_{\mathbf{B}}|$ are then determined by an algorithmic computation on the braids of the monodromy called the *braid linking*, which we present in Section 3.2. Technical details of the braid monodromy processing and computation algorithms implemented in `Sage` [Sag23] are dealt with in Section 3.3. In particular, Section 3.3.3 gives several examples of known and new Zariski tuples detected by the homology inclusion.

Link slopes and concordance

General context

A link is an embedding of disjoint polygonal curves inside S^3. The Lickorish-Wallace theorem allows to connect the study of oriented closed manifolds of dimension 3 to the study of knots and links. The *concordance relation* is a natural extension of link equivalence in dimension 4 which was designed by R. Fox and J. Milnor [Fox62]. Two links are concordant if they co-bound a set of properly embedded disjoint cylinders in S^4, and a link is *slice* if it is concordant to the unlink. This relation creates a natural connection between link theory and the study of surfaces embedded in S^4.

The character slope

The slope invariant was developed by A. Degtyarev, V. Florens and A.G. Lecuona in [DFL22b] and later refined in [DFL21; DFL22a]. It appeared as an invariant of its own during a study of the signature of the intersection form on the exterior of coloured links, and has been applied in this context. As we mentioned in the general introduction, its construction is based on the fact that the boundary of the exterior of the link M_L is merely a union of disjoint tori.

$$\partial M_L = \bigsqcup_{i=1}^{n} T_{L_i}$$

Applying the strategy similar as what we presented for line arrangements, they considered the induced application of the inclusion $i : \partial M_L \hookrightarrow M_L$ in *twisted homology*:

$$i_* : H_1(\partial M_L; \rho) \longrightarrow H_1(M_L; \rho)$$

where ρ is a representation of the fundamental group. It turns out that the twisted homology of the boundary is a vector space entirely determined by the image of the chosen representation on the peripheral structure of the link, namely its meridians and longitudes. With a one-dimensional character $\omega : \pi_1(M_L) \to \mathbb{C}^*$, the dimension of $H_1(\partial M_L; \omega)$ is a multiple of the number of non-trivial values of ω. Therefore by imposing ω to be trivial on a single component K called *distinguished*, they obtained:

Theorem D. *Let ω be an admissible character and suppose that $\dim \ker i_* = 1$. Then $\ker i_*$ is generated by a vector of $H_1(\partial M_{K\cup L}; \omega)$ of the form*

$$\ker i_* = \langle a \cdot \ell + b \cdot m \rangle$$

with $[a : b] \in \mathbb{CP}^1$. The (K/L)-slope is defined by the formula

$$s_{(K/L)}(\omega) := -\frac{b}{a} \in \mathbb{C} \cup \{\infty\} \qquad \triangleleft$$

The slope is a rational function in the values of the character and has several interesting properties, some of which are recalled in Chapter 4. In particular it does not depend on the fitting ideals of the homology group and it is a concordance invariant.

In fact the character slope generalises the single-variable η-function developed by S. Kojima and M. Yamasaki [KY79] and the invariant developed by N. Sato and J. Levine [Sat84]. The η-function can be seen as a generating function of the β-invariants of T. Cochran [Coc85]. These are in turn [Coc90] linked to certain lifts of the linking numbers of J. Milnor [Mil54].

Our objective is to generalise the main idea of the first slope, which we now call the *character slope*, to other contexts. We present two such extensions. The first, the $\mathrm{SL}_2(\mathbb{C})$-*slope*, was developed in collaboration with L. Bénard. It allows to define a slope on knots using adjoint representations over $\mathrm{SL}_2(\mathbb{C})$. The second extension, the *generalised slope*, uses complexified representations in $\mathrm{SO}_2(\mathbb{R})$ and a method due to V. Arnold [Arn67] to allow more than one component of the link to be distinguished.

Summary of Part II

Chapter 4 gives a quick presentation of the character slope and its main properties to establish the reference for the generalisations. Section 4.1 makes a quick overview of basic knot and link theory definitions. Section 4.2 gives the main Definition 4.2.4 of the character slope. Some results use new more general proofs detailed in Appendix A, notably Theorem 4.2.10 which establishes the concordance invariance. Section 4.2 explains how to compute the slope using Fox calculus [Fox54] in Theorem 4.3.1. We also present a new implementation in `GAP` [GAP22] of that method which allows to compute the slope on any compatible link diagram thanks to [CD08; DT83].

Chapter 5 is adapted from [BFR21] and is dedicated to the $\mathrm{SL}_2(\mathbb{C})$-slope on knots. The choice of $\mathrm{SL}_2(\mathbb{C})$ is motivated by the fact that the set of all representations of the group of knot K in $\mathrm{SL}_2(\mathbb{C})$ carries naturally the structure of an algebraic set. This holds also for the characters of these representations, whose set is called the $\mathrm{SL}_2(\mathbb{C})$-character variety of the knot. Given a peripheral structure of the knot, the character variety is a plane curve in $\mathbb{C}^* \times \mathbb{C}^*$, whose coordinates M and L correspond to the eigenvalues of the meridian m and the preferred longitude ℓ of K. The polynomial $A_K(L, M)$ defining this curve is an invariant of the knot, called the *A-polynomial*. This invariant contains many interesting informations on the knot; in particular, S. Boyer and X. Zhang [BZ05] and N. Dunfield and S. Garoufalidis [DG04] showed that $A_K = L - 1$ if and only if K is trivial. One of our main motivations, as explained in Section 5.1, is to harness topological information from the A-polynomial using the slope. Indeed, the A-polynomial is a high-level invariant which is difficult to compute in general, whereas the $\mathrm{SL}_2(\mathbb{C})$-slope can be computed with Fox calculus with a similar process as the character slope. The construction and general results about the A-polynomial are covered in Section 5.2.

The construction of the $\mathrm{SL}_2(\mathbb{C})$-slope is presented in Section 5.3. It is based on the fact that a perfect analogue of Theorem D exists for which all adjoint *non-parabolic* representations of the knot group in $\mathrm{SL}_2(\mathbb{C})$ are admissible, see Definition 5.3.2 and Lemma 5.3.4. The invariant is thus defined again as the slope of $\ker i_*$ with respect to the meridian-longitude basis. It should be noted that unlike the character slope, no restrictions are made on the knot K. The slope definition relies on constructions made by J. Porti [Por97] which observed that $H_1(\partial M_K; \mathrm{Ad} \circ \rho)$ has the structure of a *symplectic space* for the induced intersection form. J. Porti also defined the *Reidemeister torsion* for the $\mathrm{Ad} \circ \rho$-twisted homology, and we prove that it is linked to the slope by a simple formula given in Proposition 5.3.15. Section 5.4 contains the main Theorem 5.4.1 which connects the slope and the A-polynomial. This result is notably used in Corollary 5.4.5 which establishes that the $\mathrm{SL}_2(\mathbb{C})$-slope can detect the unknot.

Chapter 6 presents our other generalisation of the character slope. To simplify, we consider links with zero linking numbers between all components. The fundamental proposition for the construction is the following:

Proposition E. *Let $\rho_\omega : \pi_1(M_L) \to \mathrm{SO}_2(\mathbb{R})$ be the realification of a character ω of the link. Then the $\mathbb{R}$-vector space $H_1(\partial E_L, \mathbb{R}(\rho_\omega))$ endowed with the intersection form is a symplectic space freely generated by the meridians and longitudes of the components of L_i such that $\omega(m_i) = 1$. Moreover, $\ker i_*$ is a Lagrangian subspace, i.e. its own orthogonal for the symplectic form.* $\lhd$

V. Arnol'd [Arn67] has developed a method based on the complexification of a symplectic real vector space, which allows to characterise all Lagrangian subspaces with a class of complex unitary matrices. This method is briefly recalled in Section 4.2. The *generalised slope* is defined in Section 6.2 as the argument of the determinant of the matrix characterising $\ker i_*$, see Definition 6.2.7. Unlike the character slope, the generalised slope can be defined as a function on the entire character torus of the link L. In Section 6.3, we prove that it is again a concordance invariant.

PART I

HOMOLOGY OF LINE ARRANGEMENTS

CHAPTER

1

GRAPH STABILISER

Contents

1.1 Preliminaries

1.1.1 Finitely presented groups

Definition 1.1.1. Let G be a group. Let F be a finitely-generated free group and let R be a finite set of elements in F. Let $\langle R\rangle$ be the smallest normal subgroup of F generated by R. The data $\mathcal{P} = \langle F \mid R\rangle$ is called a *presentation* of G if there is an exact sequence

$$0 \longrightarrow \langle R\rangle \longrightarrow F \longrightarrow G \longrightarrow 0 \qquad \diamond$$

For A and B two $\mathbb{Z}$-modules we often make use of the equivalence

$$\mathrm{Hom}(A, B) \simeq A^* \otimes B$$

1.1.2 Braid groups

We recall some basic results about braid group presentation, which are taken from [Art47; Bir75].

The braid group on m strands $\mathbb{B}_m$ is generated by the elementary braids $\sigma_{i,j}$ that permutes the strands i and j, with the relations

$$\begin{aligned} &\sigma_{s,t}\sigma_{q,r} = \sigma_{q,r}\sigma_{s,t} && \text{if } (t-r)(t-q)(s-r)(s-q) > 0 \\ &\sigma_{s,t}\sigma_{r,s} = \sigma_{r,t}\sigma_{s,t} = \sigma_{r,s}\sigma_{r,t} && \text{for } 1 \le r < s < t \le n \end{aligned}$$

Alternatively, $\mathbb{B}_m$ can also be generated by the elementary braids σ_i for $1 \le i \le m-1$ that permutes the strands i and $i+1$, with the relations

$$\begin{aligned} \sigma_i \cdot \sigma_k &= \sigma_k \cdot \sigma_i && \text{if } |i-k| \le 2 \\ \sigma_i \cdot \sigma_{i+1} \cdot \sigma_i &= \sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1} && \text{for } 1 \le i \le n-1 \end{aligned}$$

The two presentations are connected by the inverse group isomorphisms

$$\begin{aligned} \mathbb{B}_m &\longleftrightarrow \mathbb{B}_m \\ \sigma_{i,j} &\longmapsto (\sigma_i \cdots \sigma_{j-2}) \cdot \sigma_{j-1} \cdot (\sigma_i \cdots \sigma_{j-2})^{-1} \\ \sigma_{i,i+1} &\longleftarrow\!\!\mapsfromchar\ \sigma_i \end{aligned}$$

Let $\mathfrak{S}_m$ be the permutation group on m elements. By convention, a permutations $\sigma \in \mathfrak{S}_m$ performs a right action on the elements which is denoted $i \cdot \sigma$. There is a natural epimorphism $\sigma : \mathbb{B}_m \twoheadrightarrow \mathfrak{S}_m$ defined by

$$\sigma(\sigma_{i,j}) := (i, j)$$

The *pure braid group* on m strands $\mathbb{P}_m$ is defined as the kernel of σ. It is generated by the braids $a_{i,j} := \sigma_{i,j}^2$ that performs a full twist on the strands i and j with the relations

$$a_{r,s} \cdot a_{i,j} \cdot {a_{r,s}}^{-1} = \begin{cases} a_{i,j} & \text{if } r < s < i < j \\ & \text{or } i < r < s < j \\ a_{r,j} \cdot a_{i,j} \cdot {a_{r,j}}^{-1} & \text{if } r < s = i < j \\ (a_{i,j} \cdot a_{s,j}) \cdot a_{i,j} \cdot (a_{i,j} \cdot a_{s,j})^{-1} & \text{if } r = i < j < s \\ \left(a_{r,j} \cdot a_{s,j} \cdot a_{r,j}^{-1} \cdot a_{s,j}^{-1}\right) \cdot a_{i,j} \cdot \left(a_{r,j} \cdot a_{s,j} \cdot a_{r,j}^{-1} \cdot a_{s,j}^{-1}\right)^{-1} & \text{if } r < i < s < j \end{cases}$$

For every subset $I = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$, the *full twist over* I is the pure braid defined by

$$\Delta_I^2 := \left(\sigma_{i_1,i_2} \cdots \sigma_{i_{k-1}i_k}\right)^k \tag{1.1}$$

1.1.3 Mapping class group of planar surfaces

Let Σ_r^m be a planar surface with m boundary components $D^1, \ldots, D^m$ and r punctures $x^1, \ldots, x^r$. When there are no punctures we only write Σ^m. A homeomorphism $\varphi : \Sigma_r^m \to \Sigma_r^m$ that restricts to the identity on $\partial\Sigma_r^m$ is called a *boundary-homeomorphism.* We denote by $\mathrm{Homeo}_\partial(\Sigma_r^m)$ the group boundary-homeomorphism of M up to isotopies that also restrict to the identity on the boundary.

The *mapping class group* $\mathcal{M}(\Sigma_r^m)$ is the group of isotopy classes of orientation-preserving diffeomorphisms of Σ_r^m that respect the boundary set. The *pure mapping class group* $\mathcal{P}(\Sigma_r^m)$ is the subgroup of $\mathcal{M}(\Sigma_r^m)$ consisting of elements that fix each boundary component $\partial^i D$ and each puncture x^i individually.

The results of this section are well-known, see for example [FM11] and [Bir75].

Theorem 1.1.2. *For $m \geq 2$, the group $\mathcal{M}(\Sigma^{m+1})$ (resp. $\mathcal{P}(\Sigma^{m+1})$) is isomorphic to $\mathbb{B}_m \times \mathbb{Z}^m$ (resp. $\mathbb{P}_m \times \mathbb{Z}^m$), where the generator $\sigma_{j,l} \in \mathbb{B}_m$ (resp. $a_{j,l} \in \mathbb{P}_m$) corresponds to a Dehn half-twist (resp. full twist) along the curve $\delta_{j,l}$, and the generator $d_i \in \mathbb{Z}^m$ is the full Dehn twist around a curve δ_i parallel to $\partial^i D$, as shown on Figure 1.1.1.* ◁

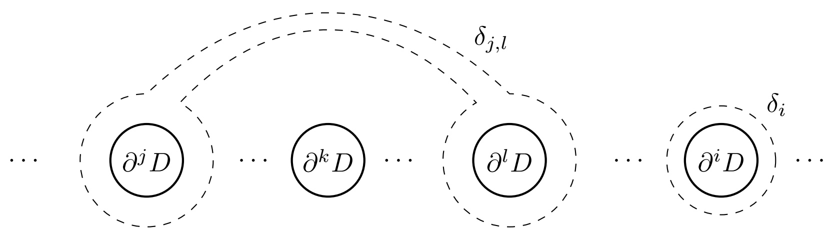


Figure 1.1.1: Dehn twist generators

Theorem 1.1.3. *Fill in a boundary component of Σ^{m+1} to obtain Σ^m. There is a surjective group homomorphism*

$$f_m : \mathcal{P}\left(\Sigma^{m+1}\right) \twoheadrightarrow \mathcal{P}(\Sigma^m)$$

that respects the action of $\mathcal{P}\left(\Sigma^{m+1}\right)$ on the sub-surface $\Sigma^{m+1} \subset \Sigma^m$. ◁

Proof. It is known from [Art47] that $\mathcal{P}\left(\Sigma_r^1\right)$ is homeomorphic to $\mathbb{P}_r$ for $r \geq 3$. The generator $a_{j,l}$ corresponds to a Dehn full twist along a curve $\delta_{j,l}$ that goes around x^j and x^l only. There is a relation that arises on mapping class groups by capping a boundary component with a punctured disk.

Lemma 1.1.4. *Fill in the boundary component D^m of Σ_r^m with a punctured disk $B^2 \smallsetminus \{x^{r+1}\}$ to get Σ_{r+1}^{m-1}. Then*

$$\mathcal{P}(\Sigma_r^m) = \mathcal{P}\left(\Sigma_{r+1}^{m-1}\right) \times \langle d_m \rangle$$

where d_m is the full Dehn twist around a curve δ_m parallel to D^m. ◁

Starting from Σ^{m+1}, capping all boundary components but one using Lemma 1.1.4 gives Theorem 1.1.2.

A similar relation arises by filling in a puncture.

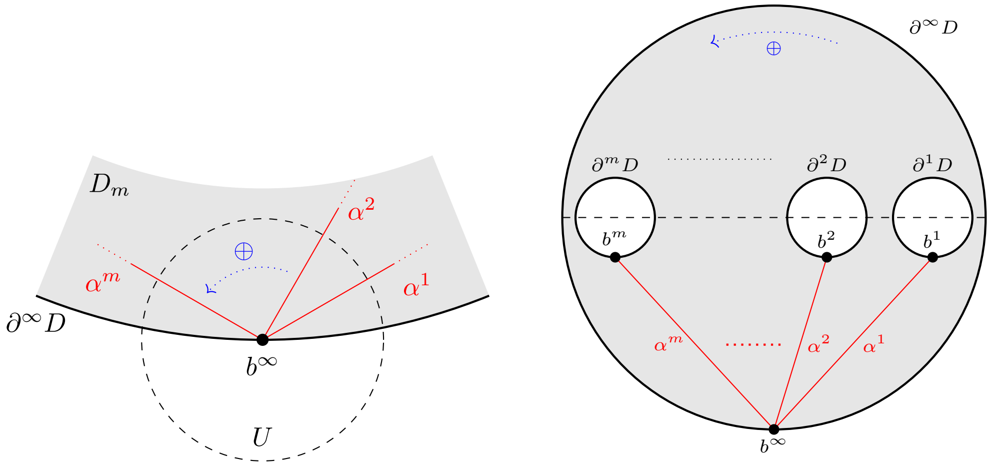
Lemma 1.1.5 ([Bir75])**.** *Fill in the puncture x^{r+1} of Σ_{r+1}^m to obtain Σ_r^m. If $m + r > 2$ then there is a short exact sequence*

$$1 \longrightarrow \pi_1(\Sigma_r^m, x^{r+1}) \longrightarrow \mathcal{P}\left(\Sigma_{r+1}^m\right) \longrightarrow \mathcal{P}(\Sigma_r^m) \longrightarrow 1$$ ◁

Fill in the boundary component D^{m+1} of Σ^{m+1} to obtain Σ^m. If $m > 2$, by combining Lemma 1.1.4 and Lemma 1.1.5, we get Theorem 1.1.3. □

1.2 Ordered stars

An *ordered star* is a set of non-intersecting paths drawn on a disc D_m with m holes which are geometrically ordered. We denote by $\mathcal{O}_m$ the set of all isotopy classes of ordered stars on D_m.



(a) Ordering of a star

(b) Standard ordered star

Figure 1.2.1: Ordered star

Ordered stars will be used in Section 1.4.2 as elementary pieces to assemble the graphed embeddings on a graph manifold. Their role is to define a way to properly embed all half-edges of the graph with a common starting vertex v_i inside the corresponding circle bundle S_i. An important choice of our construction is that the half-edges, and thus the whole ordered star, are to be drawn on a *section* $s_i : \Sigma_i \to S_i$, where Σ_i is homeomorphic to a 2-sphere with $m_i + 1$ holes.

We will also show in the beginning of Section 1.4 that one can in fact always generically remove the interior of a disc from Σ_i disjoint from its boundary. Ordered stars are thus defined as objects on a 2-disc with a finite number m of holes.

1.2.1 Definition and standard model

Let R be a closed disc in the oriented plane $\mathbb{C}$. Let $D^1, \dots D^m$ be disjoint identical closed pairwise disjoint discs enclosed within R and ordered by their descending horizontal coordinates, and let x^j be the centre of D^j. By convention, we write $\partial^\infty D$ for the boundary ∂R and $\partial^j D$ for $\partial(D^j)$. These boundary circles are often seen as looping paths, denoted $\partial^j_+ D$ (resp. $\partial^j_- D$) when travelled along in the positive (resp. negative) sense relatively to the orientation of $\mathbb{C}$. Finally, define

$$D_m := R \smallsetminus \bigcup_{j=1}^{m} \mathring{D}^j \qquad\qquad \Delta_m := R \smallsetminus \bigcup_{j=1}^{m} \{x^j\}$$

For any non-looping curve α of the complex plane, we denote by $\partial_-\alpha$ its starting point and $\partial_+\alpha$ its ending point.

Definition 1.2.1. Let $\omega \in \mathfrak{S}_m$ be a permutation. An ω*-star* on D_m is a collection of properly embedded simple curves $\alpha = \left(\alpha^j\right)_{1\leq j\leq m}$ drawn on D_m such that:

(i) for every curve α^j, $\partial_+\alpha^j := b^{j\cdot\omega} \in \partial^{j\cdot\omega} D$ and $\partial_-\alpha^j$ is a common point $b^\infty \in \partial^\infty D$.

(ii) for every pair of curves, $\alpha^j \cap \alpha^k = \{b^\infty\}$.

(iii) there exists a disc U centred on b^∞ in the complex plane and an orientation-preserving homeomorphism $\phi : U \to U$ that sends the pair $(D_m \cap U, \alpha \cap U)$ to the pair shown in Figure 1.2.1a. ◇

We denote by $\mathcal{S}_m(\omega)$ the set of all isotopy classes of ω-stars on D_m and by $\mathcal{S}_m$ the reunion of all sets $\mathcal{S}_m(\omega)$ for every $\omega \in \mathfrak{S}_m$.

Definition 1.2.2. An *ordered star* on D_m is a star of $\mathcal{S}_m$ with respect to the permutation $\omega = \mathrm{Id} \in \mathfrak{S}_m$. The set of all ordered stars is denoted $\mathcal{O}_m$. ◇

***Remark* 1.2.3.** It is easy to construct a simple ordered star on D_m as shown in Figure 1.2.1b, so the set $\mathcal{O}_m$ is not empty. ⌑

1.2.2 Action of the pure mapping class group

Let $\omega \in \mathfrak{S}_m$ and write $\xi := \omega^{-1}$. Let $\alpha \in \mathcal{S}_m(\omega)$ be a star on D_m. Up to isotopy, one can always move α such that the loop $\delta_{j,l}$ (resp. δ_k) of Figure 1.1.1 does not intersect any path from α except $\alpha^{j\cdot\xi}$ and $\alpha^{l\cdot\xi}$ (resp $\alpha^{k\cdot\xi}$), with only one transverse intersection for each. The respective actions of the Dehn twists on the curves of α is then shown on Figure 1.2.2. It is clear that the modified curves still respect the conditions from Definition 1.2.1, and thus we obtain:

Proposition 1.2.4. *There is a well-defined action of $\mathcal{M}(D_m)$ on $\mathcal{S}_m$ induced by its natural action on D_m. Additionally, for every star $\alpha \in \mathcal{S}_m(\omega)$ and every braid $\beta \in \mathbb{B}_m$, we have*

$$\beta \cdot \alpha \in \mathcal{S}_m(\omega \cdot \sigma(\beta)) \qquad \triangleleft$$

Corollary 1.2.5. *There is a well-defined action of $\mathcal{P}(D_m)$ on $\mathcal{O}_m$ induced by its natural action on D_m.* ◁

The action of the pure braid generator $a_{j,l} \in \mathbb{P}_m$ on an ordered star is shown on Figure 1.2.3.

The rest of the section is dedicated to the proof of the following result, and can be skipped on first reading.

Proposition 1.2.6. *The group $\mathcal{P}(D_m)$ acts transitively on the set $\mathcal{O}_m$.* ◁

Proof. We split the proof by considering separately the actions of the subgroups $\mathbb{P}_m$ and $\mathbb{Z}^m$ of $\mathcal{P}(D_m)$.

The fundamental group of Δ_m is a free group $\mathbb{F}_m$ with m generators. Let α be an ordered star on D_m. We say that we travel positively along the curve α^j when we go from b^0 to b^j, and negatively otherwise, which we write as $(\alpha^j)^{-1}$. Then every curve α^j of the ordered star can be associated to the closed curve of Δ_m defined by:

$$\theta(\alpha)^j := \alpha^j \cdot \partial^j_+ D \cdot \left(\alpha^j\right)^{-1} \tag{1.2}$$

where the closed path along $\partial^j D$ is followed in the positive sense with respect to the orientation of Δ_m. There is thus a map

$$\theta : \mathcal{O}_m \longrightarrow (\mathbb{F}_m)^m$$

which associates to any ordered star $\alpha \in \mathcal{O}_m$ the homotopic classes of all the curves $\theta(\alpha)^j$ inside $\pi_1(\Delta_m) = \mathbb{F}_m$. Note that the product $\prod_{j=1}^m \theta(\alpha)^j$ is equal to the class of the loop $\partial^\infty_+ D$.

Lemma 1.2.7. *The group $\mathbb{P}_m$ acts transitively on the set $\theta(\mathcal{O}_m)$.* ◁

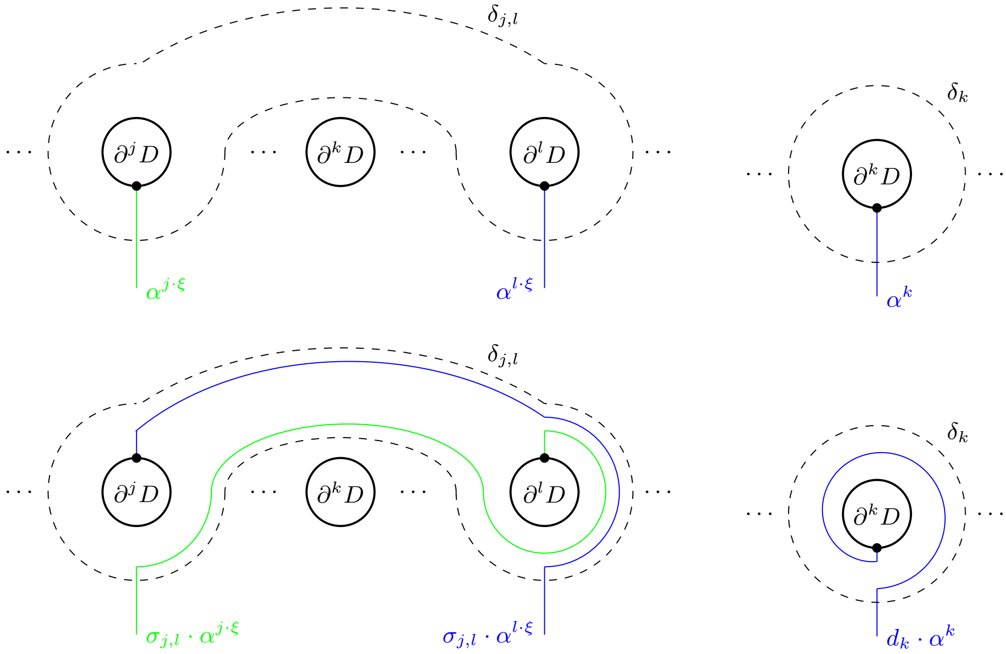
Go back to D_m by removing again the discs D^j from Δ_m. We now consider the action on $\mathcal{O}_m$ of the $\mathbb{Z}^m$ subgroup of $\mathcal{P}(D_m)$ generated by the Dehn twists along the δ_k curves. We define the equivalence relation on $\mathcal{O}_m$ by

$$\forall\, \alpha, \alpha' \in \mathcal{O}_m : \quad \alpha \simeq \alpha' \iff \theta(\alpha) = \theta(\alpha') \tag{1.3}$$

Lemma 1.2.8. *The group $\mathbb{Z}^m$ acts transitively on every equivalence class of $\mathcal{O}_m$.* ◁

Let $\alpha, \alpha' \in \mathcal{O}_m$ be two ordered stars on D_m. By Lemma 1.2.7 there exists some pure braid $\beta \in \mathbb{P}_m$ such that $\beta \cdot \theta(\alpha) = \theta(\alpha')$ for the action of $\mathcal{P}(D_m)$ on $\pi_1(\Delta_m) \simeq \pi_1(D_m)$. This action is induced by the natural action of $\mathcal{P}(D_m)$ on D_m itself, which fixes all boundary components of D_m. So from Eq. (1.2) it is clear that $\beta \cdot \theta(\alpha) = \theta(\beta \cdot \alpha)$. This means that $\beta \cdot \alpha$ and α' are equivalent in the sense of Eq. (1.3), so by Lemma 1.2.8 there exists some $d \in \mathbb{Z}^m$ such that $d \cdot (\beta \cdot \alpha) = (\beta, d) \cdot \alpha = \alpha'$, which concludes the proof. □

Proof of Lemma 1.2.7. Consider the standard ordered star α^0 on D_m shown on Figure 1.2.1b. Set $T = (t_1, \dots, t_n) := \theta(\alpha^0)$. It is clear that T is a free generating family of $\pi_1(\Delta_m) = \mathbb{F}_m$. Now consider any other ordered star $\alpha \in \mathcal{O}_m$ and set $S = (s_1, \dots, s_m) := \theta(\alpha)$. Every element s_j of S can be expressed in the basis T. It is also clear from Eq. (1.2) that each s_j is conjugated to the homotopical class of $\partial^j_+ D$, i.e t_j. There is thus a mapping $f : (\mathbb{F}_m)^m \to (\mathbb{F}_m)^m$ such that $f(t_j) = s_j = S_j t_j S_j^{-1}$ for every j, where $S_j \in \mathbb{F}_m$. By [Art47, Theorem 14], f is canonically associated to the mapping-class-group action of a pure braid $\beta \in \mathbb{P}_m$ on $\pi_1(\Delta_m)$. □



(a) Generator $\sigma_{j,l}$ (b) Generator d_k

Figure 1.2.2: Action of $\mathcal{M}(D_m)$ on a star

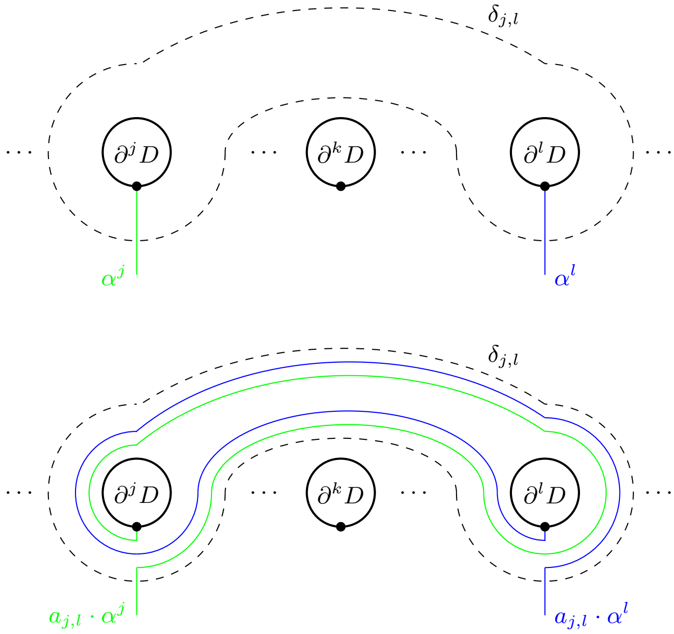


Figure 1.2.3: Action of $a_{j,l}\in\mathcal{P}(D_m)$ on an ordered star

Proof of Lemma 1.2.8. It is easy to see from Figure 1.2.2b that for every $\alpha \in \mathcal{O}_m$ and every k, $\theta(d_k \cdot \alpha)$ is homotopic to $\theta(\alpha)$ if D^k is filled in except for its centre, so the action is well-defined. Consider $\alpha, \alpha' \in \mathcal{O}_m$ equivalent and define the collection of m closed curves $\alpha \cdot \alpha'^{-1}$ by the formula

$$\left(\alpha \cdot \alpha'^{-1}\right)^k := \left(\alpha'^k\right)^{-1} \cdot \alpha^k$$

Since $(\alpha \circ \alpha'^{-1})^k$ is a closed curve in D_m and thus in Δ_m, we can see it as an element of $\pi_1(\Delta_m)$. Remember that $\theta(\omega) = (t_1, \ldots, t_m)$ is a free generating family of $\pi_1(\Delta_m)$. Using Eq. (1.2) inside the equation

$$1 = \theta(\alpha')^{-1} \cdot \theta(\alpha)$$

we obtain that $\left(\alpha \cdot \alpha'^{-1}\right)^k$ commutes with the homotopical class of $\partial_+^k D$, i.e t_k. Since t_k is a free generator of $\pi_1(\Delta_m) = \mathbb{F}_m$, there must exist some integer $r_k \in \mathbb{Z}$ such that $(\alpha \cdot \alpha'^{-1})^k = {t_k}^r$. This means exactly that up to homotopy $\alpha^k = {d_k}^{r_k} \cdot \alpha'^k$. Doing this for all k, we obtain an element $d := \prod_k {d_k}^{r_k} \in \mathbb{Z}^m$ such that $d \cdot \alpha = \alpha'$. □

1.3 Graph manifolds

The graph manifolds were introduced by Waldhausen in [Wal67a; Wal67b] and were further studied by Mumford [Mum61] and Neumann [Neu81].

The results of this section are well-known. See for example [ST80] and [Hat99] for circle bundles, and [FM97] or [JS79] for a modern approach on graph manifolds.

1.3.1 Circle bundles

Definition 1.3.1. A *circle bundle* S is a fibre bundle $p : S \to \Sigma$, where Σ is a compact 2-surface, such that for every $s \in S$, there exist an open neighbourhood U of $p(s)$ and an isomorphism $q : U \times S^1 \to p^{-1}(U)$ with $(p \circ q)_{|U}$ corresponding to the projection on the first variable. ◇

The surface Σ is called the *basis* of the bundle. The pre-image $p^{-1}(x)$ of a point $x \in \Sigma$ is homeomorphic to a circle S^1 which is called the *fibre over* x. A *section* is an embedding $s : \Sigma \hookrightarrow S$ such that $p \circ s = \mathrm{Id}_\Sigma$.

Definition 1.3.2. A homeomorphism $\Psi : S \to S'$ between two circle bundles is called *fibrewise* if it sends every fibre of S to a fibre of S'. ◇

Definition 1.3.3. An *oriented circle bundle* is an orientation-preserving fibre bundle $p : S \to \Sigma$ along with the data of two of the following three:

- an orientation on S.
- an orientation on all fibres of p.
- an orientation of Σ. ◇

***Remark* 1.3.4.** Fixing just the orientation of S does not fix the orientation on the basis and fibres. Indeed, there exists a homeomorphism ν of S that changes the orientation of the basis and all fibres at the same time, without changing the global orientation of S. ⌑

Definition 1.3.5. A fibrewise homeomorphism $\Psi : S \to S$ is said to be *fibre-positive* if the induced homeomorphisms on the basis Σ and on every fibre S^1 are all positive. ◇

We denote by $\mathrm{Homeo}_\partial^+(S)$ the group of fibre-positive boundary-homeomorphisms of S. We assign a numbering in $\{1, \ldots, m\}$ to the boundary components of S and we denote by $\partial_i S$ the i-th boundary component of S.

Definition 1.3.6. A *boundary section collection* on a circle bundle S with m components is the data of a collection of closed curves $\bar{\mu} := (\mu_i)_{1 \le i \le m}$ such that $\mu_i \subset \partial_i S$ and is transverse to the fibres of S. ◇

Circle bundles are a special case of Seifert manifolds, namely those without singular fibres. In our study we are only interested in the circle bundles that respect the following set of conditions:

Conditions 1.3.7. Let S be a circle bundle with basis Σ.

(C1) S is orientable.

(C2) Σ is planar.

(C3) Σ has $m \geq 3$ boundary components.

It must be noted that Condition (C3) can exceptionally be relaxed when we consider non-minimal graph structures.

We now describe the construction of the reference model that we use for all our circle bundles with fixed sections on the boundary.

Definition 1.3.8. Let $\varepsilon \in \mathbb{Z}$. Recall from Section 1.2.1 the model $D_m \subset \mathbb{C}$ homeomorphic to the surface Σ^{m+1}. Consider the direct product

$$T_m := D_m \times S^1$$

and a separate solid torus

$$T_\infty := D' \times S^1$$

Assign the same respective orientations on the sections and fibres of T_m and T_∞. Let $\breve{s}$ be a fixed section of T_m. It is naturally associated with a collection $\breve{\mu}$ of sections on the boundary with $\breve{\mu}_i := \breve{s}(\partial^i D_m)$. Take a point x_∞ on μ_∞ and let $\check{\lambda}_\infty$ be the fibre over x_∞. Let $\breve{s}'$ be a section of T_∞ and take $\breve{\mu}'_\infty$ the positive path $\partial D'$ and $\check{\lambda}'_\infty$ the fibre over a point x'_∞ on $\breve{\mu}'_\infty$. Glue ∂T_∞ to the toric boundary component $\partial^\infty D_m \times S^1$ of T_m using the gluing map:

$$g_\varepsilon : \begin{cases} \breve{\mu}_\infty \longmapsto -\breve{\mu}'_\infty - \varepsilon \cdot \check{\lambda}'_\infty \\ \check{\lambda}_\infty \longmapsto \check{\lambda}'_\infty \end{cases} \tag{1.4}$$

The manifold

$$S(m, \breve{s}, \varepsilon) := T_m \cup_{g_\varepsilon} T_\infty$$

thus obtained is a circle bundle with m boundary components, and ε is called its *Euler number.* ◇

Proposition 1.3.9. *The manifold $S(m, \breve{s}, \varepsilon)$ does not depend on the choice of the section $\breve{s}'$ on the solid torus T_∞.* ◁

Proof. This is a direct consequence from the fact that all fibre structures on the solid torus are isotopic. □

***Remark* 1.3.10.** The notion of Euler number is usually defined on closed fibred spaces. The choice of the fixed section $\breve{s}$ allows to extend this definition and designate ε as the Euler number of $S(m, \breve{s}, \varepsilon)$. Note that this could also have been achieved by fixing only a boundary section collection $\breve{\mu}$. ⌑

Theorem 1.3.11 ([JS79])**.** *Let S be a circle bundle with m boundary components that respects Conditions 1.3.7 and let $\bar{\mu}$ be a collection of sections on the boundary of S. Then there exists an orientation-preserving fibrewise boundary-homeomorphism $\chi : S(m, \breve{s}, \varepsilon) \to S$ such that $\chi(\breve{\mu}_i) = \mu_i$ for some unique value of $\varepsilon \in \mathbb{Z}$.* ◁

The number $\varepsilon \in \mathbb{Z}$ is called the *Euler number* of the oriented circle bundle S.

***Remark* 1.3.12.** Orienting the manifold $S(m, \breve{s}, \varepsilon)$ induces an orientation of the bundle. Indeed, changing the orientation of just (say) the sections of $T_m \subset S(m, \breve{s}, \varepsilon)$ would also change the Euler number ε into its opposite $-\varepsilon$. ⌑

***Remark* 1.3.13.** For $m = 2$, the manifold $S(2, \breve{s}, \varepsilon)$ is homeomorphic to a thickened torus $T \times I$ for any value of $\varepsilon \in \mathbb{Z}$. This means that the thickened torus has not a unique structure as a circle bundle, which justifies the requirement that $m \geq 3$ in Theorem 1.3.11. ⌑

1.3.2 Definition of the graph manifolds

We shall begin by recalling some basics about the graph manifolds, in preparation of Section 1.6. Let M be an oriented connected 3-manifold

Definition 1.3.14. A *graph structure* on M is a set Θ of pairwise disjoint tori such that $M \setminus \Theta$ is a disjoint union of Seifert manifolds. If M has a graph structure it is called a *graph manifold* and the elements of Θ are called *joining tori.* ◇

Theorem 1.3.15 ([Wal67b]). *Any graph manifold that does not belong to the exceptions listed in [Wal67b, Satz 8.1] admits a unique graph structure with a minimal number of tori.* ◁

In fact only a handful of exceptional classes arise in our applications to line arrangements. They are listed in Section 2.2.3 and we do not intend to study them. From now on we only consider non-exceptional graph manifolds, for which we only use the unique minimal structure, which we will thus refer to as 'the graph structure'.

According to Theorem 1.3.11, for every circle bundle component S there exists an orientation-preserving fibrewise boundary-homeomorphism $\chi : S(m, \check{s}, \varepsilon) \to S$ for some $m \in \mathbb{N}$ and $\varepsilon \in \mathbb{Z}$. A joining torus $T \in \Theta$ that joins two circle bundles S and S' corresponds to a component $\partial^k D_m \times S^1$ in $S(m, \check{s}, \varepsilon)$ and a component $\partial^{k'} D_{m'} \times S^1$ in $S(m', \check{s}', \varepsilon')$.

We intend to apply the graph manifold theory to regular neighbourhoods of complex line arrangements. For this reason, we only consider a subclass of non-exceptional graph manifold with specific additional properties.

Conditions 1.3.16. Let M be a graph manifold with minimal graph structure Θ.

(M1) All irreducible components of M are oriented circle bundles (i.e. Seifert manifolds with no exceptional fibres).

(M2) Every circle bundle component S of M verifies Conditions 1.3.7.

(M3) There is a choice of a section $\bar{\mu} : D_m \to S$ on every circle bundle S of M oriented as the opposite of a base section.

(M4) For every joining torus $T \in \Theta$ between S and S', let μ be the chosen oriented section of T as a boundary component of S, and let λ be the oriented fibre over a point x_0 on μ. Define μ' and λ' similarly inside S'. Then the gluing map from $T \subset S$ to $T \subset S'$ is given by

$$g_{\text{ext}} : \begin{cases} \mu \longmapsto \lambda' \\ \lambda \longmapsto \mu' \end{cases}$$

1.3.3 Ordered graphs

As the name suggests, the graph structure of a given graph manifold can be represented as a graph.

Definition 1.3.17. The *graph* Γ of a graph manifold M with graph structure Θ is given by the following description:

- every vertex v is associated to a circle bundle component S of $M \setminus \Theta$.
- every vertex v is decorated with the Euler number $\varepsilon \in \mathbb{Z}$ of the corresponding component $S \simeq S(m, \check{s}, \varepsilon)$.
- every edge e is associated to a unique joining torus in Θ that co-bounds S and S'. ◇

For a non-exceptional graph manifold M, the graph is unique since it is entirely determined by the unique minimal graph structure Θ.

The restrictions on the graph manifold given by Conditions 1.3.16 induce new restrictions on the graph itself:

Conditions 1.3.18. The graph Γ of the graph manifold M is such that:

(G1) no edge starts and ends at the same vertex.

(G2) there is at most one edge between every two vertices.

(G3) every vertex has at least three neighbours.

A graph that verifies the first two conditions is also called *simplicial*. The third condition is specific to the class of graph manifolds that we consider in this chapter.

***Remark* 1.3.19.** In the most general case of graph manifolds as defined by [Wal67a; Neu81], vertices are also weighted by the genus of S, which in our case is set as zero for all vertices with Condition (C2). The gluing map is also characterised by a slope value for each boundary similar to Eq. (1.4), which is put as a decoration on the edges. Again, in our case this slope is set as ∞ for all edges with Condition (M4). ⌑

We denote by V the set of vertices and by E the set of edges of Γ. We fix an ordering $\omega : V \to \{1, \dots, n\}$ which we denote by $V = \{v_1, \dots, v_n\}$. For every vertex v_i, the set of its neighbours is V_i with cardinal $m_i \geq 3$. By default edges are not oriented and can be seen as subsets $\{v_i, v_j\} \subset V$. The set of edges is denoted by E. We use the notation $e_{i,j}$ (or $e_{j,i}$) as a shortcut for the unique edge $\{v_i, v_j\}$. The corresponding joining torus in Θ is denoted $T_{i,j}$. We also denote by $\vec{e}_{i,j}$ and $\vec{e}_{j,i}$ the two half-edges that compose $e_{i,j}$ on the side of v_i and v_j respectively.

Definition 1.3.20. A *graph ordering* over Γ is a collection of functions $\Omega = (\omega_i)_{v_i \in V}$ where

$$\omega_i : V_i \to \{1, \dots, m_i\}$$

are bijections defined on the sets of neighbours. For every vertex $v_i \in V$, the function ω_i is called the *local order* around v_i. ◇

In some contexts we make a greater use of the inverse of ω_i, which we denote by

$$\xi_i : \{1, \dots, m_i\} \to V_i$$

A convenient way of representing a graph ordering is to put a decoration on every half-edge $\vec{e}_{i,j}$ of the graph indicating the local order $\omega_i(v_j)$ of the corresponding neighbour v_j around the starting vertex v_i, as illustrated in Figure 1.3.1.

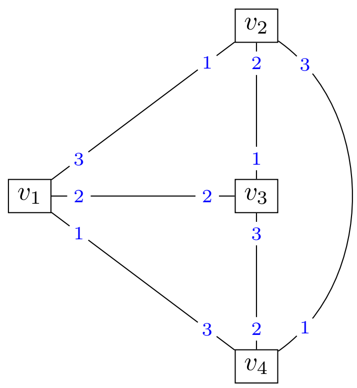


Figure 1.3.1: An ordered graph

1.3.4 Graphed homeomorphisms

Theorem 1.3.21 ([Wal67a])**.** *Let M and N be two graph manifolds with respective minimal graph structures Θ_M and Θ_N. Let $\Phi : M \to N$ be a homeomorphism. Then, if M and N do not belong to one of the exceptional cases, Φ is isotopic to a homeomorphism $\Psi : M \to N$ such that $\Psi(\Theta_M) = \Psi(\Theta_N)$.* ◁

According to Theorem 1.3.21, up to isotopy any homeomorphism of a graph manifold acts on the graph structure. Since joining tori are associated to edges of the graph, one can thus define a surjective morphism

$$G : \mathrm{Homeo}(M) \twoheadrightarrow \mathrm{Aut}(\Gamma)$$

where $\mathrm{Homeo}(M)$ stands for the group of isotopy classes of homeomorphisms of M.

Definition 1.3.22. The kernel of G is called the group of *graphed homeomorphisms* of the graph manifold M and is denoted $\mathrm{Homeo}_\Gamma(M)$. ◇

Graph manifolds are orientable as well as all of their circle bundle components. We thus take these orientations in consideration.

Definition 1.3.23. A homeomorphism $\Psi : M \to M$ is said to be *positive* if it respects the global orientation of M. It is *strongly positive* if its restriction on every component S_i is fibre-positive in the sense of Definition 1.3.5. ◇

We denote by $\mathrm{Homeo}^{+}(M)$ (resp. $\mathrm{Homeo}^{++}(M)$) the group of isotopy classes of positive (resp. strongly positive) homeomorphisms of M. The similar notations $\mathrm{Homeo}^{+}_{\Gamma}(M)$ (resp. $\mathrm{Homeo}^{++}_{\Gamma}(M)$) denotes the respective subgroups of graphed homeomorphisms of M. In fact $\mathrm{Homeo}^{++}_{\Gamma}(M)$ is a normal subgroup of index 2 of $\mathrm{Homeo}^{+}_{\Gamma}(M)$. The quotient is generated by the positive homeomorphism of M defined as follows:

- On every circle bundle component S_i there is a homeomorphism ν_i that changes the orientation of all sections and all fibres at the same time, without changing the global orientation of S_i.
- Since the gluing map given by Condition (M4) permutes fibres and sections, the action of ν_i on one component S_i of M propagates to every other component. The ensuing global homeomorphism of M is denoted by ν and does not change the global orientation of the manifold.

Proposition 1.3.24. *The group* $\mathrm{Homeo}^{++}_{\Gamma}(M)$ *is fixed by the action of* ν *and we have:*

$$\mathrm{Homeo}^{+}_{\Gamma}(M) = \mathrm{Homeo}^{++}_{\Gamma}(M) \ltimes_{\nu} \mathbb{Z}/2\mathbb{Z} \qquad \triangleleft$$

***Remark* 1.3.25.** The definition of strongly positive homeomorphisms is motivated by the fact that the action of ν on M does not change the global orientation of M but still affects the orientation of ordered stars drawn on the basis of every circle bundle component. ⌑

1.4 Ordered graphed embeddings

In this section we define a proper way to draw the graph of a graph manifold on the manifold itself. This allows us to obtain a presentation of the fundamental group of the graph manifold which is combinatorially determined up to some specific homeomorphisms. We then solve that ambiguity by giving a way to construct the subgroup of the graph manifold homeomorphisms that leave the presentation unchanged. This subgroup is called the *graph stabiliser* of the graph manifold.

1.4.1 Ordered model of a graph manifold

Let M be a graph manifold with associated (unique) graph structure Θ and graph Γ and let Ω be a graph ordering on Γ. For every vertex $v_i \in V$, we denote by S_i the associated circle bundle. Theorem 1.3.11 ensures that there exists a function

$$\chi_i : S(m_i, \check{s}_i, \varepsilon_i) \longrightarrow S_i$$

where ε_i is the Euler number of S_i. The model circle bundle decomposes as:

$$S(m_i, \check{s}_i, \varepsilon_i) = T_{m_i} \cup_{g_{\varepsilon_i}} T_{\infty}$$

where $T_{m_i} = D_{m_i} \times S^1$ for any $m \geq 0$ and g_{ε_i} is the gluing map given in Eq. (1.4). The basis D_{m_i} is a disc with m_i holes described at the beginning of Section 1.2.1. The boundary components of D_{m_i} have a *natural order* given by the horizontal coordinate of their basis circle in $\mathbb{C}$, which we extend to the boundary components of T_{m_i} and $S(m_i, \check{s}_i, \varepsilon_i)$.

Definition 1.4.1. Let M be a graph manifold with associated (unique) graph structure Θ and graph Γ. Let Ω be a graph ordering on Γ. An *ordered model* of M with respect to Ω is a collection of functions $\mathrm{X} := (\chi_i : S(m_i, \check{s}_i, \varepsilon_i) \to S_i)_{v_i \in V}$ such that:

(i) for every $v_i \in V$, χ_i is a positive fibrewise boundary-homeomorphism.

(ii) for every boundary torus $T_{i,j} \in \Theta$, the pre-image ${\chi_i}^{-1}(T_{i,j})$ is the $\omega_i(v_j)$-th boundary component of $S(m_i, \check{s}_i, \varepsilon_i)$ for the natural order. ◇

Proposition 1.4.2. *For any graph ordering Ω, there always exists an ordered model X of M with respect to Ω.* $\triangleleft$

Proof. According to Theorem 1.3.11, there exists an orientation-preserving fibrewise boundary-homeomorphism $\chi_i : S(m_i, \check{s}_i, \varepsilon_i) \to S_i$. One then just needs to make the corresponding boundary tori coincide along the ordered graph by using homeomorphisms on the boundary that permutes the components. $\square$

Notation. When

$$s : D_{m_i} \longrightarrow T_{m_i} \subset S(m_i, \check{s}_i, \varepsilon_i)$$

is a section of the trivial circle bundle and

$$\chi_i : S(m_i, \check{s}_i, \varepsilon_i) \longrightarrow S_i$$

is a local ordered model of a circle bundle component S_i, we write

$$\widehat{s_i} := \chi_i \circ s_i : D_{m_i} \longrightarrow S_i$$

The map $\widehat{s_i}$ is not a full section of S_i but rather a section of the subset $\chi_i(T_{m_i})$. For simplicity, we call $\widehat{s_i}$ a section nonetheless since we never consider objects in the $\chi_i(T_\infty)$ part of S_i.

Proposition 1.4.3. *Let $\mathrm{X} = (\chi_i)_{v_i \in V}$ and $\mathrm{X}' = (\chi_i')_{v_i \in V}$ be two ordered models of a graph manifold M with respect to the same graph ordering Ω. Define the* change of model Ψ *as the unique homeomorphism of M which restricts to $\Psi_i := \chi_i' \circ (\chi_i)^{-1}$ on every circle bundle component S_i. Then $\Psi \in \mathrm{Homeo}_\Gamma^{++}(M)$.* $\triangleleft$

Proof. We first prove that Ψ is well defined. By Definition 1.4.1, for every $v_i \in V$, χ_i and χ_i' are boundary-homeomorphisms on S_i, and thus so is Ψ_i. We can then extend Ψ_i by the identity on every gluing torus $T_{i,j}$ to define $\Psi \in \mathrm{Homeo}(M)$. It is clear that Ψ is graphed by construction. Besides, χ_i and χ_i' are fibrewise positive, and so is Ψ_i. This implies that Ψ respects the orientation of all fibres and sections of every S_i, and thus $\Psi \in \mathrm{Homeo}_\Gamma^{++}(M)$. $\square$

1.4.2 Definition of an ordered graphed embedding

Ordered graphed embeddings are defined piece by piece by tying together one ordered star on each circle bundle component of the graph manifold.

Definition 1.4.4. Let M be a graph manifold whose graph Γ is ordered by Ω. Let $\gamma : \Gamma \to M$ be a graphed embedding, i.e. a map that sends vertices of the graph to disjoint points and edges of the graph to disjoint simple curves. Suppose that there exist an ordered model X of M and, for every vertex $v_i \in V$, a section $s_i : D_{m_i} \to T_{m_i}$ and an ordered star $\alpha_i \in \mathcal{O}_{m_i}$ drawn on D_{m_i} such that $\gamma \cap S_i = \widehat{s_i}(\alpha_i)$. Then γ is called an *ordered graphed embedding* with respect to the graph ordering Ω.

The set of isotopy classes of ordered graphed embeddings with respect to the graph ordering Ω is denoted by $\mathrm{E}_\Gamma(\Omega)$. $\diamond$

***Remark* 1.4.5.** We can always tie together an ordered star α_i drawn on s_i and another ordered star α_j drawn on s_j along an edge $e_{i,j} \in E$. Indeed, since the gluing map g given by Condition (M4) permutes longitudes and meridians, the circles $g(\widehat{s_i} \cap T_{i,j})$ and $\widehat{s_j} \cap T_{i,j}$ have exactly one point of intersection. One can thus always perform an isotopy on a small neighbourhood of $\partial^{\omega_i(v_j)} D_{m_i}$ inside s_i so that the endpoint of $\alpha_i^{\omega_i(v_j)}$ coincides with the intersection point, and likewise on the other side. ⌑

Example 1.4.6. The schematic construction of a graphed embedding of the ordered graph of Figure 1.3.1 is represented on Figure 1.4.1. The fibres of each circle bundle are assigned a specific colour. By the gluing map of Condition (M4), each boundary curve of the basis of every circle bundle is identified with a fibre of the corresponding neighbour. ⬠

***Remark* 1.4.7.** Definition 1.4.4 is not the most general way of embedding the graph Γ inside M in a way that respects the graph structure. The specificity of our construction is to ask that the image of every half-edge inside each circle bundle to be drawn on a section. This allows to impose a circular order on the half-edges inside each circle bundle S_i, thus taking into account the local order ω_i. This local order is indeed required to study the homotopy class of the half-edges, as evidenced by Theorem 1.5.9. ⌑

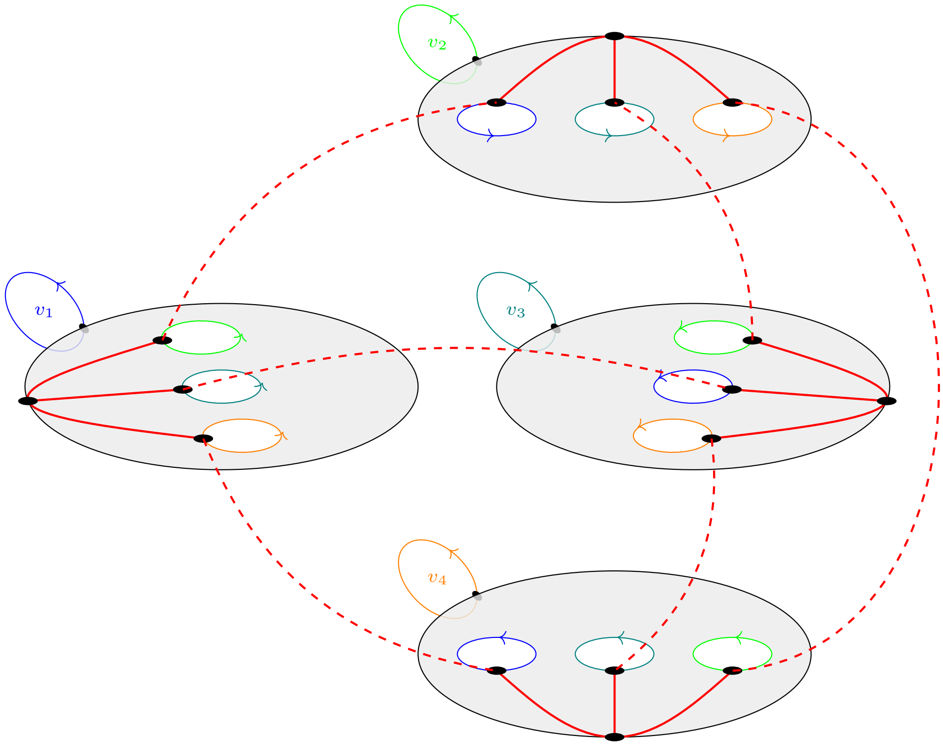


Figure 1.4.1: Construction of a graphed embedding

Proposition 1.4.8. *For any cycled graph manifold M with graph Γ and any choice of Ω, the set $\mathrm{E}_\Gamma(\Omega)$ is non-empty.* ◁

Proof. By Proposition 1.4.2, there always exists an ordered model X of M. By Remark 1.2.3, there always exists an ordered star $\alpha_i \in \mathcal{O}_{m_i}$ for every possible value of $m_i \geq 3$. Choose a section $\widehat{s_i} : D_{m_i} \to S_i$ for every $v_i \in V$. Now tie together the ordered stars $\widehat{s_i}(\alpha_i)$ along every edge $e_{i,j} \in E$ as described in Remark 1.4.5. The reunion of all the α_i for $1 \leq i \leq n$ is an ordered graphed embedding $\gamma \in \mathrm{E}_\Gamma(\Omega)$. □

1.4.3 Action of the homeomorphisms

Theorem 1.4.9. *Let M be a graphed manifold ordered whose graph Γ is ordered by Ω. There is a well-defined action of the group of strongly positive homeomorphisms $\mathrm{Homeo}_\Gamma^{++}(M)$ on $\mathrm{E}_\Gamma(\Omega)$. This action is transitive.* ◁

Proof. We prove that the action is well defined. Consider $\gamma \in \mathrm{E}_\Gamma(\Omega)$. There exists an ordered model X and for every $v_i \in V$ a section $s_i : D_{m_i} \to T_{m_i}$ and an ordered star $\alpha_i \in \mathcal{O}_{m_i}$ such that $\gamma \cap S_i = \widehat{s_i}(\alpha_i) = \chi_i \circ s_i(\alpha_i)$. Let $\Psi \in \mathrm{Homeo}_\Gamma^{++}(M)$. By Definition 1.3.22, up to isotopy Ψ restricts to $\Psi_i \in \mathrm{Homeo}_\partial^+(S_i)$ for every $v_i \in V$. Define $\Psi_i(\widehat{s_i}) := \Psi_i \circ \widehat{s_i} \circ \widetilde{\Psi_i}^{-1}$. Then $\Psi_i(\widehat{s_i}) : D_{m_i} \to S_i$ is a (partial) section, and we have

$$(\Psi_i \circ \widehat{s_i})(\alpha_i) = \Psi_i(\widehat{s_i}) \circ \widetilde{\Psi_i}(\alpha_i) \tag{1.5}$$

The full basis of S_i is $\Sigma_i \simeq \Sigma^{m_i}$ and the boundary-homeomorphism $\widetilde{\Psi_i}$ is an element of $\mathcal{P}(\Sigma^{m_i})$. By Theorem 1.1.3, there exists an element $\psi_i \in \mathcal{P}(D_{m_i})$ such that $\widetilde{\Psi_i} = f_{m_i} \circ \psi_i$. By Proposition 1.2.6, $\psi_i(\alpha_i)$ is still an ordered star on D_{m_i}. Since f_{m_i} respects the inclusion $D_{m_i} \subset \Sigma_i$ then $\widetilde{\Psi_i}(\alpha_i)$ is still an ordered star on D_{m_i} for the same order ω_i.

Thus, the action of Ψ sends every couple section/ordered star $(\widehat{s_i}, \alpha_i)$ to another well-defined couple $(\Psi_i(\widehat{s_i}), \widetilde{\Psi_i}(\alpha_i))$ on the same ordered model χ_i of S_i. All the new ordered stars can then be tied together in the same points as γ since Ψ restricts to the identity on all joining tori. We therefore have formed a new ordered graphed embedding $\Psi(\gamma) \in \mathrm{E}_\Gamma(\Omega)$.

We now prove that the action is transitive. Let $\gamma, \gamma' \in \mathrm{E}_\Gamma(\Omega)$ be two ordered graphed embeddings and let X, X' be the corresponding ordered models.

Consider $v_i \in V$ and the two couples section/ordered stars $(\widehat{s_i}, \alpha_i)$ and $(\widehat{s_i}', \alpha_i')$ corresponding to γ and γ' respectively. There always exists a $\Phi_i \in \mathrm{Homeo}_\partial^+(T_{m_i})$ such that $\Phi_i(s_i) = s_i'$ but also such that Φ_i induces the identity on the basis D_{m_i}, and thus $\widetilde{\Phi_i}(\alpha_i) = \alpha_i$.

By Proposition 1.2.6, there also exists an element $\pi_i \in \mathcal{P}(D_{m_i})$ such that $\pi_i \cdot \alpha_i = \alpha_i'$. Now extend π_i to T_{m_i} by taking the section s_i' as the basis of the fibration, to obtain $\Pi_i \in \mathrm{Homeo}_\partial^+(T_{m_i})$. Now set $\Pi_i \circ \Phi_i$ to $S(m_i, \check{s}_i, \varepsilon_i)$ by the identity on T_∞ along the gluing map g_{ε_i} and define

$$\Psi_i := \chi_i' \circ \Pi_i \circ \Phi_i \circ (\chi_i)^{-1} \in \mathrm{Homeo}_\partial^+(S_i)$$

By construction, $\Psi_i(\widehat{s_i}) = \widehat{s_i}'$ and $\widetilde{\Psi_i}(\alpha_i) = \alpha_i'$.

Now define Ψ as the reunion of all the Ψ_i's for every $v_i \in V$. Then $\Psi \in \mathrm{Homeo}_\Gamma^{++}(M)$ and $\Psi(\gamma) = \gamma'$. □

1.5 Fundamental group of a graph manifold

1.5.1 Cycles of the graph

Let M be a graph manifold with graph Γ. We reuse notations from Section 1.3.3. Recall that V is the vertex set and E is the edge set.

There are two different ways to construct the cycles of the graph: either as generators of the fundamental group $\pi_1(\Gamma)$ or as elements of the first homological group $H_1(\Gamma)$.

From now on it is convenient to orient the graph Γ.

Definition 1.5.1. An *orientation* of Γ is a function $\delta : V^2 \to \{-1, 0, 1\}$ such that

- $\delta_{i,j} \in \{\pm 1\}$ if $e_{i,j}$ is an edge of Γ.
- $\delta_{i,j} = 0$ if $e_{i,j}$ is not an edge.
- $\delta_{i,j} = -\delta_{j,i}$. ◇

Let r be a vertex. The generators of the fundamental group $\pi_1(\Gamma, r)$ can be drawn directly on the graph Γ itself using a *spanning tree* based in a vertex r as shown on Figure 1.5.1a.

Definition 1.5.2. A *spanning tree* of a graph Γ verifying Conditions 1.3.18 is a connected acyclic subgraph $\mathcal{T}$ of Γ containing all the vertices. ◇

Proposition 1.5.3. *Let $r \in V$. For every vertex $v \in V$, there always exist a unique path from r to v inside $\mathcal{T}$. The vertex r is called a* root *of the tree $\mathcal{T}$.* ◁

Proposition 1.5.4. *Let $\mathcal{T}$ be a spanning tree of a graph Γ and let $r \in V$ be a root. Then the set of edges $E \setminus \mathcal{T}$ is naturally associated with a basis of $\pi_1(\Gamma, r)$ as follows: assign to every edge $e_{i,j} \in E \setminus \mathcal{T}$ (with i, j ordered such that $\delta_{i,j} = 1$) the unique cycle $c_{i,j}$ on the graph Γ that goes along the unique path inside $\mathcal{T}$ from r to v_i, then along $e_{i,j}$, and finally along the unique path inside $\mathcal{T}$ from v_j back to r as illustrated on Figure 1.5.1b.* ◁

The alternative way to describe the cycles of Γ is to use the fact that Γ has a natural structure of a CW-complex generated by the vertices and the edges. The boundary map $\partial_1 : C_1(\Gamma) \to C_0(\Gamma)$ is defined on the basis E of $C_1(\Gamma)$ as $\partial_1(e_{i,j}) = \delta_{i,j}(v_j - v_i)$.

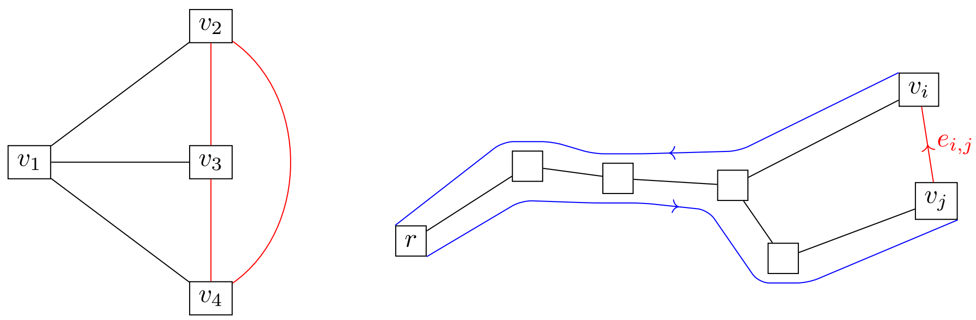
The first homology group $H_1(\Gamma)$ is defined as the kernel of ∂_1 and does not depend on the choice of the orientation. There is a natural group homomorphism

$$\zeta_\delta : H_1(\Gamma) \longrightarrow C_1(\Gamma)$$

which decomposes every cycle into the sum of its oriented edges. Note that the choice of the orientation δ only affects the signs of the decomposition.

The connection between the two definitions of the cycles of Γ is made by the natural abelianisation map

$$\mathrm{Ab} : \pi_1(\Gamma, r) \longrightarrow H_1(\Gamma)$$



(a) A spanning tree (b) $\mathcal{T}$-cycle curve

Figure 1.5.1: Spanning tree on a graph

1.5.2 Presentation of the fundamental group

Fix a graph ordering Ω, an orientation δ on Γ and a graphed embedding $\gamma \in \mathrm{E}_\Gamma(\Omega)$. Let $\mathcal{T}$ be a spanning tree with root r and let $b := \gamma(r)$.

Definition 1.5.5. For every edge $e_{i,j} \in E \smallsetminus \mathcal{T}$, let $c_{i,j}$ be the unique cycle drawn on Γ going through $e_{i,j}$ and the root r as per Proposition 1.5.4. Then the simple closed curve $\gamma_{i,j} := \gamma(c_{i,j})$ is called a *$\mathcal{T}$-cycle curve* of M. ◇

Definition 1.5.6. For every vertex $v_i \in V$, let f_i be the fibre curve of the circle bundle S_i over the point $\gamma(v_i)$, and let c_i be the unique path from the root r to v_i in $\mathcal{T}$. Then $\mu_i := \gamma(c_i) \cdot f_i \cdot \gamma(c_i)^{-1}$ is called a *$\mathcal{T}$-meridian curve* of M. ◇

Recall from Proposition 1.5.4 that the set $E \smallsetminus \mathcal{T}$ naturally determines a basis of the free group $\pi_1(\Gamma, r)$. We denote by $\mathbb{F}_V$ the group freely generated by the vertex set V.

Definition 1.5.7. A graphed embedding $\gamma \in \mathrm{E}_\Gamma(\Omega)$ naturally induces a map

$$\gamma_{\mathcal{T},\delta} : \mathbb{F}_V * \pi_1(\Gamma, r) \longrightarrow \pi_1(M, b)$$

which sends $v_i \in V$ to the class of the $\mathcal{T}$-meridian curve μ_i of Definition 1.5.6 and each cycle $c_{i,j}$ for $e_{i,j} \in E \smallsetminus \mathcal{T}$ to the class of the $\mathcal{T}$-cycle curve $\gamma_{i,j}$ of Definition 1.5.5. ◇

***Remark* 1.5.8.** If $v_i, v_j \in V$ are linked by an edge $e_{i,j}$ then by the gluing map of Condition (M4), the meridian generator μ_i can be retracted to the closed curve $\widehat{s_j}(\partial^{\omega_j(v_i)} D_{m_j})$ inside the section $\widehat{s_j}$ of S_j. ⌑

Using Remark 1.5.8 and a combination of Seifert-van Kampen's theorem and HNN-extensions along the graph Γ of the manifold, E. Westlund [Wes97] obtained a presentation of the fundamental group of M. Similar works were done in [Mum61; ACM19b; KN14]. We reformulate his result using our notations and the new concept of ordered graphed embeddings.

Theorem 1.5.9. *Let δ be an orientation of Γ, let $\mathcal{T}$ be a spanning tree of Γ and let r be the root of $\mathcal{T}$. Let $\gamma \in \mathrm{E}_\Gamma(\Omega)$ be an ordered graphed embedding and write $b := \gamma(r)$. Then there is an exact sequence*

$$0 \longrightarrow P_\Gamma(\mathcal{T}, \delta, \gamma) \longrightarrow \mathbb{F}_V * \pi_1(\Gamma, r) \xrightarrow{\gamma_{\mathcal{T},\delta}} \pi_1(M, b) \longrightarrow 0$$

where $P_\Gamma(\mathcal{T}, \delta, \gamma)$ is the subgroup of relations normally generated by

- $\forall\ v_i \in V: \quad \prod_{k=1}^{m_i} \mu_{\xi_i(k)}{}^{u(i,\xi_i(k))} = \mu_i{}^{-\varepsilon_i}$

- $\forall\ e_{i,j} \in E: \quad \left[\mu_i, \mu_j{}^{u(i,j)}\right] = 1$

$$\textit{where } u(i,j) := \begin{cases} \gamma_{i,j} & \textit{if } e_{i,j} \in E \smallsetminus \mathcal{T} \textit{ and } \delta_{i,j} = 1. \\ \gamma_{i,j}{}^{-1} & \textit{if } e_{i,j} \in E \smallsetminus \mathcal{T} \textit{ and } \delta_{i,j} = -1. \\ 1 & \textit{if } e_{i,j} \in \mathcal{T}. \end{cases}$$ ◁

For $x, y \in \pi_1(M, b)$ the notations are $[x, y] = xyx^{-1}y^{-1}$ and $x^y = y^{-1}xy$.

***Remark* 1.5.10.** This presentation can often be simplified, in particular when some Euler numbers ε_i are equal to 0, 1 or -1. ⌑

In fact the presentation $P_\Gamma(\mathcal{T}, \delta, \gamma)$ does not depend on the ordered graphed embedding.

Theorem 1.5.11. *With the spanning tree $\mathcal{T}$ and orientation δ of Γ being fixed, the presentation $P_\Gamma(\mathcal{T}, \delta, \gamma)$ depends only on the graph ordering Ω.* ◁

This result is crucial to study invariants directly derived from the fundamental group $\pi_1(M)$, as it ensures that the presentation is stable under the change of the graphed embedding induced by the action of $\mathrm{Homeo}^{++}_\Gamma(M)$. This allows to compute representations of the fundamental group and other derived invariants such as twisted homology. Since these applications exceeds the scope of our presentation, we do not prove Theorem 1.5.11 here. The remainder of this subsection is dedicated to the proof of Theorem 1.5.9.

Lemma 1.5.12. *Let S be a circle bundle with m boundary components and Euler number ε that verifies Conditions 1.3.7. Let b be a point in S. Then $\pi_1(S, b)$ admits the presentation*

Generators: $f, (x^1), \ldots, (x^m)$

Relation: $\prod_{k=1}^{m} (x^k) = f^{-\varepsilon}$ ◁

Proof. This is obtained directly by using Seifert-van Kampen's theorem on the description of the model $S(m, \check{s}, \varepsilon)$ given in Section 1.3.1 and Theorem 1.3.11. The generator f is the positive fibre over b, and for every $1 \le r \le m$, the generator (x^k) is the positive curve $\alpha^k \cdot \partial^k D_m \cdot (\alpha^k)^{-1}$ where $\alpha \in \mathcal{O}_n$ is any ordered star on the basis D_m of $T_m \subset S$. □

Proof of Theorem 1.5.9. Let $\mathcal{T}$ be a spanning tree of Γ and let $\gamma \in \mathrm{E}_\Gamma(\Omega)$. Let V^* be the set of vertices of Γ that borders an edge of $E \smallsetminus \mathcal{T}$. Consider $v_i \in V$. From Lemma 1.5.12 we get the presentation $P(m_i, \varepsilon_i)$ of $\pi_1(S_i, \gamma(v_i))$.

By Remark 1.5.8, (x_i^k) is homotopic inside M to f_j with $k = \omega_i(v_j)$, and μ_j is homotopic to f_j inside S_j. Now consider $e_{i,j} \in \mathcal{T}$ and glue together S_i and S_j along the gluing torus $T_{i,j}$. Using Seifert-van Kampen's theorem and the gluing map from Conditions 1.3.16 along all edges of $\mathcal{T}$, we obtain a sub-manifold $M_\mathcal{T}$ of M whose fundamental group $\pi_1(M_\mathcal{T}, \gamma(r))$ has the following presentation:

Generators: μ_i for $v_i \in V$ and $x_i^{\omega_i(v_j)}$ for every $e_{i,j} \in E \smallsetminus \mathcal{T}$

Relations:

- $\forall\ e_{i,j} \in E \smallsetminus \mathcal{T}: \quad [\mu_i, \mu_j] = 1$
- $\forall\ v_i \in V \smallsetminus V^* : \prod_{k=1}^{m_i} \mu_{\xi_i(k)} = {\mu_i}^{-\varepsilon_i}$
- $\forall\ v_i \in V^* : \prod_{k=1}^{m_i} (x_i^k) = {\mu_i}^{-\varepsilon_i}$ and $(x_i^k) = \mu_j$ when $\xi_i(k) \in V^*$ (which happens for only one value of k).

Let $e_{i,j} \in E \smallsetminus \mathcal{T}$, with $v_i, v_j \in V^*$. We glue back the tori $T_{i,j}$ corresponding to the edge $e_{i,j}$. This corresponds to a HNN-extension of $M_\mathcal{T}$ by gluing a handle from $S^1 \times \partial^{\omega_i(v_j)} D_{m_i}$ to $S^1 \times \partial^{\omega_j(v_i)} D_{m_j}$. The fundamental group of the extension contains a new generator which corresponds to the homological class of the cycle curve $\gamma_{i,j}$ from Definition 1.5.5. Suppose that $i < j$. Since the gluing map once again permutes the meridian and longitude of $T_{i,j}$, the new relations are

$$\left(x_i^{\omega_i(v_j)}\right) = {\gamma_e}^{-1} \cdot f_j \cdot \gamma_e \qquad \left[f_i, \left(x_i^{\omega_i(v_j)}\right)\right] = 1$$
$$f_i = {\gamma_e}^{-1} \cdot \left(x_j^{\omega_j(v_i)}\right) \cdot \gamma_e \qquad \left[f_j, \left(x_j^{\omega_j(v_i)}\right)\right] = 1$$

But f_i (resp. f_j) is homotopic to μ_i (resp. μ_j), which yields the simplified relations

$$\left(x_i^{\omega_i(v_j)}\right) = {\mu_j}^{\gamma_e} \qquad \left(x_j^{\omega_j(v_i)}\right) = {\mu_i}^{({\gamma_e}^{-1})} \qquad [\mu_i, {\mu_j}^{\gamma_e}] = 1$$

Doing the HNN-extensions of $\pi_1(M_\mathcal{T}, b)$ for all edges $e_{i,j} \in E \smallsetminus \mathcal{T}$, and then replacing all $\left(x_i^{\omega_i(v_j)}\right)$ for every $v_i \in V^*$ in the previous presentation gives the desired presentation of $\pi_1(M, b)$. □

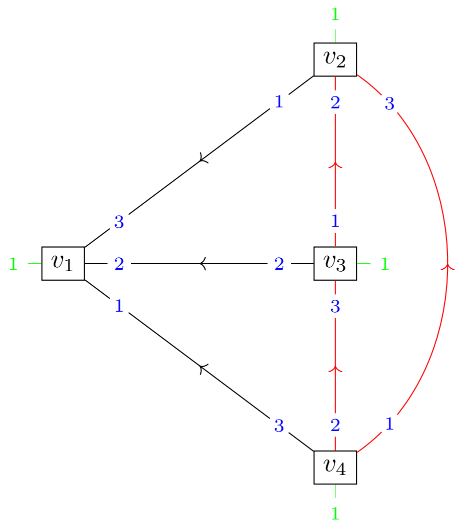


Figure 1.5.2: Minimal graph of the generic combinatorics with 4 vertices

Example 1.5.13. Consider the graph shown on Figure 1.5.2. Each vertex v_i is decorated with its Euler number in green and its local order ω_i in blue. Each edge of E is oriented and the edges of $E \setminus \mathcal{T}$ are drawn in red. The corresponding presentation $P_\Gamma(\mathcal{T}, \delta, \Omega)$ of $\pi_1(M, v_0)$ is given by:

Generators: meridian curves $\mu_1, \mu_2, \mu_3, \mu_4$; cycle curves $\gamma_{2,3}, \gamma_{3,4}, \gamma_{2,4}$.

Relations:

$$\mu_4\mu_3\mu_2 = \mu_1{}^{-1} \quad \mu_2\mu_1\mu_4 = \mu_3{}^{-1} \quad \mu_1\mu_3\mu_4 = \mu_2{}^{-1} \quad \mu_2\mu_3\mu_1 = \mu_4{}^{-1}$$
$$[\mu_1, \mu_2] = 1 \qquad [\mu_1, \mu_3] = 1 \qquad [\mu_1, \mu_4] = 1$$
$$[\mu_2, \mu_3{}^{\gamma_{2,3}}] = 1 \quad [\mu_3, \mu_4{}^{\gamma_{3,4}}] = 1 \quad [\mu_2, \mu_4{}^{\gamma_{2,4}}] = 1$$
⬡

1.5.3 First homology group of a graph manifold

Recall that every vertex $v_i \in V$ of the graph Γ is decorated with the Euler number $\varepsilon_i \in \mathbb{Z}$ of the corresponding circle bundle component S_i.

Definition 1.5.14. The *meridian homology* $V(\Gamma)$ of a graph Γ is the quotient of the free abelian vertex group $C_0(\Gamma)$ by the normally generated subgroup

$$R(\Gamma) := \left\langle \varepsilon_i \cdot v_i + \sum_{v_j \in V_i} v_j, \; v_i \in V \right\rangle$$

with the exact sequence:

$$0 \longrightarrow R(\Gamma) \longrightarrow C_0(\Gamma) \xrightarrow{\eta} V(\Gamma) \longrightarrow 0$$

For any element $v \in C_0(\Gamma)$, we also note $\overline{v}$ its class in $V(\Gamma)$. ◇

The terminology 'meridian homology' comes from the fact that $V(\Gamma)$ corresponds precisely to the contribution of the meridians of the components to the first homology group of the graph manifold M.

Example 1.5.15. The meridian homology of the graph from Figure 1.5.2 has four generators $\mu_1, \mu_2, \mu_3, \mu_4$ with the relation:

$$\mu_1 + \mu_2 + \mu_3 + \mu_4 = 0$$
⬡

When a disambiguation is necessary, for a closed curve w inside a manifold M we use the notation $[w]$ for its class inside $H_1(M, \mathbb{Z})$, which we shorten to $H_1(M)$.

Theorem 1.5.16. *Let $\gamma \in \mathrm{E}_\Gamma(\Omega)$ be a graphed embedding. Then γ induces a group isomorphism*

$$\gamma_* : V(\Gamma) \oplus H_1(\Gamma) \xrightarrow{\sim} H_1(M)$$
◁

Proof. Abelianising the exact sequence of Theorem 1.5.9 yields a new exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_\Gamma(\mathcal{T},\delta,\gamma) & \longrightarrow & \mathbb{F}_V * \pi_1(\Gamma,r) & \xrightarrow{\gamma_{\mathcal{T},\delta}} & \pi_1(M,b) & \longrightarrow & 0 \\ & & \downarrow \mathrm{Ab} & & \downarrow \mathrm{Ab} & & \downarrow \mathrm{Ab} & & \\ 0 & \longrightarrow & R(\Gamma) & \longrightarrow & C_0(\Gamma)\oplus H_1(\Gamma) & \xrightarrow{\mathrm{Ab}(\gamma)} & H_1(M,\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

which is exactly the defining exact sequence of $V(\Gamma)\oplus H_1(\Gamma)$. The morphism γ_* is defined as the morphism induced by $\mathrm{Ab}(\gamma)$ on the quotient, which does not depend on the choice of the presentation $P_\Gamma(\mathcal{T},\delta,\gamma)$. □

***Remark* 1.5.17.** Even if the morphism γ_* does not depends on the presentation $P_\Gamma(\mathcal{T},\delta,\gamma)$ and its underlying parameters, it is necessary to fix such a presentation to obtain a basis of $C_0(\Gamma)\oplus H_1(\Gamma)$ and thus compute the exact values of γ_*. ⌑

Proposition 1.5.18. *For every pair of graphed embeddings $\gamma,\gamma'\in \mathrm{E}_\Gamma(\Omega)$, we have*

$$\left({\gamma_*}^{-1}\circ\gamma'_*\right)_{|V(\Gamma)} = \mathrm{Id}_{V(\Gamma)} \qquad \triangleleft$$

Proof. Fix a spanning tree $\mathcal{T}$ and a graph ordering Ω. Let $v_i\in V$. By construction, $\gamma'_*(\overline{v_i}) = [\mu_i]$, which by Definition 1.5.6 has the same class as the fibre f_i over $\gamma(v_i)$ inside S_i. Similarly, $\gamma_*(\overline{v_i})$ is equal to the class of the fibre f'_i over $\gamma'(v_i)$ inside S_i. But by Lemma 1.5.12, all fibres of a circle bundle have the same homological class. Therefore all morphisms γ_* send $\overline{v_i}$ to the same unique class $[f_i]\in H_1(M)$ coming from $V(\Gamma)$. Thus $\gamma'_*(\overline{v_i}) = [f_i]$ and $\gamma_*^{-1}([f_i]) = \overline{v_i}$. □

***Remark* 1.5.19.** When M is the boundary of a regular neighbourhood of a complex algebraic curve embedded in the projective plane $\mathbb{CP}^2$, $V(\Gamma)$ is also isomorphic to the first homology group of the exterior of the curve. For the case of complex line arrangements this is proven in Proposition 2.4.22. ⌑

1.6 Graph stabiliser

1.6.1 Definition of the graph stabiliser

The graph stabiliser is the quotient of $\mathrm{Hom}(H_1(M),V(\Gamma))$ by the differences between every two ordered graphed embeddings. The functions of the quotient are thus *stable* by all possible changes of the generators of $H_1(M)$. Let Γ be a graph verifying Conditions 1.3.18 on page 16 and let M be the associated graph manifold. Let also Ω be an ordering on Γ.

Definition 1.6.1. The *graph stabiliser* $\mathcal{G}_\Gamma(\Omega)$ is defined as the quotient of

$$\mathrm{Hom}(H_1(\Gamma),V(\Gamma))$$

by the subgroup

$$\left\langle \phi\circ(\gamma_*-\gamma'_*)_{|H_1(\Gamma)} \;\middle|\; \begin{array}{c}\gamma,\gamma'\in \mathrm{E}_\Gamma(\Omega), \phi\in\mathrm{Hom}(H_1(M),V(\Gamma)) \\ (\phi\circ\gamma_*)_{|V(\Gamma)} = \mathrm{Id}_{V(\Gamma)}\end{array}\right\rangle \qquad \diamond$$

Our objective in the remainder of this section is to find a combinatorial presentation of the graph stabiliser.

1.6.2 Difference maps

The difference maps are used to compute the homological difference between two graphed embeddings. This difference lies exclusively on the cycle generators of $H_1(M)$ since by Proposition 1.5.18 two graphed embeddings always coincide on the homological meridian generators.

Definition 1.6.2. Let $v_i\in V$ be a vertex of the graph Γ. Let $\langle V_i\rangle$ be the submodule of $C_0(\Gamma)$ freely generated by V_i. The natural map

$$g_i : \langle V_i\rangle \longrightarrow H_1\left(D_{m_i}\right)$$

which sends v_k to the class $[\partial^{w_i(v_k)}D_{m_i}]$ for every $v_k\in V_i$ is called the *homological neighbour map*. ◇

Consider two ordered graphed embeddings $\gamma, \gamma' \in \mathrm{E}_\Gamma(\Omega)$ and let $e_{i,j} \in E$ be an edge with $\delta_{i,j} = 1$. Let α_i (resp. α'_i) be the ordered star of $\mathcal{O}_{m_i}$ associated to γ (resp. γ') for the vertex v_i. We can always suppose that α_i and α'_i have the same starting point in D_{m_i}. The branches $\alpha_i^{\omega_i(v_j)}$ and $\alpha'^{\omega_i(v_j)}_i$ both have their end points on the circular boundary component $\partial_{\omega_i(v_j)} D_{m_i}$. Let a_i^j be a simple circle arc that joins them, such that the curve

$$w_i^j(\alpha_i, \alpha'_i) := \alpha_i^{\omega_i(v_j)} \circ a_i^j \circ \left(\alpha'^{\omega_i(v_j)}_i\right)^{-1} \subset D_{m_i}$$

is closed.

Definition 1.6.3. The *edge difference map* is the morphism

$$\Delta : \mathrm{E}_\Gamma(\Omega)^2 \longrightarrow \mathrm{Hom}\left(C_1(\Gamma), C_0(\Gamma)\right)$$

defined by:

$$\Delta(\gamma, \gamma')(e_{i,j}) := g_i^{-1}\left(w_i^j(\alpha, \alpha')\right) - g_j^{-1}\left(w_j^i(\alpha, \alpha')\right) \in \langle V_i, V_j \rangle \subset C_0(\Gamma) \qquad \diamond$$

Proposition 1.6.4. *For every $\gamma, \gamma' \in \mathrm{E}_\Gamma(\Omega)$, the edge difference map Δ verifies the following properties:*

(i) $\Delta(\gamma, \gamma)$ *is the trivial group homomorphism.*

(ii) $\Delta(\gamma, \gamma')(e) = -\Delta(\gamma', \gamma)(e)$ *for every $e \in C_1(\Gamma)$.* ◁

Definition 1.6.5. Let δ be an orientation of Γ. The *cycle difference map* is the map

$$\widetilde{\Delta} : \mathrm{E}_\Gamma(\Omega)^2 \longrightarrow \mathrm{Hom}\left(H_1(\Gamma), V(\Gamma)\right)$$

defined by

$$\widetilde{\Delta}(\gamma, \gamma') := (\zeta_\delta^* \otimes \eta) \circ \Delta(\gamma, \gamma')$$

where $\zeta_\delta : H_1(\Gamma) \to C_1(\Gamma)$ is the natural inclusion map and $\eta : C_0(\Gamma) \twoheadrightarrow V(\Gamma)$ is the projection from Definition 1.5.14. ◇

***Remark* 1.6.6.** The cycle difference map $\widetilde{\Delta}$ verifies properties similar to Proposition 1.6.4 which are induced by the properties of the edge difference map Δ. ⌑

As announced, the cycle difference map gives a first reformulation of the graph stabiliser definition.

Proposition 1.6.7. *Let $\gamma, \gamma' \in \mathrm{E}_\Gamma(\Omega)$. Then*

$$\gamma_* \circ \widetilde{\Delta}(\gamma, \gamma') = (\gamma_* - \gamma'_*)_{|H_1(\Gamma)} \in \mathrm{Hom}(H_1(\Gamma), V(\Gamma)) \qquad ◁$$

Theorem 1.6.8. *There is a natural identification*

$$\mathcal{G}_\Gamma(\Omega) \simeq \mathrm{Hom}(H_1(\Gamma), V(\Gamma)) \big/ \left\langle \widetilde{\Delta}(\gamma, \gamma') \mid \gamma, \gamma' \in \mathrm{E}_\Gamma(\Omega) \right\rangle \qquad ◁$$

Proof. Let $\gamma, \gamma' \in \mathrm{E}_\Gamma(\Omega)$ and $\phi \in \mathrm{Hom}(H_1(M))$ such that $(\phi \circ \gamma_*)_{|V(\Gamma)} = \mathrm{Id}_{V(\Gamma)}$ as in Definition 1.6.1 of $\mathcal{G}_\Gamma(\Omega)$. Then by Proposition 1.6.7:

$$\phi \circ (\gamma_* - \gamma'_*)_{H_1(\Gamma)} = \phi \circ \gamma_* \circ \widetilde{\Delta}(\gamma, \gamma') = \widetilde{\Delta}(\gamma, \gamma') \qquad \square$$

The remainder of this subsection is dedicated to the proof of Proposition 1.6.7. Let $e_{i,j} \in E$ be an edge. By Definition 1.4.4 on page 19 of $\gamma \in \mathrm{E}_\Gamma(\Omega)$, there exist an ordered model X, sections s_i, s_j and ordered stars α_i, α_j such that

$$\gamma(\vec{e}_{i,j}) = \widehat{s_i}(\alpha_i^{\omega_i(v_j)}) \qquad \gamma(\vec{e}_{j,i}) = \widehat{s_j}(\alpha_j^{\omega_j(v_i)})$$

where $\vec{e}_{i,j}$ is the half-edge of Γ starting from v_i and going towards v_j (and reciprocally for $\vec{e}_{j,i}$). The junction point of $\gamma(\vec{e}_{i,j})$ and $\gamma(\vec{e}_{j,i})$ is exactly the point $\gamma \cap T_{i,j}$. We therefore have

$$\gamma(e_{i,j}) = \gamma(\vec{e}_{i,j}) \cdot \gamma(\vec{e}_{j,i})^{-1} \subset \widehat{s_i}(D_{m_i}) \cup \widehat{s_j}(D_{m_j})$$

We use similar notations for γ'.

Lemma 1.6.9. *The following diagram commutes*

$$\begin{array}{ccc} H_1(D_{m_i}) & \xrightarrow{\widehat{s_i}^*} & H_1(M) \\ {\scriptstyle g_i}\uparrow & & \uparrow{\scriptstyle \gamma_*} \\ \langle V_i \rangle & \xrightarrow[\eta]{} & V(\Gamma) \end{array}$$ ◁

Proof. As noted in Remark 1.5.8 on page 22, for any $v_k \in V_i$ the meridian curve μ_k has the same homological class as the image by $\widehat{s_i}$ of the corresponding boundary component $\partial^{w_i(v_k)} D_{m_i}$. This precisely means that $\widehat{s_i}^*(v_k) = [\mu_k]$. But by Theorem 1.5.16, we also have $\gamma_*(\overline{v_k}) = [\mu_k]$ by construction of γ_*. □

Corollary 1.6.10. $\left(\widehat{s_i}^*\right)^{-1} \circ \widehat{s'_i}^* = \mathrm{Id}_{H_1(D_{m_i})}$ ◁

Proof. This is an immediate consequence of combining Lemma 1.6.9 with Proposition 1.5.18. □

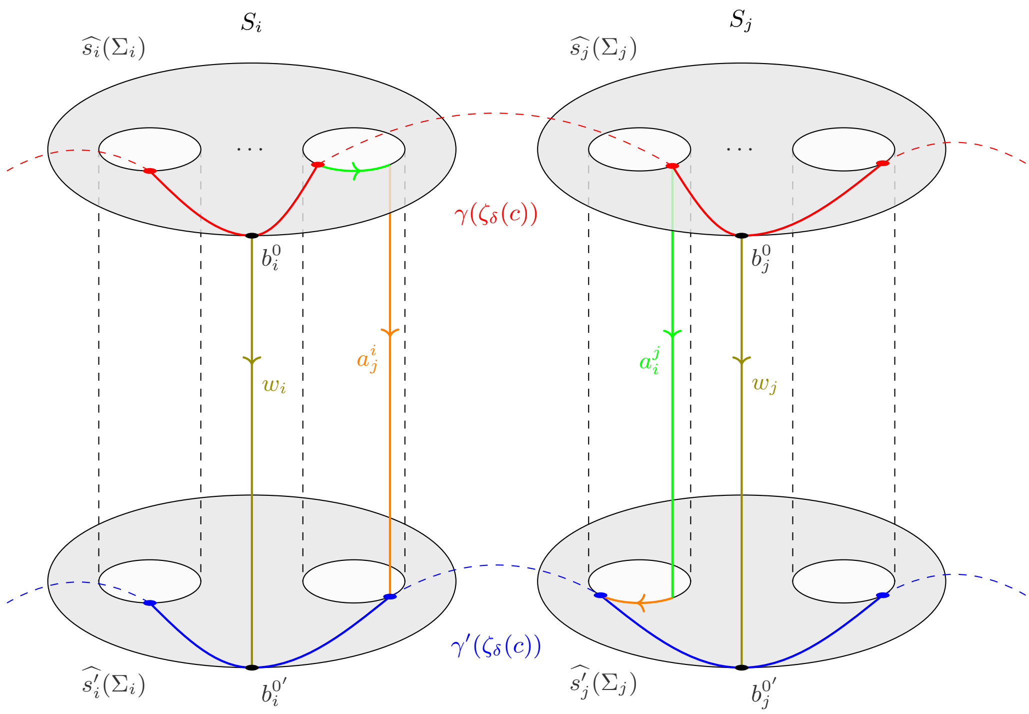


Figure 1.6.1: Difference between two graphed embeddings

Proof of Proposition 1.6.7. Up to isotopy, one can always suppose that the starting point b_i^0 of $\gamma_l(e_{i,j})$ and the starting point $b_i^{0'}$ of $\gamma'_l(e_{i,j})$ lie in the same fibre f_i of S_i, with a path w_i joining them. Then the curves

$$w_L(\gamma, \gamma')(e_{i,j}) := \gamma(\vec{e}_{i,j}) \cdot a_i^j \cdot a_j^i \cdot \gamma'(\vec{e}_{i,j})^{-1} \cdot w_i^{-1} \subset S_i$$
$$w_R(\gamma, \gamma')(e_{i,j}) := \gamma(\vec{e}_{j,i}) \cdot a_i^j \cdot a_j^i \cdot \gamma'(\vec{e}_{j,i})^{-1} \cdot w_i^{-1} \subset S_j$$

are closed, as shown on Figure 1.6.1. By retracting the vertical fibre curves within S_i and S_j, we obtain the following equalities for the homological classes:

$$[w_L(\gamma, \gamma')(e_{i,j})]_M = \widehat{s_i}^*\left(\left[w_i^j(\alpha_i, \alpha'_i)\right]_{D_{m_i}}\right)$$
$$[w_R(\gamma, \gamma')(e_{i,j})]_M = \widehat{s_j}^*\left(\left[w_j^i(\alpha_j, \alpha'_j)\right]_{D_{m_j}}\right)$$

Let $c \in H_1(\Gamma)$ and write $\zeta_\delta(c) = \sum_k \delta_k \cdot e_{i_k,i_{k+1}}$ with $\delta_k := \delta_{i_k,i_{k+1}}$. By construction of the embedding $\gamma \in \mathrm{E}_\Gamma(\Omega)$ we have

$$\gamma(\zeta_\delta(c)) = \prod_k \gamma(e_{i_k,i_{k+1}})^{\delta_k}$$

Inside $H_1(M)$, $(\gamma_* - \gamma'_*)(c)$ can be seen as the class of the closed curve

$$w(\gamma,\gamma')(c) = \gamma(\zeta_\delta(c)) \cdot w_{i_0} \cdot \gamma'(\zeta_\delta(c))^{-1} \cdot {w_{i_0}}^{-1}$$

As shown on Figure 1.6.1, a 2-chain bordering that curve inside M can be decomposed into a sum of squared 2-chains bordering each of the closed curves $w_L(\gamma,\gamma')(e_{i_k,i_{k+1}})$ and $w_R(\gamma,\gamma')(e_{i_k,i_{k+1}})$ on each reunion $S_{i_k} \cup S_{i_{k+1}}$. Inside $H_1(M)$, this yields the equation:

$$(\gamma_* - \gamma'_*)(c) = \sum_k \delta_k \left([w_L(\gamma,\gamma')(e_{i_k,i_{k+1}})]_M - [w_R(\gamma,\gamma')(e_{i_k,i_{k+1}})]_M \right)$$

Applying successively Lemma 1.6.9 and Definition 1.6.3 of the edge difference map $\Delta(\gamma,\gamma')$, we get:

$$\begin{aligned}
&[w_L(\gamma,\gamma')(e_{i_k,i_{k+1}})]_M - [w_R(\gamma,\gamma')(e_{i_k,i_{k+1}})]_M \\
&\qquad = \widehat{s_i}^* \circ g_i \circ {g_i}^{-1}\left(\left[w_i^j(\alpha_i,\alpha'_i)\right]_{D_{m_i}}\right) - \widehat{s_j}^* \circ g_j \circ {g_j}^{-1}\left(\left[w_j^i(\alpha_j,\alpha'_j)\right]_{D_{m_j}}\right) \\
&\qquad = \gamma_* \circ \eta \circ \Delta(\gamma,\gamma')(e_{i_k,i_{k+1}})
\end{aligned}$$

Replacing in the sum yields:

$$\begin{aligned}
(\gamma_* - \gamma'_*)(c) &= \sum_k \delta_k \cdot \gamma_* \circ \eta \circ \Delta(\gamma,\gamma')(e_{i_k,i_{k+1}}) \\
&= \gamma_* \circ \left((\zeta_\delta^* \otimes \eta) \circ \Delta(\gamma,\gamma')\right)(c) \\
&= \gamma_* \circ \widetilde{\Delta}(\gamma,\gamma')(c)
\end{aligned}$$ □

1.6.3 Presentation of the graph stabiliser

The results obtained in Section 1.1.3 allows us to compute explicitly the image of the cycle difference map $\widetilde{\Delta}$ and thus give a combinatorial presentation of $\mathcal{G}_\Gamma(\Omega)$.

Theorem 1.6.11. *The group $\mathcal{G}_\Gamma(\Omega)$ is finitely presented and admits the presentation:*

Generators: $c^* \otimes \widetilde{v}$ *for every* $c \in H_1(\Gamma)$ *and* $v \in V$.

Relations: *the images by* $\zeta_\delta^* \otimes \eta$ *of*

(GS1) $e_{i,j}^* \otimes v_i = 0$ *and* $e_{i,j}^* \otimes v_j = 0$ *for every edge* $e_{i,j} \in E$.

(GS2) $e_{i,j}^* \otimes v_k - \delta_{i,j}\delta_{j,k} \cdot e_{j,k}^* \otimes v_i = 0$ *for every pair of adjacent edges* $e_{i,j}$ *and* $e_{j,k}$ *in* Γ. ◁

Corollary 1.6.12. *The graph stabiliser $\mathcal{G}_\Gamma(\Omega)$ does not depend on the choice of the graph ordering Ω.* ◁

Proof of Theorem 1.6.11. Applying Definition 1.6.5, the subgroup

$$\left\langle \widetilde{\Delta}(\gamma,\gamma') \mid \gamma,\gamma' \in \mathrm{E}_\Gamma(\Omega) \right\rangle \subset H^1(\Gamma) \otimes V(\Gamma)$$

in isomorphic to the image by $\zeta_\delta^* \otimes \eta$ of the subgroup

$$\left\langle \Delta(\gamma,\gamma') \mid \gamma \in \mathrm{E}_\Gamma(\Omega), \Psi \in \mathrm{Homeo}_\Gamma^{++}(M) \right\rangle \subset C^1(\Gamma) \otimes C_0(\Gamma)$$

By Theorem 1.4.9 on page 20, this subgroup is in turn isomorphic to

$$G_\Gamma := \left\langle \Delta(\Psi(\gamma),\gamma) \mid \gamma \in \mathrm{E}_\Gamma(\Omega), \Psi \in \mathrm{Homeo}_\Gamma^{++}(M) \right\rangle$$

Therefore, we just have to determine that G_Γ is freely generated by the elements of type (GS1) and (GS2) listed above.

Let $\Psi \in \mathrm{Homeo}_\Gamma^{++}(M)$ and $\gamma \in \mathrm{E}_\Gamma(\Omega)$. Recall from the proof of Theorem 1.4.9 that the action of Ψ on γ can be seen as induced by the combined actions of elements $\psi_i \in \mathcal{P}(D_{m_i})$ on each ordered star $\alpha_i \in D_{m_i}$ that compose γ.

Let $e_{i,j} \in E$. We reuse the notations from Sections 1.2 and 1.6.2. We want to compute $\Delta(\psi_i \cdot \gamma, \gamma)(e_{i,j})$ where the action of $\psi_i \in \mathcal{P}(D_{m_i})$ is extended by the identity everywhere outside S_i. In particular, ψ_i only acts on the $\gamma(\vec{e}_{i,j})$. Applying Definition 1.6.3, $\Delta(\psi_i \cdot \gamma, \gamma)(e_{i,j})$ can be seen as the image by ${g_i}^{-1}$ of the homological class inside $H_1(D_{m_i})$ of the loop

$$(\psi_i \cdot \alpha_i)^{\omega_i(v_j)} \cdot \left(\alpha_i^{\omega_i(v_j)}\right)^{-1}$$

The generators of $\mathcal{P}(D_{m_i})$ were given in Theorem 1.1.2 on page 10 : the Dehn twists d_k and $d_{j,l}$ for $1 \le j,k,l \le m_i$. The actions of each of these generators on the path of ordered stars are shown on Figure 1.2.3 on page 13. Denote by x_k the class of $\partial_k D_{m_i}$ inside $H_1(D_{m_i})$. By superimposing the upper part of Figure 1.2.3 on the lower part, we see that for every $v_j, v_k, v_l \in V_i$ we have:

$$[(d_r \cdot \gamma)(e_{i,k}) \cdot \gamma(e_{i,k})^{-1}]_{\Sigma_i} = \begin{cases} \delta_{i,k} \cdot x_i^{\omega_i(v_k)} & \text{if } r = \omega_i(v_k). \\ 0 & \text{otherwise.} \end{cases}$$
$$[(d_{r,s} \cdot \gamma)(e_{i,j}) \cdot \gamma(e_{i,j})^{-1}]_{\Sigma_i} = \begin{cases} \delta_{i,j} \cdot \left(x_i^{\omega_i(v_j)} + x_i^{\omega_i(v_l)}\right) & \text{if } \{r,s\} = \{\omega_i(v_j), \omega_i(v_l)\} \\ 0 & \text{otherwise.} \end{cases}$$

Recall from Definition 1.6.2 that ${g_i}^{-1}(x_k) = v_k \in V_i$. Hence we get that for every vertices $v_j, v_k, v_l \in V_i$ the differences are:

$$\Delta(d_{\omega_i(v_k)} \cdot \gamma, \gamma) = \delta_{i,k} \cdot e_{i,k}^* \otimes v_k$$
$$\Delta(d_{\omega_i(v_j),\omega_i(v_l)} \cdot \gamma, \gamma) = \delta_{i,j} \cdot e_{i,j}^* \otimes (v_j + v_l) + \delta_{i,l} \cdot e_{i,l}^* \otimes (v_j + v_l)$$

Repeating this for all vertices $v_i \in V$ gives the generating set of the subgroup G_Γ. However, it can be simplified by removing the first generator from the second when $k = j$ and $k = l$, which gives:

$$G_\Gamma = \left\langle e_{i,k}^* \otimes v_k,\ e_{i,j}^* \otimes v_l - \delta_{j,i}\delta_{i,l} \cdot e_{i,l}^* \otimes v_j \;\middle|\; \forall\, v_i \in V,\ \forall\, v_j, v_k, v_l \in V_i \right\rangle$$

The group G_Γ is thus generated by the elements of types (GS1) and (GS2). □

Example 1.6.13. We apply Theorem 1.6.11 to compute a simplified combinatorial presentation of the graph stabiliser $\mathcal{G}_\Gamma$ for the data of the graph Γ, spanning tree $\mathcal{T}$ and orientation δ of Figure 1.5.2 on page 24.

The basis of $H_1(\Gamma)$ associated to $E \setminus \mathcal{T}$ is given by:

$$c_{34} = -e_{31} + e_{34} + e_{41} \qquad c_{42} = e_{21} + e_{42} - e_{41} \qquad c_{32} = e_{21} + e_{32} - e_{31}$$

The dual map $\zeta_\delta^* : C^1(\Gamma) \to H^1(\Gamma)$ is given by:

$$e_{21}^* \mapsto c_{42}^* + c_{32}^* \qquad e_{31}^* \mapsto -c_{34}^* - c_{32}^* \qquad e_{41}^* \mapsto c_{34}^* - c_{42}^*$$
$$e_{32}^* \mapsto c_{32}^* \qquad e_{42}^* \mapsto c_{42}^* \qquad e_{34}^* \mapsto c_{34}^*$$

The generators of $\mathcal{G}_\Gamma$ are thus:

$$c_{32}^* \otimes \overline{v_2} \quad c_{34}^* \otimes \overline{v_2} \quad c_{42}^* \otimes \overline{v_2}$$
$$c_{32}^* \otimes \overline{v_3} \quad c_{34}^* \otimes \overline{v_3} \quad c_{42}^* \otimes \overline{v_3}$$
$$c_{32}^* \otimes \overline{v_4} \quad c_{34}^* \otimes \overline{v_4} \quad c_{42}^* \otimes \overline{v_4}$$

Now we compute the relations. This is done in two steps: first writing the relations of G_Γ inside $C^1(\Gamma) \otimes C_0(\Gamma)$, and then taking their image by the map $\zeta_\delta^* \otimes \eta$. We use the symbol $\equiv$ to denote the equivalence of relations in the group $\mathcal{G}_\Gamma$. The first set of relations of type (GS1) is given by:

$$e_{32}^* \otimes v_3 \mapsto c_{32}^* \otimes \overline{v_3} \qquad e_{32}^* \otimes v_2 \mapsto c_{32}^* \otimes \overline{v_2}$$
$$e_{34}^* \otimes v_3 \mapsto c_{34}^* \otimes \overline{v_3} \qquad e_{34}^* \otimes v_4 \mapsto c_{34}^* \otimes \overline{v_4}$$
$$e_{42}^* \otimes v_4 \mapsto c_{42}^* \otimes \overline{v_4} \qquad e_{42}^* \otimes v_2 \mapsto c_{42}^* \otimes \overline{v_2}$$

The second set of relations of type (GS1) is given by:

$$\begin{array}{rclcl}
e^*_{21}\otimes v_2 & \mapsto & (c^*_{42}+c^*_{32})\otimes\overline{v_2} & \equiv & 0\\
e^*_{21}\otimes v_1 & \mapsto & -(c^*_{42}+c^*_{32})\otimes(\overline{v_2}+\overline{v_3}+\overline{v_4}) & \equiv & -c^*_{42}\otimes\overline{v_3}-c^*_{32}\otimes\overline{v_4}\\
e^*_{31}\otimes v_3 & \mapsto & -(c^*_{32}+c^*_{34})\otimes\overline{v_3} & \equiv & 0\\
e^*_{31}\otimes v_1 & \mapsto & (c^*_{32}+c^*_{34})\otimes(\overline{v_2}+\overline{v_3}+\overline{v_4}) & \equiv & c^*_{32}\otimes\overline{v_4}+c^*_{34}\otimes\overline{v_2}\\
e^*_{41}\otimes v_4 & \mapsto & (c^*_{34}-c^*_{42})\otimes\overline{v_4} & \equiv & 0\\
e^*_{41}\otimes v_1 & \mapsto & -(c^*_{34}-c^*_{42})\otimes(\overline{v_2}+\overline{v_3}+\overline{v_4}) & \equiv & -c^*_{34}\otimes\overline{v_2}+c^*_{42}\otimes\overline{v_3}
\end{array}$$

Similar computations show that all relations of type (GS2) are redundant for this set of data. ◯

1.6.4 Plumbing moves

In [Neu81, Proposition 2.1], Neumann gives a series of *plumbing moves* that can be performed on any graph structure Θ of a graph manifold M without changing the oriented diffeomorphism type of M. Using these moves one can reduce any decomposition of M in Seifert manifolds to the minimal graph structure of Theorem 1.3.15.

The original topological Definition 1.6.1 of the graph stabiliser is only valid as such on the minimal graph Γ of the manifold M, because it critically relies on Theorem 1.3.21 that ensures that the minimal graph structure is preserved up to homeomorphism of M. However, Neumann's theorem still allows us to compute the purely combinatorial presentation of the graph stabiliser $\mathcal{G}_\Gamma$ given in Theorem 1.6.11 using non-minimal graphs.

Neumann's plumbing calculus is defined in a more general class of graphs than the ones verifying only Conditions 1.3.18. In particular, he allows 'negative' edges which correspond to a gluing map that reverses the orientation of the meridian and longitude of the gluing torus. By this definition, our class of graphs contains only 'positive' edges. It turns out that only one flavour of the Neumann plumbing moves, the *blowing down*, can give graphs respecting Conditions 1.3.18. The two applicable moves are denoted by (R1b) *(binary case)* and (R1u) *(unary case)* respectively. They are represented on Figure 1.6.2, using the notation $V_i = \widetilde{V_i} \sqcup \{v_a\}$.

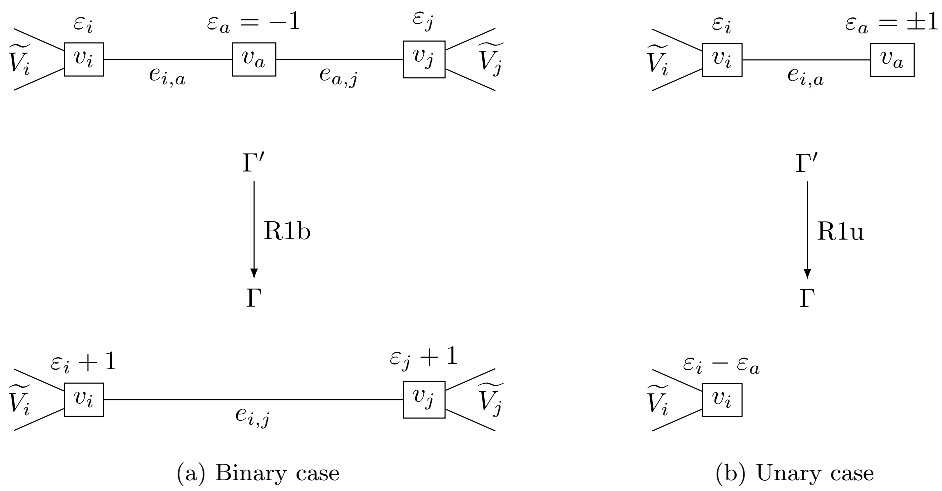


Figure 1.6.2: Blowing-down

For the subclass of graph manifolds that we consider, removing the minimality condition is therefore equivalent to alleviating Condition (G3). For the remainder of this section we thus consider graphs that verify the new set of conditions:

Conditions 1.6.14. The graph Γ' is such that:

(G1) no edge starts and ends at the same vertex.

(G2) there is at most one edge between every two vertices.

(G3') every vertex has at least three neighbours, *except*:

- vertices with $\varepsilon = -1$ can have *one or two* neighbours.
- vertices with $\varepsilon = 1$ can have *one* neighbour.

Thanks to Theorem 1.6.11, the graph stabiliser can be seen as a combinatorial object which can thus be defined on graphs verifying only the new Conditions 1.6.14. Then applying any of the blowing-down moves leave the graph stabiliser unchanged.

Theorem 1.6.15. *If Γ and Γ' are two graphs verifying Conditions 1.6.14, such that Γ is obtained from Γ' by a series of blowing-down moves, then there is a natural isomorphism $\mathcal{G}_{\Gamma'} \xrightarrow{\sim} \mathcal{G}_{\Gamma}$.* ◁

***Remark* 1.6.16.** In all generality, the (R1b) move with $\varepsilon_a = +1$ could also be applied to the family of graphs that we consider. This case is nevertheless not considered for two reasons. Firstly because it is a combinatorial exception to Theorem 1.6.15. Secondly because the non-minimal graphs which include this situation are precisely the graphs of the exceptional combinatorics of Definition 2.2.18 on page 39, which correspond to graph manifolds that do not have a unique minimal graph structure, i.e. exceptions of Theorem 1.3.15. ⌑

The remainder of this section is dedicated to the purely combinatorial proof of Theorem 1.6.15 and can be skipped on first reading.

Lemma 1.6.17. *If Γ' and Γ are related by a series of blowing-down moves then there exist a group isomorphism h and an epimorphism g such that the following diagram commutes:*

$$\begin{array}{ccc} C_0(\Gamma') & \xrightarrow{g} & C_0(\Gamma) \\ {\scriptstyle \eta'}\downarrow & & \downarrow{\scriptstyle \eta} \\ V(\Gamma) & \xrightarrow[h]{\sim} & V(\Gamma) \end{array}$$ ◁

Proof. Suppose that Γ is obtained from Γ' by a (R1b) blowing-down move on the edges $e_{i,a}$ and $e_{a,j}$. Define the morphism g by

$$v_a \longmapsto v_i + v_j \qquad\qquad v \longmapsto v \quad \text{if } v \neq v_a$$

By Definition 1.5.14 inside $V(\Gamma')$ we have the relations

$$\overline{v_a} = \overline{v_i} + \overline{v_j} \qquad -\varepsilon_i \overline{v_i} = \overline{v_a} + \sum_{v \in \widetilde{V_i}} \overline{v} \qquad -\varepsilon_j \overline{v_j} = \overline{v_a} + \sum_{w \in \widetilde{V_j}} \overline{w} \tag{1.6}$$

Replacing $\overline{v_a}$ in the other two relations of Eq. (1.6) yields

$$-(\varepsilon_i + 1)\overline{v_i} = \sum_{v \in \widetilde{V_i}} \overline{v} \qquad\qquad -(\varepsilon_j + 1)\overline{v_j} = \sum_{w \in \widetilde{V_j}} \overline{w}$$

which are exactly the corresponding relations in $V(\Gamma)$. All other generators and relations of $V(\Gamma')$ are identical in $V(\Gamma)$ and are indeed preserved by the plumbing move. Therefore we can define h by $h(\overline{v}) := \eta(g(v))$ for every $v \in V'$.

Suppose that Γ is obtained from Γ' by a (R1u) blowing-down move on the edge $e_{i,a}$. Define the morphism g by

$$v_a \longmapsto -\varepsilon_a v_i \qquad\qquad v \longmapsto v \quad \text{if } v \neq v_a$$

Inside $V(\Gamma')$ we have the relations

$$-\varepsilon_a \overline{v_a} = \overline{v_i} \qquad\qquad -\varepsilon_i \overline{v_i} = \overline{v_a} + \sum_{v \in \widetilde{V_i}} \overline{v} \tag{1.7}$$

Replacing $\overline{v_a}$ in the other two relations of Eq. (1.7) yields

$$-(\varepsilon_i - \varepsilon_a)\overline{v_i} = \sum_{v \in \widetilde{V_i}} \overline{v}$$

which is exactly the corresponding relation in $V(\Gamma)$. We can then define h by the same formula as in the (R1b) case. □

Lemma 1.6.18. *Let Γ and Γ' be two graphs verifying Conditions 1.6.14 such that Γ is obtained from Γ' by a series of blowing-down moves. Let δ' be an orientation of Γ'. Then there exist a monomorphism $f : C_1(\Gamma) \hookrightarrow C_1(\Gamma')$, an orientation δ of Γ, and an isomorphism $q : H_1(\Gamma) \xrightarrow{\sim} H_1(\Gamma')$ such that*

$$\zeta_\delta \circ q = f \circ \zeta_\delta \qquad \triangleleft$$

Proof. Suppose that Γ is obtained from Γ' by a (R1b) blowing-down move on the edges $e_{i,a}$ and $e_{a,j}$. Define the morphism f by

$$e_{i,j} \longmapsto \delta'_{a,j} \cdot e_{i,a} + \delta'_{i,a} \cdot e_{a,j} \qquad\qquad e \longmapsto e \quad \text{if } e \neq e_{i,j}$$

and define the orientation δ by $\delta_{i,j} := \delta'_{i,a}\delta'_{a,j}$ and $\delta := \delta'$ everywhere else. The morphism q sends any cycle going through $e_{i,j}$ inside Γ to the corresponding cycle going through $e_{i,a}$ and $e_{a,j}$ inside Γ', and is the identity on all other cycles. One easily verifies that $\zeta_{\delta'} \circ q = f \circ \zeta_\delta$.

Suppose that Γ is obtained from Γ' by a (R1u) blowing-down move on the edge $e_{i,a}$. Define the morphism f by the natural inclusion of E inside E' and δ' as equal to δ on E and equal to any value in $\{\pm 1\}$ on $e_{i,a}$. Any cycle $c \in H_1(\Gamma)$ going through $e_{i,a}$ does it twice in opposite directions. In other terms

$$\zeta_\delta(c) = \cdots + \delta_{j,i} \cdot e_{j,i} + \delta_{i,a} \cdot e_{i,a} - \delta_{i,a} \cdot e_{i,a} + \delta_{i,k} \cdot e_{i,k} + \cdots$$

Therefore q is the identity and we immediately have $\zeta_{\delta'} \circ q = f \circ \zeta_\delta$. □

Proof of Theorem 1.6.15. By Theorem 1.6.11, we have

$$\mathcal{G}_\Gamma(\mathcal{T}) = {H^1(\Gamma) \otimes V(\Gamma)}\big/{(\zeta_\delta^* \otimes \mathrm{Id})(G_\Gamma)}$$

where $G_\Gamma \subset C^1(\Gamma) \otimes C_0(\Gamma)$ is the free group generated by the elements of type (GS1) and (GS2) listed in Theorem 1.6.11. Combining Lemmas 1.6.17 and 1.6.18, we have the following commutative diagram

$$\begin{array}{ccc}
C^1(\Gamma) \otimes C_0(\Gamma) & \xleftarrow{\;f^*\otimes g\;} & C^1(\Gamma') \otimes C_0(\Gamma') \\
\Big\downarrow{\scriptstyle \zeta_\delta^* \otimes \eta} & & \Big\downarrow{\scriptstyle \zeta_{\delta'}^* \otimes \eta'} \\
H^1(\Gamma) \otimes V(\Gamma) & \xleftarrow[q^*\otimes h]{\sim} & H^1(\Gamma') \otimes V(\Gamma')
\end{array}$$

To prove the theorem it is enough to prove that $(f^* \otimes g)(G_{\Gamma'}) = G_\Gamma$. In both cases we only compute the image of the non trivial elements. It is understood that $f^* \otimes g$ restricts to the identity on all other elements of $G_{\Gamma'}$ not mentioned.

Suppose that Γ was obtained from Γ' by a (R1b) blowing-down move on the edges $e_{i,a}$ and $e_{a,j}$. We have $e^*_{i,j} = \delta'_{a,j} f^*(e^*_{i,a}) = \delta'_{i,a} f^*(e^*_{a,j})$ and $g(v_a) = v_i + v_j$. On the elements of type (GS1) of $G_{\Gamma'}$, we have

$$f^*(e^*_{i,a}) \otimes g(v_i) = \delta'_{a,j} \cdot e^*_{i,j} \otimes v_i \qquad f^*(e^*_{i,a}) \otimes g(v_a) = \delta'_{a,j} \cdot e^*_{i,j} \otimes (v_i + v_j)$$
$$f^*(e^*_{a,j}) \otimes g(v_j) = \delta'_{i,a} \cdot e^*_{i,j} \otimes v_j \qquad f^*(e^*_{a,j}) \otimes g(v_a) = \delta'_{i,a} \cdot e^*_{i,j} \otimes (v_i + v_j)$$

We obtain all the elements of type (GS1) of G_Γ. Now let $v_l \in \widetilde{V_i}$ and $v_k \in \widetilde{V_j}$. On the elements of type (GS2) of $G_{\Gamma'}$, we have

$$f^*(e^*_{i,a}) \otimes g(v_j) - \delta'_{i,a}\delta'_{a,j} \cdot f^*(e^*_{a,j}) \otimes g(v_i) = \delta'_{a,j}(e^*_{i,j} \otimes v_j + e^*_{i,j} \otimes v_i)$$

$$\begin{aligned}
&f^*(e^*_{l,i}) \otimes g(v_a) - \delta_{l,i}\delta'_{i,a} \cdot f^*(e^*_{i,a}) \otimes g(v_l) \\
&= e^*_{l,i} \otimes (v_i + v_j) - \delta_{l,i}\delta'_{i,a}\delta'_{a,j} \cdot e^*_{i,j} \otimes v_l \\
&= e^*_{l,i} \otimes v_i + (e^*_{l,i} \otimes v_j - \delta_{l,i}\delta_{i,j} \cdot e^*_{i,j} \otimes v_l)
\end{aligned}$$

$$\begin{aligned}
&f^*(e^*_{a,j}) \otimes g(v_k) - \delta'_{a,j}\delta_{j,k} \cdot f^*(e^*_{j,k}) \otimes g(v_a) \\
&= \delta'_{i,a} \cdot e^*_{i,j} \otimes v_k - \delta'_{a,j}\delta_{j,k} \cdot e^*_{j,k} \otimes (v_i + v_j) \\
&= -\delta'_{a,j}\delta_{j,k} \cdot e^*_{j,k} \otimes v_j + \delta'_{i,a}(e^*_{i,j} \otimes v_k - \delta_{i,j}\delta_{j,k} \cdot e^*_{j,k} \otimes v_i)
\end{aligned}$$

We obtain all the elements of type (GS2) of G_Γ.

Suppose that Γ was obtained from Γ' by a (R1u) blowing-down move on the edge $e_{i,a}$. We have $f^*(e^*_{i,a}) = 0$ and $g(v_a) = -\varepsilon_a v_i$. On the elements of type (GS1) of $G_{\Gamma'}$, we have

$$f^*(e^*_{i,a}) \otimes g(v_i) = 0 \qquad\qquad f^*(e^*_{i,a}) \otimes g(v_a) = 0$$

All other elements of type (GS1) of $G_{\Gamma'}$ are preserved and we thus obtain all the corresponding elements of $G_{\Gamma'}$. Now let $v_l \in \widetilde{V}_i$. On the elements of type (GS2) of $G_{\Gamma'}$, we have

$$f^*(e^*_{l,i}) \otimes g(v_a) - \delta_{l,i}\delta'_{i,a} \cdot f^*(e^*_{i,a}) \otimes g(v_l) = -\varepsilon_i \cdot e^*_{l,i} \otimes v_i + 0$$

which is an element of type (GS1) of $G_{\Gamma'}$. Again all other elements of type (GS2) of $G_{\Gamma'}$ are identically mapped to all of the corresponding elements in G_Γ. $\square$

CHAPTER

2

HOMOLOGY INCLUSION OF LINE ARRANGEMENTS

Contents

2.1 Presentation of line arrangements

Definition 2.1.1. A *line arrangement* is a union of complex lines

$$\mathcal{A} = \bigcup_{i=0}^{n} L_i$$

drawn on the projective complex plane $\mathbb{CP}^2$. ◇

In particular, for given homogenous coordinates $[x : y : z]$ on $\mathbb{CP}^2$, there exist n homogenous polynomials $f_i(x, y, z)$ of degree 1 such that

$$L_i = \{[x : y : z] \in \mathbb{CP}^2 \mid f_i(x, y, z) = 0\}$$

We denote by $\mathcal{L} = (L_i)_{1 \leq i \leq n}$ the set of line components of $\mathcal{A}$.

Definition 2.1.2. A *defining polynomial* f of $\mathcal{A}$ is a square-free homogenous polynomial such that

$$\mathcal{A} = \{[x : y : z] \in \mathbb{CP}^2 \mid f(x, y, z) = 0\}$$ ◇

In particular, the product $P_{\mathcal{A}} = \prod_{i=1}^{n} f_i$ is a defining polynomial of $\mathcal{A}$.

Definition 2.1.3. The restriction of a line arrangement $\mathcal{A}$ to the standard affine chart $\mathbb{C}^2 \equiv \{[x : y : 1] \in \mathbb{CP}^2\}$ is called its *affine part* $\mathcal{A}^{\text{aff}}$. ◇

Definition 2.1.4. An *affine line arrangement* is a union of complex lines

$$\mathcal{A}^{\text{aff}} = \bigcup_{i=0}^{n} L_i^{\text{aff}}$$

drawn on the complex plane $\mathbb{C}^2$. ◇

If the line at infinity $\{z = 0\}$ is not already in an arrangement $\mathcal{A}$, then by a linear change of coordinates one can always send any of the lines L_i at infinity. The resulting affine part $\mathcal{A}^{\text{aff}}$ then coincides with the sub-arrangement $(\mathcal{A} \setminus L_i)^{\text{aff}}$. We often alternate between both projective and affine points of view on arrangements. In general we will explicitly mention if the line at infinity is within the arrangement only when it is relevant.

The intersections between the lines are called *singular points*. We write

$$P_{i_1, i_2, \ldots, i_m} = L_{i_1} \cap L_{i_2} \cap \cdots \cap L_{i_m}$$

where m is called the *multiplicity* of the point. Alternatively, for a given singular point P, we write $V_P = \{L_{i_1}, \ldots, L_{i_m}\}$ the sub-arrangement composed only with the m lines that meet P. We denote by $\mathcal{Q}$ the set of all singular points.

Definition 2.1.5. Two arrangements $\mathcal{A}$ and $\mathcal{A}'$ in $\mathbb{CP}^2$ are *topologically equivalent* if there exists a homeomorphism Φ of the pair $(\mathbb{CP}^2, \mathcal{A})$ to the pair $(\mathbb{CP}^2, \mathcal{A}')$. ◇

If two arrangements are topologically equivalent then there is a bijection between their sets of lines that naturally extends into a bijection between their sets of singular points.

An *ordering* of a line arrangement $\mathcal{A}$ is a function $\theta : \mathcal{L} \to \{1, \ldots, n\}$. An *ordered line arrangement* is the data of a line arrangement along with an ordering on its lines. In particular a topological equivalence homeomorphism Φ between ordered line arrangements induces a permutation $\theta_\Phi \in \mathfrak{S}_n$ on the set of lines $\mathcal{L}$.

Definition 2.1.6. Two *ordered* line arrangements $(\mathcal{A}, \theta)$ and $(\mathcal{A}', \theta')$ are topologically equivalent if $\theta = \theta'$ and there exists a topological equivalence homeomorphism $\Phi : (\mathbb{CP}^2, \mathcal{A}) \to (\mathbb{CP}^2, \mathcal{A}')$ such that $\theta_\Phi = \text{Id}$. ◇

An *orientation* of a line arrangement is the data of an orientation on every complex line component $L_i \subset \mathbb{CP}^2$ of $\mathcal{A}$. Two oriented line arrangements are equivalent if there exists an orientation-preserving topological equivalence between them.

2.2 Combinatorics

2.2.1 Definitions

Combinatorics is a general concept that generalises incidence. It applies to all hyperplane arrangements but also to other types of algebraic curves in $\mathbb{CP}^2$. However, since we are focused on line arrangements, we introduce it in a form specific to this context.

Definition 2.2.1. A *line combinatorics* is a triple $C = (\mathcal{L}, \mathcal{Q}, \in)$ where $\mathcal{L}$ is a finite set, $\mathcal{Q}$ is a finite subset of $\mathcal{P}(\mathcal{L})$ and $\in$ is a relation from $\mathcal{Q}$ to $\mathcal{L}$ such that:

(i) For every element $P \in \mathcal{Q}$, there exist at least two distinct elements $L, L' \in \mathcal{L}$ such that $P \in L$ and $P \in L'$.

(ii) For every pair $L, L' \in \mathcal{L}$ with $L \neq L'$ there exists a unique $P \in \mathcal{Q}$ such that $P \in L$ and $P \in L'$. ◇

For an element $P \in \mathcal{Q}$ or $L \in \mathcal{L}$, we often make use of the *neighbour sets*:

$$V_P := \{L \in \mathcal{L} \mid P \in L\} \qquad\qquad V_L := \{P \in \mathcal{Q} \mid P \in L\} \tag{2.1}$$

The *multiplicity* of an element $P \in \mathcal{Q}$ or $L \in \mathcal{L}$ are defined respectively as $m(P) = \#V_P$ and $m(L) = \#V_L$. In particular, for $P \in \mathcal{Q}$, we have $2 \leq m(P) \leq n$ where $n = \#\mathcal{L}$. Since every line has to meet all the others exactly once, we have the relation:

$$\forall\ L \in \mathcal{L}:\ \sum_{P \in L} m(P) = n - 1 + m(L)$$

The extreme values of the multiplicity correspond to two unique type of line combinatorics.

Definition 2.2.2. Let $n \geq 2$.

- There is a unique line combinatorics which contains a $P \in \mathcal{Q}$ such that $m(P) = n$. In this case $\mathcal{Q}$ is a singleton $\{P\}$. This is called the *trivial* combinatorics with n lines.
- There is a unique line combinatorics where $m(P) = 2$ for every $P \in \mathcal{Q}$. This is called the *generic* combinatorics with n lines. ◇

Definition 2.2.3. An *ordered line combinatorics* is a line combinatorics $C = (\mathcal{L}, \mathcal{Q}, \in)$ along with the data of an ordering $\theta : \mathcal{L} \to \{1, \ldots, n\}$. ◇

Note that $\mathcal{Q}$ is *not* ordered in general for the usual definition of an ordered combinatorics. There is however a canonical way to order $\mathcal{Q}$ using the θ-lexicographic ordering on $\mathcal{P}(\mathcal{L})$.

There is an alternative way to describe ordered line combinatorics. Replace every $P \in \mathcal{Q}$ with the set $\{\theta(L) \mid P \in L\}$. Then the full ordered combinatorics can be retrieved from just the data of $\mathcal{Q}$ as a set of sets. For example

$$(C, \theta) = [[1, 2, 3]; [1, 4]; [2, 4]; [3, 4]]$$

It is much easier to write down explicit ordered combinatorics than unordered ones. Therefore, we use the following notation

$$C = ([1, 2, 3]; [1, 4]; [2, 4]; [3, 4])$$

to denote an unordered combinatorics where the given order of both $\mathcal{L}$ and $\mathcal{Q}$ is purely indicative.

Example 2.2.4.

- $([1, 2, \ldots, n])$ is the unique trivial combinatorics with n lines.
- $([1, 2]; [1; 3]; [2, 3])$ is the unique generic combinatorics with 3 lines.
- $([i, j] \mid 1 \leq i \neq j \leq n)$ is the unique generic combinatorics with n lines. ○

Definition 2.2.5. Two combinatorics $C = (\mathcal{L}, \mathcal{Q}, \in)$ and $C' = (\mathcal{L}', \mathcal{Q}', \in')$ are *isomorphic* if there exists a bijection $\lambda : \mathcal{L} \to \mathcal{L}'$ such that if $\mu : \mathcal{Q} \to \mathcal{Q}'$ is the bijection naturally induced by λ on the points, we have:

$$\forall\, L \in \mathcal{P},\ \forall\, P \in \mathcal{Q} : \quad P \in L \Longleftrightarrow \mu(P) \in' \lambda(L) \qquad \diamond$$

The group of automorphism of a combinatorics C is denoted Aut_C.

Example 2.2.6. The combinatorics of Example 2.2.4 have exceptionally large automorphism groups:

- The automorphism group of the trivial combinatorics with n lines is isomorphic to the permutation group $\mathfrak{S}_n$.
- The automorphism group of the generic combinatorics with n lines is also isomorphic to $\mathfrak{S}_n$.

For larger combinatorics, the automorphism group has in general a much simpler structure. ○

For a line arrangement $\mathcal{A}$, there is a natural line combinatorics $C_\mathcal{A}$ associated with $\mathcal{A}$ and given by the incidence of the lines. If the arrangement is ordered, then $C_\mathcal{A}$ inherits the ordering. However, not every combinatorics can be realised by an arrangement.

Definition 2.2.7. A *realisation* of a line combinatorics C is a line arrangement $\mathcal{A} \subset \mathbb{CP}^2$ such that $C_\mathcal{A} = C$. The combinatorics C is called *realisable.* ◇

Example 2.2.8. A realisation in $\mathbb{RP}^2$ of the ordered generic combinatorics with 4 lines is shown on Figure 2.2.1. ○

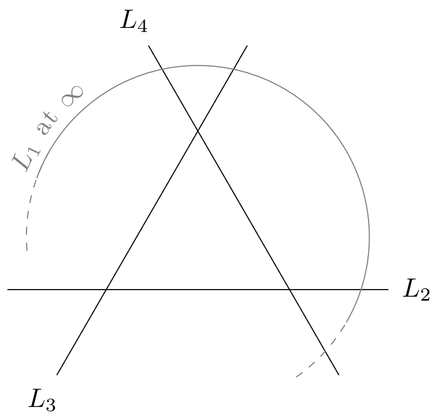


Figure 2.2.1: Realisation of the generic combinatorics with 4 lines

An ordered line arrangement with n lines can be seen as an element of $\left(\check{\mathbb{CP}^2}\right)^n$ where $\check{\mathbb{CP}^2}$ denotes the dual space. The set of all ordered realisations of a realisable combinatorics is defined by

$$\Sigma(C) = \left\{ \mathcal{A} \in \left(\check{\mathbb{CP}^2}\right)^n \;\middle|\; C_\mathcal{A} = C \right\}$$

There is a natural action of $\mathrm{PGL}_3(\mathbb{C})$ on $\mathbb{CP}^2$ which extends to its dual and then $\Sigma(C)$. The *moduli space* is defined as the quotient of $\Sigma(C)$ by this natural action.

Proposition 2.2.9. *If two line arrangements $\mathcal{A}$ and $\mathcal{A}'$ are topologically equivalent then they have isomorphic combinatorics $C_\mathcal{A}$ and $C_{\mathcal{A}'}$.* ◁

The converse is wrong. In fact, a large part of the theory of line arrangements revolves around measuring in which cases the combinatorics do not characterise the topological type.

Definition 2.2.10. A pair of line arrangements $(\mathcal{A}, \mathcal{A}')$ is called a *combinatorial pair* if $C_\mathcal{A}$ is isomorphic to $C_{\mathcal{A}'}$. A combinatorial pair where $\mathcal{A}$ and $\mathcal{A}'$ do not have the same topological type is called a *Zariski pair.* ◇

An usual way of building a combinatorial pair is to consider a defining polynomial f_ω with a cyclotomic parameter $\omega \in \mathbb{Q}(\sqrt{n})$. Then the conjugated arrangements have the same combinatorics.

2.2.2 Graphs of a combinatorics

Let $C = (\mathcal{L}, \mathcal{Q}, \in)$ be a combinatorics. Then C can be represented by a graph. If C is realisable, this graph encodes the incidence data of every realisation of C.

Definition 2.2.11. The *full incidence graph* $\Gamma(C)$ of C is defined by the following description:

Vertices: all elements of $\mathcal{L} \cup \mathcal{Q}$.

Edges: $L \in \mathcal{L}$ and $P \in \mathcal{Q}$ are linked by an edge $e_{P,L}$ if and only if $P \in L$. ◇

The full incidence graph is a *bipartite graph*, which means that its set of vertices is separated in two subsets $\mathcal{P}$ and $\mathcal{Q}$, and every edge only link a vertex of $\mathcal{P}$ to a vertex of $\mathcal{Q}$. The vertices corresponding to $\mathcal{P}$ are called *line-vertices* and the vertices corresponding to $\mathcal{Q}$ are called *point-vertices*.

If the combinatorics C is ordered then $\Gamma(C)$ inherits the ordering on the line-vertices. Point-vertices can then be labelled with the set of their line-vertex neighbours.

Since two lines always intersect, the point-vertices with only two neighbours do not carry any characterising information. They can thus be removed without any loss of generality.

Definition 2.2.12. The *reduced incidence graph* $\widetilde{\Gamma}(C)$ of C is defined by the following description:

Vertices: all elements of $\mathcal{L}$ and all elements $P \in \mathcal{Q}$ such that $m(P) \geq 3$.

Edges:

- $L \in \mathcal{L}$ and $P \in \mathcal{Q}$ such that $m(P) \geq 3$ are linked by an edge $e_{P,L}$ if and only if $P \in L$.

- two line-vertices $L, L' \in \mathcal{L}$ are linked by an edge $e_{L,L'}$ if and only if the element $P \in \mathcal{Q}$ that verifies $P \in L, P \in L'$ is such that $m(P) = 2$. ◇

Graphs may also bear additional combinatorial data, such as a graph ordering (see Definition 1.3.20 on page 17), an orientation or a spanning tree (see Definitions 1.5.1 and 1.5.2 on page 21).

Definition 2.2.13. The *standard orientation* δ^0 on $\Gamma(C)$ is defined by

$$\delta^0_{L,P} = -\delta^0_{P,L} = -1$$ ◇

Now suppose that combinatorics C is ordered by $\theta : \mathcal{L} \to \{1, \ldots, n\}$.

Definition 2.2.14. The *induced spanning tree* $\mathcal{T}^\theta$ on $\Gamma(C)$ is defined for every edge $e_{L_i,P}$ by

- $e_{L,P} \in \mathcal{T}^\theta$ if $P \in L_0$ or if $\theta(L) = \min \theta(N_P)$.
- $e_{L,P} \notin \mathcal{T}^\theta$ otherwise. ◇

The vertex v_{L_0} is the root of $\mathcal{T}^\theta$ and every vertex v_P lying at distance greater than 2 of v_{L_0} has only one adjacent edge in $\mathcal{T}^\theta$.

A graph orientation of $\Gamma(C)$ can be derived from the ordering θ on $\mathcal{L}$ and an additional ordering ν on $\mathcal{Q}$. They induce two *graph sub-orderings* $\Omega_{\mathcal{L}}$ and $\Omega_{\mathcal{Q}}$ on the disjoint sets of vertices corresponding to $\mathcal{L}$ and $\mathcal{Q}$ respectively.

Definition 2.2.15. The induced graph sub-ordering $\Omega^\theta_{\mathcal{Q}}$ on $\Gamma(C)$ is defined on every vertex v_P by $\theta_{|V_P}$ where V_P is seen as both the set of neighbour line-vertices of v_P and the subset of lines of $\mathcal{L}$ that contain P.

The induced graph sub-ordering $\Omega^\nu_{\mathcal{L}}$ is defined on every vertex v_L by $\nu_{|V_L}$ where V_L is seen as both the set of neighbour point-vertices of v_L and the subset of points of $\mathcal{Q}$ that are contained in L. ◇

Together, $\Omega^\theta_{\mathcal{Q}}$ and $\Omega^\nu_{\mathcal{L}}$ form a full graph ordering $\Omega^{\theta,\nu}$ of $\Gamma(C)$. Note that one could chose ν as the θ-lexicographic ordering and thus obtain a graph ordering Ω^θ of $\Gamma(C)$ depending only on θ.

Proposition 2.2.16. *Any graph ordering Ω of the full incidence graph $\Gamma(C)$ restricts to a graph ordering $\widetilde{\Omega}$ on the reduced graph ordering $\widetilde{\Gamma}(C)$ defined as follows:*

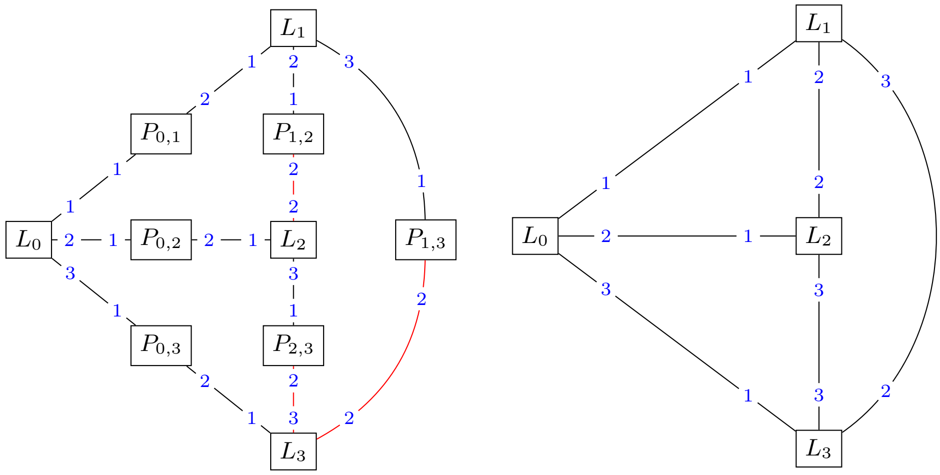
- $\widetilde{\omega}_P = \omega_P$ *for every* $P \in \mathcal{Q}$ *with* $m(P) > 2$.
- *for every* $L \in \mathcal{L}$*:*

- $\widetilde{\omega}_L(v_P) = \omega_L(v_P)$ *if* $P \in V_L$ *is such that* $m(P) > 2$.
- $\widetilde{\omega}_L(v_{L'}) = \omega_L(v_P)$ *if* $P \in V_L$ *is such that* $m(P) = 2$ *and* $V_P = \{L, L'\}$. ◁

Example 2.2.17. Consider the ordered generic combinatorics with 4 lines given by

$$(G_4, \theta) := [[1,2];[1,3];[1,4];[2,3];[2;4];[3,4]]$$

A realisation of G_4 is shown on Figure 2.2.1. The full incidence graph $\Gamma(G_4)$ ordered with Ω^θ is shown on Figure 2.2.2a. The edges of $E \setminus \mathcal{T}^\theta$ are shown in red. The reduced incidence graph $\widetilde{\Gamma}(G_4)$ ordered with the reduced ordering $\widetilde{\Omega}^\theta$ is shown on Figure 2.2.2b. ⬡



(a) Full $\Gamma(G_4)$ (b) Reduced $\widetilde{\Gamma}(G_4)$

Figure 2.2.2: Ordered incidence graphs of the generic combinatorics G_4

2.2.3 Binary vertices

In fact most of the time the reduced incidence graph coincides with the minimal graph of the graph manifold structure of the boundary manifold of a line arrangement. There are however some exceptions, which we establish in this section.

Definition 2.2.18. A combinatorics C is called *exceptional* if its reduced graph $\widetilde{\Gamma}(C)$ contains reduced vertices of multiplicity 2, called *binary vertices*. A line arrangement whose combinatorics is exceptional is also called exceptional. ◇

Note that by Definition 2.2.12 of the reduced incidence graph, binary vertices can only be line-vertices. The associated lines are also called binary. From the point of view of arrangements, a line L is binary if and only if $m(L) = 2$. We write $\mathcal{L}^{>2}$ and $\mathcal{Q}^{>2}$ the sets of non-binary lines and singular points of C.

In fact the combinatorics that contain binary vertices can be fully listed.

Theorem 2.2.19. *Let C be an exceptional combinatorics with $n \geq 3$. Then C is isomorphic to one of the following:*

(B1) *the generic combinatorics* $([1,2];[1,3];[2,3])$ *with three binary lines.*

(B2) *the* near-pencil combinatorics

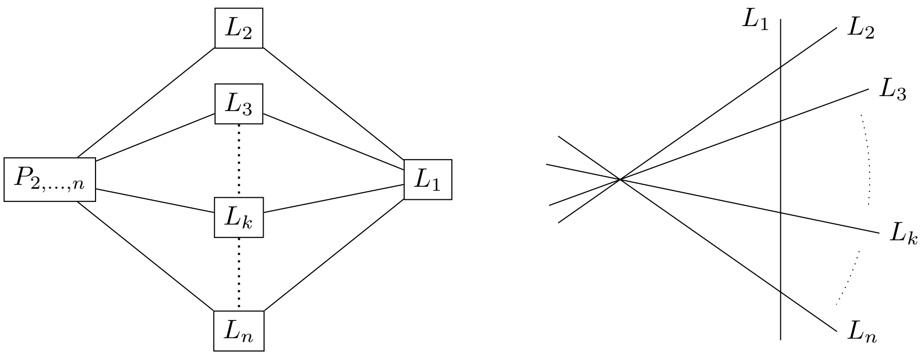
$$([1,2];[1,3];\cdots;[1,n];[2,\ldots,n])$$

with exactly $n-1$ *binary lines.*

(B3) *one of the* r-double-pencil combinatorics

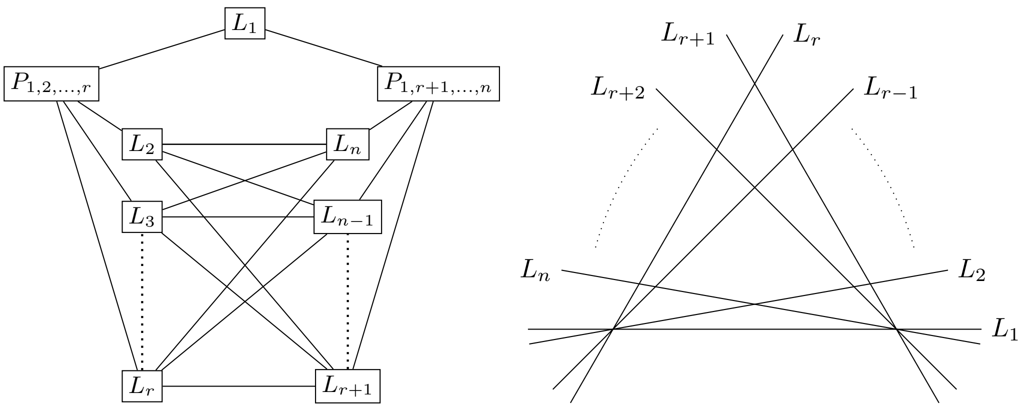
$$([1,2,\ldots,r];[1,r+1,\ldots,n]) \cup ([a,b] \mid 2 \leq a \neq b \leq n), \quad 3 \leq r \leq \frac{n}{2}$$

with exactly one binary line. ◁



(a) Reduced graph (b) Realisation in $\mathbb{RP}^2$

Figure 2.2.3: The near-pencil combinatorics



(a) Reduced graph (b) Possible realisation in $\mathbb{RP}^2$

Figure 2.2.4: The r-double-pencil combinatorics

The proof of Theorem 2.2.19 is decomposed in the following Lemmas 2.2.20 to 2.2.23.

Lemma 2.2.20. *The generic combinatorics with 3 lines* $([1,2];[1,3];[2,3])$ *is the only combinatorics that contains two binary vertices connected by an edge.* ◁

Proof. Suppose that C has two connected binary lines $L_1, L_2 \in \mathcal{L}$. Then there exists a $P_{1,2} \in \mathcal{Q}$ which meets no other line other than L_1 and L_2. Consider a third line L_3. Then L_3 has to meet both L_1 and L_2 in P_a and P_b respectively. If there is no other line in $\mathcal{L}$, then C is the generic combinatorics with three lines. Suppose that there exists a fourth line $L_4 \in \mathcal{L}$. Then L_4 has to meet all of the previous lines. But L_1 and L_2 already meet two singular points, so L_4 cannot create a third for either one of them. It cannot meet $P_{1,2}$ either, since its multiplicity is fixed. Thus L_4 has to meet L_1 in P_a and L_2 in P_b. But then L_4 would meet L_3 twice, which is impossible. □

Lemma 2.2.21. *Let C be a combinatorics with at least four lines that contains a binary line. Then any singular point that does not meet the binary line has multiplicity* 2. ◁

Proof. Let L_0 be a binary line of C. It meets exactly two singular points P_a and P_b. Suppose that there exists another singular point $P \in \mathcal{Q}^{>2}$ with $m(P) \geq 3$. Then there exist three lines $L_1, L_2, L_3 \in \mathcal{L}$ that meet P. But at least two of these three lines also need to meet L_0 in either P_a or P_b. They will thus meet twice, which is impossible. □

Lemma 2.2.22. *For $n \geq 4$, the near-pencil combinatorics is the only combinatorics with n lines that has more than one binary line.* ◁

Proof. We proceed by induction on the number of lines n.

Let C be a combinatorics with four lines that contains at least two binary lines $L_1, L_2 \in \mathcal{L}$. By Lemma 2.2.20, L_1 and L_2 meet on a point $P_0 \in \mathcal{Q}^{>2}$ with $m(P_0) \geq 3$. Then there exists at least another line L_0 that meets P_0. Let P_a (resp. P_b) be the other singular point of L_1 (resp. L_2). Then the fourth line L_3 must meet L_1 in P_a and L_2 in P_b, and it cannot meet L_0 in P_0. The result is that C is the near-pencil combinatorics with four lines.

Suppose that the proposition is true for $n \geq 4$ lines. Consider a combinatorics C with $n+1$ lines and suppose that it contains at least two binary lines $L_1, L_2 \in \mathcal{L}$. By Lemma 2.2.20, L_1 and L_2 meet on a point $P_0 \in \mathcal{Q}^{>2}$ with $m(P_0) \geq 3$. Let P_a (resp. P_b) be the other singular point of L_1 (resp. L_2). On L_1 we have $m(P_0)+m(P_a) = n+2$ and on L_2 we have $m(P_0)+m(P_b) = n+2$. Thus $m(P_a) = m(P_b)$. If $m(P_0) < n$ then $m(P_a)$, $m(P_b)$ and $m(P_0) > 2$, which by Lemma 2.2.21 is impossible. Thus $m(P_a) = n > 3$. Therefore there exists a line $L_{n+1} \in \mathcal{L}$ such that $P_a \in L_{n+1}$ but $P_b, P_c \notin L_{n+1}$. Consider the sub-combinatorics $\widehat{C}$ obtained by removing L_{n+1} from C. Then $\widehat{C}$ still has the two binary lines L_1 and L_2, so by assumption $\widehat{C}$ is the near-pencil combinatorics with n lines.

In $\widehat{C}$ the lines L_1 through L_{n-1} are binary and the line L_n meet all of them in double points. Now add back the line L_{n+1}. If L_{n+1} goes through the point $P_{2,\dots,n}$, then it meets L_1 on a new singular point, and C is exactly the near-pencil combinatorics with $n+1$ lines. If L_{n+1} does not go through $P_{2,\cdots,n}$, then it will create new singular points on $n-1$ or $n-2$ of the binary lines $L_1, \dots, L_{n-1}$, depending on whether he meets L_n at an existing singular point or not. In either case, C would have at most one binary line remaining, which is a contradiction. □

Lemma 2.2.23. *For $n \geq 4$, the r-double pencil combinatorics are the only combinatorics with exactly one binary line.* ◁

Proof. Let C be a combinatorics with $n \geq 4$ lines and exactly one binary line L_1. Let $P_a, P_b \in \mathcal{Q}$ be the two singular points that meet P_1. Denote by $r = m(P_a)$. Then $m(P_b) = n+1-r$. Since P_a and P_b have symmetric role, we can suppose that $r \leq \frac{n}{2}$. Suppose that $r = 2$, and let L_2 be the other line that meets P_a. By Lemma 2.2.21, L_2 must meet all other lines in points of multiplicity 2. Then C is isomorphic to the near-pencil combinatorics with n lines, which contradicts C having exactly one binary line. Therefore $r = m(P_a) \geq 3$ and $m(P_b) \geq 3$. By Lemma 2.2.21, all other singular points of C have multiplicity 2. The lines that meet P_a form a pencil of r lines, and the lines that meet P_b form another pencil of $n-r$ lines, with the binary line L_1 being the only line shared by both pencils. □

2.3 Boundary manifold of a line arrangement

In this section, for a given arrangement $\mathcal{A}$ in $\mathbb{CP}^2$ with $n+1$ lines, the line at infinity is part of the arrangement and is denoted by $L_0 \in \mathcal{A}$. In this case the affine part arrangement $\mathcal{A}^{\text{aff}}$ coincides with the sub-arrangement $(\mathcal{A} \setminus L_0)^{\text{aff}}$ with n lines in $\mathbb{C}^2$.

In [CS08], D. Cohen and A. Suciu give a geometrical construction of a regular neighbourhood $N_{\mathcal{A}}$ of a line arrangement $\mathcal{A}$ as follows. Choose homogenous coordinates $\mathbf{x} = [x : y : z]$ on $\mathbb{CP}^2$. A closed, regular neighbourhood of $\mathcal{A}$ may be constructed as follows. Define $\phi : \mathbb{CP}^2 \to \mathbb{R}$ by

$$\phi(x) = |P_{\mathcal{A}}(\mathbf{x})|^2 / \|\mathbf{x}\|^{2(n+1)}$$

and define $N_{\mathcal{A}} := \phi^{-1}([0,\delta])$, for $\delta > 0$ sufficiently small.

Definition 2.3.1. The *boundary manifold* $B_{\mathcal{A}}$ of a line arrangement is defined as the boundary of the regular neighbourhood $N_{\mathcal{A}}$. The *exterior manifold* $E_{\mathcal{A}}$ is defined as

$$E_{\mathcal{A}} := \mathbb{CP}^2 \setminus \widehat{\mathring{N}_{\mathcal{A}}}$$ ◇

***Remark* 2.3.2.** The exterior manifold $E_{\mathcal{A}}$ is a deformation retract of the *complementary* of the arrangement $\mathbb{CP}^2 \setminus \mathcal{A}$. ⌑

The boundary manifold of $\mathcal{A}$ is a special case of a graph manifold, which means that it is constructed algorithmically from the incidence graph, by gluing together Seifert manifolds as explained in Section 1.3.2.

In this section we want to give a somewhat more detailed construction of $B_{\mathcal{A}}$ due to [Wes97] which directly shows its graph manifold structure.

To do this we make use of the *blow-up* operation that helps to resolve the singularities of the line arrangements.

2.3.1 Blow-up

Definition 2.3.3. Let (x, y) be coordinates on $\mathbb{C}^2$. Consider $\mathbb{CP}^1$ with coordinates $[x' : y']$. The *blow-up* of the complex plane $\mathbb{C}^2$ at the point $(0, 0)$ is defined as the set

$$X_{(0,0)} := \left\{((x, y), [x' : y']) \in \mathbb{C}^2 \times \mathbb{CP}^1 \mid xy' = x'y\right\} \qquad \diamond$$

There exists a natural projection σ from $X_{(0,0)}$ to $\mathbb{C}^2$ defined by the diagram

$$\begin{array}{ccc} X_{(0,0)} & \hookrightarrow & \mathbb{C}^2 \times \mathbb{CP}^1 \\ & \overset{\sigma}{\searrow} & \downarrow \\ & & \mathbb{C}^2 \end{array}$$

By extension, the *blow-up* of an open neighbourhood U of $(0, 0)$ in $\mathbb{C}^2$ is defined as $\widehat{U} := \sigma^{-1}(U)$. Note that $\sigma : \widehat{U} \to U$ is an isomorphism everywhere outside $(0, 0)$.

Using a change of coordinates, the blow-up can then be defined around any point P of $\mathbb{C}^2$.

Definition 2.3.4. Let S be a complex surface and let $P \in S$. Consider a local chart $c : V \to U \subset \mathbb{C}^2$ on a neighbourhood V of P inside S. The blow-up $\widehat{S_P}$ of S in P is defined as

$$\widehat{S_P} := \left((S \smallsetminus \{P\}) \sqcup \widehat{U}\right)\big/_f$$

where $f = \sigma^{-1} \circ c$ is the isomorphism between $V \smallsetminus \{P\}$ and $\widehat{U} \smallsetminus \{\sigma^{-1}((0, 0))\}$. $\diamond$

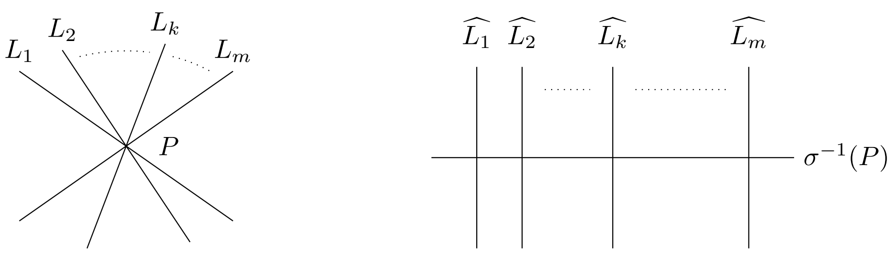
We now apply this construction to the case of line arrangements, where all non-binary singular points will be blown-up separately.

Definition 2.3.5. The *full blow-up of a line arrangement* is the data of $(X, \widehat{\mathcal{A}})$, where X is the blow-up of all *non-binary* singular points $P \in \mathcal{Q}^{>2}$ of $\mathcal{A}$ (called the *blow-up space*), and $\widehat{\mathcal{A}}$ is the pre-image of $\mathcal{A}$ inside the blow-up space.

The *total blow-up* of $\mathcal{A}$ is the data of $(X', \widehat{\mathcal{A}}^{\max})$ where X' is the blow-up of *all* singular points $P \in \mathcal{Q}$ of $\mathcal{A}$, and $\widehat{\mathcal{A}}^{\max}$ is the pre-image of $\mathcal{A}$. $\diamond$

We often called $\widehat{\mathcal{A}}$ itself the 'full blow-up of $\mathcal{A}$'.

Example 2.3.6. The blow-up pre-image of a pencil with n lines is shown on Figure 2.3.1. $\circ$



(a) Pencil of m lines

(b) Blow-up

Figure 2.3.1: Blow-up of a line arrangement

The irreducible components of $\widehat{\mathcal{A}}$ are of two types:

- The components $\widehat{L_P}$ arising from the blow-ups of the points $P \in \mathcal{Q}^{>2}$ are called *exceptional lines*.
- The components $\widehat{L_i}$ arising from the lines L_i of $\mathcal{A}$ are called *regular lines*.

We denote by $\widehat{\mathcal{L}}$ the set of irreducible components of $\widehat{\mathcal{A}}$. We have:

$$\widehat{\mathcal{L}} = \left(\widehat{L_i}\right)_{1\leq i\leq n} \cup \left(\widehat{L_P}\right)_{P\in\mathcal{Q}^{>2}}$$

By construction, all irreducible components of $\widehat{\mathcal{A}}$ intersect at binary singular points.

Proposition 2.3.7. *The pre-image $\widehat{\mathcal{A}}$ inside the blow-up is a divisor with normal crossings of the line arrangement $\mathcal{A}$.* ◁

A divisor with normal crossings is a type of algebraic variety that contains only binary singular points and smooth irreducible components. It has a combinatorial structure encoded by a graph in a similar fashion to line arrangements but with simpler rules and no need to distinguish two types of vertices.

Definition 2.3.8. Let $\widehat{\mathcal{A}}$ be the full blow-up of a line arrangement. The *dual graph* $\widehat{\Gamma}(\mathcal{A})$ of $\widehat{\mathcal{A}}$ is defined by the following description:

Vertices: v_ℓ for every irreducible component ℓ of $\widehat{\mathcal{A}}$.

Edges: v_ℓ and $v_{\ell'}$ are linked by an edge $e_{\ell,\ell'}$ if and only if $\ell\cap\ell'\neq\varnothing$ in $\widehat{\mathcal{A}}$. ◇

Proposition 2.3.9. *The dual graph $\widehat{\Gamma}(\mathcal{A})$ of the full blow-up $\widehat{\mathcal{A}}$ is identical to the reduced incidence graph $\widetilde{\Gamma}(C_\mathcal{A})$, with every point-vertex of $\widetilde{\Gamma}(C_\mathcal{A})$ replaced with the vertex of the corresponding exceptional-line component in $\widehat{\Gamma}(\mathcal{A})$.*

Similarly, the dual graph $\widehat{\Gamma}^{\max}(\mathcal{A})$ of the total blow-up $\widehat{\mathcal{A}}^{\max}$ is identical to the full incidence graph $\Gamma(C_\mathcal{A})$. ◁

2.3.2 Construction of the boundary manifold

Instead of constructing the boundary manifold on the original line arrangement $\mathcal{A}$ inside $\mathbb{CP}^2$ as was done in Definition 2.3.1, we build the boundary manifold of the full blow-up $\widehat{\mathcal{A}}$ inside the blow-up space X, using its dual graph $\widehat{\Gamma}(\mathcal{A})$. Theorem 2.3.12 justifies that the two constructions lead to homeomorphic manifolds.

Just as with the line arrangement $\mathcal{A}$ in Definition 2.3.1, the boundary manifold of $\widehat{\mathcal{A}}$ is obtained by taking the boundary of a regular neighbourhood of $\widehat{\mathcal{A}}$. This neighbourhood is obtained by gluing together local regular neighbourhoods of each irreducible component of $\widehat{\mathcal{A}}$.

For every line $\ell\in\widehat{\mathcal{L}}$ of the full blow-up (exceptional or regular), let N_ℓ be a regular neighbourhood of ℓ inside the full blow-up space X. One can always take them small enough such that $N_\ell\cap N_{\ell'}=\varnothing$ if $\ell\cap\ell'=\varnothing$.

Definition 2.3.10. Let

$$N_{\widehat{\mathcal{A}}} := \bigcup_{\ell\in\widehat{\mathcal{L}}} N_\ell$$

Then $N_{\widehat{\mathcal{A}}}$ is a *regular neighbourhood* of $\widehat{\mathcal{A}}$ inside the blow-up space X. ◇

Definition 2.3.11. The manifold $B_{\widehat{\mathcal{A}}} := \partial N_{\widehat{\mathcal{A}}}$ is called the *boundary manifold* of the full blow-up $\widehat{\mathcal{A}}$. ◇

The usefulness of the full blow-up construction is justified by the following

Theorem 2.3.12. *There is a homeomorphism between $B_{\widehat{\mathcal{A}}}$ and $B_\mathcal{A}$.* ◁

Proof. Contracting each exceptional line $\widehat{L_P}\in\widehat{\mathcal{L}}$ back into a point gives an isotopy from $B_{\widehat{\mathcal{A}}}$ to $B_\mathcal{A}$. This can be seen by thickening the lines of Figure 2.3.1: the 'thick star' around the pencil is homeomorphic to the 'thick comb' around its blow-up (recall that $\sigma^{-1}(P)$ is compact). □

2.3.3 Graph structure

To make the graph manifold structure of $B_{\widehat{\mathcal{A}}}$ (and $B_\mathcal{A}$) explicit we re-decompose it as a union of local boundary manifolds around each line of the blow-up.

Definition 2.3.13. For every line $\ell \in \widehat{\mathcal{L}}$, define

$$\Sigma_\ell := \ell \smallsetminus \bigcup_{\ell \cap \ell' \neq \varnothing} \ell \cap N_{\ell'}$$
$$S_\ell := \partial N_\ell \smallsetminus \bigcup_{\ell \cap \ell' \neq \varnothing} (\partial N_\ell \cap N_{\ell'})$$ ◇

For every line $\ell \in \widehat{\mathcal{L}}$, the surface Σ_ℓ is homeomorphic to $\Sigma^{m(\ell)}$.

Definition 2.3.14. For a regular line $L_i \in \mathcal{L}$, define $V_i^{>2} := \{P \in \mathcal{Q}^{>2} \mid P \in L_i\}$ and let $b(L_i) := \#V_i^{>2}$ be the *blow-up number* of L_i. ◇

The Euler number of a circle bundle coincide with its self-intersection number. The regular neighbourhood of a regular complex line in $\mathbb{CP}^2$ has self-intersection 1. An exceptional line looks like a regular line in the neighbourhood $\widehat{U}$ of the blow-up, but the change of charts f gives it a self-intersection -1. Since this also affects the line components that meet the exceptional line, their own self-intersection is reduced by 1 for each blow-up. One then obtains the following description of the structure of the local boundary manifolds.

Theorem 2.3.15. *For every regular line $L_i \in \mathcal{L}$, the manifold $S_{L_i} := \partial N_{\widehat{L_i}}$ is a circle bundle over $\Sigma^{L_i} \simeq \Sigma^{m(L_i)}$ with Euler number $\varepsilon_i = 1 - b(L_i)$.*

For every exceptional line $\widehat{L_P}$ with $P \in \mathcal{Q}^{>2}$, the manifold $S_P := \partial N_{\widehat{L_P}}$ is a circle bundle over $\Sigma^{P_l} \simeq \Sigma^{m(P)}$ with Euler number $\varepsilon_P = -1$.

In particular, S_{L_i} and S_{P_l} verify Conditions 1.3.7 on page 14. ◁

The local boundary neighbourhoods assemble along their boundary components to form a graph manifold which is exactly $B_{\mathcal{A}}$. It is however necessary to identify a collection of sections on the boundary (see Definition 1.3.6 on page 14) on each of the circle bundles so that the graph manifold respects Conditions 1.3.16 on page 16.

Proposition 2.3.16. *For every line $\ell \in \widehat{\mathcal{L}}$, $(\ell \cap \partial N_{\ell'})_{\ell\cap\ell' \neq \varnothing}$ is a collection of section on the boundary of S_ℓ.* ◁

Theorem 2.3.17 ([JY93])**.** *Let $\mathcal{A}$ be a non-exceptional line arrangement in the sense of Definition 2.2.18. The boundary manifold $B_{\mathcal{A}}$ is a graph manifold whose minimal graph structure is given by the reduced incidence graph $\widetilde{\Gamma}(C_{\mathcal{A}})$ decorated with the Euler numbers given in Theorem 2.3.15, and it verifies Conditions 1.3.16.* ◁

Corollary 2.3.18. *Let $\mathcal{A}$ and $\mathcal{A}'$ be two non-exceptional line arrangements with the same combinatorics C. Then there exists a homeomorphism $\Phi : B_{\mathcal{A}} \to B_{\mathcal{A}'}$.* ◁

We can state a similar theorem when both arrangements are endowed with the same ordering θ.

Theorem 2.3.19. *Let $(\mathcal{A}, \theta)$ and $(\mathcal{A}', \theta)$ be two non-exceptional ordered line arrangements with the same combinatorics C. Then there exists a strongly positive graphed homeomorphism $\Phi : B_{\mathcal{A}} \to B_{\mathcal{A}'}$.* ◁

Proof. By Corollary 2.3.18, there exists a homeomorphism $\Phi : B_{\mathcal{A}} \to B_{\mathcal{A}'}$. We thus write $B = B_{\mathcal{A}} \simeq B_{\mathcal{A}'}$. The graph structure of B is given by $\Gamma = \widetilde{\Gamma}(C)$. By Theorem 1.3.21, Φ induces a permutation $G(\Phi)$ on the graph Γ. However, since the ordered arrangements are equivalent, by Definition 2.1.6 the homeomorphism Φ induces the identity permutation on $\mathcal{L}$, and by extension on the whole combinatorics C. This means that $G(\Phi) = \mathrm{Id}_\Gamma$, and thus Φ is a graphed homeomorphism on B. Now take the image of Φ in $\mathrm{Homeo}_\Gamma^{++}(B)$ by the quotient map of Proposition 1.3.24 to obtain a strongly positive graphed homeomorphism from $B_{\mathcal{A}}$ to $B_{\mathcal{A}'}$. □

One can make a similar construction of the boundary manifold of the total blow-up $\widehat{\mathcal{A}}^{\max}$ of $\mathcal{A}$. Theorem 2.3.12 extends to show that $B_{\widehat{\mathcal{A}}^{\max}}$ is also homeomorphic to $B_{\mathcal{A}}$. However, the graph structure of $B_{\widehat{\mathcal{A}}^{\max}}$, which by Proposition 2.3.9 is given by the full incidence graph $\Gamma(C_{\mathcal{A}})$, is not minimal.

As described in Section 1.6.4, graphs for non-minimal structures can be obtained combinatorially from the minimal graph by performing *blowing-down* moves represented on Figure 1.6.2 on page 30. It occurs that the binary blowing-down move corresponds exactly to the inverse of the action of the blow-up operation on the graph.

Proposition 2.3.20. *Let $\mathcal{A}$ be a non-exceptional line arrangement. Then the reduced incidence graph $\widetilde{\Gamma}(C_{\mathcal{A}})$ is obtained from the full incidence graph $\Gamma(C_{\mathcal{A}})$ by blowing down all vertices corresponding to binary singular points.* ◁

Proof. By Theorem 2.3.15, the Euler numbers of the vertices of $\Gamma(C_{\mathcal{A}})$ are: -1 for every vertex v_P with $P \in \mathcal{Q}$ and $1 - \#N_i$ for every vertex v_i with $L_i \in \mathcal{L}$. Let $P \in \mathcal{Q}$ be a binary point and write $P = L_i \cap L_j$. Blowing-down the vertex v_P as described on Figure 1.6.2 will replace v_P by an edge exactly as in $\widetilde{\Gamma}(C_{\mathcal{A}})$, and add 1 to the Euler numbers of v_i and v_j. Now consider a line component L_i and the associated vertex v_i. We have $N_i = N_i^* \sqcup N_i'$ where N_i' is the set of all binary singular points $P \in L_i$. Blowing down all vertices v_P for $P \in N_i'$ leaves $1 - \#N_i^*$ as the new Euler number of v_i. This is exactly the corresponding Euler number of v_i in $\widetilde{\Gamma}(C_{\mathcal{A}})$. □

2.4 Exterior of a line arrangement

Let $\mathcal{A}$ be a line arrangement with n lines and let $E_{\mathcal{A}}$ be the complement in $\mathbb{CP}^2$ of an open regular neighbourhood of $\mathcal{A}$ as explained in Definition 2.3.1. Suppose that L_0 is the line at infinity in the standard affine chart. Throughout this section, $\mathbb{C}^2$ is thus assumed to be $\mathbb{CP}^2 \setminus L_0$. The complementary $\mathbb{C}^2 \setminus \mathcal{A}^{\text{aff}}$ is naturally homeomorphic to $\mathbb{CP}^2 \setminus \mathcal{A}$. We define

$$\mathcal{L}^* := \mathcal{L} \setminus \{L_0\} \qquad\qquad \mathcal{Q}^* := \mathcal{Q} \setminus V_{L_0}$$

2.4.1 Wiring diagram

The wiring diagram is a construction due to W. Arvola [Arv92] which allows to fully encode the topology of an ordered line arrangement on a planar diagram.

We consider linear projections $\pi : \mathbb{C}^2 \to \mathbb{C}$ such that for every $L \in \mathcal{L}^*$, the restriction $\pi_{|L^{\text{aff}}}$ is a homeomorphism of the complex plane.

Definition 2.4.1. A linear projection $\pi : \mathbb{C}^2 \to \mathbb{C}$ is said to be *generic* if each multiple point $P \in \mathcal{Q}^*$ lie in a different fibre of π. ◇

Let $\nu : \mathcal{Q}^* \to \{1, \dots, r\}$ be an ordering on $\mathcal{Q}^*$. We denote by $x_l := \pi(P_l)$ the ordered images of the singular points. Let R^0 be an open rectangle in $\mathbb{C}$ containing all the points $x_1, \dots, x_r$ and let x_0 and $x_{r+1} \in \mathbb{C}$ be points such that

$$\operatorname{Re}(x_0) \leq \inf \operatorname{Re}(R_0) \qquad\qquad \operatorname{Re}(x_{r+1}) \geq \sup \operatorname{Re}(R_0)$$

Definition 2.4.2. A smooth path $\gamma : [0,1] \to \mathbb{C}$ is said to be *ν-admissible* if $\gamma(0) = x_0$, $\gamma(1) = x_{r+1}$, and γ goes through $x_1 \dots x_r$ in order. ◇

Definition 2.4.3. Let π be a generic linear projection and let γ be a ν-admissible path. The *wiring diagram* $W_{\mathcal{A}}(\pi, \gamma)$ associated to π and γ is defined as the graph inside $[0,1] \times \mathbb{C}$ of the multivalued function

$$\pi_{\mathcal{A}} : t \in [0,1] \longmapsto \pi^{-1}(\gamma(t)) \cap \mathcal{A}$$

Each continuous value of $\pi_{\mathcal{A}}$ which corresponds to the trace of

$$w_L : t \in [0,1] \longmapsto \pi^{-1}(\gamma(t)) \cap L$$

is called the *wire* associated with the line L. ◇

Since $W_{\mathcal{A}}$ is a real one-dimensional object inside a real three-dimensional space, it can then be projected back onto a real plane. By convention, we take the projection $\pi_0 : [0,1] \times \mathbb{C} \to \mathbb{R}^2$ given by

$$\pi_0(t, x + iy) := (t, y)$$

There are values of $t \in [0,1]$ for which several projected wires $\pi_0(w_L(t))$ might cross. If $\pi_{\mathcal{A}}(t) = P$ for some $P \in \mathcal{Q}^*$, then all the wires $w_L(t)$ corresponding to the lines that contain P already merged into a single point on $W_{\mathcal{A}}$ *before* the projection π_0. The image $\pi_0(P)$ on the real plane is thus called an *actual crossing*. However, if $\pi_{\mathcal{A}}(t)$ do *not* correspond to a singular point of $\mathcal{A}$, then the crossing on the real plane was caused by π_0 only. In this case, we replace it with a *virtual crossing* in a similar fashion as on a knot diagram.

The genericity of the projection π ensures that all crossings on the real plane (actual and virtual) correspond to different values of t. In practice a non-generic wiring diagram can always be slightly deform to produce a generic one, with a new ordering on $\mathcal{Q}^*$. The admissible path γ can also always be adjusted so that only two projected wires meet on any virtual crossing.

The projection $\pi_0(W_{\mathcal{A}})$ with the addition of virtual crossings as described above is used as the common representation of the wiring diagram $W_{\mathcal{A}}$.

2.4.2 Braid monodromy

The braid monodromy was first introduced for complex algebraic curves by O. Chisini [Chi33] and O. Zariski [Zar29] and was later redefined by B. Moishezon [Moi81]. For the case of line arrangements, it is a very similar construction to the wiring digram and is in fact an equivalent way of presenting the same topological information. The construction of the braid monodromy we present was developed by E. Artal, J. Carmona and J.I. Cogolludo in [ACC03].

First we introduce another geometrical interpretation of the braid group. Let $\mathbf{y} \subset \mathbb{C}$ be a subset of n points in the complex plane. The points of $\mathbf{y}$ are naturally ordered by their ascending real parts. Fix a polygonal path $p_{\mathbf{y}}$ in $\mathbb{C}$ that joins the points of $\mathbf{y}$ in order. Let $\{\gamma_i : [0,1] \to \mathbb{C}\}_{1\leq i\leq n}$ be a set of n paths that start and end at $\mathbf{y}$ such that for every $t \in [0,1]$, the points $\{\gamma_1(t), \dots, \gamma_n(t)\}$ are all distinct. Define the group $\mathbb{B}_{\mathbf{y}}$ to be the set of homotopy classes of all such path sets $\{\gamma_1, \dots, \gamma_n\}$ in $\mathbb{C} \times [0,1]$ relatively to $p^0_{\mathbf{y}}$ and $p^1_{\mathbf{y}}$. Similarly, given two sets $\mathbf{y}_1, \mathbf{y}_2$ one can define the groupoid $\mathbb{B}_{\mathbf{y}_1,\mathbf{y}_2}$ as the set of homotopy classes of paths joining $\mathbf{y}_1$ to $\mathbf{y}_2$ relatively to the paths $p^0_{\mathbf{y}_1}$ and $p^1_{\mathbf{y}_2}$.

Proposition 2.4.4. *There are natural isomorphisms*

$$\begin{aligned} I_{\mathbf{y}} :& \quad \mathbb{B}_{\mathbf{y}} \longrightarrow \mathbb{B}_n \\ I_{\mathbf{y}_1,\mathbf{y}_2} :& \quad \mathbb{B}_{\mathbf{y}_1,\mathbf{y}_2} \longrightarrow \mathbb{B}_n \end{aligned} \qquad \triangleleft$$

The isomorphism $I_{\mathbf{y}}$ restricts to the pure braid group $\mathbb{P}_n$ for the subset of paths $\{\gamma_1, \dots, \gamma_n\}$ that all start and end at the same respective point of $\mathbf{y}$. Consider a projection $\pi : \mathbb{C}^2 \to \mathbb{C}$, which is not necessarily generic in the sense of Definition 2.4.1. Several singular points can thus be sent to a same point in $\mathbb{C}$.

Definition 2.4.5. The projection $\pi : \mathbb{C}^2 \to \mathbb{C}$ defines an *assignation function*

$$X_\pi : \pi(\mathcal{Q}^*) \longrightarrow \mathcal{P}(\mathcal{Q}^*)$$

that sends each point $x \in \pi(\mathcal{Q}^*)$ to the subset $\pi^{-1}(x) \cap \mathcal{Q}^*$. $\diamond$

The projection points $x \in \pi(\mathcal{Q}^*)$ such that $X_\pi(x)$ is a singleton are called *generic*. The others are called *non-generic*. By assumption the projection never sends a line component $L \in \mathcal{L}^*$ to a single point. Therefore, two singular points sent to a common projection point $x \in \pi(\mathcal{Q}^*)$ cannot be on the same component line of $\mathcal{A}$. In other words, if $P, P' \in X_\pi(x)$ then $V_P \cap V_{P'} = \varnothing$.

We write $r_\pi := \#\pi(\mathcal{Q}^*)$ and define $\mathbb{C}_{\mathcal{A}} := \mathbb{C} \setminus \pi(\mathcal{Q}^*)$. The set $\mathbb{C}_{\mathcal{A}}$ is a complex plane with r_π punctures.

The restriction $\pi : \mathbb{C}^2 \setminus \mathcal{A}^{\text{aff}} \to \mathbb{C}_{\mathcal{A}}$ is a locally trivial bundle. For any point $b \in \mathbb{C}_{\mathcal{A}}$, the fibre $\pi^{-1}(b)$ is isomorphic to $\mathbb{C}$ with a set of n punctures $\mathbf{y}(b)$ corresponding to the points $w_L(b) := \pi^{-1}(b) \cap L$ for every $L \in \mathcal{L}^*$.

Definition 2.4.6. For every point $b \in \mathbb{C}_{\mathcal{A}}$ there is a corresponding ordering

$$\theta^b : \mathcal{L}^* \longrightarrow \{1, \dots, n\}$$

which orders the set of points $\{w_L(b) \mid L \in \mathcal{L}^*\}$ by their ascending horizontal coordinate in the complex plane $\pi^{-1}(b)$. $\diamond$

Definition 2.4.7. Let b^- and b^+ be two points in $\mathbb{C}_{\mathcal{A}}$ and let $\gamma : [0,1] \to \mathbb{C}_{\mathcal{A}}$ be a path joining them. Consider the set of paths $\{\gamma_L \mid L \in \mathcal{L}^*\}$ joining $\mathbf{y}(b^-)$ to $\mathbf{y}(b^+)$ defined by taking the wires

$$\gamma_L : t \in [0,1] \longmapsto w_L(\gamma(t))$$

as shown on Figure 2.4.1. Then the *lifted braid* of γ is defined by

$$\rho_{\mathcal{A},\pi}(\gamma) := I_{\mathbf{y}(b^-),\mathbf{y}(b^+)}\left(\{\gamma_L \mid L \in \mathcal{L}^*\}\right) \in \mathbb{B}_n \qquad \diamond$$

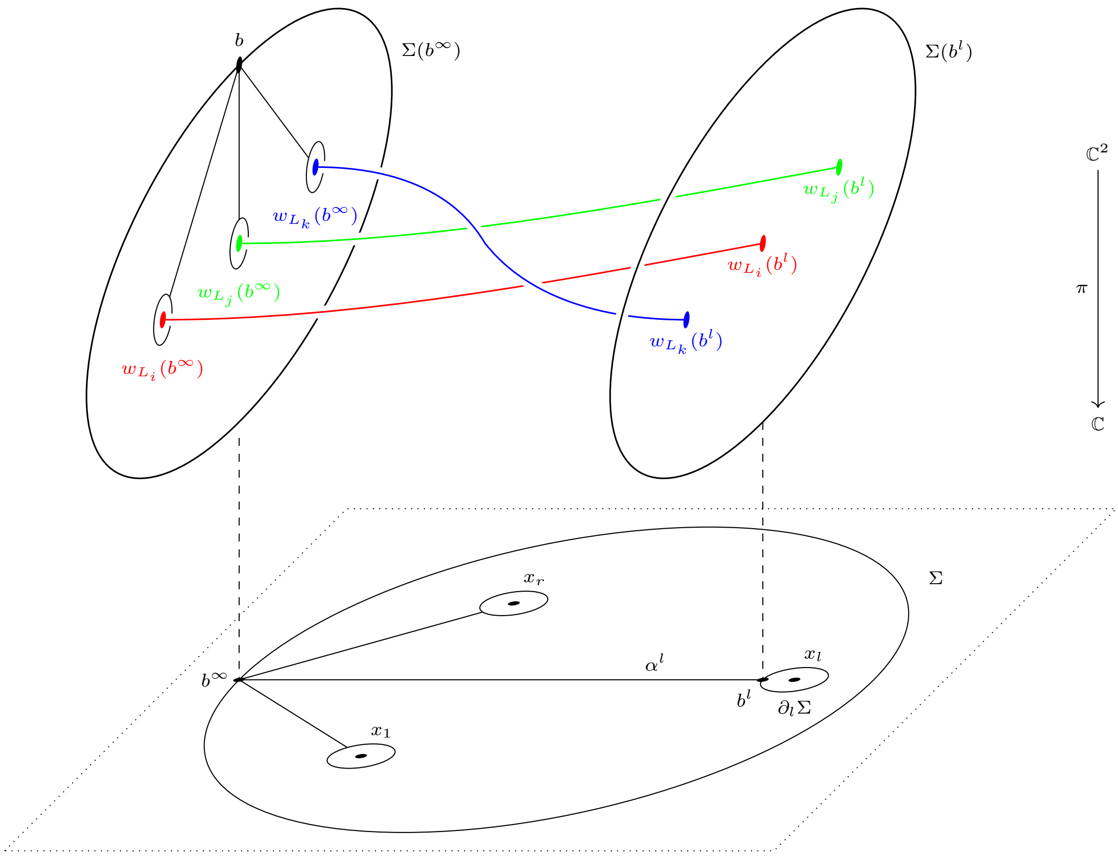


Figure 2.4.1: Braid over the path $\alpha^l \subset \Sigma$

By construction, $\rho_{\mathcal{A},\pi}(\gamma)$ only depends on the homotopic class of γ relatively to b^- and b^+. Moreover, $\rho_{\mathcal{A},\pi}$ restricts to $\mathbb{P}_n$ when $b^- = b^+$.

Now recall that N_L (resp. N_P) is a regular neighbourhood of $L \in \mathcal{L}^*$ (resp. $P \in \mathcal{Q}^*$) inside $\mathbb{CP}^2$. We have $N_P \simeq N_P^{\text{aff}}$ and the image $\pi(N_P)$ is homeomorphic to a 2-disc. Let D be a closed disc in $\mathbb{C}$ containing all the images $\pi(N_P)$ for every $P \in \mathcal{Q}^*$. The basis

$$\Sigma := D \smallsetminus \bigcup_{P \in \mathcal{Q}^*} \overset{\circ}{\widehat{\pi(N_P)}} \subset \mathbb{C}_{\mathcal{A}}$$

is homeomorphic to a 2-disc with r_π holes.

Let $b^\infty \in \partial D$ be a base point. For any closed curve γ based in b^∞, the lifted braid $\rho_{\mathcal{A},\pi}$ is pure and only depends on the homotopic class of γ in $\pi_1(\Sigma, b^\infty)$. We therefore have a map

$$\rho_{\mathcal{A},\pi} : \pi_1(\Sigma, b^\infty) \longrightarrow \mathbb{P}_n$$

Changing the base point from b^∞ to $b^{\infty\prime}$ is done by an isomorphism of $\pi_1(\Sigma)$. This corresponds to conjugating the map $\rho_{\mathcal{A},\pi}$ with a braid $\beta \in \mathbb{B}_n$ such that $\theta^{\infty\prime} = \sigma(\beta) \circ \theta^\infty$, where $\sigma(\beta) \in \mathfrak{S}_n$ is the permutation associated with β.

Definition 2.4.8. The map

$$\rho_{\mathcal{A},\pi} : \pi_1(\Sigma) \longrightarrow \mathbb{P}_n$$

defined up to conjugation in $\mathbb{B}_n$ is called the *braid monodromy* of the line arrangement $\mathcal{A}$. ◇

In general we do not use the morphism $\rho_{\mathcal{A},\pi}$ itself, but rather its image on a set of generators of $\pi_1(\Sigma, b^\infty)$. This set is called a *representative* and depends on additional parameters which can be linked to the choice of a star on the basis Σ.

Definition 2.4.9. Let $\pi : \mathbb{C}^2 \to \mathbb{C}$ be a linear projection. Up to a change of coordinates in $\mathbb{C}^2$, we suppose that all the points of $\pi(\mathcal{Q}^*)$ have distinct real parts. The projection ordering $\nu_\pi : \pi(\mathcal{Q}^*) \to \{1, \dots, r_\pi\}$ is defined such that the points $x_1, \dots, x_{r_\pi}$ of $\pi(\mathcal{Q}^*) \subset \mathbb{C}$ verify

$$\operatorname{Re}(x_1) < \cdots < \operatorname{Re}(x_{r_\pi}) \qquad \diamond$$

Let θ be an ordering on $\mathcal{L}^*$ and let $b^\infty \in \partial^\infty \Sigma$ such that $\theta^\infty = \theta$. Let ν be an ordering on $\pi(\mathcal{Q}^*)$ and let $\alpha \in \mathcal{S}_{r_\pi}$ be a star on Σ associated with the ordering $\nu \circ {\nu_\pi}^{-1}$ (see Definition 1.2.1 on page 11). Let also $x_l \in \pi(\mathcal{Q}^*)$ be a projection point. The *local braid monodromy around* x_l is $\delta_l := \rho_{\mathcal{A},\pi}(\partial_l \Sigma)$.

Proposition 2.4.10. *For every $P_l \in \mathcal{Q}^*$, the local braid monodromy δ_l is positive and*

$$\rho_{\mathcal{A},\pi}(\partial^l \Sigma) = \prod_{P \in X_\pi(x_l)} \Delta^2_{\theta^l(V_P)}$$

where θ^l is the order above the point $b^l := \partial_+ \alpha^l$ at the extremity of the l-th branch of α (see Figure 2.4.1). $\triangleleft$

Note that since the sets V_P are disjoint for all $P \in X_\pi(x_l)$, the full twists that compose $\rho_{\mathcal{A},\pi}(\partial_l \Sigma)$ act on disjoint sets of strands and commute.

Proof. Write $X_\pi(x_l) = \{P_1, \dots, P_k\}$. Let (x, y) be a coordinate system of $\mathbb{C}^2$ centred in P_1. Up to a change of coordinates, we can always suppose that π is the projection on the coordinate x. The points $P_2, \dots, P_k$ have coordinates $(0, y_2), \dots, (0, y_k)$. Up to isotopy, the local equation of $\mathcal{A}^{\text{aff}}$ around the singular point P_i with multiplicity m_i is of the form $(y - y_i)^{m_i} - x^{m_i}$. The lifted braid $\rho_{\mathcal{A},\pi}(\partial^l_+ \Sigma)$ is therefore a product of full twists Δ^2 over the m_i strands corresponding to the lines $L \in V_{P_i}$ that contain P_i. The corresponding set of indices is $\theta^l(V_{P_i}) \subset \{1, \dots, n\}$. $\square$

Definition 2.4.11. The *shift braid to* x_l is defined by $\tau_l := \rho_{\mathcal{A},\pi}(\alpha^l) \in \mathbb{B}_n$. $\diamond$

Definition 2.4.12. The pure braid given by

$$\beta_l := \tau_l \cdot \delta_l \cdot (\tau_l)^{-1} \in \mathbb{P}_n$$

is called a *star braid monodromy around* x_l. The set of braids

$$\mathrm{B}_{\mathcal{A}}(\pi, \theta, \nu, \alpha) = (\beta_l)_{1 \leq l \leq r_\pi} \in (\mathbb{P}_n)^{r_\pi}$$

is called a *representative* of the braid monodromy of $\mathcal{A}$. $\diamond$

Proposition 2.4.13. *The* braid at infinity *of the representative* $\mathrm{B}_{\mathcal{A}}(\pi, \theta, \nu, \alpha)$ *is defined by the equivalent formulas:*

$$\beta_0 := \Delta^2_{\theta^\infty(\mathcal{L}^*)} \cdot \left(\prod_{l=1}^{r_\pi} \beta_l \right)^{-1} = \prod_{P \in V_{L_0}} \Delta^2_{\theta^\infty(V_P)} \qquad \triangleleft$$

Proof. Choose a system of coordinates (x, y) of $\mathbb{C}^2$ such that π is the projection on the horizontal plane $\{y = 0\}$ and $(0, 0)$ is the centre of D^0. Remember that the space $\mathbb{C}^2$ is seen as the *standard* affine chart of $\mathbb{CP}^2$ given by $\{[x : y : 1] \in \mathbb{CP}^2\}$. In the exterior of $D \subset \mathbb{CP}^2$ (where $x \neq 0$) apply the change of coordinates

$$[x : y : 1] \longmapsto [Z^{-1} : YZ^{-1} : 1]$$
$$[1 : y/x : 1/x] \longleftarrow [1 : Y : Z]$$

The new chart $\{[1 : Y : Z] \in \mathbb{CP}^2\}$ is called the ∞-chart. Write $D = \{|x| \leq r^0\}$. The exterior $D^\infty := \mathbb{CP}^1 \setminus \mathring{D}$ becomes the 2-disc $\{|Z| \leq 1/r^0\}$ in the ∞-chart. Similary, the line $L_0 = \{x = \infty\}$ becomes $\{Z = 0\}$ and the projection $\pi : (x, y) \to (x, 0)$ becomes $\pi : (Y, Z) \to (0, Z)$. The component L_0 is now a vertical line for π, and all singular points of V_{L_0} are all sent to a same point $x_0 = (Y = 0, Z = 0)$. Let $D^{\infty'}$ be a disc inside the interior of D^∞ that does not meet any of the $\pi(N_P)$ for every $P \in V_{L_0}$. Note $x_{\infty'}$ the centre of $D^{\infty'}$. By construction, the line $L_\infty := \pi^{-1}(x_\infty)$ is generic to the arrangement $\mathcal{A}$, in the sense that all the intersection points

$L_\infty \cap L_i$ for $L_i \in \mathcal{L}^*$ are double points. Now apply a new change of coordinates by making the translation

$$[1 : Y : Z] \longmapsto [1 : Y' : Z'] := [1 : Y : Z] - x_{\infty'}$$
$$[x : y : 1] \longmapsto [x' : y' : 1] := [x : y : 1] - x_{\infty'}$$

This defines two new charts which we call the ∞'-chart and the standard prime chart respectively. The projection π is not affected. We also restrict the ∞'-chart to the interior of $D^{\infty'}$. Inside the standard prime chart, the line L_∞ is the new line at infinity, but L_0 is still vertical. Let $D' := \mathbb{CP}^1 \setminus D^{\infty'}$. Up to homeomorphism, we can suppose that all $\pi(N_P)$ for $P \in V_{L_0}$ are equal to a same disc D^0. We therefore have $D^0 \sqcup D \subset D'$ as shown on Figure 2.4.2. Now consider the path

$$\alpha^* := \prod_{l=1}^{r_\pi} \alpha^l \cdot \partial_+^l \Sigma \cdot (\alpha^l)^{-1}$$

which has the same homotopy type as ∂D. The lifted braid $\rho_{\mathcal{A},\pi}(\partial_+ D)$ is therefore equal to the product of all β_l for $1 \leq j \leq m$. Let α^0 be an extra branch to the star α that joins b^∞ to ∂D^0. Up to homeomorphism, one can increase the diameter of D^0 so that $\rho_{\mathcal{A},\pi}(\alpha^0) = 1$. Let $\gamma_0 := \alpha^0 \cdot \partial_+ D^0 \cdot (\alpha^0)^{-1}$. By Proposition 2.4.10, we have

$$\rho_{\mathcal{A},\pi}(\gamma^0) = \rho_{\mathcal{A},\pi}(\partial_+ D^0) = \prod_{P \in V_{L_0}} \Delta^2_{\theta^\infty(V_P)}$$

Similarly, let α' be another extra branch that joins b^∞ to $\partial D'$ and let $\gamma' := \alpha' \cdot \partial_+ D' \cdot (\alpha')^{-1}$. Since $\pi^{-1}(D')$ contains all singular points of $\mathcal{Q}$, including those of V_{L_0}, then in an exterior neighbourhood of D' the local equation of the arrangement $\mathcal{A} \setminus L_0$ is of the form $(x')^n - (y')^n$. This means that

$$\rho_{\mathcal{A},\pi}(\partial_+ D') = \Delta^2_{\theta^\infty(\mathcal{L}^*)}$$

This full twist braid over all strands is central in the braid group $\mathbb{B}^n$. Therefore $\rho_{\mathcal{A},\pi}(\gamma') = \rho_{\mathcal{A},\pi}(\partial_+ D')$. To conclude the proof we only need to say that γ' has the same homotopy type as $\gamma^0 \cdot \partial_+ D$. By unicity of the lifting braid and using the previous results, we get

$$\rho_{\mathcal{A},\pi}(\partial_+ D') \cdot \rho_{\mathcal{A},\pi}(\gamma^*)^{-1} = \rho_{\mathcal{A},\pi}(D^0) \qquad \square$$

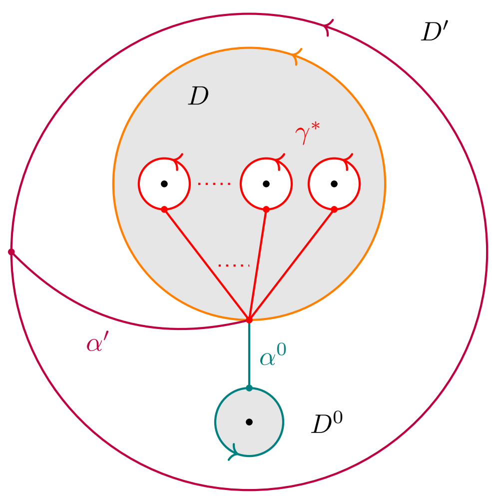


Figure 2.4.2: Construction of the braid at infinity in the standard prime chart

***Remark* 2.4.14.** The path γ^* is also almost an admissible path in the sense of Definition 2.4.2. The lifted braid over the branch parts of γ^*, seen as a geometrical object, corresponds exactly to the wiring diagram $W_\mathcal{A}(\pi, \gamma)$ in $\mathbb{C} \times [0, 1]$ with all crossings removed. This is the core argument to prove that the wiring diagram and the braid monodromy are topologically equivalent. ⌑

It is possible to change the parameters θ, ν and α of a braid monodromy representative using two separate actions on the group $(\mathbb{P}_n)^{r_\pi}$. First consider the Hurwitz action of $\mathbb{B}_{r_\pi}$ on the Cartesian product $(\mathbb{B}_n)^{r_\pi}$.

Definition 2.4.15. Let G be a group. The *right Hurwitz action* of $\mathbb{B}_{r_\pi}$ on the Cartesian product G^{r_π} is defined on any element $(a_1, \dots, a_{r_\pi}) \in G^{r_\pi}$ by

$$a_j \cdot \sigma_i^* := \begin{cases} a_j a_{j+1} {a_j}^{-1} & \text{if } j = i \\ a_{j-1} & \text{if } j = i+1 \\ a_j & \text{otherwise} \end{cases}$$

The action of each elementary braid σ_i is called a *Hurwitz move.* ◇

Next consider the action of $\mathbb{B}_n$ on $(\mathbb{B}_n)^{r_\pi}$ by conjugation:

$$\forall\, \tau \in \mathbb{B}_n : \quad (\beta_1, \dots, \beta_{r_\pi})^\tau := ({\beta_1}^\tau, \dots, {\beta_{r_\pi}}^\tau)$$

This action corresponds to the change of base point in $\pi_1(\Sigma)$.

Proposition 2.4.16 ([Moi81; ACC03])**.** *Let $\mathrm{B}_\mathcal{A}(\pi, \theta, \nu, \alpha) = (\beta_l)_{1\leq l\leq r_\pi}$ be a representative of the braid monodromy of $\mathcal{A}$. For every couple $(\kappa, \tau) \in \mathbb{B}_{r_\pi} \times \mathbb{B}_n$, define*

- $\theta' := \sigma(\tau) \circ \theta$.
- $\nu' := \sigma(\kappa) \circ \nu$.
- $\alpha' := \kappa \cdot \alpha$ *for the action of Proposition 1.2.4 on page 12.*
- $(\beta'_l)_{1\leq l\leq r_\pi} = (\kappa, \tau) \cdot (\beta_l)_{1\leq l\leq r_\pi}$ *for the Hurwitz and conjugation actions.*

Then $\mathrm{B}_\mathcal{A}(\pi, \theta', \nu', \alpha')$ coincides with $(\beta'_l)_{1\leq l\leq r_\pi}$ and is another representative of the braid monodromy of $\mathcal{A}$. ◁

2.4.3 Fundamental group of the exterior

Either the wiring diagram or the braid monodromy can be used to obtain a presentation of the fundamental group of the exterior of a line arrangement $\mathcal{A}$, and both presentations are equivalent [CS97]. The method using the wiring diagram was introduced by W. Arvola [Arv92]. The method using the braid monodromy was first introduced by O. Zariski, E.R. van Kampen and B. Moishezon, and was later readapted by A. Libgober [Lib86]. It is this method that we recall here. For the similar wiring diagram method, see for example [FGM15].

Definition 2.4.17. Let $\mathbb{F}_r$ be the free group generated by r elements $(f_1, \dots, f_r)$. The *right Hurwitz action* of $\mathbb{B}_r$ on $\mathbb{F}_r$ is defined by

$${f_j}^{\sigma_i} := \begin{cases} f_j f_{j+1} {f_j}^{-1} & \text{if } j = i \\ f_{j-1} & \text{if } j = i+1 \\ f_j & \text{otherwise} \end{cases}$$ ◇

We reuse notations from Section 2.4.2. The arrangement $\mathcal{A}$ is projected in $\mathbb{C}^2$ with the line L_0 at infinity. We fix $\pi : \mathbb{C}^2 \to \mathbb{C}$ a projection. Let ω be an ordering on $\mathcal{L}^*$ and ν be an ordering on $\pi(\mathcal{Q}^*)$.

For every $x \in \mathbb{C}_\mathcal{A}$, let $D(x)$ be a closed disc in $\pi^{-1}(x)$ containing all the intersections $D_L(x) := \pi^{-1}(x) \cap N_L^{\mathrm{aff}}$ for $L \in \mathcal{L}^*$ and define

$$\Delta(x) := D(x) \smallsetminus \{w_L(x) \mid L \in \mathcal{L}^*\}$$
$$\Sigma(x) := D(x) \smallsetminus \bigcup_{L\in\mathcal{L}^*} \pi^{-1}(x) \cap \widehat{\mathring{N}_L^{\mathrm{aff}}}$$

as shown on Figure 2.4.1.

Definition 2.4.18. Let $\alpha_x \in \mathcal{O}_n$ be an ordered star drawn on $\Sigma(x)$. Then the curves

$$\mu_i := \alpha_x^i \cdot \partial^i\Sigma(x) \cdot (\alpha_x^i)^{-1} \qquad \text{for } 1 \leq i \leq n$$

are called *exterior meridians* of the line arrangement $\mathcal{A}$. Each meridian μ_i is associated with the line $(\theta^x)^{-1}(i) \in \mathcal{L}^*$ for the ordering above x. ◇

Theorem 2.4.19 ([Zar29; Moi81]). *Let* $\mathrm{B}_{\mathcal{A}}(\pi,\theta,\nu,\alpha) = (\beta_l)_{1\leq l\leq r_\pi}$ *be a representative of the braid monodromy of* $\mathcal{A}^{\mathrm{aff}} = \mathcal{A} \setminus L_0$. *Let* b^∞ *be the base point of* α *with* $\theta^\infty = \theta$. *Let* $\mu_1,\dots,\mu_n$ *be exterior meridian curves of* $\mathcal{A}$ *drawn on* $\Sigma(b^\infty)$. *For every* $x_l \in \pi(\mathcal{Q}^*)$, *let* $I_k \subset \{1,\dots,n\}$ *be the reunion of the sets* $\theta(V_P)$ *for every* $P \in X_\pi(x_l)$. *The fundamental group of* $\mathbb{C}^2 \setminus \mathcal{A}^{\mathrm{aff}}$ *admits the following presentation:*

Generators: $\mu_1,\cdots,\mu_n$.

Relations: $\left({\mu_i}^{\delta_l}\cdot{\mu_i}^{-1}\right)^{{\tau_l}^{-1}} = \mu_i$ *for every* $1\leq l\leq r_\pi$ *and every* $i \in I_l$. ◁

***Remark* 2.4.20.** The product $\mu_1\cdots\mu_n$ has the same homotopy type as $\partial_+ D(b^\infty)$ and corresponds to the inverse of a meridian μ_0 of the line at infinity. ⌑

Corollary 2.4.21. *The first homology group of the exterior* $H_1(E_{\mathcal{A}},\mathbb{Z})$ *is a* $\mathbb{Z}$*-module of finite type given by the presentation*

$$H_1(E_{\mathcal{A}},\mathbb{Z}) = \left\langle \mu_0,\mu_1,\dots,\mu_n \;\middle|\; \sum_{i=0}^{n}\mu_i = 0 \right\rangle \qquad ◁$$

Recall the *meridian homology* $V(\Gamma)$ of a graph Γ from Definition 1.5.14 on page 24. Theorem 1.5.16 on page 24 states that $V(\Gamma)$ represents the contribution of the meridians to the first homology group of the graph manifold associated with Γ. It turns out that in the case of line arrangements that exact same construction also corresponds to the first homology group of the exterior in $\mathbb{CP}^2$. This result is crucial for the definition of the *homology inclusion map* in Section 2.5.

Proposition 2.4.22. $H_1(E_{\mathcal{A}},\mathbb{Z}) = V(\widetilde{\Gamma}(C_{\mathcal{A}}))$. ◁

Proof. By Lemma 1.6.17 on page 31, one can compute $V(\widetilde{\Gamma}(C_{\mathcal{A}}))$ on a bigger graph which reduces to $\widetilde{\Gamma}(C_{\mathcal{A}})$ by blowing-down moves. By Proposition 2.3.20 on page 45, the full incidence graph $\Gamma(C_{\mathcal{A}})$ is such a graph. Therefore it is enough to prove that $H_1(E_{\mathcal{A}},\mathbb{Z}) = V(\Gamma(C_{\mathcal{A}}))$. Recall that $\Gamma(C_{\mathcal{A}})$ is the graph structure of the total blow-up boundary manifold $B_{\widehat{\mathcal{A}}^{\max}}$, which by Theorem 2.3.12 on page 43 is homeomorphic to $B_{\mathcal{A}}$. By Definition 1.5.14, $V(\Gamma(C_{\mathcal{A}}))$ is a free abelian module generated by one meridian curve for each vertex of the graph. In the case of $B_{\mathcal{A}}$ these are the meridians $\mu_0,\mu_1\dots,\mu_n$ of Definition 1.5.6 on page 22 corresponding to the line-vertices $L_i \in \mathcal{L}$, but also the meridians μ_P for every $P \in \mathcal{Q}$, where μ_P is a fibre curve inside the local boundary manifold S_P around P. The Euler number values of each vertex were given by Theorem 2.3.15 on page 44. In the total blow-up $\widehat{\mathcal{A}}^{\max}$, the blow-up number $b(L_i)$ of a line line $L_i \in \mathcal{L}$ is equal to its multiplicity $m(L_i)$. Using the neighbour sets induced by the Definition 2.2.11 on page 38 of $\Gamma(C_{\mathcal{A}})$, the relations of $V(\Gamma(C_{\mathcal{A}}))$ become

$$\begin{aligned} \forall\ L_i \in \mathcal{L}: &\quad \sum_{P\in V_i}\mu_P = (m(L_i)-1)\cdot\mu_i \\ \forall\ P \in \mathcal{Q}: &\quad \sum_{L_i\in V_P}\mu_i = \mu_P \end{aligned}$$

Fix $L_i \in \mathcal{L}$. Replacing μ_P in the first relation for every $P \in V_i$ yields:

$$(m(L_i)-1)\cdot\mu_i = \sum_{P\in V_i}\sum_{L_k\in V_P}\mu_k = m(L_i)\cdot\mu_i + \sum_{P\in V_i}\sum_{\substack{L_k\in V_P\\ k\neq i}}\mu_k$$

since $L_i \in V_P$ for every $P \in V_i$ and $m(L_i) = \#V_i$. Thus we get:

$$0 = \mu_i + \sum_{P\in V_i}\sum_{\substack{L_k\in V_P\\ k\neq i}}\mu_k$$

But L_i meets every other line $L_j \in \mathcal{L}$ exactly once. In other words:

$$\mathcal{L} = \{L_i\} \sqcup \bigsqcup_{P\in V_i}\{L_k \in \mathcal{L} \mid L_k \in V_P, k\neq i\}$$

For every $L_i \in \mathcal{L}$, the previous relation thus becomes

$$\sum_{L_j \in \mathcal{L}} \mu_j = 0$$

Since $\mathcal{Q} = \bigcup_{L_i \in \mathcal{L}} V_i$, we have replaced all the meridians μ_P for $P \in \mathcal{Q}$, leaving only the μ_i for $L_i \in \mathcal{L}$ as generators. This simplification of the presentation of $V(\Gamma(C_{\mathcal{A}}))$ thus gives exactly the presentation of $H_1(E_{\mathcal{A}}, \mathbb{Z})$ given by Corollary 2.4.21. □

2.5 Homology inclusion map

Let $\mathcal{A}$ a non-exceptional line arrangement in $\mathbb{CP}^2$ and $\Gamma := \widetilde{\Gamma}(C_{\mathcal{A}})$ its reduced incidence graph. Let $B_{\mathcal{A}}$ be the boundary manifold, as given in Definition 2.3.1 on page 41. As per Theorem 2.3.17 on page 44, $B_{\mathcal{A}}$ is a graph manifold whose unique minimal graph structure is given by Γ. Order the arrangement $\mathcal{A}$ with ω and let Ω^θ be the reduced ordering on Γ induced by θ as per Definition 2.2.15 and Proposition 2.2.16 on page 38.

Consider the inclusion map

$$i_{\mathcal{A}} : B_{\mathcal{A}} \hookrightarrow E_{\mathcal{A}}$$

and the induced map on the first homology groups

$$i^*_{\mathcal{A}} : H_1(B_{\mathcal{A}}, \mathbb{Z}) \longrightarrow H_1(E_{\mathcal{A}}, \mathbb{Z})$$

By Theorem 1.5.16 on page 24, every graphed embedding $\gamma \in \mathrm{E}_\Gamma\left(\Omega^\theta\right)$ induces an isomorphism

$$\gamma_* : V(\Gamma) \oplus H_1(\Gamma) \xrightarrow{\sim} H_1(B_{\mathcal{A}}, \mathbb{Z})$$

where the image of the subgroup $V(\Gamma)$ is generated by the boundary meridian curves from Definition 1.5.6 on page 22. Separately, by Corollary 2.4.21 the homology of the exterior manifold $E_{\mathcal{A}}$ is generated by the exterior meridian curves defined in Section 2.4.3 and there is a natural group isomorphism

$$H_1(E_{\mathcal{A}}, \mathbb{Z}) \simeq V(\Gamma)$$

Therefore, we have a map

$$i^*_{\mathcal{A}} \circ \gamma_* : V(\Gamma) \oplus H_1(\Gamma) \longrightarrow V(\Gamma)$$

Lemma 2.5.1. *For every graphed embedding $\gamma \in \mathrm{E}_\Gamma\left(\Omega^\theta\right)$, we have*

$$(i^*_{\mathcal{A}} \circ \gamma_*)_{|V(\Gamma)} = \mathrm{Id}_{V(\Gamma)} \qquad \triangleleft$$

Proof. By construction, for every line L_i of $\mathcal{A}$ both the boundary meridian curve μ_i and the exterior meridian curve μ'_i have the same homology class as a the boundary $\partial_i D$ of a disc transverse to L_i. □

Now consider the restriction

$$(i^*_{\mathcal{A}} \circ \gamma_*)_{|H_1(\Gamma)} \in \mathrm{Hom}(H_1(\Gamma), V(\Gamma))$$

which we simply write $i^*_{\mathcal{A}} \circ \gamma_*$ for short. Remember the graph stabiliser $\mathcal{G}_\Gamma$ from Definition 1.6.1 on page 25, which by Corollary 1.6.12 on page 28 only depends on the graph Γ. Lemma 2.5.1 ensures that the class

$$|i^*_{\mathcal{A}} \circ \gamma_*| \in \mathcal{G}_\Gamma$$

is well-defined and by construction does not depend on the choice of $\gamma \in \mathrm{E}_\Gamma\left(\Omega^\theta\right)$. Our main result states that this class element is a topological invariant of ordered line arrangements.

Theorem 2.5.2. *Let $\mathcal{A}, \mathcal{A}' \subset \mathbb{CP}^2$ be two non-exceptional line arrangements with the same combinatorics C. Endow $\mathcal{A}$ and $\mathcal{A}'$ with the same ordering θ on their set of lines. If $(\mathcal{A}, \theta)$ and $(\mathcal{A}', \theta)$ are topologically equivalent then for every $\gamma \in \mathrm{E}_\Gamma\left(\Omega^\theta\right)$, we have $|i^*_{\mathcal{A}} \circ \gamma_*| = |i^*_{\mathcal{A}'} \circ \gamma_*|$ inside the graph stabiliser $\mathcal{G}_\Gamma$.* ◁

Proof. By Theorem 2.3.19 on page 44, there exists a homeomorphism $\Psi : B_{\mathcal{A}} \to B_{\mathcal{A}'}$ such that $\Psi \in \mathrm{Homeo}_{\Gamma}^{++}(B)$ where $B = B_{\mathcal{A}} \simeq B_{\mathcal{A}'}$. We thus have $i_{\mathcal{A}'} = i_{\mathcal{A}} \circ \Psi$. Let Ω be a graph ordering on Γ and let $\gamma \in \mathrm{E}_{\Gamma}\left(\Omega^{\theta}\right)$ be an ordered graph embedding. By Theorem 1.4.9 on page 20, the image $\Psi(\gamma)$ is again an element of $\mathrm{E}_{\Gamma}\left(\Omega^{\theta}\right)$. The map Ψ induces a group automorphism $\Psi^* : H_1(B, \mathbb{Z}) \to H_1(B, \mathbb{Z})$. By construction we have $\widetilde{\Psi(\gamma)} = \Psi^* \circ \gamma_*$ and $(i_{\mathcal{A}} \circ \Psi)^* = i_{\mathcal{A}}^* \circ \Psi^*$. Then, in restriction to $H_1(\Gamma)$, we have

$$i_{\mathcal{A}'}^* \circ \gamma_* = i_{\mathcal{A}}^* \circ \Psi^* \circ \gamma_* = i_{\mathcal{A}} \circ \widetilde{\Psi(\gamma)}$$

By Definition 1.6.1 on page 25 of the graph stabiliser $\mathcal{G}_{\Gamma}$, this implies that in the quotient

$$0 = \left| i_{\mathcal{A}}^* \circ \left(\widetilde{\Psi(\gamma)} - \gamma_* \right) \right| = |i_{\mathcal{A}'}^* \circ \gamma_*| - |i_{\mathcal{A}}^* \circ \gamma_*|$$ □

***Remark* 2.5.3.** By definition of the graph stabiliser, the class $|i_{\mathcal{A}}^* \circ \gamma_*| \in \mathcal{G}_{\Gamma}$ does not depend on the graphed embedding $\gamma_* \in \mathrm{E}_{\Gamma}\left(\Omega^{\theta}\right)$. However, it does depend on the graph ordering Ω^{θ} and thus on the ordering on the combinatorics θ. The homology inclusion is therefore an *ordered* line arrangement invariant. However, some combinatorics have trivial automorphism groups (see Definition 2.2.5 on page 37). The restriction of the homology inclusion to this subclass of line arrangements becomes an unordered topological invariant. ⌑

***Remark* 2.5.4.** The graph stabiliser does not quotient the homology differences caused by the application of the complex conjugation inside $\mathbb{CP}^2$, since it is not a positive homeomorphism of the boundary manifold. A line arrangement $\mathcal{A}$ and its conjugate $\overline{\mathcal{A}}$ might thus have different homology inclusion values. If one imposes the same orientation on both arrangements, then $(\mathcal{A}, \overline{\mathcal{A}})$ becomes an *oriented* Zariski pair. ⌑

CHAPTER

3

COMPUTATIONS OF THE HOMOLOGY INCLUSION

Contents

Our objective is to compute the homology inclusion map of a non-exceptional line arrangement $\mathcal{A}$. To achieve this, we explain in Section 3.1 how to build a graph embedding $\gamma_{\mathbf{B}} \in \mathrm{E}_{\mathcal{T}}(\Omega)$ using a standard geometrical method on a representative $\mathbf{B}$ of the braid monodromy of a line arrangement $\mathcal{A}$. We then use this graph embedding in Section 3.2 to compute the value of the morphism $i^*_{\mathcal{A}} \circ \gamma^*_{\mathbf{B}}$ of the embedded graph cycles using a simple tool called the braid linking. Finally in Section 3.3 we explain how to modify the braid monodromies of two line arrangements $\mathcal{A}, \mathcal{A}'$ with the same combinatorics C in order to associate them with a common graph ordering Ω and spanning tree $\mathcal{T}$. The homology inclusion maps can then be computed in the same basis of the graph stabiliser module, and Theorem 2.5.2 allows us to compare them to determine if the pair $(\mathcal{A}, \mathcal{A}')$ is Zariski. We then describe the final algorithm written in Sage [Sag23] that sums up all these computations. Finally, we give our main examples of Zariski pairs, which were obtained in collaboration with B. Guerville-Ballé.

3.1 Standard ordered graphed embedding

Let $\mathcal{A}$ be a non-exceptional line arrangement with combinatorics $C_{\mathcal{A}}$. As explained in Section 2.3.3, the boundary manifold $B_{\mathcal{A}}$ has several graph structures. The unique minimal one is given by the reduced incidence graph $\widetilde{\Gamma}(C_{\mathcal{A}})$, and another non-minimal one is given by the complete incidence graph $\Gamma(C_{\mathcal{A}})$.

Our objective is to build a *standard graphed embedding* of the full incidence graph $\Gamma := \Gamma(C_{\mathcal{A}})$ using a representative of the braid monodromy of $\mathcal{A}$ as defined in Definition 2.4.8 on page 47. This graphed embedding is designed specifically to allow an algorithmic computation of the homology inclusion using only the braid monodromy representative as base data.

3.1.1 Combinatorial data

We reuse all notations from Section 2.4 which we recall briefly. Most of them are also summed up on Figure 2.4.1 on page 47. Fix a component $L_0 \in \mathcal{A}$ as the line at infinity. We work with the affine part $\mathcal{A}^{\mathrm{aff}} \simeq \mathcal{A} \setminus L_0$ in the standard affine chart $\mathbb{C}^2 \equiv \{[x : y : 1] \in \mathbb{CP}^2\}$. We use the sets $\mathcal{L}^* = \mathcal{L} \setminus \{L_0\}$ and $\mathcal{Q}^* = \mathcal{Q} \setminus V_{L_0}$. The set $\mathcal{L}^*$ is ordered by θ.

Let $\pi : \mathbb{C}^2 \to \mathbb{C}$ be a projection. We choose coordinates (x, y) on the standard affine chart such that $\pi(x, y) = x$. To simplify the descriptions, we also suppose that π is *generic* as in Definition 2.4.1 on page 45 which means that $\pi(\mathcal{Q}^*) \simeq \mathcal{Q}^*$. In practice one can always adjust the projection π to make it generic. We write $r := \#\mathcal{Q}^*$ and $\mathbb{C}_{\mathcal{A}} := \mathbb{C} \setminus \pi(\mathcal{Q}^*)$. For every point $x \in \mathbb{C}_{\mathcal{A}}$, the strand corresponding to L_i is $w_{L_i}(x) = \pi^{-1}(x) \cap L_i$. Let $D(x)$ be a closed disc in $\pi^{-1}(x)$ containing all the intersections $D_{L_i}(x) := \pi^{-1}(x) \cap N_{L_i}$ for every $L_i \in \mathcal{L}^*$. Define

$$\Sigma(x) := D(x) \setminus \bigcup_{L_i \in \mathcal{L}^*} \overset{\circ}{\widehat{D_{L_i}(x)}}$$

Let D be a closed disc in $\mathbb{C}$ containing all the images $\pi(N_P)$ for $P \in \mathcal{Q}^*$. Up to isomorphism, $D(x)$ can be chosen to have the same shape for all values of $x \in D$. We define

$$\Sigma := D \setminus \bigcup_{P \in \mathcal{Q}^*} \overset{\circ}{\widehat{\pi(N_P)}} \subset \mathbb{C}_{\mathcal{A}}$$

The projection π induces its own ordering $\nu_\pi : \mathcal{Q}^* \to \{1, \ldots, r\}$. Let $b^\infty \in \partial D$ be a base point and let $\alpha \in \mathcal{S}_{r_\pi}$ be a star on Σ with respect to the permutation $\nu \circ {\nu_\pi}^{-1}$, where ν is any ordering on $\pi(\mathcal{Q}^*)$.

These data allow to build a *representative of the braid monodromy*

$$\mathrm{B}_{\mathcal{A}}(\pi, \theta, \nu, \alpha) = (\beta_l)_{1 \le l \le r}$$

as explained in Definition 2.4.12 on page 48.

3.1.2 Boundary manifold and braid monodromy

Before giving the definition of the standard graphed embedding we need to identify the graph structure of the boundary manifold $B_{\mathcal{A}}$ associated with the complete incidence graph $\Gamma(C_{\mathcal{A}})$ within the same construction of the braid monodromy representative we have fixed. To achieve this, we first need to identify the local regular neighbourhoods N_{L_i} and N_{P_l} for each $L_i \in \mathcal{L}^*$ and $P_l \in \mathcal{Q}^*$, and we will then consider their boundaries. The cases of the line at infinity L_0 and the singular points it meets must be treated separately.

Let $V^*_{L_i} := V_{L_i} \setminus \{L_i \cap L_0\}$. For every $L_i \in \mathcal{L}^*$, let

$$\Sigma^{L_i} := D \setminus \bigcup_{P_l \in V^*_{L_i}} \overset{\circ}{\widehat{\pi(N_{P_l})}}$$

Note that we have $\Sigma \subset \Sigma_{L_i}$. Separately, for every $P_l \in \mathcal{Q}^*$ and every $x \in \Sigma$, let

$$\Sigma^{P_l}(x) := \Sigma(x) \cap N_{P_l}$$

By construction, $\Sigma^{P_l}(x)$ is a subsurface of $\Sigma(x)$ that contains only the boundary components $\partial D_{L_i}(x)$ for $L_i \in V_{P_l}$. Recall that

$$S^{\mathrm{aff}}_{L_i} = \partial N^{\mathrm{aff}}_{L_i} \setminus \bigcup_{P_l \in V^*_{L_i}} \overset{\circ}{\widehat{N_{P_l}}} \qquad\qquad S_{P_l} = \partial N_{P_l} \setminus \bigcup_{L_i \in V_{P_l}} \overset{\circ}{\widehat{N^{\mathrm{aff}}_{L_i}}}$$

Proposition 3.1.1. *There are homeomorphisms*

$$S^{\text{aff}}_{L_i} \simeq \bigcup_{x\in\Sigma^{L_i}} \partial D_{L_i}(x) \quad \cup \quad \bigcup_{x\in\partial D^0} D_{L_i}(x)$$
$$S_{P_l} \simeq \bigcup_{x\in\partial\pi(N_{P_l})} \Sigma^{P_l}(x) \quad \cup \quad \bigcup_{x\in\pi(N_{P_l})} \partial^\infty\Sigma^{P_l}(x)$$

for each $L_i \in \mathcal{L}^*$ *and* $P_l \in \mathcal{Q}^*$. ◁

Both manifold types are divided between a trivial circle bundle which contains all boundary components, and a solid torus. The fibres of S_{P_l} are *horizontal* circles corresponding to the intersection of $\{\Sigma^{P_l}(x) \mid x \in \partial\pi(N_{P_l})\}$ with an horizontal section of π. A section of S_{P_l} is simply $\Sigma^{P_l}(x)$ for any choice of a $x \in \partial\pi(N_{P_l})$. Remember that we only consider sections of the sub-bundle that contains the boundary. This structure of S_{P_l} is represented schematically on Figure 3.1.1b on the following page. The situation is similar for $S^{\text{aff}}_{L_i}$. The fibres of $S^{\text{aff}}_{L_i}$ are *vertical* circles $\partial D_{L_i}(x)$ for $x \in \Sigma^{L_i}$. The intersection of a horizontal section of π with those vertical circles gives a partial section of $S^{\text{aff}}_{L_i}$ in the sense that it lacks the part at infinity. Again we ignore the second solid torus. This structure of $S^{\text{aff}}_{L_i}$ is represented schematically on Figure 3.1.1a on the next page.

Proof of Proposition 3.1.1. Let $D^{P_l}(x)$ be the disc obtained by filling in the ∂D_{L_i} boundary components of $\Sigma^{P_l}(x)$. There are natural homeomorphisms

$$N^{\text{aff}}_{L_i} \simeq \bigcup_{x\in D} D_{L_i}(x) \qquad\qquad N_{P_l} \simeq \bigcup_{x\in\pi(N_{P_l})} D^{P_l}(x)$$

Consider the boundaries

$$\partial N^{\text{aff}}_{L_i} \simeq \bigcup_{x\in D} \partial D_{L_i}(x) \quad \cup \quad \bigcup_{x\in\partial D} D_{L_i}(x)$$
$$\partial N_{P_l} \simeq \bigcup_{x\in\partial\pi(N_{P_l})} D^{P_l}(x) \quad \cup \quad \bigcup_{x\in\pi(N_{P_l})} \partial D^{P_l}(x)$$

As expected, both boundaries are homeomorphic to S^3 as the reunion of two solid tori. Suppose that $P_l \in L_i$. Then

$$T_{P_l,L_i} \simeq \partial N_{P_l} \cap \partial N^{\text{aff}}_{L_i} = \bigcup_{x\in\pi(N_{P_l})} \partial D_{L_i}(x)$$

is the joining torus corresponding to the edge e_{L_i,P_l} for the $\widetilde{\Gamma}(C_{\mathcal{A}})$ graph structure. Removing the joining tori from the boundaries $\partial N^{\text{aff}}_{L_i}$ and ∂N_{P_l} give the expected description of $S^{\text{aff}}_{L_i}$ and S_{P_l}. □

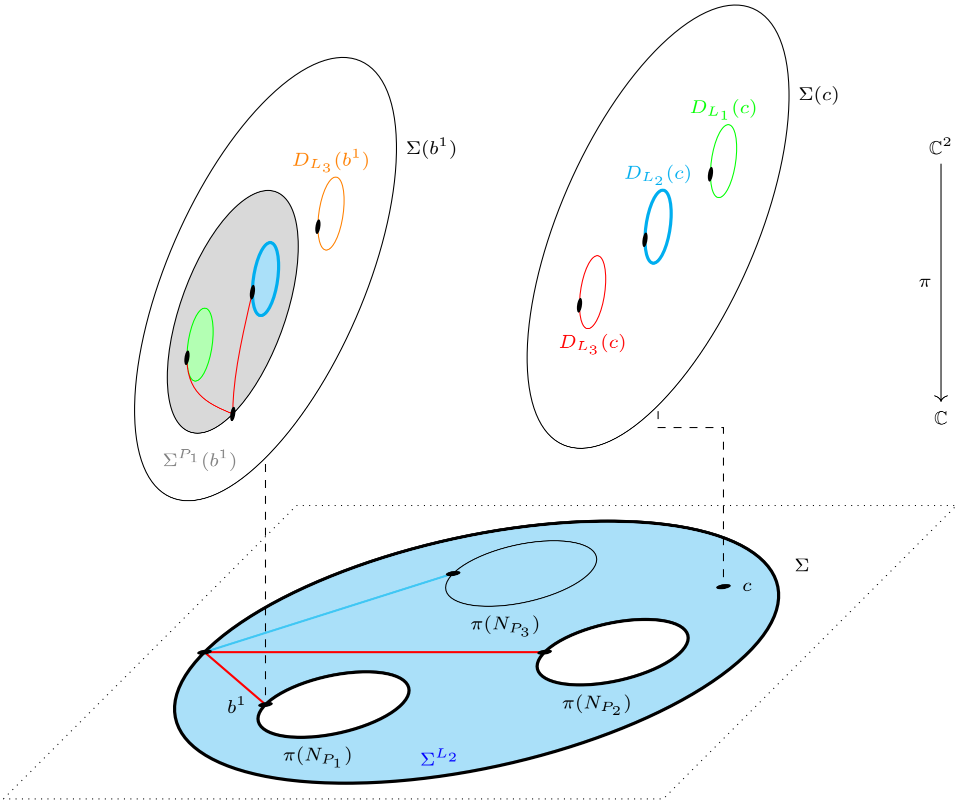
To complete $S^{\text{aff}}_{L_i}$ into S_{L_i} we must now attend the situation at infinity. This is done by switching to a new chart called the ∞-chart, as described in the proof of Proposition 2.4.13 on page 48. Denote by (Y, Z) the coordinates of the ∞-chart. The exterior $D^\infty := \mathbb{CP}^1 \setminus \mathring{D}$ is a 2-disc in the ∞-chart. The line L_0 corresponds to $\{Z = 0\}$ and the projection becomes $\pi : (Y, Z) \to (0, Z)$. This means that L_0 is now a vertical line for π, and all singular points of V_{L_0} are all sent to a same point $x_0 = (Y = 0, Z = 0)$, and thus the extension of π to the ∞-chart is not generic.

***Remark* 3.1.2.** The *braid at infinity* associated to the representative is the lifted braid over ∂D^∞ in the ∞-chart. But by Proposition 2.4.13 it can also be computed from the product of all other braids of the representative. This means that in practice it is not necessary to consider direct transformations of the graph structure near L_0 since they are automatically induced by the transformations in the standard chart. ⌑

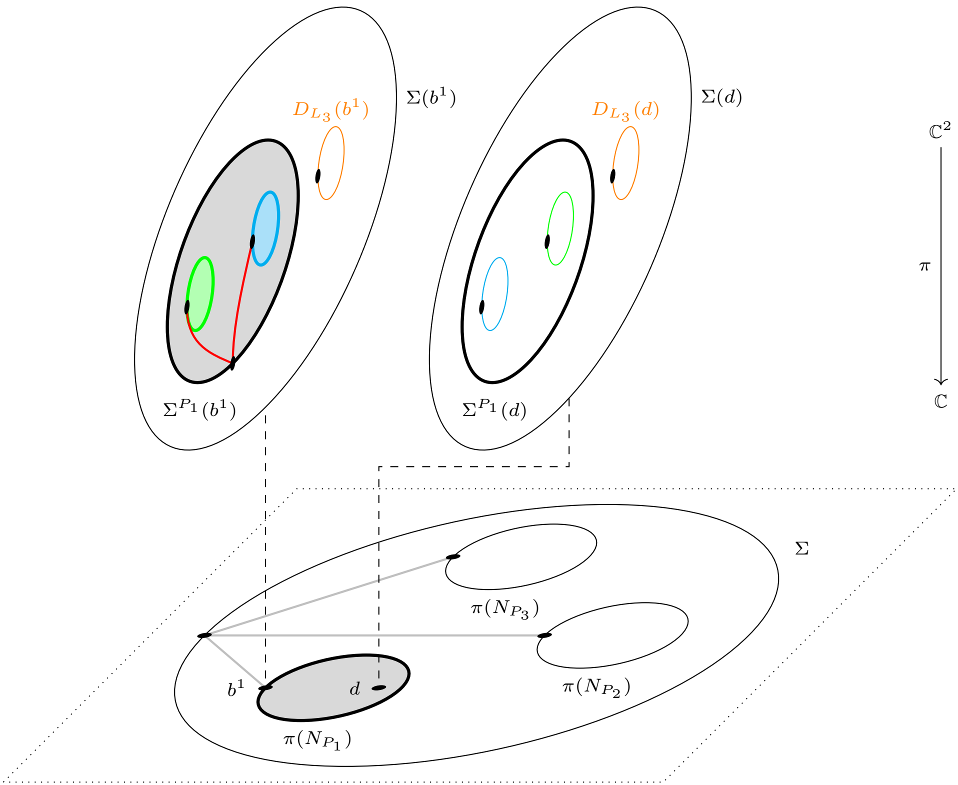
We now describe the intersection of the boundary manifold with the ∞-chart. Let

$$N^\infty_{L_i} := N_{L_i} \cap \pi^{-1}(D^\infty)$$

Extending the notations of the standard chart, for every every $L_i \in \mathcal{L}^*$, $P_l \in V_{L_0}$ and $Y \in D^\infty$ we define $D(Y)$, $D_{L_i}(Y)$, $\Sigma(Y)$ and $\Sigma^{P_l}(Y)$. The situation is different for the basis of the projection



(a) Structure of $S_{L_2}^{\text{aff}}$ with $P_1, P_2 \in L_1$



(b) Structure of S_{P_1} with $P_1 = L_1 \cap L_2$

Figure 3.1.1: Local boundary manifolds

since π is no longer generic. By construction, for every $P \in V_{L_0}$ we have $N_P \smallsetminus N_{L_0} \neq \varnothing$. This means that up to homeomorphism we can suppose that $\pi(N_{L_0})$ and all $\pi(N_P)$ for all $P \in V_{L_0}$ are a common disc D^0. We also define

$$\Sigma^\infty := D^\infty \smallsetminus \overset{\circ}{\widehat{D^0}} \qquad\qquad \Sigma^0 := D^0 \smallsetminus \overset{\circ}{\widehat{\pi(N_{L_0})}}$$

Proposition 3.1.3. *There are homeomorphisms*

$$S^\infty_{L_i} \simeq \bigcup_{x\in\Sigma^\infty} \partial D_{L_i}(Y) \quad\cup\quad \bigcup_{x\in\partial D^\infty} D_{L_i}(Y)$$

$$S_P \simeq \bigcup_{x\in\partial D^0} D^P(Y)$$

for every $P \in V_{L_0}$ and every $L_i \in \mathcal{L}^*$. ◁

The structure of $S^\infty_{L_i}$ and S_P are similar to their counterparts in the standard chart. However, for S_P the decomposition contains another horizontal torus $\{\partial D^P(Y) \mid Y \in \partial D^0\}$ which corresponds to the joining torus with S_{L_0}. A section of S_P with $P \in L_0$ is thus composed of the vertical surface $\Sigma(Y)$ for $Y \in \partial D^0$, but unlike in the standard chart the additional boundary torus do not correspond to the Euler gluing map.

The two parts $S^\infty_{L_i}$ and $S^{\text{aff}}_{L_i}$ reconnect to form S_{L_i} along the torus

$$T^\infty_{L_i} := \bigcup_{x\in\partial D^\infty} \partial D_{L_i}(x)$$

A complete section of S_{L_i} is therefore determined by the choice of a point on $D_{L_i}(x)$ and $D_{L_i}(Y)$ for every $x \in \Sigma^{L_i}$ and every $Y \in \Sigma^\infty$.

Finally, the circle bundle S_{L_0} is a special case since L_0^{aff} is vertical.

Proposition 3.1.4. *There is an homeomorphism*

$$S^{\text{aff}}_{L_0} \simeq \bigcup_{x\in\partial D^0} \Big(\pi^{-1}(Y) \smallsetminus \bigcup_{P\in V_{L_0}} D^P(Y)\Big)$$ ◁

A partial section of $S^{\text{aff}}_{L_0}$ is thus the vertical surface

$$D_{L_0}(Y) := D(Y) \smallsetminus \bigcup_{P\in V_{L_0}} D^P(Y)$$

for the choice of a point $Y \in \partial\pi(N_{L_0})$.

3.1.3 Construction of the standard graphed embedding

As explained in Definition 1.4.4 on page 19, a graphed embedding is made of ordered stars drawn on sections of each of the circle bundles S_L and S_P for every $L \in \mathcal{L}$ and every $P \in \mathcal{Q}$. Using the description of the graph structure of $B_\mathcal{A}$ established in Section 3.1.2, we can now explain how to build the standard graphed embedding associated with the representative $\mathbf{B} = \mathrm{B}_\mathcal{A}(\pi, \theta, \nu, \alpha)$ of the braid monodromy of the line arrangement $\mathcal{A}$.

A graphed embedding also depends on the choice of a graph ordering that must be fixed. Extend the ordering θ defined on $\mathcal{L}^*$ to $\mathcal{L}$ by assigning 0 to L_0. Then Definition 2.2.15 on page 38 allows to define two graph sub-orderings $\Omega^\theta_\mathcal{Q}$ and $\Omega^\nu_{\mathcal{L}^*}$ on Γ. The only neighbour set remaining to be ordered is V_{L_0}, which can be done using the (extended) θ-lexicographic ordering. We thus get a full graph ordering $\Omega^{\theta,\nu}$ of $\Gamma(C_\mathcal{A})$.

Recall that $\alpha \in \mathcal{S}_r(\nu \circ \nu_\pi{}^{-1})$ is a star drawn on $\Sigma \subset \mathbb{C}$ with $b^\infty \in \partial^\infty\Sigma$ as its base point. Extend α to Σ^∞ by adding a final branch α^0 joining b^∞ to ∂D^0 to get the new star $\widetilde{\alpha} \in \mathcal{S}_{n+1}$. We first need to adjust the local geometrical orderings θ^l for each $P_l \in \mathcal{Q}^*$ so that it coincides with the graph ordering $\Omega^{\theta,\varphi}$.

Lemma 3.1.5. *For every $P_l \in \mathcal{Q}^*$, one can always make the sub-orderings*

$$\theta, \theta^l : V_{P_l} \longrightarrow \{1, \dots, m(P_l)\}$$

coincide without changing the representative $\mathbf{B}$. ◁

Proof. Let $\sigma_l \in \mathfrak{S}_n$ be the permutation such that $\sigma_l \circ \theta^l(V_{P_l}) = \theta(V_{P_l})$ and leave all other indices unchanged. Let $\chi_l \in \mathbb{B}_n$ such that $\sigma(\chi_l) = \sigma_l$. Then the braid χ_l commute with $\Delta^2_{\theta^l(V_{P_l})}$. We can thus write

$$\beta_l = \tau_l \cdot \Delta^2_{\theta^l(V_{P_l})} \cdot (\tau_l)^{-1} = (\tau_l \cdot \chi_l) \cdot \Delta^2_{\theta(V_{P_l})} \cdot (\tau_l \cdot \chi_l)^{-1}$$

Redefining the shift braid τ_l as $\tau_l \cdot \chi_l$ gives the intended result without changing the braid β_l. □

Now we can describe the pair section-ordered star that compose the standard graph ordering for each kind of vertices:

- For every $L_i \in \mathcal{L}^*$, consider the section s_{L_i} of S_{L_i} defined as such: for every $x \in \Sigma_{L_i}$, $s_{L_i}(x)$ is the point of $\partial D_{L_i}(x)$ that minimises the value of $\mathrm{Re}(y)$. The section is completed by using the same definition on Σ^∞ in the ∞-chart. Now consider the ordered star $\alpha_{L_i} \in \mathcal{O}_{m(L_i)}$ drawn on s_{L_i} and defined by $\alpha_{L_i} := \pi^{-1}(\widetilde{\alpha}) \cap s_{L_i}$.
- For every $P_l \in \mathcal{Q}^*$, consider the section s_{P_l} of S_{P_l} defined as the vertical surface $\Sigma^{P_l}(b^l)$, where $b^l := \partial_+\alpha_l$. Let $\alpha_{P_l} \in \mathcal{O}_{m(P_l)}$ be an ordered star drawn on s_{P_l} that joins a base point to each of the points $s_{P_l} \cap s_{L_i}$ for every $L_i \in V_{P_l}$.
- For L_0, consider the section s_{L_0} of S_{L_0} defined as the vertical surface $\Sigma(b^0)$ where $b^0 = \partial_+\alpha^0$. Let $\alpha_{L_0} \in \mathcal{O}_{m(L_0)}$ be a star on s_{L_0}.
- For every $P_l \in V_{L_0}$, consider the section s_{P_l} of S_{P_l} defined as the surface $\Sigma^{P_l}(b^0)$, where $b^0 := \partial_+\alpha_0$. Let $\alpha_{P_l} \in \mathcal{O}_{m(P_l)}$ be an ordered star drawn on $\Sigma^{P_l}(b^0)$ that joins a base point to each of the points $s_{P_l} \cap s_{L_i}$ for every $L_i \in V_{P_l}$. Note that the centre of α_{P_l} is *not* on ∂D^{P_l} as usual but must be placed in the interior of the section.

Definition 3.1.6. The *standard ordered graphed embedding* $\gamma_{\mathbf{B}} \in \mathrm{E}_{\Gamma(C_\mathcal{A})}\left(\Omega^{\theta,\nu}\right)$ is defined as the reunion of the ordered stars $\alpha_{L_i} \in \mathcal{O}_{m_i}$ and $\alpha_{P_l} \in \mathcal{O}_{m_l}$ described above for every $L_i \in \mathcal{L}$ and $P_l \in \mathcal{Q}$. ◇

***Remark* 3.1.7.** By construction, the image by $\gamma_{\mathbf{B}}$ of the half-edge $\vec{e}_{L_i,P_l}$ going from $L_i \in \mathcal{L}^*$ to $P_l \in V^*_{L_i}$ is homotopic to the i-th strand of the shift braid τ_l of the representative $\mathbf{B}$ (see Definition 2.4.11 on page 48). ⌑

***Remark* 3.1.8.** Since π is generic and therefore $\pi(\mathcal{Q}^*) = \mathcal{Q}^*$, the ordering ν could have been fixed as the θ-lexicographic ordering. We made this choice for the sake of simplicity in Theorem 2.5.2 on page 52 which defines the homology inclusion of $\mathcal{A}$. The reasons to keep ν as a separate value for the explicit computation are explained in Section 3.3.1. ⌑

3.2 Values of the homology inclusion map

The specific geometrical construction of the standard graphed embedding $\gamma_{\mathbf{B}}$ allows to compute the morphism $i^*_\mathcal{A} \circ \gamma^*_{\mathbf{B}}$. This reduces to the computation of linking numbers between strands of the shift braids extracted from the representative $\mathbf{B}$ of the braid monodromy of the line arrangement $\mathcal{A}$.

3.2.1 Braid linking

The braid linking is a linking number computed between two strands of a braid according to specific rules.

Let $\mathbf{y} \subset \mathbb{C}$ be a set of n disjoint points ordered by their descending horizontal coordinates. Recall from Proposition 2.4.4 on page 46 that a braid $\beta \in \mathbb{B}_n$ can be seen as the homotopy class of n non-crossing paths $\gamma_1, \ldots, \gamma_n$ inside $\mathbb{C} \times [0,1]$ that start and end at $\mathbf{y}$. We write $\mathbb{C}(t) := \mathbb{C} \times \{t\}$, where the parameter t is called the *height*. Fix a braid $\beta := \{\gamma_1, \ldots, \gamma_n\} \in \mathbb{B}_\mathbf{y}$ and a braid decomposition

$$\beta = \prod_{s=1}^{m} \sigma_{i_s}$$

There is a set of heights $H = \{t_1, \ldots, t_s\}$ where two points $\gamma_i(t)$ and $\gamma_j(t)$ have the same horizontal coordinate in $\mathbb{C}(t)$. For every $t \in [0,1] \setminus H$, the ordering

$$\omega^t_\beta : \{\gamma_1(t), \ldots, \gamma_n(t)\} \longrightarrow \{1, \ldots, n\}$$

corresponds to the decreasing horizontal coordinates of the points in $\mathbb{C}(t)$. The index i of a strand γ_i is therefore the order of its starting point inside $\mathbb{C}(0)$. By convention we suppose that every elementary braid σ_{i_s} that permutes two consecutive strands corresponds to a positive rotation of their projection points in the base plane.

Let $1 \leq i < j \leq n$ be two fixed indices. For every $t \in [0,1]$, let $D_i(t) \subset$ be a small disc such that $\gamma_i(t)$ is the point of $D_i(t)$ with the minimum horizontal coordinate. Finally, for every $t \in [0,1] \setminus H$, let $S_{i,j}(\beta,t)$ be the path drawn on $\mathbb{C}(t)$ as shown on Figure 3.2.1. Note that the *left support strand* and the *right support strand* of the path $S_{i,j}(\beta,t)$ play non-symmetric roles. When the left and right support strands reverse their position, the modification of the path $S_{i,j}(\beta,t)$ is shown on Figure 3.2.3 on the following page. This is called a *reversal crossing*. Every other permutation of two strands involving only *one* of the support strands is called a *regular crossing*.

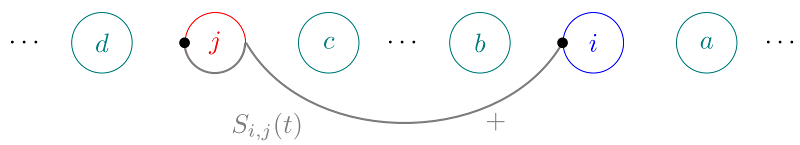


Figure 3.2.1: Section of the (i,j)-braid linking surface

Definition 3.2.1. The reunion

$$S_{i,j}(\beta) := \bigcup_{t\in[0,1]} S_{i,j}(\beta,t)$$

is an orientable surface within $\mathbb{C} \times [0,1]$ called the (i,j)-*braid linking surface* of the braid β. $\diamond$

By convention, $S_{i,j}(\beta)$ is oriented positively on the side that face the lowest vertical coordinates in the section $\mathbb{C}(0)$. At every reversal crossing, the positive and negative sides switch their positions, as illustrated on Figure 3.2.3 on the next page.

For every height $t \in [0,1]$, define

$$\mathbb{C}(\beta,t) := \mathbb{C}(t) \setminus \bigcup_{i=1}^{n} D_i(t)$$

and $\mathbb{C}(\beta) := \bigcup_{t\in[0,1]} \mathbb{C}(\beta,t)$. For any height $t \in [0,1]$, there is a natural group isomorphism

$$\begin{aligned} \mathbb{Z}^n &\longrightarrow H_1(\mathbb{C}(\beta,t),\mathbb{Z}) \\ v_i &\longmapsto [D_i(t)] \end{aligned}$$

These isomorphisms extend into a global group isomorphism $H_1(\mathbb{C}(\beta),\mathbb{Z}) \simeq \mathbb{Z}^n$ with the same generators.

Definition 3.2.2. Let $1 \leq i < j \leq n$ be two strand indices. The (i,j)-*braid linking function*

$$\lambda_{i,j} : \mathbb{B}_n \longrightarrow \mathbb{Z}^n$$

associates to a braid β the homological class of the boundary $\partial S_{i,j}(\beta)$ inside the free abelian group $H_1(\mathbb{C}(\beta),\mathbb{Z}) \simeq \mathbb{Z}^n$. $\diamond$

Example 3.2.3. The $(1,3)$-braid linking surface of the braid $\sigma_2^{-1}\sigma_1 \in \mathbb{B}_3$ is shown on Figure 3.2.2 on the following page. The corresponding $(1,3)$-braid linking value is

$$\lambda_{1,3}(\sigma_2^{-1}\sigma_1) = v_2 + v_3 \qquad \bigcirc$$

Proposition 3.2.4. *The (i,j)-braid linking of a braid $\beta \in \mathbb{B}_n$ can be computed directly from any crossings decomposition*

$$\beta = \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_r}^{\varepsilon_r}, \quad 1 \leq i_s \leq n, \quad \varepsilon_{i_s} \in \{\pm 1\}$$

by adding up the values corresponding to each crossing using the rules of Figures 3.2.3 and 3.2.4 on the next page and on page 62. The indicated signs must be reversed after each reversal crossing. $\triangleleft$

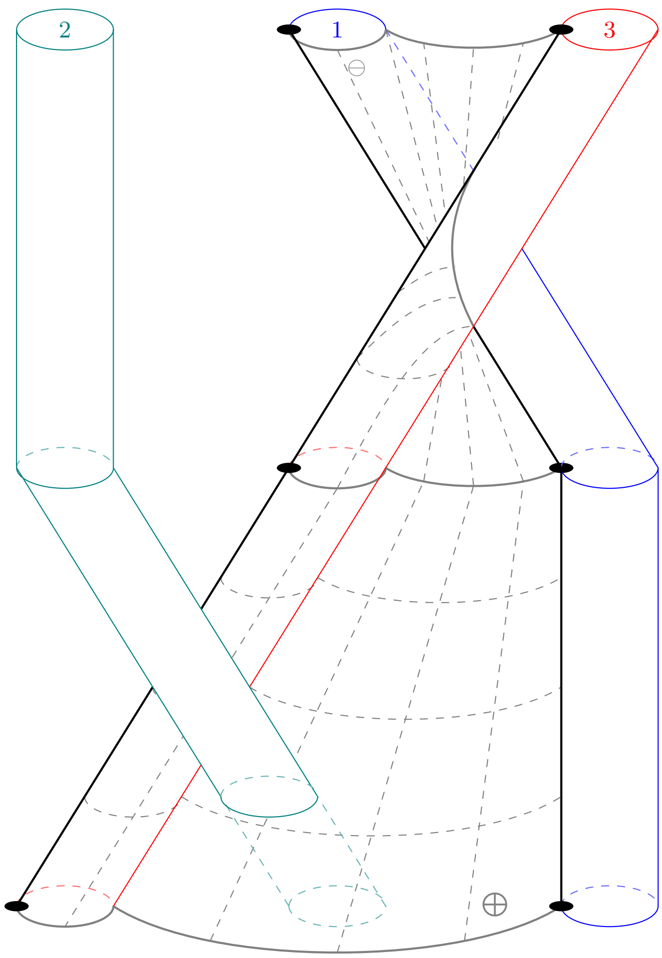


Figure 3.2.2: (1, 3)-braid linking surface of $\sigma_2^{-1}\sigma_1$

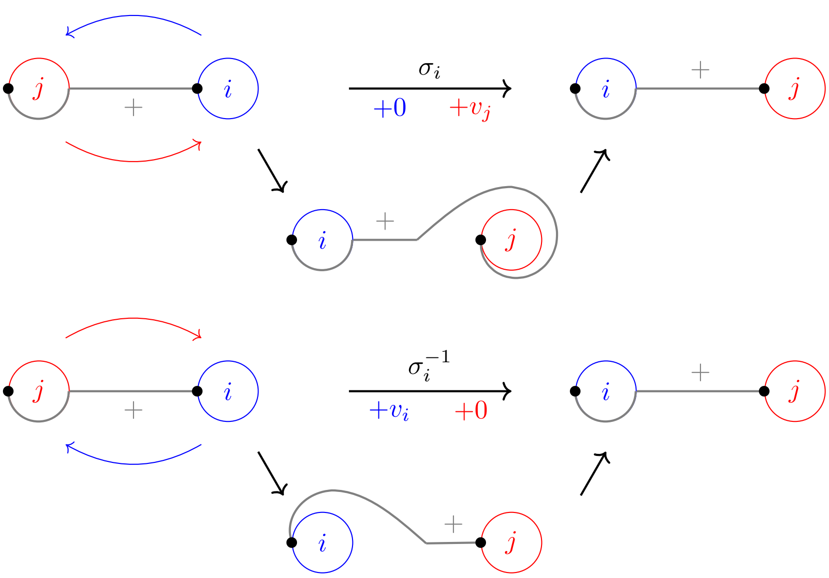
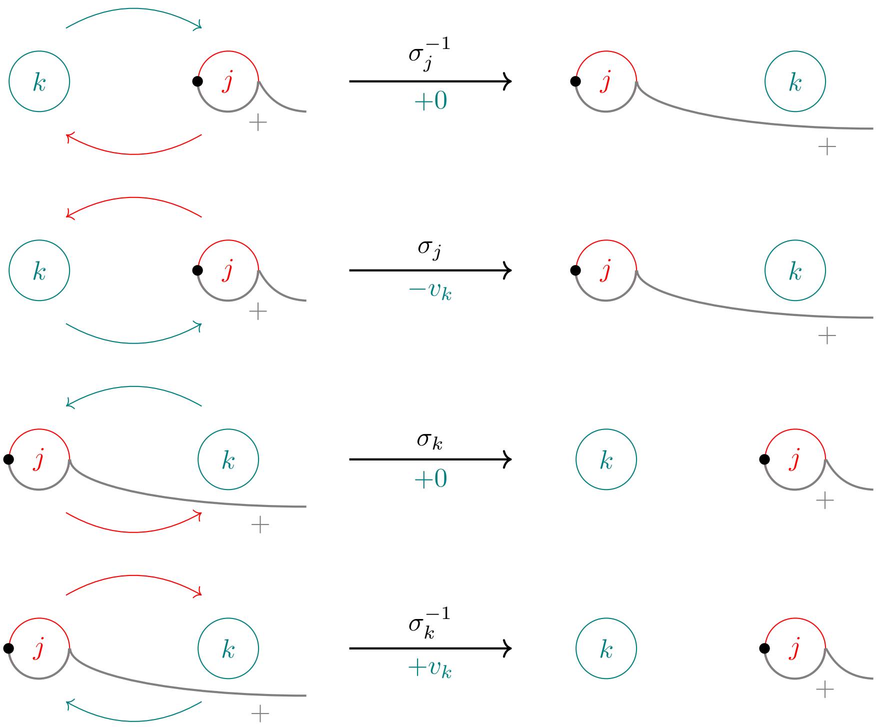
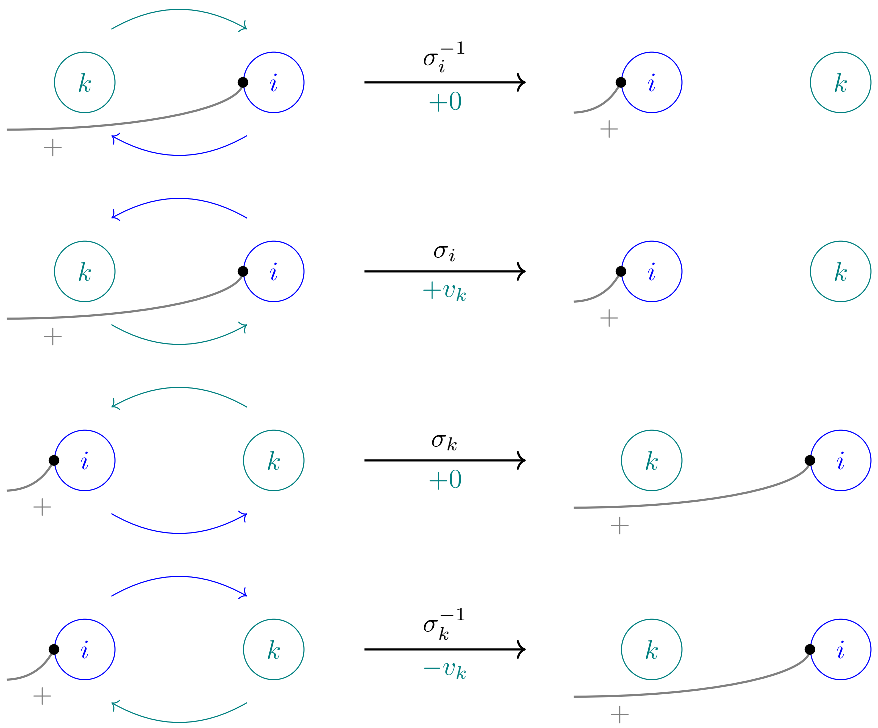


Figure 3.2.3: Braid linking surface at reversal crossings



(a) Regular crossings with left support strand



(b) Regular crossings with right support strand

Figure 3.2.4: Braid linking surface at regular crossings

Example 3.2.5. The braid linking values of the braid $\beta = \sigma_2^{-1}\sigma_1 \in \mathbb{B}_3$ are

$$\lambda_{1,2}(\beta) = v_3 \qquad \lambda_{2,3}(\beta) = v_2 \qquad \lambda_{1,3}(\beta) = v_2 + v_3$$ ○

3.2.2 Main computation theorem

This section is dedicated to the statement and proof of Theorem 3.2.6 which gives the main formula that links the braid linking function given in Section 3.2.1 and the homology inclusion map defined in Section 2.5. The statement of the theorem uses many objects and concepts introduced in the previous chapter, in addition of the specific construction of the standard graphed embedding detailed in Section 3.1. We thus begin with a quick summary of these concepts before stating the theorem.

Recall from Section 2.5 that the homology inclusion map is the morphism

$$i_{\mathcal{A}}^* \circ \gamma^* : H_1(\widetilde{\Gamma}(C_{\mathcal{A}})) \longrightarrow V(\widetilde{\Gamma}(C_{\mathcal{A}}))$$

inside the graph stabiliser $\mathcal{G}_{\widetilde{\Gamma}(C_{\mathcal{A}})}$, where $\gamma \in \mathrm{E}_{\widetilde{\Gamma}(C_{\mathcal{A}})}\left(\Omega^\theta\right)$ is any ordered graphed embedding. The class value depends on the graph ordering Ω^θ which itself depends on the ordering θ of the line component set $\mathcal{L}$.

By Proposition 2.3.20 on page 45, the reduced incidence graph $\widetilde{\Gamma}(C_{\mathcal{A}})$ is obtained from the full incidence graph $\Gamma(C_{\mathcal{A}})$ by blowing-down moves. We can thus use Theorem 1.6.15 on page 31 which gives a natural group isomorphism

$$\mathcal{G}_{\widetilde{\Gamma}(C_{\mathcal{A}})} \xrightarrow{\sim} \mathcal{G}_{\Gamma(C_{\mathcal{A}})}$$

This means that both the graph stabiliser and the homology inclusion map can be computed using the full incidence graph $\Gamma := \Gamma(C_{\mathcal{A}})$ rather than the reduced one $\widetilde{\Gamma}(C_{\mathcal{A}})$. Following Remark 3.1.8 on page 59, we also prefer to use the graph ordering $\Omega^{\theta,\nu}$ from Definition 2.2.15 on page 38 which depends on an additional ordering ν on the set of singular points $\mathcal{Q}$.

Recall the induced spanning tree $\mathcal{T}^\theta$ on Γ from Definition 2.2.14 on page 38. By Proposition 1.5.4 on page 21, the edges of $E \smallsetminus \mathcal{T}^\theta$ are naturally associated with a basis of the free abelian group $H_1(\Gamma)$. For every $P_l \in \mathcal{Q}^*$, define $n_l := \min \theta(V_{P_l})$ and

$$V_{P_l}^* := V_{P_l} \smallsetminus \{L_{n_l}\}$$

The set of edges outside $\mathcal{T}^\theta$ is then given by

$$E \smallsetminus \mathcal{T}^\theta := \{e_{L,P} \mid P \in \mathcal{Q}^*, L \in V_P^*\}$$

For any such edge $e \in E \smallsetminus \mathcal{T}^\theta$, let $c_e \in H_1(\Gamma)$ be the corresponding graph cycle. On the other end of the homology inclusion map, recall the quotient map

$$\eta : C_0(\Gamma) \longrightarrow V(\Gamma)$$

where $C_0(\Gamma) \simeq \mathbb{Z}^{n+1}$ is the free abelian group generated by the vertices of Γ.

The standard graphed ordering $\gamma_{\mathbf{B}} \in \mathrm{E}_\Gamma\left(\Omega^{\theta,\nu}\right)$ of the graph Γ is built with a geometrical method from a representative of the braid monodromy

$$\mathbf{B} := \mathrm{B}_{\mathcal{A}}(\pi, \theta, \nu, \alpha) = (\beta_l)_{1 \le l \le r}$$

The n strands of the braid β_l are naturally associated with the lines of $\mathcal{L}^* := \mathcal{L} \smallsetminus \{L_0\}$. Moreover, each braid of the representative is a conjugate of the form

$$\beta_l = \tau_l \cdot \delta_l \cdot (\tau_l)^{-1} \in \mathbb{P}_n$$

where $\tau_l \in \mathbb{B}_n$ is called the *shift braid* and δ_l is a pure twist.

Finally, recall the braid linking function $\lambda_{i,j} : \mathbb{B}_n \to \mathbb{Z}^n$ from Section 3.2.1. The free abelian group $\mathbb{Z}^n$ can be seen as the subgroup of $C_0(\Gamma)$ generated by the set of vertices $\{v_L \mid L \in \mathcal{L}^*\}$.

All these constructions allow us to compute the values of the morphism $i_{\mathcal{A}}^* \circ \gamma_{\mathbf{B}}^*$ on each element of the cycle basis given above.

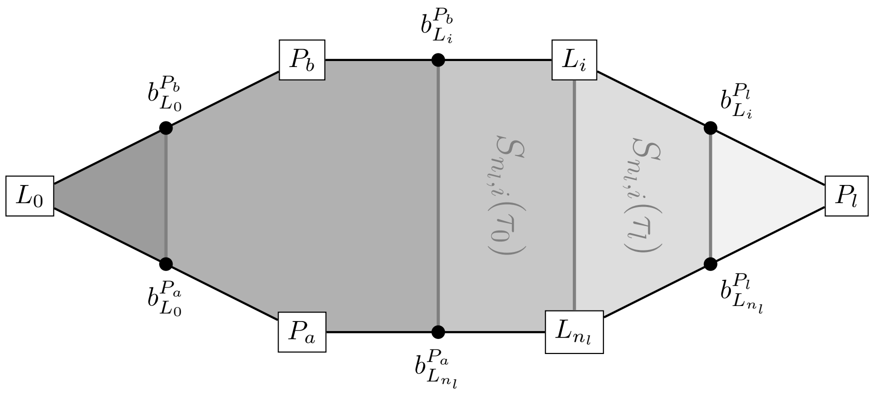


Figure 3.2.5: Decomposition of the exterior cycle surface

Theorem 3.2.6. *The homology inclusion map $i^*_{\mathcal{A}} \circ \gamma^*_{\mathbf{B}}$ admits the following value for every edge $e_{L_i,P_l} \in E \setminus \mathcal{T}^\theta$ on the associated graph cycle:*

$$i^*_{\mathcal{A}} \circ \gamma^*_{\mathbf{B}}(c_e) = \eta \circ \lambda_{m(P_l),i}(\tau_l) + h_{0,\mathbf{B}}(e)$$

where $h_{0,\mathbf{B}}(e) \in V(\Gamma)$ only depends on the braid at infinity β_0 of the representative $\mathbf{B}$ of the braid monodromy of $\mathcal{A}$. ◁

Lemma 3.2.7. *Let $e_{L_i,P_l} \in E \setminus \mathcal{T}^\theta$. Let $P_a := L_{n_l} \cap L_0$ and $P_b := L_i \cap L_0$. Then the corresponding cycle c_e drawn on Γ is made of exactly six edges:*

$$e_{P_l,L_i} + e_{L_i,P_b} + e_{P_b,L_0} + e_{L_0,P_a} + e_{P_a,L_{n_l}} + e_{L_{n_l},P_l} \quad ◁$$

Proof of Theorem 3.2.6. Fix an edge $e_{L_i,P_l} \in E \setminus \mathcal{T}^\theta$. We reuse once more the notations from Sections 2.3.3 and 3.1. Recall that α is the star on the basis Σ of π (see Figure 2.4.1 on page 47). One can add an extra branch α^0 joining the base point b^∞ to ∂D^0 as explained in the proof of Proposition 2.4.13 on page 48. The $\mathcal{T}^\theta$-cycle curve γ_e, which is the image of the cycle $c_e \in H_1(\Gamma)$ by the ordered graphed embedding $\gamma_{\mathbf{B}}$ borders a disc D_e in the exterior manifold $E_{\mathcal{A}}$. This disc can be decomposed into five parts as shown on Figure 3.2.5. From left to right:

- The first part of the surface is contained within the vertical section s_{L_0}.
- The second part is a vertical surface contained within $\pi^{-1}(b^0)$ that links the branch $\alpha^i_{P_a}$ in the section s_{P_a} and the branch $\alpha^{n_l}_{P_b}$ in the section s_{P_b}.
- The third part corresponds to the braid-linking surface $S_{n_l,i}(\tau_0)$ of the shift braid τ_0 over the branch α^0. As explained in the proof of Proposition 2.4.13, τ_0 can always be chosen as the trivial braid.
- The fourth part corresponds to the braid-linking surface $S_{n_l,i}(\tau_l)$ of the shift braid τ_l over the branch α^l. Indeed, the set $\pi^{-1}(\alpha^l)$ can be identified as $\mathbb{C} \times [0,1]$. The path $S_{n_l,i}(\beta_l, 0)$ drawn on $\Sigma(b^\infty)$ joins the two points $\gamma(v_{L_i}) = \gamma_{\mathbf{B}} \cap T^\infty_{L_i}$ and $\gamma(v_{L_{n_l}}) = \gamma_{\mathbf{B}} \cap T^\infty_{L_{n_l}}$. On the other end, the path $S_{n_l,i}(\beta_l, 1)$ drawn on $\Sigma(b^l)$ joins the two points $b^{P_l}_{L_i} := \gamma_{\mathbf{B}} \cap T^{\text{aff}}_{L_i,P_l}$ and $b^{P_l}_{L_{n_l}} := \gamma_{\mathbf{B}} \cap T^{\text{aff}}_{L_{n_l},P_l}$.
- The fifth part is contained within the vertical section s_{P_l}.

Now we evaluate what are the contributions of each part to the homological value of the boundary.

- The first and fifth part are contained within sections of a line component circle bundle. This means they do not meet other line components and therefore do not contribute to the value.
- The second part's contribution corresponds to $h_{0,\mathbf{B}}(e)$. It is entirely determined by the local orderings around the line vertex v_{L_0} and the singular points vertices v_P for $P \in V_{L_0}$. This combinatorial information is contained within the braid at infinity by Proposition 2.4.13 on page 48.

- The third part is a braid linking surface of a trivial braid, which does not meet other components by construction and do not contribute to the value.

The morphism

$$\eta : C_0(\Gamma) \simeq H_1(\mathbb{C}(\tau_l)) \longrightarrow V(\Gamma)$$

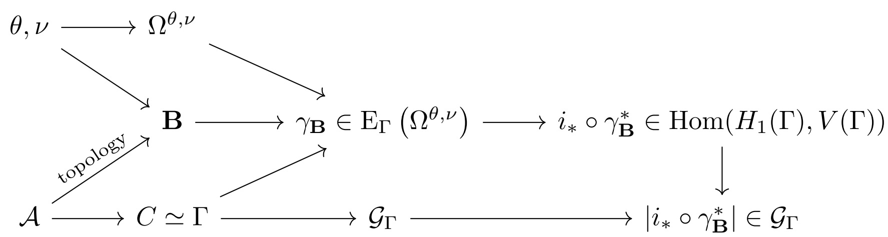
corresponds to the induced morphism in homology of the inclusion

$$\pi^{-1}(\alpha^l) \simeq \mathbb{C}(\tau_l) \hookrightarrow E_{\mathcal{A}}$$

The fourth part's contribution can then be seen as the image by η of the homological value of the braid linking surface $S_{n_l,i}(\tau_l)$ inside $\mathbb{C} \times [0,1] \simeq \pi^{-1}(\alpha^l)$, which is precisely equal to $\lambda_{m(P_l),i}(\tau_l)$ by Definition 3.2.2. □

3.3 Comparison of a combinatorial pair

The following diagram sums up the combinatorial dependencies of all the objects involved in the computation of the homology inclusion.



From now on we consider $(\mathcal{A}, \mathcal{A}')$ a combinatorial pair of line arrangements with the same combinatorics C.

3.3.1 Adjustments of the orderings

As noted in Remark 2.5.3 on page 53, the individual value of the homology inclusion of a line arrangement depends on the choice of $\Omega^{\theta,\nu}$ and therefore θ, ν. However, the difference between the two values on the combinatorial pair does not as long as the parameters θ, ν are the same in both computations. In this situation the difference is only determined by the topological types of both line arrangements.

In practice the computational algorithms used to obtain the braid monodromy representative $\mathbf{B}$ of a line arrangement $\mathcal{A}$ take $\nu = \nu_\pi$ to simplify computations, and π itself might be chosen randomly. Moreover, these algorithms do not decompose the individual star braid monodromies β_l, and do not explicitly give the orderings θ, ν associated with $\mathbf{B}$. We will thus use Proposition 2.4.16 on page 50 to modify *a posteriori* the value of θ and ν, once they are determined.

E. Artal has written a function in Sage [Sag23] that provides a braid monodromy representative $\mathbf{B}^{(0)}$ of a line arrangement $\mathcal{A}$ as well as the associated initial ordering $\theta^{(0)} : \mathcal{L}^* \to \{1, \ldots, n\}$.

Algorithm 3.3.1: Raw braid monodromy computation

Data: Defining polynomial of the affine arrangement $\mathcal{A}^{\text{aff}}$
Result: Braid monodromy representative $\mathbf{B}^{(0)} = (\beta_l^{(0)})_{1 \leq l \leq r}$
Result: Ordering $\theta^{(0)}$

In the following we describe the algorithms to process the braid monodromy representative $\mathbf{B}^{(0)}$ to prepare it for the final comparison. First we conjugate all star braid monodromies $\beta_l^{(0)}$ to get the desired ordering θ.

Algorithm 3.3.2: Adjustment of the ordering θ

Data: Braid monodromy representative $\mathbf{B}^{(0)}$
Data: Desired ordering θ
Result: New braid monodromy representative $\mathbf{B}^{(1)}$ with $\theta^{(1)} = \theta$

The next algorithm uses a method due to N. Franco and J. González Meneses [FG03] to compute a decomposition of $\beta_l^{(1)} \in \mathbb{P}_n$ of the form

$$\beta_l^{(1)} = \tau_l^{(1)} \cdot \Delta^2_{\theta^l(V_{P_l})} \cdot \left(\tau_l^{(1)}\right)^{-1}$$

Algorithm 3.3.3: Star braid assignation
Data: Star braid monodromy $\beta_l^{(1)}$ **Result:** Central full twist $\Delta^2_{\theta^l(V_{P_l})}$ **Result:** Shift braid $\tau_l^{(1)}$

In fact this first decomposition is done only to associate each braid $\beta_l^{(1)}$ with its corresponding singular point $P \in \mathcal{Q}^*$. Since the braids $(\beta_l^{(1)})$ are already ordered, this also gives the initial ordering ν_π associated with $\mathbf{B}^0$. The next algorithm applies Hurwitz moves on the representative $\mathbf{B}^{(1)}$ to get the desired ordering $\nu : \mathcal{Q}^* \to \{1, \dots, r\}$.

Algorithm 3.3.4: Adjustment of the ordering ν
Data: Braid monodromy representative $\mathbf{B}^{(1)}$ **Data:** Desired ordering ν **Result:** New braid monodromy representative $\mathbf{B}^{(2)}$ with $\nu_\pi = \nu$

We now compute the new shift braids $\tau_l^{(2)}$ by performing a second decomposition.

Algorithm 3.3.5: Star braid decomposition
Data: Star braid monodromy $\beta_l^{(2)}$ **Result:** Central full twist $\Delta^2_{\theta^l(V_{P_l})}$ **Result:** Shift braid $\tau_l^{(2)}$

There is no guarantee that the ordering θ^l coincides with the desired local ordering $\theta_{|V_{P_l}}$. The next step will multiply the shift braids on the right with a (non-full) twist braid to adjust the local ordering. This does not affect $\beta_l^{(2)}$ since the central full twist can be conjugated by the inverse twist braid.

Algorithm 3.3.6: Adjustment of the local ordering θ^l
Data: Shift braid $\tau_l^{(2)}$ **Data:** Desired ordering θ **Result:** New shift braid $\tau_l^{(3)}$ with $\theta^l = \theta_{|V_{P_l}}$

Algorithms 3.3.1 to 3.3.6 must be repeated for the other line arrangement $\mathcal{A}'$, thus giving a second set of shift braids $\mathbf{T}^{(3)'}$.

The braid at infinity (see Proposition 2.4.13 on page 48) encodes all the local orderings for v_{L_0} and every v_P for $P \in V_{L_0}$. Individual adjustment of each of these local orderings is not required. Instead it is enough to make sure that the two braids at infinity b^0 and $b^{0'} \in \mathbb{P}_n$ of the combinatorial pair are equal. This is done by finding a conjugating braid and then multiplying all shift braids $\mathbf{T}^{(3)'}$ of $\mathcal{A}'$ on the left. The set of shift braids $\mathbf{T}^{(3)}$ of $\mathcal{A}$ is not modified.

Algorithm 3.3.7: Adjustment of the infinity braids
Data: Decomposed representative $\mathbf{B}^{(3)}$ **Data:** Decomposed representative $\mathbf{B}^{(3)'}$ **Result:** New decomposed representative $\mathbf{B}^{(4)'}$ with $\beta_0^{(4)'} = \beta_0^{(3)}$

***Remark* 3.3.1.** The Hurwitz moves used on the braids of the representative to change ν are computationally expensive because they require two separate decompositions, with the conjugation performed in-between by Algorithm 3.3.4 significantly increasing the length of the braids $(\beta_l^{(1)})$. This explains why we choose to define the standard graphed embedding $\gamma_{\mathbf{B}}$ using ν as an independent parameter, in order to minimise the number of such moves required to reach a common value. ⌑

3.3.2 Comparison algorithm

The following procedure presents all the steps to compare the homology inclusion values of a combinatorial pair.

Algorithm 3.3.8: Comparison of a combinatorial pair

Data: Combinatorial pair $(\mathcal{A}, \mathcal{A}')$
Data: Orderings θ, ν
Perform Algorithms 3.3.1 to 3.3.6 on $\mathcal{A}$ and $\mathcal{A}'$ to get $\mathbf{B}^{(3)}$ and $\mathbf{B}^{(3)'}$.
Perform Algorithm 3.3.7 on $\mathbf{B}^{(3)}$ and $\mathbf{B}^{(3)'}$ to get $\mathbf{B}^{(4)'}$.
`// Shift braids are now ready.`
Compute the full incidence graph Γ.
Compute the graph stabiliser $\mathcal{G}_\Gamma$ and the projection map using Theorem 1.6.11.
`// Graph stabiliser is now ready.`
forall $e_{L_i,P_l} \in E \setminus \mathcal{T}^\theta$ **do**
  Compute the values $\lambda_{m(P_l),i}(\tau_l^{(3)})$ and $\lambda_{m(P_l),i}(\tau_l^{(4)'})$ using Proposition 3.2.4.
  Compute $i_* \circ \gamma^*_{\mathbf{B}^{(3)}}$ and $i_* \circ \gamma^*_{\mathbf{B}^{(4)'}})(c_e)$ using Theorem 3.2.6.
  Take the image by the projection map.
end
Combine the images.
Result: $\left|i_* \circ \gamma^*_{\mathbf{B}^{(3)}}\right| - \left|i_* \circ \gamma^*_{\mathbf{B}^{(4)'}}\right| \in \mathcal{G}_\Gamma$

Note that the graph stabiliser $\mathcal{G}_\Gamma$ of a combinatorics is a free abelian group since it is a quotient of the homomorphism group $H_1(H_1(\Gamma), V(\Gamma))$ between two free abelian groups. As a $\mathbb{Z}$-module it admits a decomposition as a product of cyclic groups in an adequate basis, the so-called *Smith normal form*. It is this basis that we use to actually compare the two values of the homology inclusions.

3.3.3 Examples of Zariski pairs

We now give several examples of Zariski pairs which are identified by the homology inclusion.

Example 3.3.2 (MacLane arrangements)**.** The following lexicographically ordered combinatorics were discovered by S. MacLane [Mac36]:

$$[[0,1,2],[0,3,4],[0,5,6],[0,7],[1,3],[1,5,7],[1,4,6],[2,3,5],[2,4,7],[2,6],[3,6,7],[4,5]]$$

The automorphism group of the MacLane combinatorics is isomorphic to $\mathrm{GL}_2(\mathbb{F}_3)$. It is known (see [Bjö+99]) that this constitutes the smallest possible combinatorics that does not admit a realisation in $\mathbb{RP}^2$. It admits two conjugated realisations M^+ and M^- with 8 lines. The defining (ordered) equations in $\mathbb{CP}^2$ are

$$\begin{aligned}
&L_0 : 0 = z && L_3 : 0 = y && L_6 : 0 = -x - \omega^2 y + z \\
&L_1 : 0 = -x + z && L_4 : 0 = \omega^2 x + \omega y + z && L_7 : 0 = \omega y + z \\
&L_2 : 0 = x && L_5 : 0 = -x + y &&
\end{aligned}$$

where $\omega = e^{\frac{2i\pi}{3}}$ for M and $\omega = e^{-\frac{2i\pi}{3}}$ for $\overline{M}$. Each of the combinatorics automorphisms can be realised as projective automorphisms of $\mathbb{CP}^2$. The ones in $\mathrm{SL}_2(\mathbb{F}_3)$ preserve the order. We now give the results of the computations of the graph stabiliser and the homology inclusion of M and $\overline{M}$ using Algorithm 3.3.8.

The Smith normal form of the graph stabiliser is

$$\mathcal{G}_\Gamma \simeq \mathbb{Z}/_{3\mathbb{Z}} \times \mathbb{Z}^{35}$$

The two values of the homology inclusion in the corresponding basis are given in Figure 3.3.1.

The difference is non-zero in the torsion part of $\mathcal{G}_\Gamma$. The line arrangements $(M, \overline{M})$ therefore form an ordered oriented Zariski pair (see Remark 2.5.4 on page 53).

◇

$|i_* \circ \gamma^*_{\mathbf{B}}|$: $\bar{0}$ 0 1 2 0 0 0 1 0 1 0 1 2 0 1 0 1 1 0 1 1 1 0 0 -1 1 -1 -1 0 1 0 0 0 0 1 0

$\left|i_* \circ \gamma^*_{\overline{\mathbf{B}}}\right|$: $\bar{2}$ 0 1 2 0 0 0 1 0 1 0 1 2 0 1 0 1 1 0 1 1 1 0 0 -1 1 -1 -1 0 1 0 0 0 0 1 0

Difference: $\bar{2}$ 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

Figure 3.3.1: Homology inclusion values of the MacLane arrangements

Example 3.3.3 (New Zariski quadruplet)**.** Consider the polynomial

$$P = X^4 + 2X^3 + 4X^2 + 3X + 1$$

and the following equations given by

$L_0 : 0 = z$

$L_1 : 0 = \omega^2 x - y - \omega(\omega+1)z$

L_2: $0 = (3\omega^2 + 3\omega + 1)x + (\omega+1)^2 y - (\omega^3 + 5\omega^2 + 5\omega + 2)z$

$L_3 : 0 = \omega(\omega^2 + \omega + 1)x + y + \omega(\omega+1)z$

$L_4 : 0 = \omega x + y$

$L_5 : 0 = \omega x + y - (\omega+1)z$

$L_6 : 0 = x$

$L_7 : 0 = x - z$

$L_8 : 0 = y$

$L_9 : 0 = y - z$

$L_{10} : 0 = y + \omega(\omega^2 + 2\omega + 2)z$

where $\omega = -\frac{1}{2} \pm \frac{1}{2} i\sqrt{5 \pm 2\sqrt{5}}$ take the values of the four roots of P. This defines four conjugated arrangements with 11 lines $B_1, B_2, \overline{B_1}, \overline{B_2}$ whose common ordered combinatorics is given by

[[0, 1, 2], [0, 3], [0, 4, 5], [0, 6, 7], [0, 8, 9, 10], [1, 3, 6], [1, 4, 7], [1, 5, 8], [1, 9], [1, 10], [2, 3, 5], [2, 4], [2, 6, 10], [2, 7], [2, 8], [2, 9], [3, 4, 9], [3, 7, 10], [3, 8], [4, 6, 8], [4, 10], [5, 6], [5, 7, 9], [5, 10], [6, 9], [7, 8]]

The Smith normal form of the graph stabiliser is:

$$\mathcal{G}_\Gamma \simeq \mathbb{Z}/_{5\mathbb{Z}} \times \mathbb{Z}^{119}$$

The values of the homology inclusion of the four realisations are identical on the free part but differ on the torsion part=

$B_1 : \bar{1}$ $B_2 : \bar{4}$ $\overline{B_1} : \bar{3}$ $\overline{B_2} : \bar{0}$

The automorphism group of the combinatorics is trivial, which means that the four realisations form an unordered oriented Zariski quadruplet. The two pairs (B_1, B_2) and $(B_1, \overline{B_2})$ and their respective conjugates are unordered unoriented Zariski pairs. ○

Example 3.3.4 (Rybnikov quadruplet)**.** This is the first Zariski pair of line arrangements identified by G. Rybnikov in [Ryb11]. Consider the equations:

$L_0 : 0 = z$

$L_1 : 0 = x$

$L_2 : 0 = x - z$

$L_3 : 0 = y$

$L_4 : 0 = y - z$

$L_5 : 0 = -x + \omega y + z$

$L_6 : 0 = -(\omega+1)x - y + z$

$L_7 : 0 = -(\omega+1)x + \omega x + z$

$L_8 : 0 = -4\eta x + 2\eta y + z$

$L_9 : 0 = -2x + y + 3z$

$L_{10} : 0 = (1 - 5\eta)x + 2\eta y + \eta z$

$L_{11} : 0 = (1 - 5\eta)x + 2\eta + 6\eta z$

$L_{12} : 0 = x + 2y + z$

where

$$(\omega, \eta) = \left(e^{\frac{2i\pi}{3}}, \frac{5\omega + 6}{31}\right) \text{ or } \left(e^{\frac{2i\pi}{3}}, \frac{5\bar{\omega} + 6}{31}\right)$$

This defines four conjugated line arrangements with 13 lines $R_1, R_2, \overline{R_1}, \overline{R_2}$, whose common ordered combinatorics is given by

[[0, 1, 2], [0, 3, 4], [0, 5, 6], [0, 7], [0, 8, 9], [0, 10, 10], [0, 11], [1, 3], [1, 4, 6], [1, 5, 7], [1, 8], [1, 9, 10], [1, 10, 11], [2, 3, 5], [2, 4, 7], [2, 6], [2, 8, 10], [2, 9, 11], [2, 10], [3, 6, 7], [3, 8], [3, 9], [3, 10], [3, 11], [4, 5], [4, 8], [4, 9], [4, 10], [4, 10], [4, 11], [5, 8], [5, 9], [5, 10], [5, 10], [5, 11], [6, 8], [6, 9], [6, 10], [6, 10], [6, 11], [7, 8], [7, 9], [7, 10], [7, 10], [7, 11], [8, 10, 11], [9, 10]] The Smith normal form of the graph stabiliser is:

$$\mathcal{G}_\Gamma \simeq \left(\mathbb{Z}/3\mathbb{Z}\right)^2 \times \mathbb{Z}^{219}$$

The homology inclusion values differ not only on the torsion part but also on the free abelian part=

$$R_1 : (\bar{1}, \bar{1}, f_1) \qquad \overline{R_1} : (\bar{1}, \bar{0}, f_1) \qquad R_2 : (\bar{0}, \bar{0}, f_2) \qquad \overline{R_2} : (\bar{0}, \bar{1}, f_2)$$

where $f_1, f_2 \in \mathbb{Z}^{219}$. This means that the four realisations $(R_1, R_2, \overline{R_1}, \overline{R_2})$ form an ordered oriented Zariski quadruplet. Moreover, the two pairs (R_1, R_2) and $(R_1, \overline{R_2})$ and their respective conjugates are unoriented ordered Zariski pairs. ○

***Remark* 3.3.5.** For all the Zariski pairs we obtain, the graph stabiliser contains a torsion part (this is not always the case) and the value of the difference of the homology inclusions lies at least partly in the torsion part of the graph stabiliser. B. Guerville-Ballé has conjectured that only Zariski pairs whose graph stabiliser contains a torsion part can be distinguished by the homology inclusion itself. Even if this is true, a torsion-free graph stabiliser could nevertheless be used as an intermediary step to compute the *twisted* homology inclusion values. This new difference value might be able to detect the evading Zariski pairs. ⌑

PART II

SLOPE INVARIANTS OF LINKS

CHAPTER

4

CHARACTER SLOPE OF LINKS

Contents

4.1 Preliminaries on link theory

4.1.1 Generalities

Let S^3 be the oriented 3-sphere.

Definition 4.1.1. A *knot* is an embedding $K : S^1 \hookrightarrow S^3$ such that $K(S^1)$ is a simple polygonal curve.

A *link* is an embedding

$$L : \bigsqcup_{i=1}^{n} L_i \hookrightarrow S^3$$

of several components $L_1, \dots, L_n$ whose images are disjoint simple polygonal curves. ◇

By extension, we identify the embeddings with their images and call them link and knot respectively.

Definition 4.1.2. Let L and L' be two links. An *ambient isotopy* between L and L' is an application:

$$\begin{aligned} G : \quad S^3 \times [0,1] &\longrightarrow S^3 \\ (x,t) &\longmapsto g_t(x) \end{aligned}$$

such that G is a piecewise linear homeomorphism of $S^3 \times [0,1]$ into itself, with $g_0 = \mathrm{Id}_{S^3}$ and $g_1(L) = L'$. ◇

Definition 4.1.3. Two links L and L' are said to be *equivalent* if there exists an ambient isotopy of S^3 that sends L to L'. ◇

The name 'link' now designate an equivalence class of links. For the remainder of this section, 'link' also include knots (i.e. $n = 1$) unless specified otherwise.

Definition 4.1.4. A *link orientation* is a function

$$\delta_L : (L_1, \dots, L_n) \longrightarrow \{\pm 1\}^n$$

which assigns an orientation to every link component. ◇

Definition 4.1.5. A regular link projection is an application $p : L \to \mathbb{R}^2$ such that for every point x of the *link diagram* $p(\mathcal{E})$, the fibre $p^{-1}(x)$ has no more than two element, and has exactly two elements for a finite number of point in the diagram. ◇

The points of the plane which have exactly two antecedents are called *crossings*. Each crossing has an *upper* and *lower* antecedent in the link relatively to the projection p. The link L is therefore divided in *strands* which are continuous paths in L which joins two lower points.

Proposition 4.1.6. *Two links are equivalent if and only if their respective digrams obtained by the same regular projection on $\mathbb{R}^2$ can be transformed into each other by a finite series of Reidemeister moves.* ◁

Definition 4.1.7. Let $1 \le \mu \le n$. A μ-coloured link is an oriented link in S^3 equipped with a surjective map

$$c : \{1, \dots, n\} \longrightarrow \{1, \dots, \mu\}$$ ◇

Let $\chi := c^{-1}$ such that $\xi(j) \subset \{1, \dots, n\}$ for $j \in \{1, \dots, \mu\}$. We write $L_{\chi(j)}$ for the union of individual components L_i of the link L that are j-coloured.

Definition 4.1.8. Two μ-coloured links L^0 and L^1 are *concordant* if there exists a collection of properly embedded disjoint locally flat cylinders $A := A_1 \sqcup \cdots \sqcup A_\mu$ such that

$$\partial A_j \cap (S^3 \times 0) = -L^0_{\chi(j)} \text{ and } \partial A_j \cap (S^3 \times 1) = L^1_{\chi(j)}$$

for all $1 \le j \le \mu$. A n-coloured link concordant to the unlink is called *slice*. ◇

4.1.2 Link group

Definition 4.1.9. Let $K \subset S^3$ be a knot. A *tubular neighbourhood* $T(K)$ is a regular neighbourhood of K which is homeomorphic to a solid torus $K \times D^2$. The *exterior* of the knot M_K is defined as

$$M_K := S^3 \smallsetminus \widehat{\mathring{T}(K)}$$ ◇

This definition is naturally extended to links. Let b be a base point in ∂M_L. The *link group* π_L is the fundamental group $\pi_1(M_L, b)$.

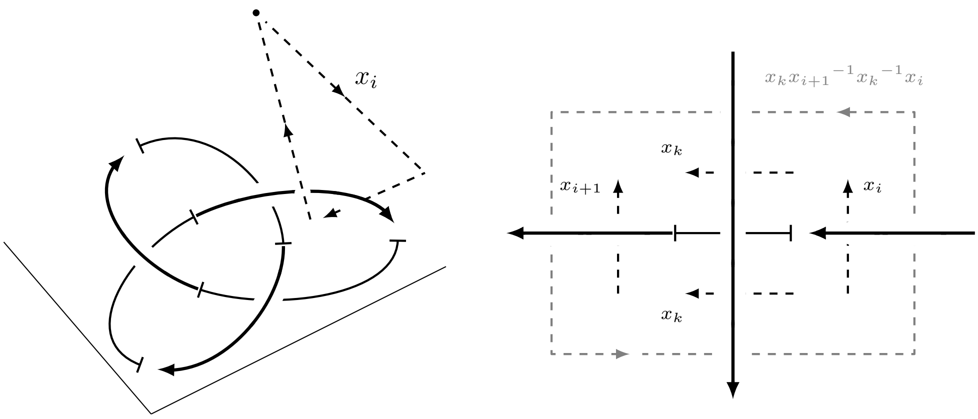
Definition 4.1.10. A *meridian* of a knot K is a curve m whose class is null in $H_1(T(K))$ but not in $H_1(M_L)$. A *longitude* of a knot K is a curve ℓ is the intersection of $T(K)$ with an orientable oriented surface S embedded in S^3 with $\partial S = K$. ◇

Since we mostly work inside the fundamental group of either M_L or $\partial T(K)$, the terms 'meridian' and 'longitude' almost always designates the homotopy classes of these two types of curves.

Definition 4.1.11. The *peripheral system* of a knot K is the triple (π_L, ℓ, m) where ℓ and m are the homotopy classes of a longitude and meridian of K respectively such that $m \cdot \ell = \ell \cdot m$. These are called the *preferred* longitude and meridians of K. The pair (m, ℓ) is unique up to conjugation of a common element in π_K. ◇

Theorem 4.1.12 (Waldhausen)**.** *Two knots K, K' are equivalent if and only if there exists a group isomorphism $\varphi : \pi_K \to \pi_{K'}$ such that $(\varphi(m), \varphi(\ell)) = (m', \ell')$.* ◁

Definition 4.1.13. The *Wirtinger presentation* of the oriented link group π_L is obtained from an link diagram of L in the following way: every strand is associated with a meridian generator as shown on Figure 4.1.1a. Every crossing of the diagram is associated with the relation as shown on Figure 4.1.1b. The *extended Wirtinger presentation* adds a peripheral couple (ℓ_i, m_i) of preferred longitude and meridian for each knot component L_i of the link L. ◇



(a) Generating meridian (b) Crossing relation

Figure 4.1.1: Wirtinger presentation of the link group

The abelianisation map

$$\mathrm{Ab} : \pi_L \longrightarrow H_1(M_L)$$

induces a group isomorphism between $H_1(M_L)$ and the free abelian group generated by a meridian for each link component. The preferred longitudes are all sent to the identity element 1.

Definition 4.1.14. Let K, K' be two knot components embedded in the same S^3. Then $\mathrm{lk}(K, K')$ is defined as the class of the curve ℓ_K inside $H_1(E_{K'}) \simeq \mathbb{Z}$. ◇

4.2 Definition of the character slope

The construction presented in the section was developed by A. Degtyarev, V. Florens and A.G. Lecuona in [DFL22b; DFL21; DFL22a]. It is the basis of the generalisations that we will present in the next chapters.

Let L be a μ-coloured link. We denote by (m_i, ℓ_i) the peripheral pairs of each component of L. The group $H_1(E_L)$ is free abelian and generated by the classes of the meridians $m_1, \ldots, m_\mu$. All meridian generators of a component L_i are conjugated to each other. A character of the link group $\omega : \pi_1(E_L) \to \mathbb{C}^*$ is therefore determined by its values on the preferred meridian of each component and can then be seen as an element

$$(\omega(m_1), \dots, \omega(m_\mu)) \in (\mathbb{C}^*)^\mu$$

with $\omega(m_i) = \omega(m_j)$ if L_i and L_j have the same colouring. We define

$$\omega^{-1} := \left(\omega_1{}^{-1}, \dots, \omega_\mu{}^{-1}\right) \qquad \overline{\omega} := (\overline{\omega_1}, \dots, \overline{\omega_\mu}) \qquad \omega^\dagger := (\overline{\omega})^{-1}$$

A character ω is called *unitary* if $\omega^\dagger = \overline{\omega}$.

4.2.1 Twisted homology of the boundary

Let $K \cup L$ be a $(1, \mu)$-coloured link where $L = L_1 \cup \cdots \cup L_\mu$ is a sublink and K is a special component called *distinguished*. We use the convention that $L_0 = K$.

Definition 4.2.1. A character $\omega \in (\mathbb{C}^*)^{\mu+1}$ is called *admissible* if $\omega(m_0) = 1$ and *non-vanishing* if $\omega(m_i) \neq 1$ for every $1 \leq i \leq \mu$. The variety of admissible characters is denoted by $\mathcal{A}(K/L)$. The subvariety of non-vanishing admissible characters is denoted by $\mathcal{A}^\circ(K/L)$. ◇

We are interested in the restriction of the character ω to $H_1(\partial E_K)$ seen as a subgroup of $H_1(E_L)$. The subgroup $H_1(\partial E_K)$ is free abelian generated by the classes of m_0 and ℓ_0. By Definition 4.1.14, we have

$$\forall\, i \in \{1, \dots, \mu\} : \quad \omega(\ell_0) = \omega(m_i)^{\mathrm{lk}(K, L_i)} \tag{4.1}$$

From now on, we suppose that the linking numbers $\mathrm{lk}(K, L_i)$ are zero for every $1 \leq i \leq \mu$. This means that $\omega(\ell_0) = 1$ and

$$\mathcal{A}(K/L) = (\mathbb{C}^*)^\mu \qquad \mathcal{A}^\circ(K/L) = (\mathbb{C}^* \setminus \{1\})^\mu$$

Proposition 4.2.2. *There is a natural vector space isomorphism*

$$H_1(\partial E_L; \omega) \simeq \bigoplus_{\omega(m_i) = 1} \langle \ell_i, m_i \rangle \otimes \mathbb{C}(\omega) \qquad \triangleleft$$

Proof. We have the decomposition

$$H_1(\partial E_L; \omega) \simeq \bigoplus_{\omega(m_i) \neq 1} H_1(\partial E_{L_i}; \omega) \quad \oplus \quad \bigoplus_{\omega(m_i) = 1} H_1(\partial E_{L_i}; \omega)$$

For the components L_i where $\omega(m_i) - 1 \neq 0$, Corollary A.2.2 gives $H_1(\partial E_{L_i}; \omega) = \{0\}$. For the components L_i where $\omega(m_i) = 1$, Corollary A.2.3 gives a natural isomorphism

$$H_1(\partial E_{L_i}; \omega) \simeq \langle \ell_i, m_i \rangle \otimes \mathbb{C}(\omega) \qquad \square$$

Corollary 4.2.3. *If $\omega \in \mathcal{A}^\circ(K/L)$ then $H_1(\partial E_L; \omega)$ is a $\mathbb{C}$-vector space of dimension 2 which admits the canonical basis (ℓ_0, m_0).* ◁

4.2.2 Definition of the character slope

Consider the inclusion map

$$i : \partial M_{K \cup L} \longrightarrow M_{K \cup L}$$

and the induced map on the first twisted homology groups of $(M_{K \cup L}, \omega)$:

$$i_* : H_1(\partial M_{K \cup L}; \omega) \longrightarrow H_1(M_{K \cup L}; \omega)$$

We consider the space $\mathcal{Z}(\omega) := \ker i_*$.

Definition 4.2.4. Let $\omega \in \mathcal{A}^\circ(K/L)$ and suppose that $\dim \mathcal{Z}(\omega) = 1$. The space $\mathcal{Z}(\omega)$ is generated by a vector of $H_1(\partial M_{K \cup L}; \omega)$ of the form

$$\mathcal{Z}(\omega) = \langle a \cdot \ell + b \cdot m \rangle$$

with $[a : b] \in \mathbb{CP}^1$. The (K/L)-slope is defined by the formula

$$s_{(K/L)}(\omega) := -\frac{b}{a} \in \mathbb{C} \cup \{\infty\} \qquad \diamond$$

Proposition 4.2.5. *If the slope $s_{(K/L)}(\omega)$ is well defined for $\omega \in \mathcal{A}(K/L)$, then so are the slopes for $\overline{\omega}$ and $\omega^\dagger$, and one has*

$$s_{(K/L)}(\omega^\dagger) = s_{(K/L)}(\omega) \qquad s_{(K/L)}(\overline{\omega}) = \overline{s_{(K/L)}(\omega)} \qquad \triangleleft$$

Proof. These results are immediate consequences of Poincaré duality as stated in Lemma A.1.5 on page 104. □

Proposition 4.2.6. *If ω is unitary then the slope is well-defined and*

$$s_{(K/L)}(\omega) \in \mathbb{R} \cup \{\infty\} \qquad \triangleleft$$

Proof. The slope is well-defined as a direct application of Theorems A.1.8 and A.1.9 on page 104 and on page 105. The slope is real because of Proposition 4.2.5. □

4.2.3 Properties of the slope

We recall some of the known properties of the slope. The proofs can be found in [DFL22b; DFL21; DFL22a].

Definition 4.2.7. The *characteristic varieties* of the $(1, \mu)$-coloured link L are the jump loci

$$\mathcal{V}_r(L) := \{\omega \in (\mathbb{C}^*)^\mu \mid \dim H_1(M_L; \omega) \geq r\}$$

They are nested algebraic sub-varieties of the character torus:

$$(\mathbb{C}^*)^\mu = \mathcal{V}_0(L) \supset \mathcal{V}_1(L) \supset \cdots$$

A characteristic variety is called *proper* if $\mathcal{V}_r(L) \neq (\mathbb{C}^*)^\mu$. The first proper characteristic variety is denoted by $\mathcal{V}_{\max}(L)$. ◇

Theorem 4.2.8 ([DFL22b, Theorems 3.19 and 3.21])**.** *If $\operatorname{lk}(K, L) = 0$, the slope is a rational function, possibly identical to $\{\infty\}$, on $\mathcal{A}^\circ(K/L) \smallsetminus \mathrm{V}_{\max}(L)$. If $\mathrm{V}_{\max}(L) = \mathrm{V}_1(L)$, one has*

$$s_{K/L}(\omega) = -\frac{\nabla'(1, \sqrt{\omega})}{2\nabla_L(\sqrt{\omega})} \in \mathbb{C} \cup \{\infty\}$$

where ∇' is the derivative of the Conway polynomial $\nabla_{K\cup L}(t, \cdot)$ with respect to t. ◁

Definition 4.2.9. Consider the subset of Laurent polynomials defined by

$$U := \left\{P \in \mathbb{Z}\left[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}\right] \mid P(1, \ldots, 1) = \pm 1\right\}$$

An element $\omega \in \mathcal{A}(K/L)$ is called a *concordance root* if there is a polynomial $P \in U$ such that $P(\omega) = 0$. The set of *non*-concordance roots is denoted by $\mathcal{A}_c(K/L)$ and the subset of non-vanishing non-concordance roots by $\mathcal{A}_c^\circ(K/L)$. ◇

Theorem 4.2.10 ([DFL22a, Theorem 3.2])**.** *Let $K \cup L^0$ and $K \cup L^1$ be two concordant $(1, \mu)$-coloured links. Then*

$$\mathcal{A}_c(K^0/L^0) = \mathcal{A}_c(K^1/L^1)$$

and the slope functions $s_{(K^0/L^0)}(\omega)$ and $s_{(K^1/L^1)}(\omega)$ are equal on $\mathcal{A}(K/L)$. ◁

Corollary 4.2.11. *If $K \cup L$ is slice then $s_{K/L}(\omega) = 0$ for every $\omega \in \mathcal{A}_c(K/L)$.* ◁

4.3 Slope computation

The slope computation is done using Fox calculus, whose definition is recalled in Appendix A.3. Let $K \cup L$ be a $(1, \mu)$-coloured link and let $\omega \in \mathcal{A}^\circ(K/L)$ be a non-vanishing admissible character.

Theorem 4.3.1. *Let $\mathcal{P}$ be a presentation of the link group $\pi_1(E_{K\cup L})$ that contains m_K and ℓ_K. Let A^ω be the twisted Alexander matrix of $M_{K\cup L}$ associated with ω and $\mathcal{P}$. Let A^ω_K be the sub-matrix of A^ω containing only the columns associated with ℓ_K and m_K, and let A^ω_C be its complementary sub-matrix. Then*

$$A^\omega_K \cdot (\ker A^\omega_C)$$

is a generating matrix of $\ker i_*$ *and in particular has rank 1. If v is a generating vector of the row-space with*

$$v = a \cdot d\ell_k + b \cdot dm_K$$

then $s_{K/L}(\omega) = -\frac{b}{a}$. ◁

Proof. This is a direct application of Theorem A.4.1 since $M_{K\cup L}$ and $\omega \in \mathcal{A}^\circ(K/L)$ verify Eq. (A.2). □

The algorithm used to compute the slope, which is written in `GAP` [GAP22]. Theorem 4.3.1 reduces the computation of the slope to Fox calculus and elementary linear algebra. It is however necessary to obtain the extended Wirtinger presentation of the link $K \cup L$ first.

To encode links, we use the oriented DT-code (which stands for C.H. Dowker and M.B. Thistlethwaite [DT83]). This code attributes a number to each strand of the link in-between two crossings, starting from any crossing. The DT strands therefore do not coincide directly with the generators of the Wirtinger presentation, since over-strands are counted multiple times. The DT code also induces an ordering on the link components, dubbed the *DT order.*

Algorithm 4.3.1: Extended Wirtinger presentation
Data: Oriented DT code of $K \cup L$ **Data:** DT order of the distinguished component K **Result:** Extended Wirtinger presentation of $K \cup L$ with ℓ_K

***Remark* 4.3.2.** It is crucial for all our studies of slope invariants to be able to compute the *extended* Wirtinger presentation, in such a way that the generators are *explicitly* associated with their corresponding strands in the link diagram. It turns out that amongst the many knot theory computer programs available, none meets these precise requirements. We therefore designed our own system to achieve this. It is based on the program `PLink` created by M. Culler and N. Dunfield [CD08], which is a lightweight graphical link editor which allows to draw a link by hand and can also give the oriented DT code of the link with an explicit display of the number assigned to each strand. Our own Algorithm 4.3.1 will then compute the presentation. The initial DT numbers are kept inside the generators' names throughout all our computations, including when simplifications are performed. ⌑

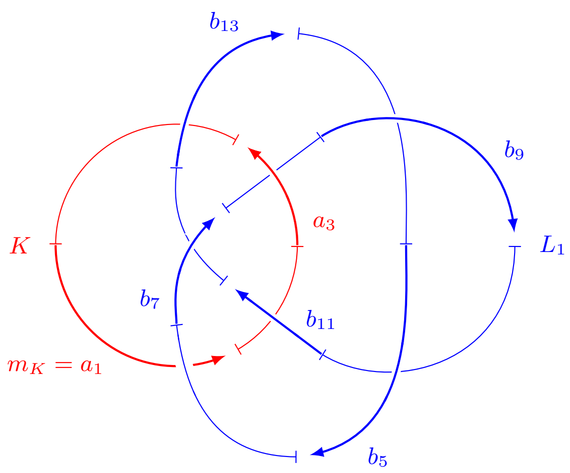


Figure 4.3.1: Link `L7a1`

Example 4.3.3 (`L7a1`)**.** The link `L7a1` represented on Figure 4.3.1 has two components K, L with $\operatorname{lk}(K,L) = 0$. A simplified extended Wirtinger presentation of the link group is given by

$$\pi_1(M_{K\cup L}) = \left\langle b_9, b_{13}, m_K, \ell_K \;\middle|\; \begin{array}{l} r_0 : \ell_K^{-1} b_9^{-1} b_{13}^{-1} b_9 b_{13} b_9 b_{13}^{-1} \\ r_1 : \ell_K b_{13} m_K^{-1} b_9^{-1} b_{13}^{-1} b_9 m_K b_{13}^{-1} m_K^{-1} b_9^{-1} b_{13} b_9 m_K \\ r_2 : [\ell_K, m_K] \end{array} \right\rangle$$

Let ω be a unitary character with $\omega_{|K} = 1$ and $\omega_{|L} = b \in S^1$. The twisted Alexander matrix A^ω

is given by

$$\begin{array}{c} \\ dr_0 \\ dr_1 \\ dr_2 \end{array} \begin{array}{c} \begin{array}{cccc} d\ell_K & dm_K & db_9 & db_{13} \end{array} \\ \begin{bmatrix} -1 & 0 & -b^{-1}\left(1-b^{-1}-b\right) & b^{-1}\left(1-b^{-1}-b\right) \\ 1 & -b+2-b^{-1} & b^{-1}\left(-b+2-b^{-1}\right) & -b^{-1}\left(-b+2-b^{-1}\right) \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

We have

$$\ker A_C^\omega = \begin{array}{c} \begin{array}{ccc} dr_0 & dr_1 & dr_2 \end{array} \\ \begin{bmatrix} 1+\left(1-b^{-1}-b\right)^{-1} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

and

$$A_K^\omega\left(\ker A_C^\omega\right) = \begin{array}{c} \begin{array}{cc} d\ell_K & dm_K \end{array} \\ \begin{bmatrix} -(1-b^{-1}-b)^{-1} & -b+2-b^{-1} \\ 0 & 0 \end{bmatrix} \end{array}$$

The value of the slope is then

$$s_{K/L}(\omega) = (1-b)\left(1-b^{-1}\right)\left(1-\left(b+b^{-1}\right)\right)$$

which is a real number as expected. ◯

The following result is a reformulation using the slope of a property of abelian Fox calculus already observed by R. Crowell [Cro71].

Proposition 4.3.4. *If $K \cup L$ is a boundary link then for any $(1,\mu)$-colouring $K \cup L = K \cup L$, the slope function $s_{(K/L)}$ is identically zero.* ◁

Proof. It is known that if $K \cup L$ is a boundary link, then all its longitudes are commutators of commutators, i.e. for every component K of $K \cup L$ there exist meridians $a, b, c, d \in \pi_1(M_{K\cup L})$ such that

$$\ell_K = [[a,b],[c,d]]$$

An elementary computation shows that inside $\mathbb{Z}[H_1(M_{K\cup L})]$ the derived vector $d\ell_K$ is zero. In particular this is still the case after composition by a character $\omega : \mathbb{Z}[H_1(K\cup L)] \to \mathbb{C}^*$. Therefore

$$d\ell_K \in \operatorname{im}\left(A_K^\omega \cdot (\ker A_C^\omega)\right)$$

By Theorem 4.3.1 this matrix has rank 1 and therefore $d\ell_K$ generates its row-space. Then necessarily $s_{(K/L)}(\omega) = 0$. □

CHAPTER

5

THE $SL_2(\mathbb{C})$-SLOPE OF KNOTS

Contents

5.1 Introduction

The character set of all representations of the group of a knot K in $\mathrm{SL}_2(\mathbb{C})$ carries naturally the structure of an algebraic set. Given a peripheral structure of the knot, the character variety is a plane curve in $\mathbb{C}^* \times \mathbb{C}^*$, whose coordinates M and L correspond to the eigenvalues of the meridian m and the preferred longitude ℓ. The polynomial $A_K(L, M)$ defining this curve is an invariant of the knot, called the A-polynomial.

In this chapter, our motivations come, among others, from the following result of Boden:

Theorem 5.1.1 ([Bod14]). *If the M-degree $\deg_M A_K(L, M)$ of the A-polynomial is zero, then K is the trivial knot.* ◁

This result motivates the systematic study of the *logarithmic Gauss map* of the A-polynomial

$$\frac{M}{L} \cdot \frac{\partial_M A_K(L, M)}{\partial_L A_K(L, M)} \tag{5.1}$$

where ∂_M and ∂_L denote the partial derivatives. By Theorem 5.1.1, this rational function vanishes identically on $\{A_K = 0\}$ if and only if K is trivial.

The logarithmic Gauss map has introduced in [GKZ94] by I.M. Guelfand, M.M. Kapranov and A.V. Zelevinsky in order to study some determinantial varieties. Then it has been used for instance by G. Mikhalkin in [Mik00] for studying the topology of arrangements of real plane curves. In [GM21], A. Guilloux and J. Marché showed it is related with the volume function of the A-polynomial of knots, or more generally of exact polynomials.

Our proposal is to develop a homological point of view on this function, by extending the constructions of A. Degtyarev, V. Florens and A.G. Lecuona [DFL22b; DFL21] already presented in Section 4.2 to the setting of non-abelian representations. Let K be an oriented knot in the 3-sphere S^3 with exterior M_K. Denote by $R(M_K)$ and $X(M_K)$ the $\mathrm{SL}_2(\mathbb{C})$-representation and character varieties of the knot K. We consider representations $\rho\colon \pi_1(M_K) \to \mathrm{SL}_2(\mathbb{C})$ composed with the adjoint action of $\mathrm{SL}_2(\mathbb{C})$ on the Lie algebra $\mathrm{Ad}\colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Aut}(\mathfrak{sl}_2(\mathbb{C}))$, and show that there is a non-empty Zariski open subset of $X(M_K)$ such that for all ρ in this subset

- there is an element $v_\rho \in \mathfrak{sl}_2(\mathbb{C})$ such that $(v_\rho \otimes \ell, v_\rho \otimes m)$ is a basis of the homology group $H_1(\partial M_K; \mathrm{Ad} \circ \rho) \simeq \mathbb{C}^2$ with coefficients twisted by $\mathrm{Ad} \circ \rho$, and
- the kernel of the homomorphism induced by the inclusion:

$$\mathcal{Z}(\partial M_K; \mathrm{Ad} \circ \rho) := \ker \left(H_1(\partial M_K; \mathrm{Ad} \circ \rho) \xrightarrow{i_*} H_1(M_K; \mathrm{Ad} \circ \rho) \right)$$

  is generated by a single vector of the form $a\, v_\rho \otimes \ell + b\, v_\rho \otimes m$ for some element $[b : a] \in \mathbb{CP}^1$.

The representations which verify these conditions are called *admissible*. We define the slope of K at the admissible representation ρ by

$$s_K(\rho) := -\frac{b}{a} \in \mathbb{CP}^1$$

We prove that representations which restrict to *non-parabolic* representations of the boundary ∂M_K of M_K are admissible, see Lemma 5.3.4. If ρ is a boundary-parabolic representation, we define the slope $s_K(\rho)$ as the modulus of the euclidean structure induced by the restricted representation on $\pi_1(\partial M_K)$, see Section 5.3.3. It turns out that these two different definitions fit well and that the following holds.

Proposition 5.1.2. *The slope depends only on the conjugacy classes of the representations and induces a rational function*

$$s_K : X \subset X(M_K) \longrightarrow \mathbb{CP}^1$$

on each irreducible component X of the character variety. ◁

Note that if the representation is real or unitary, then s_K takes values in $\mathbb{RP}^1$ (see Proposition 5.3.13). For any knot, the function s_K can be computed by Fox calculus, see Section 5.3.5. We illustrate the method in the case of the trefoil knot, and further compute the slope of the figure-eight knot.

The following theorem relates s_K to the original motivation; a precise statement is given in Theorem 5.4.1.

Theorem 5.1.3. *The slope function s_K equals minus the logarithmic Gauss map of the A-polynomial defined in Eq. (5.1).* ◁

We also relate s_K to the change of curve factor for the Reidemeister torsion. Let $\mathbb{T}_{M_K,\ell}(\rho)$ and $\mathbb{T}_{M_K,m}(\rho)$ be the Reidemeister torsions according to homology bases induced by the choices of the curves ℓ and m in ∂M_K, see Section 5.3.4.

Proposition 5.1.4. *The slope coincides with the quotient of Reidemeister torsion:*

$$s_K(\rho) = \frac{\mathbb{T}_{M_K,\ell}(\rho)}{\mathbb{T}_{M_K,m}(\rho)}$$

for all ρ such that this formula is well-defined. ◁

J. Porti had already observed ([Por97, Corollary 4.9]) that the logarithmic Gauss map of the A-polynomial could be expressed as a ratio of torsions -up to a sign-, and that this ratio of torsions is equal to the modulus of ρ when it is a boundary-parabolic representation ([Por97, Proposition 4.7]). Our point of view permits to fix and compute the sign ambiguity. Moreover, our results Proposition 5.1.2 and Theorem 5.1.3 are more general, since they do not require the Reidemeister torsion to be well-defined, for instance they hold for high dimensional components of the character variety.

Finally, we consider *ideal* points of the A-polynomial, those are points added at infinity in a compactification of the curve $\{A(L,M) = 0\}$ in $\mathbb{C}^2$. In [CS83], M. Culler and P. Shalen constructed incompressible surfaces in M_K associated to such points. Those surfaces have a non-empty boundary, whose slope is determined by a rational number p/q. We prove the following theorem=

Theorem 5.1.5. *Let y be an ideal point in a one-dimensional component Y of the A-polynomial. Then the value of the slope function at the ideal point y equals* minus *the boundary slope of an incompressible surface corresponding to y or* minus *the slope of the corresponding side of the Newton polygon of the A-polynomial.* ◁

This theorem sheds some light on the main theorem of [Coo+94], which states that the boundary slopes of the Culler–Shalen surfaces are boundary slopes of the Newton polygon of the A-polynomial. Indeed it is well-known that the logarithmic Gauss map converges at those ideal points to the value of the slope of the corresponding boundary of the Newton polygon.

To conclude this introduction, we mention that the slope invariant can be extended to orthogonal (real) representations of link groups. In this more general setting, the first twisted homology space $H_1(\partial M_K, \rho)$ can have an arbitrary dimension higher than 2 and the kernel $\mathcal{Z}(K,\rho)$ might not be a line any more. However, the space $H_1(\partial M_K, \rho)$ carries a natural symplectic structure given by the (twisted) intersection form on ∂M_K, and $\mathcal{Z}(K,\rho)$ is still a Lagrangian subspace. A construction of V.I. Arnol'd [Arn67] related to the Maslov index allows to construct a *generalised slope* for this context, lying in $S^1 \subset \mathbb{C}^*$. As it turns out, in the case of a representation $\rho\colon \pi_1(M_K) \to \mathrm{SU}_2(\mathbb{C})$, both theories coincide via the natural isomorphism $\mathbb{RP}^1 \simeq S^1$.

In Section 5.2 we collect basic definitions on character varieties and A-polynomials. In Section 5.3 we define the slope invariant and we prove Proposition 5.1.2 and Proposition 5.1.4. In Section 5.4 we prove Theorem 5.1.3. Only the A-polynomial is concerned by Section 5.4.2, where we prove Theorem 5.1.1. Finally, in Section 5.4.3 we prove Theorem 5.1.5.

This chapter is adapted from work in collaboration with L. Bénard and V. Florens [BFR21].

5.2 Representation varieties and A-polynomial

This section is devoted to definitions and properties of representations spaces and character varieties (Section 5.2.1). We compute the character variety of the group $\mathbb{Z}^2$ in Section 5.2.2 and define the A-polynomial of knots in Section 5.2.3. References for character varieties are [Sha01; Sik12], the A-polynomial was first defined in [Coo+94], see also [CL98].

5.2.1 Representation and character varieties

Let Γ be a finitely generated group. The *representation variety* is the affine algebraic set

$$R(\Gamma) = \mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$$

If Γ is generated by n elements, the representation variety is an algebraic subset of $\mathrm{SL}_2(\mathbb{C})^n$ given by polynomial relations corresponding to the relations of the group Γ. Two different presentations yield naturally isomorphic algebraic sets.

A representation $\rho : \Gamma \to \mathrm{SL}_2(\mathbb{C})$ is *abelian* is $\rho(\Gamma)$ is an abelian subgroup of $\mathrm{SL}_2(\mathbb{C})$. A representation ρ is *reducible* if there exists a proper subspace of $\mathbb{C}^2$ invariant under the action of $\rho(\Gamma)$. Equivalently, $\rho(\Gamma)$ is conjugated to a subgroup of the group of upper-triangular matrices in $\mathrm{SL}_2(\mathbb{C})$. Abelian representations are reducible, but the converse does not hold. Non-reducible representations are *irreducible*.

Two representations ρ and ρ' in $R(\Gamma)$ are equivalent if they have the same trace:

$$\rho \sim \rho' \text{ if and only if } \operatorname{Tr}\rho(\gamma) = \operatorname{Tr}\rho'(\gamma), \text{ for any } \gamma \in \Gamma.$$

The set of equivalence classes of representations coincides with the algebro-geometric quotient of $R(\Gamma)$ by the action of $\mathrm{SL}_2(\mathbb{C})$ by conjugation. This quotient is usually constructed through invariant theory, and is denoted

$$X(\Gamma) = R(\Gamma) /\!/ \mathrm{SL}_2(\mathbb{C})$$

Points of the character variety are called *characters*. The equivalence class of a representation ρ (the character of ρ) is denoted by $\chi_\rho : \Gamma \to \mathbb{C}$ with $\chi_\rho(\gamma) = \operatorname{Tr}(\rho(\gamma))$ for $\gamma \in \Gamma$. If Γ is the fundamental group of a manifold W, we simply write $R(W)$ and $X(W)$ for the representation and character varieties of the manifold W.

Despite being abelian is not a well-defined notion on the character variety, the notion of being reducible makes sense there, since a reducible representation $\rho \colon \Gamma \to \mathrm{SL}_2(\mathbb{C})$ can be characterised by the fact that for any $\gamma, \delta \in \Gamma$, the following equality holds (see for instance [CS83, Lemma 1.2.1]):

$$\operatorname{Tr}\rho\left(\gamma\delta\gamma^{-1}\delta^{-1}\right) = 2. \tag{5.2}$$

The character variety $X(\Gamma)$ can be decomposed as

$$X(\Gamma) = X^{\mathrm{irr}}(\Gamma) \cup X^{\mathrm{red}}(\Gamma)$$

where $X^{\mathrm{red}}(\Gamma)$ is the set of reducible characters, and its complement $X^{\mathrm{irr}}(\Gamma)$ is the set of irreducible characters. Eq. (5.2) implies that $X^{\mathrm{red}}(\Gamma)$ is a Zariski closed subset of $X(\Gamma)$.

An algebraic set is *reducible* if it can be written as a union of two proper algebraic subset, else it is *irreducible*. An irreducible component of an algebraic set is a maximal irreducible algebraic subset.

***Remark* 5.2.1.** Despite $R(\Gamma)$ or $X(\Gamma)$ are called *varieties*, they are not quite algebraic varieties in general: they are actually reducible, and might not be reduced as schemes (some points or subspaces might have multiplicity). On the other hand, any irreducible component is irreducible, and in particular reduced, by definition. ⌑

Two representations ρ and ρ' are *conjugate* if there exists a matrix $M \in \mathrm{SL}_2(\mathbb{C})$ such that $\rho(\gamma) = M\rho'(\gamma)M^{-1}$ for every $\gamma \in \Gamma$. Two conjugate representations define the same character; the converse is false in general, but true for elements of $X^{\mathrm{irr}}(\Gamma)$. More precisely, the following holds.

Theorem 5.2.2 ([CS83, Proposition 1.5.2])**.** *If ρ and ρ' are two representations $\Gamma^A \to \mathrm{SL}_2(\mathbb{C})$ with ρ irreducible and $\chi_\rho = \chi_{\rho'}$, then ρ and ρ' are conjugate (and ρ' is irreducible as well).* ◁

Two non-conjugate representations having the same character in $X(\Gamma)$ must be reducible. If Γ is a knot group, G. Burde and G. de Rham [Bur67; Rha67] showed that the set of characters containing non-conjugate representations is finite.

5.2.2 The character variety of $\mathbb{Z}^2$

We describe explicitly the character variety of a 2-torus $S^1 \times S^1$. Pick a basis m, ℓ of $\pi_1(S^1 \times S^1) = \mathbb{Z}^2$. Any representation in $\mathrm{SL}_2(\mathbb{C})$ is conjugate to a representation ρ given by two commuting matrices of the form

$$\rho(m) = \begin{bmatrix} M & * \\ 0 & M^{-1} \end{bmatrix} \qquad\qquad \rho(\ell) = \begin{bmatrix} L & * \\ 0 & L^{-1} \end{bmatrix}$$

for $M, L \in \mathbb{C}^*$. Each point of the character variety $X(S^1 \times S^1)$ has a pre-image in $R(S^1 \times S^1)$ of the form

$$\rho(m) = \begin{bmatrix} M & 0 \\ 0 & M^{-1} \end{bmatrix} \qquad \rho(\ell) = \begin{bmatrix} L & 0 \\ 0 & L^{-1} \end{bmatrix} \qquad M, L \in \mathbb{C}^* \tag{5.3}$$

This pre-image is unique up to the involution σ of $(\mathbb{C}^*)^2$ which sends (L, M) to (L^{-1}, M^{-1}), and $X(S^1 \times S^1)$ can be identified with the singular affine complex surface $(\mathbb{C}^*)^2/\sigma$. It embeds in $\mathbb{C}^3$ as the zeros of the polynomial

$$\Delta = x^2 + y^2 + z^2 - xyz - 4$$

Indeed, the function algebra of $X(S^1 \times S^1)$ naturally identifies with the σ-invariant sub-algebra $\mathbb{C}[M + M^{-1}, L + L^{-1}]$ of $\mathbb{C}[L^{\pm 1}, M^{\pm 1}]$. This algebra of invariant functions is isomorphic with $\mathbb{C}[x, y, z]/(\Delta)$ through

$$M + M^{-1} \longmapsto x \qquad L + L^{-1} \longmapsto y \qquad ML + (ML)^{-1} \longmapsto z$$

From this description, one sees that the singular locus of $X(S^1 \times S^1)$ consists on the four points $\{(L, M) = (\pm 1, \pm 1)\}$.

5.2.3 The *A*-polynomial

Let K be an oriented knot in S^3 with exterior M_K. The inclusion $\partial M_K \subset M_K$ induces an injective group homomorphism $\pi_1(\partial M_K) \hookrightarrow \pi_1(M_K)$. Let r be the restriction map:

$$r \colon X(M_K) \longrightarrow X(\partial M_K) \simeq X(S^1 \times S^1)$$

For short we denote by $\rho_\partial = r(\rho)$ the restriction of ρ to $\pi_1(\partial M_K)$. By Section 5.2.2, the choice of the longitude ℓ and the meridian m induces an identification of $X(S^1 \times S^1)$ with a quotient of $(\mathbb{C}^*)^2$. The image of r is a union of points and curves, possibly with multiplicities, see for instance [DG04, Lemma 2.1]. Discarding the 0-dimensional components, the *A-polynomial* of K is the unique polynomial $A_K(L, M)$ in $\mathbb{C}[L, M]$ whose zero locus in $\mathbb{C}^2$ is exactly mapped onto the image of the algebraic map r Note that $A_K(L, M)$ is always divisible by $L - 1$. This factor corresponds to the curve of reducible characters. S. Boyer, X. Zhang, N. Dunfield and S. Garoufalidis have shown the following result.

Theorem 5.2.3 ([BZ05; DG04]). *Let K be a knot in S^3. The A-polynomial $A_K(L, M)$ is equal to $(L - 1)^k$ for some k, if and only if K is the trivial knot (and in this case $k = 1$).* ◁

5.3 The $\mathrm{SL}_2(\mathbb{C})$-slope invariant

In this section we define the slope of an admissible representation (Section 5.3.1), and observe that generic $\mathrm{SL}_2(\mathbb{C})$-representations are admissible. In Section 5.3.2 we show that the slope is invariant by conjugation of the representation. We prove in Section 5.3.3 that it yields a rational function on irreducible components of the character variety and that the slope of a real representation is a real number. Then we prove in Section 5.3.4 that the slope can be written as a quotient of Reidemeister torsions. Finally, in Section 5.3.5 we describe a procedure to compute the slope with an Alexander matrix.

5.3.1 Admissible representations

Let V be a finite dimensional $\mathbb{C}$-vector space, and $\rho \colon \pi_1(M_K) \to \mathrm{GL}(V)$ be a representation. The representation extends to a ring homomorphism and V can be viewed as a right $\mathbb{Z}[\pi_1(M_K)]$-module denoted by $V(\rho)$. Let $H_*(M_K; \rho)$ be the ρ-twisted homology spaces of M_K as defined in Appendix A.

Definition 5.3.1. A representation $\rho \colon \pi_1(M_K) \to \mathrm{GL}(V)$ is *admissible* if it satisfies:

- there exists $v_\rho \in V$ such that $\{\ell \otimes v_\rho, m \otimes v_\rho\}$ is a basis of $H_1(\partial M_K; \rho) \simeq \mathbb{C}^2$,

- the kernel of the homomorphism induced by the inclusion:
$$\mathcal{Z}(\partial M_K;\rho) := \ker\left(H_1(\partial M_K;\rho) \xrightarrow{i_*} H_1(M_K;\rho)\right)$$
has dimension one. ◇

We restrict to representations $\rho\colon \pi_1(M_K) \to \mathrm{SL}_2(\mathbb{C})$. The composition of ρ with the adjoint action Ad of $\mathrm{SL}_2(\mathbb{C})$ on $\mathfrak{sl}_2(\mathbb{C})$ induces the following representation:

$$\begin{aligned}\mathrm{Ad}\circ\rho:\quad \pi_1(M_K) &\longrightarrow \mathrm{Aut}(\mathfrak{sl}_2(\mathbb{C}))\\ \gamma &\longmapsto \left(v \mapsto \rho(\gamma)v\rho(\gamma)^{-1}\right)\end{aligned}$$

Definition 5.3.2. Let $\rho\colon \pi_1(M_K) \to \mathrm{SL}_2(\mathbb{C})$ be such that $\mathrm{Ad}\circ\rho$ is admissible. Let

$$a\,(\ell \otimes v_\rho) + b\,(m \otimes v_\rho)$$

be a generator of $\mathcal{Z}(\partial M_K;\mathrm{Ad}\circ\rho)$ for some $[a:b] \in \mathbb{CP}^1$. The *slope* of the knot K at the representation ρ is

$$s_K(\rho) := -\frac{b}{a} \in \mathbb{C}\cup\infty$$ ◇

Definition 5.3.3. A representation $\rho\colon \pi_1(M_K) \to \mathrm{SL}_2(\mathbb{C})$ is *boundary-parabolic* if the restriction $\rho_\partial : \pi_1(\partial M_K) \to \mathrm{SL}_2(\mathbb{C})$ is parabolic, that is $\mathrm{Tr}\,\rho(\gamma) = \pm 2$ for any $\gamma \in \pi_1(\partial M_K)$.

A boundary-parabolic character is the character of a boundary-parabolic representation. ◇

Lemma 5.3.4. *Let $\rho\colon \pi_1(M_K) \to \mathrm{SL}_2(\mathbb{C})$ be a non-parabolic representation. The vector space $H_1(\partial M_K;\mathrm{Ad}\circ\rho)$ is isomorphic to $\mathbb{C}^2$, and the kernel $\mathcal{Z}(M_K;\mathrm{Ad}\circ\rho)$ has dimension 1. Moreover, if ρ is not boundary-parabolic, then $\mathrm{Ad}\circ\rho$ is admissible.* ◁

Proof. The group $\pi_1(M_K)$ is generated by pairwise conjugate meridians. If ρ is non-parabolic, then the image of a meridian must differ to $\pm I_2$, otherwise we would have $\rho(\pi_1(M_K)) \subset \{\pm I_2\}$. Since $\mathrm{Ad}\circ\rho \in \mathrm{Aut}(\mathfrak{sl}_2(\mathbb{C})) \simeq \mathrm{SO}_3(\mathbb{C})$ is unitary, a direct application of Lemma A.2.1 on page 106 gives the dimension of $H_1(\partial M_K;\mathrm{Ad}\circ\rho)$. Moreover, when ρ is not boundary-parabolic, for $v_\rho \in \mathfrak{sl}_2(\mathbb{C})$ invariant by $\mathrm{Ad}\circ\rho_\partial$, the pair of vectors $(\ell\otimes v_\rho, m\otimes v_\rho)$ forms a basis of $H_1(\partial M_K;\mathrm{Ad}\circ\rho)$. The constructions of the Appendices A.1.3 and A.1.4 can also be applied to $\mathrm{Ad}\circ\rho$ using the $\mathbb{C}$-bilinear Killing form on $\mathfrak{sl}_2(\mathbb{C})$ as the standard vector product, see [Por97, Section 0.3]. Theorem A.1.9 thus gives the dimension of the subspace $\mathcal{Z}(M_K;\mathrm{Ad}\circ\rho)$. □

As an example, we compute the slope for abelian non boundary-parabolic representations. Let $\varphi\colon \pi_1(M_K) \to H_1(M_K) = \mathbb{Z}$ be the abelianisation. For any $\lambda \in \mathbb{C}^*$, there is an abelian representation

$$\begin{aligned}\rho_\lambda\colon\ \pi_1(M_K) &\longrightarrow \mathrm{SL}_2(\mathbb{C})\\ \gamma &\longmapsto \begin{bmatrix}\lambda^{\varphi(\gamma)} & 0\\ 0 & \lambda^{-\varphi(\gamma)}\end{bmatrix}\end{aligned} \tag{5.4}$$

and any abelian, non boundary-parabolic representation is conjugate to a representation of this form.

Lemma 5.3.5. *For any $\lambda \neq \pm1$, the slope at the abelian representation ρ_λ vanishes:*

$$s_K(\rho_\lambda) = 0$$ ◁

Proof. Up to conjugation, the representation $\mathrm{Ad}\circ\rho$ has the form

$$\mathrm{Ad}\circ\rho(\gamma) = \begin{bmatrix}\lambda^{2\varphi(\gamma)} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \lambda^{-2\varphi(\gamma)}\end{bmatrix}$$

and $\mathfrak{sl}_2(\mathbb{C})$ splits as $\mathbb{Z}[\pi_1(M_K)]$-module as

$$\mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}_{\lambda^2} \oplus \mathbb{C} \oplus \mathbb{C}_{\lambda^{-2}}$$

This yields a splitting in twisted homology (with abelian coefficients), for $U = \partial M_K$ or $U = M_K$:

$$H_1(U;\mathrm{Ad}\circ\rho) = H_1\left(U;\mathbb{C}_{\lambda^2}\right) \oplus H_1(U;\mathbb{C}) \oplus H_1\left(U;\mathbb{C}_{\lambda^{-2}}\right)$$

Since $\lambda \neq \pm 1$, by Corollaries A.2.2 and A.2.3 on page 106 for $U = \partial M_K$ the only non-trivial summand is $H_1(\partial M_K, \mathbb{C})$, and the map $H_1(\partial M_K; \mathrm{Ad} \circ \rho) \xrightarrow{i_*} H_1(M_K; \mathrm{Ad} \circ \rho)$ coincides with the map induced by the inclusion in homology with trivial coefficients $H_1(\partial M_K, \mathbb{C}) \to H_1(M_K, \mathbb{C})$, whose kernel is generated by ℓ. □

5.3.2 The slope of characters

By the following lemma, the slope does not depend on the conjugacy class of an irreducible representation. Combined with Theorem 5.2.2, it follows that the slope of an irreducible representation depends only on its character.

Lemma 5.3.6. *Let ρ and $\rho'\colon \pi_1(M_K) \to \mathrm{SL}_2(\mathbb{C})$ be two irreducible, non boundary-parabolic representations. If ρ and ρ' are conjugate, then $s_K(\rho) = s_K(\rho')$.* ◁

Proof. Let A be a matrix in $\mathrm{GL}_2(\mathbb{C})$ such that $\rho' = A\,\rho\,A^{-1}$. Any $\mathrm{Ad} \circ \rho$-invariant vector $v_\rho \alpha^\infty \in \mathfrak{sl}_2(\mathbb{C})$ yields an $\mathrm{Ad} \circ \rho'$-invariant vector $v'_\rho = A\, v_\rho\, A^{-1}$, and the conjugation by A induces an isomorphism

$$H_1(\partial M_K; \mathrm{Ad} \circ \rho) \longrightarrow H_1(\partial M_K, \mathrm{Ad} \circ \rho')$$

sending the basis $\{v_\rho \otimes \ell, v_\rho \otimes m\}$ to $\{v'_\rho \otimes \ell, v'_\rho \otimes m\}$ and the subspace $\mathcal{Z}(\partial M_K; \mathrm{Ad} \circ \rho)$ to $\mathcal{Z}(K, \mathrm{Ad} \circ \rho')$. Hence $s_K(\rho) = s_K(\rho')$. □

***Remark* 5.3.7.** There exist pairs of reducible, non-conjugate representations with the same character. Indeed, let χ be an arbitrary reducible character in $X(M_K)$. Consider a representation ρ of the form $\left[\begin{smallmatrix} \lambda(\gamma) & * \\ 0 & \lambda^{-1}(\gamma) \end{smallmatrix}\right]$, where $\lambda\colon \pi_1(M_K) \to \mathbb{C}^*$ is a group homomorphism, chosen such that $\chi(\rho) = \chi$. Note that λ can further be written $\lambda(\gamma) = \lambda^{\varphi(\gamma)}$ for some $\lambda \in \mathbb{C}^*$ and $\varphi : \pi_1(M_K) \to \mathbb{Z}$. Hence the abelian representation ρ_λ defined in Eq. (5.4) has also character χ, but is not conjugated in general to ρ. It turns out that they can have different slope values.

For example, consider the right-handed trefoil knot T in S^3. The character variety $X(M_T)$ is the union of a line X^{red} and a conic X^{irr} in the plane. The line contains only reducible characters, and any character in the conic is irreducible except the two intersection points $X^{\mathrm{red}} \cap X^{\mathrm{irr}}$. Let χ be a point in $X^{\mathrm{red}} \cap X^{\mathrm{irr}}$. Since χ is reducible, there exists a $\lambda \in \mathbb{C}^*$ such that the abelian representation ρ_λ has character χ. By Lemma 5.3.5, one has $s_T(\rho_\lambda) = 0$. However, we show in Example 5.3.19 that the slope defines a constant function on X^{irr}, everywhere equal to -6. ⌑

5.3.3 Regularity and properties of the slope

We extend the slope to a rational function –locally a quotient of polynomials– on the character variety $X(M_K)$.

There is a component $X^{\mathrm{red}} \subset X(M_K)$ of reducible characters only. By Remark 5.3.7 any character in X^{red} is the character of an abelian representation. Hence the slope is identically zero on X^{red}, see Lemma 5.3.5. Suppose now that $X \subset X(M)$ is an irreducible component containing an irreducible character.

Proposition 5.3.8. *Let $X \subset X(M)$ be an irreducible component which contains an irreducible character. The slope extends to a rational function on X, still denoted s_K. Moreover, if $\chi \in X$ is a boundary-parabolic character then*

$$s_K(\chi) = \tau(\chi), \tag{5.5}$$

where the modulus $\tau(\chi) \in \mathbb{C}$ is defined by taking the representative ρ of χ satisfying

$$\rho(m) = \begin{bmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{bmatrix} \qquad \rho(\ell) = \begin{bmatrix} \pm 1 & \tau(\chi) \\ 0 & \pm 1 \end{bmatrix}$$ ◁

***Remark* 5.3.9.** If χ is the character of an irreducible representation and lies at the intersection of several irreducible components, then the value of the slope at χ is well-defined. ⌑

The rest of the section is devoted to the proof of Proposition 5.3.8. Lemma 5.3.10 asserts that the slope is a rational function in the neighbourhood of any irreducible, non boundary-parabolic character. For boundary parabolic characters χ, we *define* the slope by the relation in Eq. (5.5) and we show that the result is still a rational function on X in Lemma 5.3.12.

Lemma 5.3.10. *Let χ_0 an irreducible, non-boundary-parabolic character in X. The slope is a rational function in a neighbourhood of χ_0 in X.* ◁

Proof. Let ρ_0 in $R(M_K)$ be a representation with character χ_0. By Lemma 5.3.4 one has $H_1(\partial M_K; \mathrm{Ad} \circ \rho) \simeq \mathbb{C}^2$. The set of complex lines

$$\mathbb{P}(\rho) = \mathbb{P}(H_1(\partial M_K; \mathrm{Ad} \circ \rho))$$

is a complex algebraic variety isomorphic to $\mathbb{C}P^1$. If ρ and ρ' are conjugate, then there is a natural algebraic isomorphism $\mathbb{P}(\rho) \simeq \mathbb{P}(\rho')$. It defines an algebraic $\mathbb{CP}^1$-fibration on a neighbourhood of χ_{ρ_0}, and for any χ, the complex line $Z(M_K; \mathrm{Ad} \circ \rho)$ is an algebraic section of this fibration, independent of the choice of representation ρ with character χ.

It remains to show that the identification $\mathbb{P}(\rho) \simeq \mathbb{CP}^1$ is algebraic, in other words, that the choice of the basis $(v_\rho \otimes \ell, v_\rho \otimes m)$ depends algebraically on ρ. Since ρ_0 is not boundary-parabolic, we can shrink the chosen neighbourhood so that no representation ρ near ρ_0 is boundary-parabolic. Then, since ρ_∂ is conjugated to a diagonal representation, there is a unique $\mathrm{Ad} \circ \rho_\partial$-invariant vector v_ρ with norm 1 in $\mathfrak{sl}_2(\mathbb{C})$. This choice depends polynomially on the entries of the matrix $\mathrm{Ad} \circ \rho(m)$, and then the basis $(v_\rho \otimes \ell, v_\rho \otimes m)$ depends algebraically on ρ. □

We now consider the case of boundary-parabolic characters.

Lemma 5.3.11. *Let ρ_0 be a boundary-parabolic representation whose character χ_{ρ_0} lies in X. Then ρ_0 is irreducible, in particular $\rho_0(m) \neq \pm I_2$.* ◁

Proof. For ρ reducible in X, [Bur67; Rha67] implies that $\rho(m)$ has eigenvalues λ, λ^{-1} in $\mathbb{C}$, whose square is a root of the Alexander polynomial $\Delta_{M_K}(t)$, in particular $\lambda \neq \pm 1$, and ρ is not boundary-parabolic. Now for irreducible ρ, the image of any meridian must be different of I_2, since meridians generate the group $\pi_1(M_K)$. □

Lemma 5.3.12. *Let $X \subset X(M)$ be an irreducible component containing an irreducible character, and $\chi_0 \in X$ a boundary-parabolic character. Then the slope function s_K is rational in a neighbourhood of χ_0.* ◁

Proof. Suppose first that X contains only boundary-parabolic characters. Any $\chi \in X$ is the character of a representation ρ such that $\rho(m) = \left[\begin{smallmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{smallmatrix}\right]$. Hence $\chi \mapsto \tau(\chi)$ is rational.

Now we assume that X contains a non boundary-parabolic character. By definition, boundary-parabolic characters form a Zariski closed subset of X. By Lemma 5.3.10, the slope function is rational on the open, non-empty subset of X consisting of non boundary-parabolic characters. By analytic continuation, it is enough to show that

$$\lim_{\chi \to \chi_0} \alpha^\infty s_K(\chi) = \tau(\chi_0)$$

By Lemma 5.3.11 any boundary-parabolic representation ρ_0 with character χ_0 is irreducible. Moreover, since $\rho_0(m)$ can not be trivial, we can chose such a ρ_0 satisfying

$$\rho_0(m) = \begin{bmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{bmatrix}$$

For any χ close to χ_0, we chose similarly a representation ρ with character χ such that

$$\rho(m) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix}$$

with M close to ± 1 in $\mathbb{C}^*$. For such ρ, let

$$v_\rho = \begin{bmatrix} M - M^{-1} & 2 \\ 0 & M^{-1} - M \end{bmatrix}$$

be an $\mathrm{Ad} \circ \rho_\partial$-invariant vector. The limit at ρ_0 of v_ρ is the $(\mathrm{Ad} \circ \rho_0)_\partial$-invariant vector $v_{\rho_0} = \left[\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right]$. However, a direct computation shows that $v_{\rho_0} \otimes \ell$ and $v_{\rho_0} \otimes m$ are linearly dependent in $H_1(\partial M_K, \mathrm{Ad} \circ \rho_0)$, and we cannot compute the slope of the boundary parabolic representation ρ_0 by means of Definition 5.3.2. Nevertheless, the subspace $\mathcal{Z}(K, \mathrm{Ad} \circ \rho_0)$ is one-dimensional by Lemma 5.3.4.

It implies that the map $i_*\colon H_1(\partial M_K;\mathrm{Ad}\circ\rho)\to H_1(M_K;\mathrm{Ad}\circ\rho)$ has rank one at any representation ρ near ρ_0, and at ρ_0 as well. In particular, for any ρ near ρ_0, the slope can be computed as the ratio of $i_*(v_\rho\otimes\ell)$ and $i_*(v_\rho\otimes m)$ in $i_*(H_1(\partial M_K;\mathrm{Ad}\circ\rho))$. This actually makes sense for $\rho=\rho_0$ as well. An explicit computation of the boundary operator $\partial_1\colon C_2(\partial M_K,\mathrm{Ad}\circ\rho_0)\to C_1(\partial M_K,\mathrm{Ad}\circ\rho_0)$ shows that the vector $v_{\rho_0}\otimes\ell-\tau(\chi_0)\cdot v_{\rho_0}\otimes m$ belongs to $\mathrm{im}\,\partial_2$, and the equality

$$v_{\rho_0}\otimes\ell=\tau(\chi_0)\cdot v_{\rho_0}\otimes m$$

holds in $H_1(\partial M_K,\mathrm{Ad}\circ\rho_0)$. This implies that the ratio of $i_*(v_{\rho_0}\otimes\ell)$ and $i_*(v_{\rho_0}\otimes m)$ coincides with the modulus $\tau(\chi_0)$. This proves the lemma, and achieves the proof of Proposition 5.3.8. □

We end this section with the following observation.

Proposition 5.3.13. *Let $X\subset X(M)$ be an irreducible component which contains a non boundary-parabolic representation. If $\rho\in X$ is a* real *representation $\rho\colon\pi_1(M_K)\to\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SU}_2(\mathbb{C})$, then the slope is a real number in $\mathbb{RP}^1$.* ◁

Proof. First assume that ρ is non-boundary-parabolic. If ρ is real, denoting by $\mathrm{Ad}\circ\rho_{\mathbb{R}}$ the action of ρ on the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ (resp. $\mathfrak{su}(2)$) of $\mathrm{SL}_2(\mathbb{R})$ (resp. $\mathrm{SU}_2(\mathbb{C})$), then obviously the Lagrangian $Z(M_K;\mathrm{Ad}\circ\rho)\subset H_1(\partial M_K;\mathrm{Ad}\circ\rho)$ is the complexification of the real Lagrangian $\mathcal{Z}(M_K,\mathrm{Ad}\circ\rho_{\mathbb{R}})$ in the real symplectic vector space $H_1(\partial M_K,\mathrm{Ad}\circ\rho_{\mathbb{R}})$ and the slope of this real Lagrangian is the slope of its complexification, a real number. If ρ is boundary-parabolic and re al, then it takes value into $\mathrm{SL}_2(\mathbb{R})$ and the proposition follows from the definition of the modulus τ. □

5.3.4 Slope and Reidemeister torsion

In this section we show that the slope coincides with the 'change of curve term' for the Reidemeister torsion as stated in Proposition 5.1.4.

If ρ is an irreducible representation in $X(M_K)$, we consider the torsion of the complex $C_*(M_K;\mathrm{Ad}\circ\rho)$ defined in Section 5.3.1. This complex is naturally based from a cell decomposition of M_K and a choice of a basis of $\mathfrak{sl}_2(\mathbb{C})$, but not acyclic. The Reidemeister torsion is usually defined for acyclic complexes. In the case we are considering, one needs to make some additional choices to define it, namely a basis of each homology group $H_*(M_K;\mathrm{Ad}\circ\rho)$.

According to [Por97], one can still define the Reidemeister torsion of the cellular complex $C_*(M_K;\mathrm{Ad}\circ\rho)$ for representations ρ in $R(M_K)$ such that $H_1(M_K;\mathrm{Ad}\circ\rho)$ has dimension 1. For a given curve $\gamma\in\pi_1(\partial M_K)$, the representation ρ is γ*-regular* if there exists a vector $v_\rho\in\mathfrak{sl}_2(\mathbb{C})$ such that $v_\rho\otimes\gamma$ spans $H_1(M_K;\mathrm{Ad}\circ\rho)$. In this case, since there is a natural choice of a basis of $H_2(M_K;\mathrm{Ad}\circ\rho)$, the curve γ determines a homology basis of the complex $C_*(M_K;\mathrm{Ad}\circ\rho)$ and the torsion $\mathbb{T}_{M_K,\gamma}(\mathrm{Ad}\circ\rho)\in\mathbb{C}^*$ is defined. Note that this torsion depends only on the conjugacy class of ρ, as well as the property of being γ-regular.

Let $X\subset X(M_K)$ the component containing χ, the torsion function is the rational function

$$\mathbb{T}_{M_K,\gamma}\colon X\longrightarrow\mathbb{C}$$

defined as the Reidemeister torsion of the complex $C_*(M_K,\mathrm{Ad})$ if χ is γ-regular, and by $T_{M_K,\gamma}(\chi)=0$ otherwise.

We start with the following lemma, which provides the genuine setting to define the Reidemeister torsion.

Lemma 5.3.14. *If X has dimension one and contains the character of a scheme-smooth representation ρ, then* $\dim H_1(M_K;\mathrm{Ad}\circ\rho)=1$. ◁

Proof. The proof of Lemma 5.3.14 follows from the isomorphism between the Zariski tangent space of $X(M_K)$ at ρ and the module $H^1(M_K;\mathrm{Ad}\circ\rho)$, see [Sik12, Theorem 1]. Scheme-smoothness implies that the Zariski tangent space is the actual tangent space, which is one-dimensional because X is. □

Note that scheme-smoothness is a Zariski open condition.

It turns out that the character variety $X(M_K)$ of a knot exterior is often one-dimensional. This is the case if the knot is *small* (if it does not contains a closed incompressible surface

[Coo+94, Proposition 2.4]). This is also the case for any component $X \subset X(M_K)$ containing the character of a lift of the holonomy representation $\overline{\rho}\colon \pi_1(M_K) \to \mathrm{PSL}_2(\mathbb{C})$, provided that the interior of M_K admits a hyperbolic structure.

The following proposition is the main result of this section.

Proposition 5.3.15. *Let $X \subset X(M)$ be an irreducible one-dimensional component which contains a scheme-smooth, non-boundary parabolic character. For all $\chi \in X$ the following holds*

$$s_K(\chi) = \frac{\mathbb{T}_{M_K,\ell}(\chi)}{\mathbb{T}_{M_K,m}(\chi)} \qquad \triangleleft$$

We provide two different proofs of this result: one uses the natural definition of the torsion while the other relies directly on some results on the *torsion form* proved by L. Bénard in [Bén20].

Torsion and chain complexes

This section is devoted to the proof of Proposition 5.3.15 by using the chain complex of M_K. The proof is very similar to [DFL22b, Theorem 3.21] or [DFL22b, Theorem 6.7]. We use the following technical lemma.

Lemma 5.3.16. *Let γ be a curve in $\pi_1(\partial M_K)$, and χ be an irreducible γ-regular character in $X(M_K)$. There exists a Zariski open neighbourhood of χ such that any character in this neighbourhood is irreducible and γ-regular.* $\triangleleft$

Proof. Being irreducible is a Zariski open condition, see Eq. (5.2). The γ-regularity follows from lower semi-continuity of the rank of a linear map. Indeed the dimension of $H_1(M_K; \mathrm{Ad}\circ\rho)$ is upper semi-continuous. It is at least one (the dimension of X) again because it is isomorphic to the Zariski tangent space hence it is locally constant equal to one. On the other hand, the rank of the linear map $H_1(\gamma, \mathrm{Ad}\circ\rho) \to H_1(M_K; \mathrm{Ad}\circ\rho)$ sending $v_\rho \otimes \gamma$ to itself is lower semi-continuous. It is at most one (the dimension of $H_1(\gamma, \mathrm{Ad}\circ\rho)$ and it cannot decrease on a neighbourhood of χ. We deduce that $H_1(\gamma, \mathrm{Ad}\circ\rho) \to H_1(M_K; \mathrm{Ad}\circ\rho)$ is an isomorphism on a Zariski open subset. ☐

Proof of Proposition 5.3.15. Let χ be an irreducible, scheme-smooth, non boundary-parabolic character and let ρ be a representation in $R(M_K)$ with character χ. We first assume that ρ is ℓ and m-regular, that is for $v \in \mathfrak{sl}_2(\mathbb{C})$ an $\mathrm{Ad}\circ\rho_\partial$-invariant vector, both $v \otimes \ell$ and $v \otimes m$ provide a basis of $H_1(M_K; \mathrm{Ad}\circ\rho)$.

The calculation of the torsions $\mathbb{T}_{M_K,\ell}(\chi)$ and $\mathbb{T}_{M_K,m}(\chi)$ involves different choices of homology basis of $C_*(M_K; \mathrm{Ad}\circ\rho)$. By [Por97, Proposition 3.18], the bases of $H_2(M_K; \mathrm{Ad}\circ\rho)$ are determined by the fundamental class of $H_2(\partial M_K; \mathbb{C})$ and can be chosen to be the same. Hence, if b_1 is a basis of $\mathrm{im}(\partial_1)$, the ratio of torsions corresponding to the choice of m or of ℓ is reduced to

$$\frac{\mathbb{T}_{M_K,\ell}(\chi)}{\mathbb{T}_{M_K,m}(\chi)} = \frac{\det(b_1 \oplus (v \otimes \ell), c_1)}{\det(b_1 \oplus (v \otimes m), c_1)}.$$

In parallel, consider the affine equation in $C_1(M_K; \mathrm{Ad}\circ\rho)$:

$$y\, b_1 + x\, v \otimes m = v \otimes \ell$$

with at least a solution $y = 0$ and $x = s_K(\rho)$. The Cramer determinants expressed in the common basis c_1 show that $s_K(\rho)$ coincides with the ratio of torsions.

If there exist a character in X which is ℓ-regular and a character in X which is m-regular, then Proposition 5.3.15 holds on the whole component X by Lemma 5.3.16.

Assume that X contains only characters that are not (say) ℓ-regular. Since the map $H_1(\partial M_K; \mathrm{Ad}\circ\rho) \to H_1(M_K; \mathrm{Ad}\circ\rho)$ is not trivial (by Lemma 5.3.4), it is onto on a Zariski open subset $U \in X$, again because $H_1(M_K; \mathrm{Ad}\circ\rho)$ has dimension one generically. Thus all characters in U must be m-regular, and it follows from the definition that the slope and the quotient of torsions are identically zero on X. A similar argument works replacing ℓ by m and zero by infinity. ☐

The torsion form

In this paragraph, we present an alternative proof of Proposition 5.3.15. We follow a slightly different point of view on the torsion, as a volume form on the character variety. The following lemma asserts that the cotangent space of the character variety [Sik12, Section 8] is isomorphic to the first $\mathrm{Ad}\circ\rho$-twisted homology group.

Lemma 5.3.17. *Let χ be an irreducible character in $X(M_K)$, and a representation ρ with character χ. Let $T^*_\chi X(M_K)$ be the Zariski tangent space of $X(M_K)$ at χ. There is a natural isomorphism*

$$H_1(M_K;\mathrm{Ad}\circ\rho)\simeq T^*_\chi X(M_K)$$

Moreover, if χ is not boundary-parabolic, then

$$H_1(\partial M_K;\mathrm{Ad}\circ\rho)\simeq T^*_{r(\chi)}X(\partial M_K) \qquad \lhd$$

The proof of Lemma 5.3.17 follows from [Sik12, Theorem 1]. Note that through the isomorphism, the space $\mathcal{Z}(\partial M_K;\mathrm{Ad}\circ\rho)$ is the Zariski co-normal bundle of $r(X(M_K))$ in $X(\partial M_K)$.

If $X\subset X(M_K)$ is a one-dimensional component of the character variety which contains a scheme-smooth character, L. Bénard proved in [Bén20, Proposition 5.1] that the *torsion form* can be written as

$$\mathrm{tor}(M_K)=\frac{1}{\mathbb{T}_{M_K,\ell}}\,r^*\left(\frac{dL}{L}\right)=\frac{1}{\mathbb{T}_{M_K,m}}\,r^*\left(\frac{dM}{M}\right) \tag{5.6}$$

where r^* is the cotangent map

$$r^*\colon T^*X(\partial M_K)\longrightarrow T^*X(M_K)$$

Proof of Proposition 5.3.15. By Eq. (5.6), the ratio of torsions can be written as

$$\frac{\mathbb{T}_{M_K,\ell}}{\mathbb{T}_{M_K,m}}=\frac{r^*(dL/L)}{r^*(dM/M)}$$

If χ is a non boundary-parabolic character, the character variety $X(\partial M_K)$ is diffeomorphic to $(\mathbb{C}^*)^2$ in a neighbourhood of $r(\chi)$. A local chart of $X(\partial M_K)$ is given by taking $\mathfrak{l},\mathfrak{m}\in\mathbb{C}$ satisfying $\exp\mathfrak{l}=L$ and $\exp\mathfrak{m}=M$. The latter ratio of torsions can be written

$$\frac{\mathbb{T}_{M_K,\ell}}{\mathbb{T}_{M_K,m}}=\frac{r^*(d\,\mathfrak{l})}{r^*(d\,\mathfrak{m})}$$

Lemma 5.3.17 implies that the cotangent map $r^*\colon T^*_{r(\chi)}X(\partial M_K)\to T^*_\chi X(M_K)$ coincides with the homomorphism in homology $H_1(\partial M_K;\mathrm{Ad}\circ\rho)\to H_1(M_K;\mathrm{Ad}\circ\rho)$, thus by Lemma 5.3.4 the range of the map r^* is one-dimensional, and the images of the elements $d\,\mathfrak{l},d\,\mathfrak{m}$ are collinear. It turns out that the ratio $\frac{r^*(d\,\mathfrak{l})}{r^*(d\,\mathfrak{m})}$ coincides with the slope by its very definition.

Finally, the formula extends to the whole X since irreducible, non boundary-parabolic character are Zariski dense in X. ☐

5.3.5 Compute the slope

In this section we compute the slope $s_K(\rho)$ when ρ is an irreducible non-boundary parabolic representation, with Fox calculus, similarly to Section 4.3 and [DFL22b]. Note that for the boundary-parabolic case, the slope can be computed directly from the representation using Proposition 5.3.8.

The $\mathrm{Ad}\circ\rho$-twisted homology group $H_1(M_K;\mathrm{Ad}\circ\rho)$ can be computed using the chain complex $S_*(\mathrm{Ad}\circ\rho)$ and the Alexander matrix derived from an extended Wirtinger presentation of the knot group $\pi_1(M_K)$, as explained in Appendix A.3.

The specific computation of the slope is achieved with the following result:

Proposition 5.3.18. *If ρ is irreducible and non-boundary parabolic, then there exist $a,b\in\mathbb{C}$ and an $\mathrm{Ad}\circ\rho_\partial$-invariant vector $v_\rho\in\mathfrak{sl}_2(\mathbb{C})$ such that*

$$\mathrm{im}\,(\partial_1(\rho))\cap\langle v_\rho\otimes d\ell, v_\rho\otimes dm\rangle=\langle a\,(v_\rho\otimes d\ell)+b\,(v_\rho\otimes dm)\rangle$$

and the slope is $s_K(\rho)=-\frac{b}{a}$. $\lhd$

Proof. Set a base point p on ∂M_K. The subcomplex $S_*(\rho_\partial)$ defined by considering only the generators $x_1 = m$, $x_2 = \ell$ and the relation $[m, \ell] = 1$ computes the space $H_1(\partial M_K, p; \mathrm{Ad} \circ \rho)$. There are natural identifications

$$H_1\left(\partial M_K, p; \mathrm{Ad}\circ\rho\right) = H_1(\partial M_K; \mathrm{Ad}\circ\rho)$$
$$H_1(\partial M_K, p; \mathrm{Ad}\circ\rho) \hookrightarrow H_1(\partial M_K; \mathrm{Ad}\circ\rho)$$

and the following diagram commutes:

$$\begin{array}{ccccc}
S_1^{\partial}(\rho) & \hookrightarrow & & \longrightarrow & S_1(\rho) \\
\Big\downarrow{\scriptstyle h_{\partial M_K}} & & & & \Big\downarrow{\scriptstyle h} \\
H_1(\partial M_K; \mathrm{Ad}\circ\rho) & \xrightarrow{i_*} & H_1(M_K; \mathrm{Ad}\circ\rho) & \hookrightarrow & H_1(M_K, p; \mathrm{Ad}\circ\rho)
\end{array}$$

where h and $h_{\partial M_K}$ are the quotient maps.

Let $u \in \mathfrak{sl}_2(\mathbb{C})$ be an $\mathrm{Ad}\circ\rho$-invariant vector and $\gamma \in \pi_1(M_K)$. Any element $u \otimes \gamma$ of $H_1(\partial M_K; \mathrm{Ad}\circ\rho)$ can be lifted to $u \otimes dw$ in $S_1^{\partial}(\rho)$. Since ρ is admissible, there exist $a, b \in \mathbb{C}$ such that $\mathcal{Z}(\partial M_K; \mathrm{Ad}\circ\rho) = \ker i_* = \langle a\,(v_\rho \otimes \ell) + b\,(v_\rho \otimes m)\rangle$. Then $a\,(v_\rho \otimes d\ell) + b\,(v_\rho \otimes dm) \in \ker(h) = \mathrm{im}\,(\partial_1(\rho))$.

Reciprocally, suppose that there exist complex numbers $a, b \in \mathbb{C}$ such that $dz := a\,(v_\rho \otimes d\ell) + b\,(v_\rho \otimes dm)$ is a non-zero vector belonging to $\mathrm{im}\,(\partial_1(\rho))$. Then $h_{\partial M_K}(dz) = a\,(v_\rho \otimes \ell) + b\,(v_\rho \otimes m)$ must be non-zero since $(v_\rho \otimes \ell, v_\rho \otimes m)$ is a free basis of $H_1(\partial M_K; \mathrm{Ad}\circ\rho)$. However, $h(dz) = i_*\left(h_{\partial M_K}(dz)\right) = 0$; hence $h_{\partial M_K}(dz) \in \ker i_*$. Since $\ker i_*$ is one-dimensional, then $\ker i_* = \langle h_{\partial M_K}(dz)\rangle$, and the slope is $-\frac{b}{a}$. □

Example 5.3.19 (Trefoil knot)**.** Let T be the exterior of the right-handed trefoil knot, with group $\pi_1(M_K) = \langle u, v \mid uvu = vuv\rangle$. Any irreducible representation is conjugate to ρ with

$$\rho(u) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix} \qquad\qquad \rho(v) = \begin{bmatrix} M^{-1} & 0 \\ -1 & M \end{bmatrix}$$

where $M \in \mathbb{C}$. If $\ell = vuv^{-1}uvu^{-3}$ is the preferred longitude with corresponding meridian $m = u$, we obtain

$$\rho(\ell) = \begin{bmatrix} -M^{-6} & M^5 + M^3 + M + M^{-1} + M^{-3} + M^{-5} \\ 0 & -M^6 \end{bmatrix}$$

Whenever $M \neq \pm 1$, the vector $v_\rho = \left[0, 1, \frac{1}{M - M^{-1}}\right]$ is right $\mathrm{Ad}\circ\rho_\partial$-invariant. The Alexander matrix (acting on the right on the coefficients) whose row-space is generating $\mathrm{im}\,(\partial_1)$ is given by

$$\begin{array}{c} \\ dr_1 \\ \\ \\ dr_2 \\ \\ \end{array}
\begin{array}{c}
\begin{array}{ccc:ccc:ccc}
\multicolumn{3}{c}{\mathfrak{sl}_2(\mathbb{C})\otimes d\ell} & \multicolumn{3}{c}{\mathfrak{sl}_2(\mathbb{C})\otimes dm} & \multicolumn{3}{c}{\mathfrak{sl}_2(\mathbb{C})\otimes dv}
\end{array} \\
\left[\begin{array}{ccc:ccc:ccc}
0 & 0 & 0 & 1 & 0 & 0 & -M^{-2} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -M^{-1} & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 2M & -M^2 \\
\hdashline
-1 & 0 & 0 & -2\left(2-M^{-2}\right) & 0 & 0 & 1+M^{-2} & 0 & 0 \\
0 & -1 & 0 & 2M^{-1} & -2 & 0 & M^{-1} & 2 & 0 \\
0 & 0 & -1 & -2 & -4M & 2M^2-4 & -1 & -2M & M^2+1
\end{array}\right]
\end{array}$$

where r_1 is $uvu = vuv$ and r_2 is the longitude definition. By Proposition 5.3.18, the space $Z(M_K; \mathrm{Ad}\circ\rho)$ has generator

$$\left[0, 1, \tfrac{1}{M - M^{-1}}, 0, 6, \tfrac{6}{M - M^{-1}}, 0, 0, 0\right]$$

in the 2-dimensional subspace spanned by

$$\left\{ \left[0, 1, \tfrac{1}{M - M^{-1}}, 0, 0, 0, 0, 0, 0\right], \left[0, 0, 0, 0, 1, \tfrac{1}{M - M^{-1}}, 0, 0, 0\right] \right\}$$

and the slope is $s_T(\mathrm{Ad}\circ\rho) = -6$. In particular it does not depend on ρ. ○

Example 5.3.20 (Figure-eight knot)**.** Let K be the figure-eight knot. There is a unique component $X \subset X(M_K)$ containing irreducible characters (see for instance [Bén20, Examples 1.6.2 and 5.5]). This component is a plane curve given by the equation

$$\{2x^2 + y^2 - x^2y - y - 1 = 0\} \subset \mathbb{C}^2$$

where x it the coordinate function given by $\chi \mapsto \chi(m)$. Note that the coordinate function of the longitude is $\chi \mapsto \chi(\ell) = x^4 - 5x^2 + 2$. Using [Por97, Théorème 4.1 (ii)] and Proposition 5.3.15 we compute

$$s_K(x,y)^2 = \frac{x^2-4}{(x^4-5x^2+2)^2-4}(4x^3-10x)^2 = \frac{4(2x^2-5)^2}{(x^2-5)(x^2-1)}$$

Expanding the denominator with the relation $x^2 = \frac{y^2-y-1}{y-2}$, we obtain, up to sign

$$s_K(x,y) = \pm\frac{2(2x^2-5)(y-2)}{(y-1)(y-3)} \qquad \diamond$$

5.4 The $\mathrm{SL}_2(\mathbb{C})$-slope and the A-polynomial

In this section, we express the slope function in terms of the A-polynomial of the knot. As mentioned in Section 5.2.3, $r(X)$ might have 0-dimensional components but they are omitted in the definition of the A-polynomial.

5.4.1 The derivation formula

Theorem 5.4.1. *Let $X \subset X(M_K)$ be an irreducible component such that $r(X)$ has dimension 1. For all $\chi \in X$ with $r(\chi) = (L, M)$, the following holds*

$$s_K(\chi) = -\frac{M}{L} \cdot \frac{\partial_M A(L,M)}{\partial_L A(L,M)},$$

where $A(L,M) = A_K(L,M)$ and ∂_L and ∂_M are the partial derivatives. ◁

***Remark* 5.4.2.** Combining Proposition 5.3.15 with [Por97, Corollaire 4.9], the result of Theorem 5.4.1 follows directly, up to sign, in the case where X has itself dimension 1. We resolve those two issues. Moreover Theorem 5.4.1 does not require the characters in X to be scheme-reduced, and the factors of the A-polynomial might have multiplicities greater than 1. ⌑

Proof. From Lemma 5.3.17 it follows that the Lagrangian $Z(M_K; \mathrm{Ad} \circ \rho)$ generically identifies with the Zariski co-normal bundle of $r(X(M_K))$ in $X(\partial M_K)$. Picking local coordinates $\mathfrak{l} = \log L, \mathfrak{m} = \log M$ around $r(\chi)$, the kernel of the cotangent map is generated by

$$dA(e^{\mathfrak{l}}, e^{\mathfrak{m}}) = \partial_{\mathfrak{l}} A(e^{\mathfrak{l}}, e^{\mathfrak{m}})d\mathfrak{l} + \partial_{\mathfrak{m}} A(e^{\mathfrak{l}}, e^{\mathfrak{m}})d\mathfrak{m}$$

in $\mathbb{C}^2 = \langle d\mathfrak{l}, d\mathfrak{m}\rangle$. Using the chain rule, we obtain that it is generated by the vector

$$\left(L\frac{\partial A(M,L)}{\partial L}, M\frac{\partial A(M,L)}{\partial M}\right)$$

and the proposition follows. □

***Remark* 5.4.3.** Let T be the right-handed trefoil knot, with $A_T(L,M) = 1 + LM^6$. Theorem 5.4.1 gives

$$s_T = -\frac{M}{L} \cdot \frac{6M^5L}{M^6} = -6$$

Compare to Example 5.3.19. ⌑

5.4.2 Detecting the unknot

Let K be an oriented knot in S^3, and $A_K(L, M)$ be the A-polynomial of K. In this subsection we prove the following theorem, whose proof is a refinement of arguments by Boyer and Zhang [BZ05]:

Theorem 5.4.4. *If $\deg_M(A_K(L, M)) = 0$, then K is the trivial knot.* ◁

The following corollary asserts that the slope detects the trivial knot:

Corollary 5.4.5. *Let K be a knot in S^3 such that the slope s_K is identically zero. Then K is the trivial knot.* ◁

Proof. Since $s_K(\mathrm{Ad} \circ \rho) \equiv 0$, by Theorem 5.4.1 the A-polynomial consists of a collection of lines $L = \alpha_i$. This is prohibited by Theorem 5.4.4, unless K is the trivial knot. □

The rest of the section is devoted to the proof of Theorem 5.4.4.

Proof of Theorem 5.4.4. We assume that the M-degree of the A-polynomial is zero, and we will prove that K must be the trivial knot. The A-polynomial of K can be written as a finite product

$$A_K(M, L) = \prod_i (L - \alpha_i)$$

Claim. *The α_i are roots of unity.* ◁

Proof of the claim. By [Coo+94, Proposition 3.1], compactifying the line $L - \alpha_i$ in $\mathbb{CP}^2$ yields an *ideal* point which produces an incompressible surface S in M whose boundary curves are parallel to the longitude ℓ. Moreover, by the *root of unity phenomenon* ([Coo+94, Theorem 5.7]), there is an associated representation $\rho\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ such that the eigenvalues $\rho(\ell)$ are roots of unity. By construction, those eigenvalues are precisely $\alpha_i^{\pm 1}$, proving the claim. □

Assume now that K is not the trivial knot. Let $M_K(r)$ the 3-manifold obtained by Dehn surgery on K, with coefficient $r \in \mathbb{Q}$. We use the following result of P. Kronheimer and T. Mrowska:

Theorem 5.4.6 ([KM04])**.** *For any non-zero integer n, there is an irreducible representation $\rho_{1/n}\colon \pi_1(M_K(1/n)) \to \mathrm{SU}_2(\mathbb{C})$ with non-cyclic image.* ◁

Composing with the epimorphism $\pi_1(M_K) \to \pi_1(M_K(1/n)) = \pi_1(M_K)/\langle\langle m\ell^n\rangle\rangle$, the representations $\rho_{1/n}$ yield a family of irreducible representations still denoted by

$$\rho_{1/n}\colon \pi_1(M_K) \longrightarrow \mathrm{SU}_2(\mathbb{C}) \subset \mathrm{SL}_2(\mathbb{C})$$

whose character is denoted $\chi_{1/n} \in X(M_K)$. For all i, the image $r(\chi_{1/n})$ of the character $\chi_{1/n}$ in $X(\partial M_K)$ belongs to $\{L - \alpha_i = 0\}$ if $\rho_{1/n}$ is conjugated to

$$\rho(\ell) = \begin{bmatrix} \alpha_i & 0 \\ 0 & {\alpha_i}^{-1} \end{bmatrix} \qquad \rho(m) = \begin{bmatrix} {\alpha_i}^{-n} & 0 \\ 0 & {\alpha_i}^{n} \end{bmatrix}$$

By the claim above, α_i is a root of unity. Let d_i be its order.

Claim. *For any i, for any $k \in d_i\mathbb{Z}$, the image $r(\chi_{1/k})$ does not belong to $\{L - \alpha_i = 0\}$.* ◁

Proof of the claim. Indeed $d_i | k$ implies $\rho_k(m) = \mathrm{Id}$, contradicting the irreducibility of ρ_k. □

Finally, we have proved the intermediate result:

Lemma 5.4.7. *For $d = \prod d_i$, the characters $\{r(\chi_{\rho_{1/dn}})\}_{n \geq 1}$ do not belong to any of the lines $\{L - \alpha_i = 0\}$. In particular, this whole family of characters collapse to a* finite *family of isolated points in $r(X(M_K)) \subset X(\partial M_K)$.* ◁

By similar arguments, the following holds:

Claim. *It there are $k \neq k' \in d\mathbb{Z}$ such that $r(\chi_{\rho_{1/k}}) = r(\chi_{\rho_{1/k'}})$ in $X(\partial M_K)$, then $\rho_{1/k}(\ell)$ is conjugated to $\left[\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right]$ where α is a root of unity of order $p_0 \mid (k - k')$, but p_0 does not divide k neither k'.* ◁

Proof of the claim. Since $r(\chi_{\rho_{1/k}}) = r(\chi_{\rho_{1/k'}})$, we have

$$\rho_{1/k}(m) = \rho_{1/k'}(m) \text{ and } \rho_{1/k}(\ell) = \rho_{1/k'}(\ell)$$

Moreover

$$\rho_{1/k}(m\ell^k) = \rho_{1/k'}(m\ell^{k'}) = \mathrm{Id}.$$

Hence $\rho_{1/k}(\ell)^k = \rho_{1/k}(\ell)^{k'}$ and $\rho_{1/k}(\ell)$ has order dividing $k - k'$. This implies that the order of the eigenvalue α divides $k - k'$ as well. As above, this order p_0 cannot divide k, neither k', otherwise the representation $\rho_{1/k}$ would be trivial. □

Now, all isolated points x_i of $X(\partial M_K)$ in $r\left(\{\rho_{1/dn}\}_{n\geq 1}\right)$ yield an eigenvalue α_i of finite order p_i. We showed that if $\rho_{1/dn}, \rho_{1/d'n}$ are mapped to x_i, then $dn \equiv d'n \not\equiv 0 \mod p_i$. In particular, with $p = \prod p_i$, the representation ρ_{np} is not mapped on any of the isolated points x_i. Since it is neither mapped into one of the lines $\{L - \alpha_i = 0\}$, this gives a contradiction and proves that K is the trivial knot. □

5.4.3 The slope at an ideal point

In this section we prove Theorem 5.1.5. The context of this result is the work of M. Culler and P. Shalen (see for instance [Sha01]) which associates incompressible surfaces in M_K to ideal points of curves of $X(M_K)$.

Let $X \subset X(M_K)$ be an irreducible component whose image $r(X) = Y$ is a curve in $X(\partial M_K)$, defined as the zero locus of an irreducible factor P of $A_K(L, M)$. Its function ring is usually denoted by $\mathbb{C}[Y] = \mathbb{C}[L, M]/(P)$, and its function field is $\mathbb{C}(Y) = \mathrm{Frac}(\mathbb{C}[Y])$.

To any point y in Y one can associate a *discrete valuation* v on the multiplicative group $\mathbb{C}(Y)^*$ in the field $\mathbb{C}(Y)$ of rational functions on Y. A discrete valuation $v\colon \mathbb{C}(Y)^* \to \mathbb{Z}$ is a group epimorphism satisfying $v(f + g) \geq \min(v(f), v(g))$. The valuation associated to a smooth point y is simply the map

$$f \longmapsto v_y(f) = \mathrm{ord}_y\, f$$

given the vanishing order of f at the point y. More generally, the smooth projective model $\overline{Y}$ of Y is smooth compact curve bi-rational to Y, canonically defined up to isomorphism, and the points of $\overline{Y}$ are bijectively associated to discrete valuations on the function field $\mathbb{C}(Y) \simeq \mathbb{C}(\overline{Y})$.

An *ideal* point y of Y is a point added 'at infinity' in the smooth projective model $\overline{Y}$, it corresponds to a valuation v_y on $\mathbb{C}(Y)$ such that not every regular function $f \in \mathbb{C}[Y]$ has non-negative valuation $v_y(f)$. In other words, some regular functions (at least one) should have poles at y.

In [CS83], M. Culler and P. Shalen gave a procedure to construct an *incompressible* surface Σ in M_K from the data of an ideal point x in a sub-curve C of $X(M_K)$ together with the valuation $v_x\alpha^\infty\colon \mathbb{C}(C)^* \to \mathbb{Z}$. Not any ideal point $x \in X(M_K)$ yields an ideal point $y = r(x) \in X(\partial M_K)$.

In this special case, the ideal point y in Y gives an incompressible surface in M_K of a particular kind: as observed in [Coo+94, Proposition 3.1], the incompressible surface Σ must have non-empty boundary $\partial\Sigma \subset \partial M_K$. The curve $\partial\Sigma$ is a finite union of parallel circles in ∂M_K and uniquely determines a *boundary slope* in $\mathbb{Q} \cup \{\infty\}$: the slope of $a\ell + bm$ in $H_1(\partial M_K; \mathbb{Z})$ is the rational number $\frac{b}{a}$.

On the other hand, the Newton polygon of $A_K(L, M) = \sum_{i,j} a_{i,j} L^i M^j$ is the convex hull in $\mathbb{C}^2$ of the points $\{(i, j) \in \mathbb{Z}^2 \mid a_{i,j} \neq 0\}$. It is a convex polygon of $\mathbb{C}^2$ with integral vertices, whose sides have a slope in $\mathbb{Q} \cup \{\infty\}$. In [Coo+94], D. Cooper, M. Culler, H. Gillet, D. Long and P. Shalen proved the following result:

Theorem 5.4.8 ([Coo+94, Theorem 3.4])**.** *The slopes of the sides of the Newton polygon of the* A*-polynomial* $A_K(L, M)$ *are boundary slopes of incompressible surfaces in* M_K *which correspond to ideal points of one-dimensional components of* $r^*(X(M_K))$ *in* $X(\partial M_K)$*.* ◁

Our next statement states that the slope invariant studied in this chapter coincides with the slopes of [Coo+94] at ideal points.

Theorem 5.1.5. *Let* y *be an ideal point in a one-dimensional component* Y *of the* A*-polynomial. Then the value of the slope function at the ideal point* y *equals* minus *the boundary slope of an incompressible surface corresponding to* y *or* minus *the slope of the corresponding side of the Newton polygon of the* A*-polynomial.* ◁

Proof. The coordinate functions L, M define rational functions on Y, in particular their valuations $v_y(L)$ and $v_y(M)$ are well-defined. Since y is an ideal point and L, M generate the coordinate ring $\mathbb{C}[Y]$ of the curve Y, at least one of this valuation must be negative, and at least one of these coordinate functions must have a pole at y.

Lemma 5.4.9. *The value of* s_K *at the ideal point* y *is* $\frac{v_y(L)}{v_y(M)}$. ◁

Now the theorem follows directly from Lemma 5.4.9, because it is proven in [Coo+94, Proposition 3.1] that the quantity $-\frac{v_y(L)}{v_y(M)}$ is the boundary slope of an incompressible surface corresponding to y. □

Proof of Lemma 5.4.9. From the proof of Proposition 5.3.15, we deduce that the value of the slope at y is given by

$$s_K(y) = \lim_{(L,M)\to y} \frac{r^*(dL/L)}{r^*(dM/M)}$$

The following argument is an algebraic analogue of taking Taylor expansion of the functions L and M around the ideal point y. We pick t a local coordinate around y. It is characterised by $v_y(t) = 1$, and we can write

$$L = u_1 t^{v_y(L)}$$

for $u_1 \in \mathbb{C}(Y), v_y(u_1) = 0$, and similarly

$$M = u_2 t^{v_y(M)}$$

for $u_2 \in \mathbb{C}(Y), v_y(u_2) = 0$. Moreover, near y it follows that

$$\frac{r^*(dL/L)}{r^*(dM/M)} = \frac{v_y(L)/t}{v_y(M)/t} = \frac{v_y(L)}{v_y(M)}$$

and the claim follows. □

CHAPTER

6

GENERALISED SLOPE OF LINKS

Contents

6.1 Slope of a Lagrangian

A *symplectic space* is a couple (V, q_s) where V is a $\mathbb{R}$-vector space and q_s is a definite positive antisymmetric quadratic form on V.

Definition 6.1.1. A Lagrangian subspace U of (V, q_s) is a subspace such that $q_s(x, y) = 0$ for all $x, y \in U$. ◇

Any symplectic space (V, q_s) is of even dimension, i.e. $\dim V = 2r$ for some $r \in \mathbb{N}$, and any Lagrangian U of V has dimension r. The Lagrangian Grassmannian of (V, q_s) is denoted $\Lambda(V)$.

Definition 6.1.2. A basis $\mathcal{B} = (x_1, \ldots, x_r \,;\, y_1, \ldots, y_r)$ of (V, q_s) is *symplectic* if :

$$\forall\, i, j \in \{1, \ldots, r\} : \quad \begin{cases} q_s(x_i, y_j) = \delta_{i,j} \\ q_s(x_i, x_j) = q_s(y_i, y_j) = 0 \end{cases}$$ ◇

The characterisation of Lagrangian subspaces using complexification was introduced by V.I. Arnol'd [Arn67]. Readers can refer to [ACL03] for the proofs of the standard results relating to Lagrangians recalled in Sections 6.1.1 and 6.1.2. The objective of this section is to make the following new construction:

Theorem 6.1.3. *Let (V, q_s) be a real symplectic space and let $\mathcal{B}$ be a symplectic basis. Then the triple $(V, q_s, \mathcal{B})$ gives rise to a function*

$$s : \Lambda(V) \longrightarrow \left]-\frac{\pi}{2}; \frac{\pi}{2}\right]$$

called the slope of a Lagrangian*.* ◁

6.1.1 Complexification

Let (V, q_s) be a real symplectic vector space of dimension $2r$, and let $\mathcal{B}$ be a given symplectic basis. The basis $\mathcal{B}$ is naturally divided into two families $\mathcal{B}^R = (b_1, \ldots, b_r)$ and $\mathcal{B}^I = (b_{r+1}, \ldots, b_{2r})$.

Definition 6.1.4. Let J be the automorphism of V defined by:

$$J(x_i) := y_i \qquad\qquad J(y_i) := -x_i$$

for every $1 \le i \le r$. Then J is called the *canonical complex structure* on (V, q_s) with respect to $\mathcal{B}$. ◇

Note that $J^2 = -\mathrm{Id}_V$. The space V is decomposed as $V = V^I \oplus V^R$ where $\mathcal{B}^R$ (resp. $\mathcal{B}^I$) acts as a basis for V^R (resp. V^I). Since J is an isomorphism from V^R to V^I, every vector $v \in V$ can be uniquely be decomposed as $v = J\left(v^I\right) + v^R$ where v^R and v^I are both vectors of rank r belonging to V^R. Now denote by j the multiplicative right action of J on V. Then V can be seen as a complex vector space $V^{\mathbf{C}}$ of complex dimension r, called the *complexification of* V. Moreover, there are two inverse $\mathbb{R}$-linear applications $p_{\mathbf{R}}$ and $p_{\mathbf{C}}$ defined by:

$$V \underset{p_{\mathbf{C}}}{\overset{p_{\mathbf{R}}}{\rightleftarrows}} V^{\mathbf{C}}$$

$$v = \left(v^I\right) J + v^R \longleftrightarrow v^{\mathbf{C}} = v^I \cdot j + v^R$$

Note that $p_{\mathbf{R}}$ is $\mathbb{R}$-linear but is *not* $\mathbb{C}$-linear, since the multiplication by a complex scalar is not defined directly on V. Subsequently, one can also define complexification and realification for endomorphisms:

$$P_{\mathbf{C}} : \begin{array}{ccc} \mathrm{E}(V) & \longrightarrow & \mathrm{E}\left(V^{\mathbf{C}}\right) \\ f & \longmapsto & p_{\mathbf{C}} \circ f \circ p_{\mathbf{R}} \end{array} \qquad P_{\mathbf{R}} : \begin{array}{ccc} \mathrm{E}\left(V^{\mathbf{C}}\right) & \longrightarrow & \mathrm{E}(V) \\ g & \longmapsto & p_{\mathbf{R}} \circ g \circ p_{\mathbf{C}} \end{array}$$

and the complexified basis $\mathcal{B}^{\mathbf{C}}$ of $V^{\mathbf{C}}$ by

$$\mathcal{B}^{\mathbf{C}} := (y_i \cdot j + x_i)_{1 \le i \le r}$$

6.1.2 Characterisation of Lagrangians

Definition 6.1.5. Let q_e be the Euclidean form on V defined by

$$q_e(x, y) := q_s((x)J, y) \tag{6.1}$$

Let q_h be the Hermitian form defined on $V^{\mathbf{C}}$ by

$$q_h(z, w) := q_e\left(p_{\mathbf{R}}(z), p_{\mathbf{R}}(w)\right) + q_s\left(p_{\mathbf{R}}(z), p_{\mathbf{R}}(w)\right) \cdot j \tag{6.2}$$

These two forms are induced by the symplectic form q_s. ◇

Definition 6.1.6. Each of the three forms q_s, q_e, q_h has a corresponding subgroup of stable endomorphisms.

- The subgroup of $\mathrm{E}(V)$ respecting q_s is called the *symplectic group* $\mathrm{Sp}(V)$.
- The subgroup of $\mathrm{E}(V)$ respecting q_e is called the *orthogonal group* $\mathrm{O}(V)$.
- The subgroup of $\mathrm{E}\left(V^{\mathbf{C}}\right)$ respecting q_h is called the *unitary group* $\mathrm{U}\left(V^{\mathbf{C}}\right)$. ◇

These three groups are linked together by the following result:

Theorem 6.1.7. $P_{\mathbf{C}}\left(\mathrm{Sp}(V) \cap \mathrm{O}(V)\right) = \mathrm{U}\left(V^{\mathbf{C}}\right)$ ◁

Theorem 6.1.7 is the key to characterise every Lagrangian subspace. Indeed, any basis of a Lagrangian U can be extended into a symplectic basis $\mathcal{B}_U$ of V. This basis is naturally associated to a symplectic endomorphism f_U, namely the change of basis from $\mathcal{B}$. One can then orthonormalise $\mathcal{B}_U$ without affecting its symplectic property. According to Theorem 6.1.7, the endomorphism f_U can then be complexified into an Hermitian endomorphism of $V^{\mathbf{C}}$. Reciprocally, the realification of any Hermitian homeomorphism is symplectic and thus sends any Lagrangian to another Lagrangian. From these ideas one obtains the characterisation theorem=

Theorem 6.1.8. *The group* $\mathrm{U}\left(V^{\mathbf{C}}\right)$ *acts transitively on* $\Lambda(V)$*. Moreover, the subgroup* $\mathrm{O}\left(V^{\mathbf{C}}\right)$ *of* $\mathrm{U}\left(V^{\mathbf{C}}\right)$ *is the stabiliser of the* standard Lagrangian V^R. ◁

Corollary 6.1.9. *There is a bijection between sets:*

$$\begin{array}{rccl} G: & \Lambda(V) & \xrightarrow{\sim} & \mathrm{U}\left(V^{\mathbf{C}}\right) / \mathrm{O}\left(V^{\mathbf{C}}\right) \\ & U & \longmapsto & [g_U] \ \textit{such that } V^R \cdot g_U = U \end{array}$$ ◁

***Remark* 6.1.10.** The characterising class $G(U)$ of a Lagrangian does not depend on the full symplectic basis $\mathcal{B}$ but only on the decomposition $\mathcal{B} = \mathcal{B}^R \oplus \mathcal{B}^I$. In other words, if $g \in \mathrm{O}(V)$ is such that $\left\langle g(\mathcal{B}^R)\right\rangle = \left\langle \mathcal{B}^R \right\rangle$ and $\left\langle g(\mathcal{B}^I)\right\rangle = \left\langle \mathcal{B}^I \right\rangle$ then the map G built from the symplectic basis $g(\mathcal{B})$ is identical to the one built from $\mathcal{B}$. ⌑

Rather than using a class of Hermitian endomorphisms to fully characterises any Lagrangian, we define the slope as the determinant of this class. Since $\det(g_U) \in S^1/\{\pm 1\}$, by convention we only take the argument of the left representative in the unit circle.

Definition 6.1.11. The *slope of a Lagrangian* is the function

$$\begin{array}{rccl} s_V: & \Lambda(V) & \longrightarrow & \left]-\frac{\pi}{2}; \frac{\pi}{2}\right] \\ & U & \longmapsto & \arg\left(\det\left(g_U\right)\right) \end{array}$$ ◇

Remark that the slope does not fully characterises a Lagrangian unlike G.

The following result explains how one can explicitly compute the slope of a Lagrangian using a generating matrix on $\mathcal{B}$.

Proposition 6.1.12. *Let* $U \in \Lambda(V)$ *and* M_U *be a generating matrix of* U *in the basis* $\mathcal{B}$*, of size* $(r, 2r)$*. Suppose that the rows of* M_U *are* q_e*-orthonormal. Consider the two halves* M_U^R *and* M_U^I*, which are real square matrices of size* r*. Then the slope is given by*

$$s_V(U) = \arg\left(\det\left(M_U^R + M_U^I \cdot j\right)\right)$$ ◁

Proof. Consider the real square matrix $\overline{M_U}$ of size $(2r, 2r)$ given by the block description:

$$\overline{M_U} := \begin{array}{c} \\ V^I \\ V^R \end{array}\begin{array}{c} V^I \quad V^R \\ \left[\begin{array}{c} (M_U)\,J \\ \hdashline M_U \end{array}\right] \end{array} = \begin{array}{c} \\ V^I \\ V^R \end{array}\begin{array}{c} V^I \qquad V^R \\ \left[\begin{array}{c:c} M_U^R & -M_U^I \\ \hdashline M_U^I & M_U^R \end{array}\right] \end{array}$$

By assumption, $\overline{M_U} \in \mathrm{O}(V)$. One easily checks that $\overline{M_U} \in \mathrm{Sp}(V)$. Then by Theorem 6.1.7, we have $P_{\mathbf{R}}(\overline{M_U}) = M_U^R - M_U^I \cdot j \in \mathrm{U}\left(V^{\mathbf{C}}\right)$. The matrix

$$S_r := \begin{array}{c} V^I \quad V^R \\ \left[\begin{array}{c:c} 0_r & I_r \end{array}\right] \end{array}$$

generates the standard Lagrangian V^R. It is clear that $S_r \overline{M_U} = M_U$. Therefore $P_{\mathbf{R}}(\overline{M_U}) = g_U$ as in the definition of $s_V(U)$. □

Corollary 6.1.13. *If r is even then $s_V(V^R) = s_V(V^I) = 0$.* ◁

Proof. The matrix $S_r \cdot J = [I_r : 0_r]$ generates V^I in the basis $\mathcal{B}$. Applying Proposition 6.1.12, one gets

$$s_V(V^I) = \arg\left(\det(I_r \cdot j)\right) = \arg\left(j^r \det I_r\right) = \arg(\det I_r) = s_V(V^r) = 0$$ □

6.2 Generalised slope construction

6.2.1 Character torus

Let K be a knot with peripheral system $(\pi_1(M_K), \ell, m)$. Consider a unitary character $\omega : \pi_1(M_K) \to S^1 \subset \mathbb{C}^*$. The character is determined by $\omega(m) := e^{i\theta_m}$. Denote $\omega(\ell) := e^{i\theta_l}$. Let ϕ be the realification morphism of $\mathbb{C}$ and define $\rho_\omega := \phi(\omega)$. The values of ρ_ω are in $\mathrm{SO}_2(\mathbb{R})$ and are given by

$$\rho_\omega(m) = \begin{bmatrix} \cos\theta_m & -\sin\theta_m \\ \sin\theta_m & \cos\theta_m \end{bmatrix} \qquad \rho_\omega(\ell) = \begin{bmatrix} \cos\theta_l & -\sin\theta_l \\ \sin\theta_l & \cos\theta_l \end{bmatrix} \tag{6.3}$$

The following lemma describes the complete computation of the first twisted homology group of the torus for the representation ρ_ω.

Lemma 6.2.1. *Let $\mathbb{R}(\rho_\omega)$ be $\mathbb{R}^2$ seen as a $\mathbb{R}$-module for the action of ρ_ω. Then*

$$H_1\left(\partial M_K; \rho_\omega\right) = \begin{cases} \langle m, \ell\rangle \otimes \mathbb{R}^2 & \text{if } \omega(m) = \omega(\ell) = 1 \\ \{0\} & \text{otherwise} \end{cases}$$ ◁

Proof. This is a direct application of Lemma A.2.1 on page 106 with the additional remark that

$$V_{\rho_\omega} = \ker(\rho_\omega(\ell) - I_2) \cap \ker(\rho_\omega(m) - I_2) = \begin{cases} \mathbb{R}^2 & \text{if } \omega(m) = \omega(\ell) = 1 \\ \{0\} & \text{otherwise.} \end{cases}$$ □

Let now $L = L_1 \cup \cdots \cup L_n$ be a link. We reuse some notations from Section 4.2. We suppose that all linking numbers are zero, i.e. $\mathrm{lk}(L_i, L_j) = 0$ for every pair $i \neq j$.

The group $H_1(M_L)$ is free abelian, generated by the classes m_i of the meridians of the components L_i. Let $\omega : \pi_1(M_L) \to S^1 \subset \mathbb{C}^*$ be a unitary character. Since ω is abelian, it is fully determined by its value ω_i on m_i for all $1 \le i \le n$. Let $\mathbb{T}^n := \mathrm{Hom}\left(\pi_1(M_L), S^1\right)$ be the set of unitary characters. Then we have a natural identification

$$\mathbb{T}^n \simeq \{\omega = (\omega_1, \ldots, \omega_n) \mid \omega_i \in S^1\} \subset (\mathbb{C}^*)^n$$

As the notation suggests, this set only depends on the number of components n of the link L.

6.2.2 Symplectic structure of the twisted homology

We consider a symplectic structure on the first homology of ∂M_L.

For a given character $\omega \in \mathbb{T}^n$, we define

$$D(\omega) := \{i \mid \omega_i = 1\} \qquad\qquad d(\omega) := \#D(\omega)$$

The components L_i where $i \in D(\omega)$ are called *generating for* ω.

Proposition 6.2.2. *The space $H_1(\partial E_L, \mathbb{R}(\rho_\omega))$ endowed with the intersection form is a symplectic space of dimension $4d(\omega)$, and there is a canonical isomorphism*

$$H_1(\partial E_L; \rho_\omega) = \bigoplus_{i \in D(\omega)} H_1(\partial E_{L_i}; \rho_{\omega_i}) \qquad \triangleleft$$

Proof. Each $\mathbb{R}$-vector space $H_1(\partial E_{L_i}; \rho_{\omega_i})$ is symplectic as a direct application of Theorem A.1.8 on page 104. The intersection form on $H_1(\partial E_L; \rho_\omega)$ is given by the direct sum of all the forms since $\omega(\ell_i) = 1$ and $\rho_\omega(\ell_i) = I_2$ (as $\mathrm{lk}(L_i, L_j) = 0$, see Eq. (4.1) on page 74). By Lemma A.2.1 on page 106, $H_1(\partial E_{L_i}; \rho_{\omega_i})$ has dimension 4 if $\omega(m_i) = 1$, i.e. if $i \in D(\omega)$, and dimension 0 otherwise. $\square$

Definition 6.2.3. Let $\langle c_1, c_2 \rangle$ be a fixed basis of $\mathbb{R}^2$. Let $\mathcal{B}_L$ be the function:

$$\begin{aligned} \mathcal{B}_L : \qquad \mathbb{T}^n &\longrightarrow \mathbb{R}^{4n} \\ \omega = (\omega_1, \ldots, \omega_n) &\longmapsto (a_1, \ldots, a_{2n}\ ;\ b_1, \ldots, b_{2n}) \end{aligned}$$

where

$$\forall\, i \in D(\rho),\ \forall\, j \in \{1,2\} : \begin{cases} a_{2(i-1)+j-1} & = m_i \otimes c_j \\ b_{2(i-1)+j-1} & = \ell_i \otimes c_j \end{cases}$$
$$\forall i \notin D(\rho),\ \forall\, j \in \{1,2\} : a_{2(i-1)+j-1} = b_{2(i-1)+j-1} = 0 \qquad \diamond$$

The rank of the vector family $\mathcal{B}_L(\omega)$ is $4d(\omega) \leq 4n$ and thus depends on ω. Nevertheless, we slightly abuse notation and also call $\mathcal{B}_L(\omega)$ the full-rank basis of the subspace of $\mathbb{R}^{4n}$ it generates.

Proposition 6.2.4. *The family $\mathcal{B}_L(\omega)$ is a symplectic basis of $H_1(\partial M_L; \rho_\omega)$.* $\triangleleft$

Proof. The family is a basis thanks to Lemma A.2.1. Using the notations of Appendix A.1, on every component $H_1(\partial M_{L_i}; \rho_{\omega_i})$ with $i \in D(\omega)$ the intersection form particularises as

$$\begin{aligned} \langle \ell_i \otimes c_r \mid m_i \otimes c_s \rangle &= (c_r \mid c_s) = \delta_{r,s} \\ \langle m_i \otimes c_r \mid \ell_i \otimes c_s \rangle &= (c_r \mid c_s) = -\delta_{r,s} \\ \langle \ell_i \otimes c_r \mid \ell_i \otimes c_s \rangle &= 0 \\ \langle m_i \otimes c_r \mid m_i \otimes c_s \rangle &= 0 \end{aligned} \qquad \square$$

***Remark* 6.2.5.** It occurs that the coefficients of $H_1(\partial M_L; \rho_\omega)$ come from the realification of $\mathbb{C}(\omega)$ into $\mathbb{R}(\rho_\omega) \simeq \mathbb{R}^2$ using ϕ. However this symplectic structure on $\mathbb{R}^2$ does not extend to the tensor product in general. As shown in Theorem A.1.8, only the specific property of the intersection form on ∂M_L ensures that $H_1(\partial M_L; \rho_\omega)$ is indeed symplectic, with the longitudes and meridians as the 'real' and 'imaginary' parts respectively. When a disambiguation is necessary, we call ϕ the *coefficient* realification and $p_{\mathbf{R}}$ the *intersection* realification. ⌑

6.2.3 Definition of the generalised slope

Consider the inclusion map

$$i : \partial M_L \hookrightarrow M_L$$

and the induced map on the first twisted homology groups of (M_L, ρ_ω):

$$i_* : H_1(\partial M_L; \rho_\omega) \longrightarrow H_1(M_L; \rho_\omega)$$

We consider the space $\mathcal{Z}(\rho_\omega) := \ker i_*$.

Lemma 6.2.6. *The subspace $\mathcal{Z}(\rho_\omega)$ is a Lagrangian subspace of the symplectic vector space $H_1(\partial M_L; \rho_\omega)$ endowed with the intersection form.* ◁

Proof. This is a direct application of Theorem A.1.9 on page 105. □

Definition 6.2.7. The *generalised slope* of the link L is the function

$$\begin{aligned} s_L : \quad \mathbb{T}^n &\longrightarrow \left]-\tfrac{\pi}{2}; \tfrac{\pi}{2}\right] \\ \omega &\longmapsto s_{H_\omega}(\mathcal{Z}(\rho_\omega)) \end{aligned}$$

where $H_\omega := H_1(\partial M_L; \rho_\omega)$. By convention, when we have $\omega_i \neq 1$ for all $i \in \{1, \dots, n\}$ and thus $d(\omega) = 0$, we set $s_L(\omega) := 0$. ◇

The generalised slope is computed in two steps: the first is to use Algorithm 4.3.1 and Theorem A.4.1 on page 76 and on page 108 to determine a generating matrix of $\mathcal{Z}(\rho_\omega)$, which is then orthonormalised. Then Proposition 6.1.12 is used to compute the slope $s_L(\omega)$ proper.

***Remark* 6.2.8.** By Corollary 6.1.13, the generalised slope $s_L(\omega)$ cannot distinguish between the case where $\ker i_*$ is generated only by the longitudes (i.e. $\mathcal{Z}(\rho_\omega) = H^R_\omega$) and the case where $\ker i_*$ is only generated by the meridians (i.e. $\mathcal{Z}(\rho_\omega) = H^I_\omega$). However in practice the generating matrix of $\mathcal{Z}(\rho_\omega)$ will contains only zeros in its L-side in the first case, and in its R-side in the second. Since computing this matrix is necessary to get $s_L(\omega)$, one can immediately identify these two cases before computing the Lagrangian slope. ⌑

6.2.4 Relation with the character slope

We explain the relation between the generalised slope s_L of Definition 6.2.7 and the 'actual' link character slope presented in Chapter 4. Let $L = L_1 \cup L'$ be a link with a single component L_1 distinguished and such that $\operatorname{lk}(L_1, L_i) = 0$ for every $2 \le i \le n$. Reusing notations from Section 4.2 we say that the character slope is defined on the subset of $\mathbb{T}^n$ composed of non-vanishing unitary characters:

$$\mathcal{A}^\circ_u(L_1/L') := \{\omega = (1, \omega_2, \dots, \omega_n) \mid \omega_i \in S^1, \omega_i \neq 1\} \subset \mathbb{T}^n$$

The next proposition show how the generalised slope restricts to the character slope, for some specific characters.

Proposition 6.2.9. *Let $\omega \in \mathcal{A}^\circ_u(L_1/L')$ be an admissible non-vanishing unitary character. Then $s_{(L_1/L')}(\omega) \in \mathbb{R} \cup \{\infty\}$ and we have*

$$s_L(\omega) = \begin{cases} 2\arctan s_{(L_1/L')}(\omega) - \pi & \text{if } s_{(L_1/L')}(\omega) > 1. \\ 2\arctan s_{(L_1/L')}(\omega) & \text{if } -1 < s_{(L_1/L')}(\omega) \le 1. \\ 2\arctan s_{(L_1/L')}(\omega) + \pi & \text{if } s_{(L_1/L')}(\omega) \le -1. \\ 0 & \text{if } s_{(L_1/L')}(\omega) = \infty. \end{cases}$$ ◁

Proof. By construction, the *coefficient* realification morphism ϕ makes the following diagram commute:

$$\begin{array}{ccc} H_1(\partial M_L; \omega) & \xrightarrow{i_*} & H_1(M_L; \omega) \\ \phi \downarrow & & \downarrow \phi \\ H_1(\partial M_L; \rho_\omega) & \xrightarrow[i_*]{} & H_1(M; \rho_\omega) \end{array}$$

One has therefore $\phi(\mathcal{Z}(\omega)) = \mathcal{Z}(\rho_\omega)$. Let $\sigma := s_{(L_1/L')}(\omega)$ and suppose that $\sigma \neq \infty$. By Definition 4.2.4 on page 74, one has

$$\mathcal{Z}(\omega) = \langle \sigma \cdot m_1 - \ell_1 \rangle$$

and thus

$$\mathcal{Z}(\rho_\omega) = \langle (\sigma \cdot m_1 - \ell_1) \otimes c_1 \,,\, (\sigma \cdot m_1 - \ell_1) \otimes c_2 \rangle$$

since σ is a real number by Proposition 4.2.6. An orthonormalised generating matrix of $\mathcal{Z}(\rho_\omega)$ in the basis $\mathcal{B}_L$ of $H_1(\partial M_L; \rho_\omega)$ is given by

$$\frac{1}{\sqrt{1+\sigma^2}} \begin{bmatrix} \overset{m_1 \otimes c_1}{\sigma} & \overset{m_1 \otimes c_2}{0} & \vdots & \overset{\ell_1 \otimes c_1}{-1} & \overset{\ell_1 \otimes c_2}{0} \\ 0 & \sigma & \vdots & 0 & -1 \end{bmatrix}$$

Applying Proposition 6.1.12, we get that

$$s_{H_\omega}(\mathcal{Z}(\rho_\omega)) = \arg \frac{1}{1+\sigma^2}\begin{vmatrix} \sigma\cdot j+1 & 0 \\ 0 & \sigma\cdot j+1 \end{vmatrix} = \arg \frac{(1-\sigma^2)+2\sigma\cdot j}{1+\sigma^2}$$

Using the tangent half-angle, one has $\arctan\left(\frac{2\sigma}{1-\sigma^2}\right) = 2\sigma$. Evaluating the argument of $s_{H_\omega}(\mathcal{Z}(\rho_\omega))$ inside $\left]-\frac{\pi}{2};\frac{\pi}{2}\right]$ using arctan gives the desired result thanks to this formula.

Finally, if $\sigma = \infty$ then $\mathcal{Z}(\omega) = \langle m_1\rangle$ and $\mathcal{Z}(\rho_\omega) = H^I_\omega$. By Corollary 6.1.13, we obtain $s_{H_\omega}(\mathcal{Z}(\rho_\omega)) = 0$. □

6.3 Concordance invariance

In this section we show that the generalisation of the character slope from Chapter 4 is still a concordance invariant.

Recall from Definition 4.2.9 on page 75 that $\omega \in \mathbb{T}^n$ is a concordance root if there exists no polynomial $P \in U$ such that $P(\omega) = 0$. Let $\mathbb{T}^n_!$ be the subset of $\mathbb{T}^n$ composed of characters that are *not* concordance roots. We also exclude the characters ω such that $d(\omega) = 0$. The set $\mathbb{T}^n_!$ can be stratified to account for which values of the character are equal to 1. Let $\mathcal{P}_n$ be the set of non-empty subsets of $\{1,\dots,n\}$. For every $S \in \mathcal{P}_n$, define

$$\mathbb{T}^S_! := \{\omega \in \mathbb{T}^n_! \mid D(\omega) = S\}$$

Then we have

$$\mathbb{T}^n_! = \{(1,\dots,1)\} \sqcup \bigsqcup_{S\in\mathcal{P}_n} \mathbb{T}^S_!$$

Definition 6.3.1. Let $S \in \mathcal{P}_n$ and write $\nu = \#S$. Let also $\mu \in \{1,\dots,n-\nu\}$. A (S,μ)-*colouring* on a link $L = L_1 \sqcup \dots \sqcup L_n$ is a surjective function

$$c : \{1,\dots,n\} \longrightarrow \{d_1,\dots,d_\nu\} \sqcup \{c_1,\dots,c_\mu\}$$

such that $c(i) = d_i$ if and only if $i \in S$. ◇

For every $j \in \{1,\dots,\mu\}$, we write $\chi(j) := c^{-1}(d_j)$ and $L_{\chi(j)}$ the union of individual components of L that are c_j-coloured. As with Definition 4.1.8, (S,μ)-colouring induces a notion of concordance whose definition depends on the colouration.

Theorem 6.3.2. *Let L^0, L^1 be links with n components such that $\operatorname{lk}(L^k_i, L^k_j) = 0$ for every $1 \le i \ne j \le n$ and $s = 0,1$. Fix $S \in \mathcal{P}_n$ and $\mu \in \{1,\dots,n-\nu\}$ and make the links (S,μ)-coloured. If L^0 and L^1 are concordant then for every $\omega \in \mathbb{T}^S_!$, one has*

$$s_{L^0}(\omega) = s_{L^1}(\omega) \qquad \triangleleft$$

The proof is mostly similar to the proof of the corresponding Theorem 4.2.10 on page 75 for the character slope.

Proof. Up to reordering we suppose that $S = \{1,\dots,\nu\}$. For $k = 0,1$ we write $L^k = L^k_d \sqcup L^k_c$ with

$$L^k_d = L^s_1 \sqcup \dots \sqcup L^s_\nu \qquad\qquad L^k_c = L^s_{\nu+1} \sqcup \dots \sqcup L^s_\mu$$

Fix $\omega \in \mathbb{T}^S_!$ and let $D \cup C \subset S^3 \times [0,1]$ be the concordance. We have $D = D_1 \sqcup \dots \sqcup D_\nu$ and for every $i \in \{1,\dots,\nu\}$, $\partial D_i = -L^0_i \sqcup L^1_i$. Let $T_{D\cup C}$ be a tubular neighbourhood of $D \cup C$. Define

$$E := S^3 \times [0,1] \smallsetminus T_{D\cup C} \qquad\qquad E_0 := S^3 \times [0,1] \smallsetminus T_C$$

Then $E \cap (S^3 \times \{k\}) = M_{L^k}$ and $E_0 \cap (S^3 \times \{k\}) = M_{L^k_c}$. In particular the preferred meridian of each component of the links is sent to the preferred meridian of the corresponding cylinder inside $D \cup C$. Writing down the relative Mayer-Vietoris sequence in $\mathbb{C}$-homology of the pairs

$$(S^3 \times [0,1], S^3 \times \{k\}) = (E, M_{L^k}) \cup (\bar{T}_{D\cup C}, \bar{T}_{L^k}) = (E_0, M_{L^k_c}) \cup (\bar{T}_C, \bar{T}_{L^k_c})$$

gives

$$H_*(E, M_{L^k}) = H_*(E_0, M_{L^k_c}) = 0 \qquad (6.4)$$

Using this in the homology exact sequence of the pair (E, M_{L^k}) gives that the inclusions $M_{L^k} \hookrightarrow E$ induce an isomorphism $H_1(M_{L^k}) \simeq H_1(E)$ that sends the meridians of the link L^k to the corresponding meridians of the cylinders that composes $D \cup C$.

Now write again a relative Mayer-Vietoris sequence in twisted homology for the decomposition

$$(E_0, M_{L_d^k}) = (E, M_L) \cup (\bar{T}_D, \bar{T}_{L_d^k})$$

This gives

$$\begin{array}{c} \cdots \longrightarrow H_1(D \times S^1, L_d^k \times S^1; \rho_\omega) \\ \downarrow \\ H_1(E, M_L; \rho_\omega) \oplus H_1(\bar{T}_D, \bar{T}_{L_d^k}; \rho_\omega) \longrightarrow H_1(E_0, M_{L_d^k}; \rho_\omega) \longrightarrow \cdots \end{array}$$

Since ω is trivial on all the meridians of L_d^k and also on the meridians of D by the previous discussion, then $H_1(D \times S^1, L_d^k \times S^1; \rho_w) = 0$. Moreover, by [CNT20, Lemma 2.16] (see also [NP17]) the $\mathbb{C}$-homology of both pairs of Eq. (6.4) being null implies that their ρ_ω-twisted homology is also null:

$$H_*(E, M_{L^k}; \rho_\omega) = H_*(E_0, M_{L_c^k}; \rho_\omega) = 0$$

Replacing this in the sequence above yields $H_1(E, M_{L^k}; \rho_\omega) = 0$. Once again using this in the ρ_ω-twisted homology sequence of the pair gives an isomorphism

$$\psi : H_1(E; \rho_\omega) \xrightarrow{\simeq} H_1(M_{L^k}; \rho_\omega)$$

which preserves the meridians. Separately, since L^0 and L^i are both (S, μ)-coloured, there is a cylinder D_i joining each component L_i^0 of L_c^0 to the corresponding component L_i^0 of L_c^1. Since $\ell_i^0 = D_i \cap T_{L_i^0}$, the cylinder D_i induces an homotopy between ℓ_i^0 and ℓ_i^1. Both curves therefore have the same in $H_1(E; \rho_\omega)$, and therefore ψ sends $\ell_i^0 \in H_1(M_{L^0}; \rho_\omega)$ to $\ell_i^1 \in H_1(M_{L^1}; \rho_\omega)$. Every element of the symplectic basis $\mathcal{B}_{L^0}(\omega)$ is therefore sent to the corresponding element in $\mathcal{B}_{L^1}(\omega)$. Any vector of $\mathcal{Z}^0(\rho_\omega)$ is also homologically null in $\mathcal{Z}^1(\rho_\omega)$ and vice versa, so ψ restrains into an isomorphism $\mathcal{Z}^0(\rho_\omega) \simeq \mathcal{Z}^1(\rho_\omega)$. Therefore, the generalised slopes of the links L^0 and L^1 are equal. □

Corollary 6.3.3. *Let L^0 and L^1 be two n-coloured links with n components and liking numbers 0. If L^0 and L^1 are n-concordant then the restrictions of the generalised slope functions*

$$s_{L^k} : \mathbb{T}_!^n \longrightarrow \left]-\frac{\pi}{2}; \frac{\pi}{2}\right]$$

for $k = 0, 1$ are equal. ◁

Proof. If L^0 and L^1 are n-concordant then for every $S \in \mathcal{P}_n$ they are $(S, n-\nu)$-concordant. By Theorem 6.3.2 the generalised slope functions coincide on $\mathbb{T}_!^S$. The slopes thus coincide on $\mathbb{T}_!^n$. □

APPENDIX

A

TWISTED (CO-)HOMOLOGY

Contents

A.1 General results

References for these constructions include [Por97; DFL22b].

A.1.1 Definition

Let M be a smooth compact connected oriented n-dimensional manifold with boundary. Up to isotopy, M admits a unique p.l.-linear structure, which induces a triangulation and a CW-complex structure. Fix a base point $b \in \partial M$. Let $\pi := \pi_1(M, b)$ be the fundamental group of M and denote by $\widetilde{M}$ the universal covering of M. The group π has an action on the lifted cells from M inside $\widetilde{M}$. One can thus define the chain complex $C_*(M; \mathbb{Z}[\pi])$ generated by the lifted cells over the group ring $\mathbb{Z}[\pi]$. If N is a submanifold of M, the relative chain complex $C_*(M, N; \mathbb{Z}[\pi])$ is defined on the cells of $\widetilde{M}$ with boundary operators relative to the cells of $\widetilde{N}$.

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. Fix a basis $\mathcal{B}$ of the $\mathbb{K}$-vector space $\mathbb{K}^r$. By convention we consider vectors as rows and the matrix action on the right. Consider $\rho : \pi \to \mathrm{GL}_r(\mathbb{K})$ a representation of π. The group homomorphism ρ extends into a ring homomorphism on $\mathbb{Z}[\pi]$ and induces a right action of $\mathbb{Z}[\pi]$ on $\mathbb{K}^r$. The vector space $\mathbb{K}^r$ can thus be seen as a right $\mathbb{Z}[\pi]$-module denoted $\mathbb{K}(\rho)$.

There is a canonical involutive anti-automorphism $\dagger$ on $\mathbb{Z}[\pi]$ given by

$$\dagger : \sum_{g \in \pi} n_g g \longmapsto \sum_{g \in \pi} n_g g^{-1}$$

Define $\rho^\dagger := ({}^{\mathrm{t}}\rho)^{-1}$ where ${}^{\mathrm{t}}$ is the matrix transposition. The image of $\dagger$ is a left module for the action of $\rho^\dagger$, and up to transposition we have $\mathbb{K}(\rho)^\dagger \simeq \mathbb{K}(\rho^\dagger)$.

Definition A.1.1. Consider the (co-)chain complexes of $\mathbb{K}$-vector spaces:

$$C_*(M; \rho) := C_*(M; \mathbb{Z}[\pi]) \otimes_{\mathbb{Z}[\pi]} \mathbb{K}(\rho)$$
$$C^*(M; \rho) := \mathrm{Hom}_{\mathbb{Z}[\pi]}\left(C_*(M; \mathbb{Z}[\pi]), \mathbb{K}(\rho^\dagger)\right)$$

The *ρ-twisted (co-)homology* $H_*(M; \rho)$ (resp. $H^*(M; \rho^\dagger)$) is the (co-)homology of the (co-)chain complex $C_*(M; \rho)$ (resp. $C^*(M; \rho^\dagger)$), which have the structure of $\mathbb{K}$-vector spaces. ◇

The first module $H_1(M; \rho)$ is also called the *twisted Alexander module of* (M, ρ).

Definition A.1.2. The *twisted Alexander polynomial of* (M, ρ) is the first order of the Alexander module:

$$\Delta_M(\rho) := \mathrm{ord}\, H_1(M; \rho) \in \mathbb{Z}\left[\mathrm{GL}_r(\mathbb{K})\right] / \{\pm I_r\}$$ ◇

A.1.2 Duality

Let $(\cdot \mid \cdot)$ be a vector product on the basis $\mathcal{B}$. The product can be bilinear if $\mathbb{K} = \mathbb{R}$, or bilinear *or* sesquilinear (i.e. $(v \mid w) := v \cdot {}^{\mathrm{t}}\overline{w}$) if $\mathbb{K} = \mathbb{C}$. The representation ρ is called *unitary* if

$$\begin{cases} \rho^\dagger = \bar{\rho} & \text{if } (\cdot \mid \cdot) \text{ is sesquilinear.} \\ \rho^\dagger = \rho & \text{if } (\cdot \mid \cdot) \text{ is bilinear.} \end{cases}$$

Definition A.1.3. The *Kronecker product* is defined by

$$\begin{aligned} [\cdot \mid \cdot] : \quad C^k(M, \partial M; \rho) \times C_k(M, \partial M; \rho^\dagger) &\longrightarrow \mathbb{K} \\ (f, x \otimes v) &\longmapsto ({}^{\mathrm{t}}f(x) \mid v) \end{aligned}$$ ◇

The Kronecker product is invariant by the diagonal action of $\rho \otimes \rho^\dagger$. It is a perfect pairing and therefore induces a 'universal coefficients' formula:

Lemma A.1.4. *There is a natural vector-space isomorphism*

$$H^k(M, \partial M; \rho^\dagger) \simeq H_k(M, \partial M; \rho)^\vee$$

where $\vee$ *designates the dual vector space.* ◁

The pair $(M, \partial M)$ is a *simple Poincaré pair* as defined by C.T.C. Wall in [Wal99, Section 2]. This implies that Poincaré duality works as expected even for ρ-twisted coefficients.

Lemma A.1.5. *There are canonical Poincaré duality isomorphisms*

$$\begin{aligned} D_M : \quad & H^{n-k}(M, \partial M; \rho^\dagger) \xrightarrow{\sim} H_k(M;\rho) \\ D_M : \quad & H^{n-k}(M;\rho^\dagger) \xrightarrow{\sim} H_k(M,\partial M;\rho) \\ & f \longmapsto [M] \frown f \end{aligned}$$

where $[M] \in H_n(M, \partial M; \rho)$ *is the fundamental class of* M *and* $\frown$ *is the usual cap product.* ◁

A.1.3 Intersection form

In this section we suppose that ρ is unitary.

The standard cup product on M is naturally defined on the twisted chain complex spaces

$$\smile : C^k(\partial M;\rho^\dagger) \times C^{n-k}(\partial M;\rho^\dagger) \longrightarrow C^n(M,\partial M;\rho^\dagger \otimes \rho^\dagger)$$

where $\rho^\dagger \otimes \rho^\dagger$ designates $\mathbb{K}(\rho^\dagger) \otimes \mathbb{K}(\rho^\dagger)$ seen as a $\mathbb{Z}[\pi]$-module for the diagonal action. When ρ is unitary, the vector product is invariant by this action and can thus be seen as a morphism

$$(\cdot \mid \cdot) : \mathbb{K}(\rho^\dagger) \otimes \mathbb{K}(\rho^\dagger) \longrightarrow \mathbb{K}$$

Definition A.1.6. The *cup product form* on ρ-twisted cohomology is defined by the formula

$$\begin{aligned} \langle \cdot \mid \cdot \rangle : \quad H^k(M,\partial M;\rho^\dagger) \times H^{n-k}(M;\rho^\dagger) & \longrightarrow H^n(M,\partial M;\mathbb{K}) \simeq \mathbb{K} \\ (e,f) & \longmapsto (e \smile f)([M]) \end{aligned}$$ ◇

One can also define the intersection form on ρ-twisted homology using the cap product.

Definition A.1.7. The *intersection form* on ρ-twisted homology is defined by the formula

$$\begin{aligned} \langle \cdot \mid \cdot \rangle : \quad C_{n-k}(M;\rho) \times C_k(M,\partial M;\rho) & \longrightarrow \mathbb{K} \\ (x \otimes v, y \otimes w) & \longmapsto \sum_{\alpha \in \pi} (x\alpha \cdot y) \cdot (x\rho(\alpha) \mid y) \end{aligned}$$

where $(\cdot)$ designates the algebraic intersection number. ◇

The intersection form and the cup product are linked together by the Poincaré duality isomorphisms of Lemma A.1.5, which make the following diagram commute:

$$\begin{array}{ccc} H^k(M,\partial M;\rho^\dagger) \times H^{n-k}(M;\rho^\dagger) & \longrightarrow & H^n(M,\partial M;\mathbb{K}) \\ \downarrow{\scriptstyle D_M \times D_M} & & \downarrow{\scriptstyle \wr} \\ H_{n-k}(M;\rho) \times H_k(M,\partial M;\rho) & \longrightarrow & \mathbb{K} \end{array}$$

The properties of the cup product inherited by the intersection form give the fundamental result

Theorem A.1.8. *Suppose that* $n = 2k+1$ *and* ρ *is unitary. Then* $H_k(\partial M;\rho)$ *endowed with the intersection form is a symplectic* $\mathbb{K}$*-vector space.* ◁

A.1.4 Kernel of i_*

Consider the portion of the homology exact sequence of the pair $(M, \partial M)$

$$\cdots \longrightarrow H_{k+1}(M,\partial M;\rho) \xrightarrow{\partial} H_k(\partial M;\rho) \xrightarrow{i_*} H_k(M;\rho) \longrightarrow \cdots$$

where i_* is induced by the inclusion $i : \partial M \hookrightarrow M$. Define

$$\mathcal{Z}_k(\partial M;\rho) := \ker i_* = \operatorname{im} \partial$$

Definition 6.1.1 gives the definition of a Lagrangian subspace of a symplectic space. The fundamental result of this section is the following:

Theorem A.1.9. *Suppose that $n = 2k+1$ and that ρ is unitary. Then $\mathcal{Z}_k(\partial M;\rho)$ is a* Lagrangian subspace *of $H_k(\partial M;\rho)$ for the intersection form. In particular,*

$$\dim \mathcal{Z}_k(\partial M;\rho) = \frac{1}{2}\dim H_k(\partial M;\rho) \qquad \lhd$$

Proof. Using Lemma A.1.5 between the homology and cohomology exact sequence of the pair $(M,\partial M)$, we get that the following diagram is commutative

$$\begin{array}{ccc} H^k(M;\rho^\dagger) & \xrightarrow{i^*} & H^k(\partial M;\rho^\dagger) \\ {\scriptstyle D_M}\downarrow\wr & & \wr\downarrow{\scriptstyle D_{\partial M}} \\ H_{k+1}(M,\partial M;\rho) & \xrightarrow{\partial} & H_k(\partial M;\rho) \end{array}$$

In addition, Lemma A.1.4 implies that $i^* = (i_*)^\vee$ for the Kronecker product. On ∂M, the usual duality between the cup and cap products can be stated for $f,e \in H^k(M,\partial M;\rho^\dagger)$ as

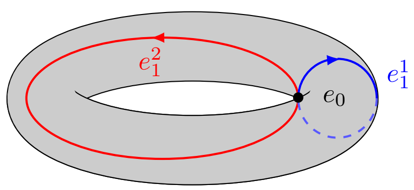
$$\langle f \mid e\rangle = (f \smile e)([\partial M]) = [f \mid [\partial M] \frown e] = [f \mid D_{\partial M}(e)]$$

Now consider two vectors $a,b \in \mathcal{Z}_k(\partial M;\rho)$. Since $\ker i_* = \operatorname{im}\partial$, we have $a = \partial A$ with $A \in H_{k+1}(M,\partial M;\rho^\dagger)$. Using the above diagram, we get

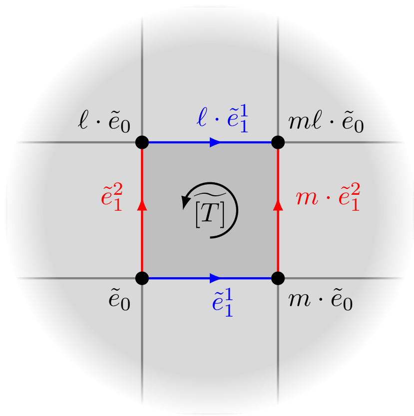
$$\begin{aligned} \langle a \mid b\rangle &= (D_{\partial M}^{-1}(a) \frown D_{\partial M}^{-1}(b))([\partial M]) \\ &= \left[D_{\partial M}^{-1}(a) \mid b\right] \\ &= \left[D_{\partial M}^{-1} \circ \partial(A) \mid b\right] \\ &= \left[i^* \circ D_M^{-1}(A) \mid b\right] \\ &= \left[D_M^{-1}(A) \mid i_*(b)\right] = 0 \end{aligned} \qquad \square$$

A.2 Twisted homology of the torus

The value of the twisted homology of the 2-torus T is the fundamental case for the construction of all slope invariants we study, as the torus represents a boundary component of the exterior M_L of a link L. We make this computation in detail from the CW-complex of T.



(a) CW-complex of T



(b) Universal covering $\widetilde{T}$

Figure A.2.1: Chain complex of the 2-torus

The generators of the CW-complex structure of T are shown on Figure A.2.1a. The longitude curve is denoted e_1^1 and the meridian curve is denoted e_1^2. The fundamental group $\pi_1(T)$ is free abelian and generated by the classes m and ℓ of the meridian and longitude of T respectively. The universal covering of the torus $\widetilde{T}$ is a $\pi_1(T)$-module homeomorphic to $\mathbb{K}^2$. The twisted chain complex of $\widetilde{T}$ as shown on Figure A.2.1b is given by:

$$\widetilde{C_2} = [\widetilde{T}] \otimes \mathbb{K}(\rho) \xrightarrow{\partial_1} \widetilde{C_1} = \langle \tilde e_1^2, \tilde e_1^1\rangle \otimes \mathbb{K}(\rho) \xrightarrow{\partial_0} \widetilde{C_0} = \{\tilde e_0\} \otimes \mathbb{K}(\rho) \longrightarrow 0$$

From Figure A.2.1b we immediately get that the matrices of the boundary operators ∂_1 and ∂_0 are given by

$$D_1 = \begin{array}{c} \quad \tilde{e}_1^1 \otimes \mathcal{B} \qquad \tilde{e}_1^2 \otimes \mathcal{B} \\ \left[I_r - \rho(\ell) \;\vdots\; \rho(m) - I_r \right] \end{array} \qquad D_0 = \begin{bmatrix} \rho(m) - I_r \\ \cdots\cdots\cdots \\ \rho(\ell) - I_r \end{bmatrix} \begin{array}{c} \tilde{e}_1^1 \otimes \mathcal{B} \\ \\ \tilde{e}_1^2 \otimes \mathcal{B} \end{array}$$

Lemma A.2.1. *Suppose that ρ is unitary and let*

$$V_\rho := \ker(\rho(\ell) - I_r) \cap \ker(\rho(m) - I_r)$$

Then there is a natural $\mathbb{K}$-vector space isomorphism

$$H_1(T;\rho) \simeq \langle \tilde{e}_1^1, \tilde{e}_1^2 \rangle \otimes V_\rho \qquad \triangleleft$$

Proof. Since $[\ell, m] = 1$ inside $H_1(T;\rho)$ and $\rho(\ell)$ and $\rho(m)$ are unitary then the two matrices are co-diagonalisable (over $\mathbb{C}$ if $\mathbb{K} = \mathbb{R}$). In the adequate basis, the module $\mathbb{K}(\rho)$ splits over $\mathbb{C}$ into

$$\mathbb{K}(\rho) = \bigoplus_{i=1}^{r} \mathbb{C}(\omega_i) \tag{A.1}$$

where $\omega_i : \pi_1(T) \to S^1$ for $1 \leq i \leq r$ are characters corresponding to the (complex) eigenvalues of ρ. Let $x \in \ker(\partial_0)$. There exists $a, b \in \mathbb{C}(\omega_i)$ such that $x = \tilde{e}_1^1 \otimes a + \tilde{e}_1^2 \otimes b$ and :

$$\begin{aligned} \partial_0(x) &= (m-1) \cdot \tilde{e}_0 \otimes a + (\ell - 1) \cdot \tilde{e}_0 \otimes b \\ &= \tilde{e}_0 \otimes (a(\omega_i(m) - 1) + b(\omega_i(\ell) - 1)) = 0 \end{aligned}$$

Then:

$$\begin{aligned} \partial_1\left(\widetilde{[T]} \otimes a\right) &= \tilde{e}_1^1 \otimes a(1 - \omega_i(\ell)) + \tilde{e}_1^2 \otimes a(\omega_i(m) - 1) \\ &= \left(\tilde{e}_1^1 \otimes a + \tilde{e}_1^2 \otimes b\right)(1 - \omega_i(\ell)) \\ &= x(1 - \omega_i(\ell)) \\ \partial_1\left(\widetilde{[T]} \otimes b\right) &= x(\omega_i(m) - 1) \end{aligned}$$

If (say) $\omega_i(\ell) \neq 1$, then every $x \in \ker(\partial_0)$ can be written as

$$x = \partial_1\left(\widetilde{[T]} \otimes a(1 - \omega_i(\ell))^{-1}\right)$$

Therefore $\ker(\partial_0) = \operatorname{im}(\partial_1)$ and $H_1(T;\omega_i) = \{0\}$. Conversely, if $\omega_i(\ell) = \omega_i(m) = 1$, then ∂_1 and ∂_0 are zero and there is a canonical isomorphism $\langle \tilde{e}_1^1, \tilde{e}_1^2 \rangle \otimes \mathbb{C}(\omega_i)$. If $\mathbb{K} = \mathbb{R}$ then $H_1(T;\omega_i)$ has the same dimension as $H_1(T;\overline{\omega_i})$, so the recomposition of $H_1(T;\rho)$ with Eq. (A.1) preserves the property. $\square$

Corollary A.2.2. *If either one of the $\mathbb{K}$-endomorphisms $(\rho(m) - I_r)$ or $(\rho(\ell) - I_r)$ is bijective then $H_1(T;\rho) = \{0\}$.* $\triangleleft$

Corollary A.2.3. *If both $\rho(m) = I_r$ and $\rho(\ell) = I_r$ then there is a natural $\mathbb{K}$-vector space isomorphism $H_1(T;\rho) = \langle \tilde{e}_1^1, \tilde{e}_1^2 \rangle \otimes \mathbb{K}(\rho)$.* $\triangleleft$

A.3 Fox calculus

In general the ρ-twisted Alexander module of a CW-complex is computed using Fox calculus, a combinatorial tool developed by R. Fox in [Fox54] which we present briefly.

Let $\mathbb{F}_p$ be the free group generated by $x_1, \ldots, x_p$ and let $\mathbb{Z}[\mathbb{F}_p]$ be its group module. Let the augmentation morphism be defined by

$$\begin{aligned} \operatorname{aug} : \quad \mathbb{Z}[\mathbb{F}_p] &\longrightarrow \mathbb{Z} \\ \sum_{f \in \mathbb{F}_p} n_f \cdot f &\longmapsto \sum_{f \in \mathbb{F}_p} n_f \end{aligned}$$

Definition A.3.1. For every generator $x_i \in \mathbb{F}_p$, the *i-th Fox derivative* is the unique linear function

$$\frac{\partial}{\partial x_i} : \mathbb{Z}[\mathbb{F}_p] \longrightarrow \mathbb{Z}[\mathbb{F}_p]$$

such that

$$\begin{cases} \forall a, b \in \mathbb{Z}[\mathbb{F}_p] : \dfrac{\partial(ab)}{\partial x_i} = \operatorname{aug}(b) \cdot \dfrac{\partial a}{\partial x_i} + a \dfrac{\partial b}{\partial x_i} \\ \dfrac{\partial x_i}{\partial x_i} = 1 \end{cases} \qquad \diamond$$

Now consider the complex of $\mathbb{Z}[\mathbb{F}_p]$-modules

$$S_* := S_2 \xrightarrow{\partial_1} S_1 \xrightarrow{\partial_0} S_0$$

where

$$S_2 := \bigoplus_{j=1}^{q} \mathbb{Z}[\mathbb{F}_p] \otimes r_j \qquad S_1 := \bigoplus_{j=1}^{p} \mathbb{Z}[\mathbb{F}_p] \otimes dx_i \qquad S_0 := \mathbb{Z}[\mathbb{F}_p]$$

and dx_i is a formal generator corresponding to x_i. For every $1 \leq i \leq p$ and $1 \leq j \leq q$ the boundary operators are defined by

$$\partial_1 : r_j \longmapsto dr_j \qquad \partial_0 : dx_i \longmapsto x_i,$$

where dw is the Fox differential of the word $w \in \mathbb{Z}[\mathbb{F}_p]$ defined by

$$dw := \sum_{i=1}^{p} \frac{\partial w}{\partial x_i} dx_i \in S_1$$

We explain how to use Fox calculus to compute $H_1(M; \rho)$. Consider a presentation $\mathcal{P}$ of the group π given by

$$0 \longrightarrow \langle r_1, \ldots, r_q \rangle \longrightarrow \mathbb{F}_p \xrightarrow{\zeta} \pi \longrightarrow 0$$

where $r_i \in \mathbb{F}_p$ for every $1 \leq i \leq q$. The morphism ζ can naturally be extended over the group modules as $\mathbb{Z}[\mathbb{F}_p] \to \mathbb{Z}[\pi]$. Similarly, the representation ρ can be extended into $\mathbb{Z}[\pi] \to \mathcal{M}_n(\mathbb{K})$. Now consider the ρ-twisted chain complex over $\mathbb{K}(\rho)$ defined by

$$S_*(\rho) := \mathbb{K}(\rho) \otimes_{\mathbb{Z}[\pi]} \zeta(S_*)$$

Theorem A.3.2 ([Cro71])**.** *There is a natural vector space isomorphism between the ρ-twisted Alexander module $H_1(M; \rho)$ and the first homology group $H_1(S_*(\rho))$.* $\triangleleft$

Definition A.3.3. The *ρ-twisted Alexander matrix* of M associated with the presentation $\mathcal{P}$ is the matrix of $\partial_1(\rho)$ with coefficients in $\mathbb{K}$ given by

$$A^\rho := \left[(\rho \circ \zeta) \left(\frac{\partial r_i}{\partial x_j} \right) \right]_{1 \leq i \leq q,\, 1 \leq j \leq p} \qquad \diamond$$

A.4 Computation of $\ker i_*$

In this section we suppose that ∂M is a union of disjoint tori:

$$\partial M = \bigsqcup_{k=1}^{\mu} \partial_k M$$

We also suppose that $\rho : \pi_1(M) \to \mathbb{K}(\rho)$ is such that

$$\begin{array}{rl} \forall\ i \in \{1, \ldots, d\} : & \rho(m_i) = \rho(\ell_i) = I_r \\ \forall\ j \in \{d+1, \ldots, \mu\} : & \begin{cases} \operatorname{rank}(\rho(m_j) - I_r) < r \\ \operatorname{rank}(\rho(\ell_j) - I_r) < r \end{cases} \end{array} \tag{A.2}$$

We note $\partial_D M = \sqcup_{i=1}^{d} \partial_i M$ the subset of the boundary ∂M where ρ is the identity.

Finally, let $\mathcal{P}$ be a presentation of the fundamental group $\pi_1(M)$. Up to standard Tietze movements, one can always suppose that $\mathcal{P}$ contains the generators m_i, ℓ_i associated to ∂M_i for every $1 \leq i \leq d$.

Theorem A.4.1. *Let A^ρ be the ρ-twisted Alexander matrix of M associated with $\mathcal{P}$. Let A^ρ_D be the sub-matrix of A^ρ containing only the columns associated with the generators of $\partial_D M$, and let A^ρ_C be its complementary sub-matrix. Then*

$$\ker i_* \simeq \operatorname{im}(A^\rho_D \cdot (\ker A^\rho_C)) \cap H(\partial_D M;\rho) \qquad \triangleleft$$

Note that one can use the Zassenhaus algorithm to compute the intersection of the row-spaces of two matrices written in the same basis.

Proof. By Theorem A.3.2, all computation of twisted homology for M and the sub-sets of ∂M can be made with Fox calculus and the associated chain complex $S_*(\rho)$. There is a commutative diagram

$$\begin{array}{ccc} S_1^\partial(\rho) & \xhookrightarrow{i_\#} & S_1(\rho) \\ {\scriptstyle h_\partial}\downarrow & & \downarrow{\scriptstyle h} \\ H_1(\partial M;\rho) & \xrightarrow[i_*]{} & H_1(M;\rho) \end{array}$$

By Corollaries A.2.2 and A.2.3 and Eq. (A.2), we have

$$H_1(\partial M;\rho) \simeq \bigoplus_{i=1}^{d} H_1(\partial_i M;\rho) \simeq \bigoplus_{i=1}^{d} \langle \ell_i, m_i\rangle \otimes \mathbb{K}(\rho)$$

Define

$$\begin{aligned} S_1^D(\rho) &:= \langle d\ell_i, dm_i \mid 1 \le i \le d\rangle \otimes \mathbb{K}(\rho) \\ S_1^C(\rho) &:= \langle dx \mid x\rangle \otimes \mathbb{K}(\rho) \end{aligned}$$

where x goes through every generator of $\mathcal{P}$ that is not ℓ_i, m_i for $1 \le i \le d$. We particularise the previous diagram to each component $i \in \{1, \ldots, d\}$:

$$\begin{array}{ccc} S_1^D(\rho) & \xhookrightarrow{i_\#} & S_1(\rho) = S_1^D(\rho) \oplus S_1^C(\rho) \\ {\scriptstyle h_i}\downarrow & & \downarrow{\scriptstyle h} \\ H_1(\partial M_i;\rho) & \xrightarrow[i_*]{} & H_1(M;\rho) \end{array}$$

Let us note $\mathcal{C}$ a complementary space of $\ker(\partial_1)$ inside $S_1^D(\rho)$. We have the following decomposition:

$$S_1^D(\rho) = H \oplus \operatorname{im}(\partial_2) \oplus \mathcal{C}$$

where $H \simeq H_1(\partial M;\rho)$. More precisely, we set an isomorphism

$$\theta : H_1(\partial M;\rho) \xrightarrow{\sim} H$$

such that for every $W \in H_1(\partial M_i;\rho)$, we have $q^{-1}(W) = \theta(W) + \operatorname{im}(\partial_2)$.

Lemma A.4.2. $(i_\# \circ \theta)(\ker i_*) = \ker h \cap \operatorname{im}(i_\#) \cap i_\#(H)$ $\triangleleft$

Proof. Consider $a \in \ker i_*$. We have

$$\begin{aligned} i_*(a) = 0 &\iff h \circ i_\# \circ h^{-1}(a) = \{0\} \\ &\iff i_\# \circ h^{-1}(a) \subset \ker h \cap \operatorname{im}(i_\#) \\ &\iff i_\#(\theta(a) + \operatorname{im}(\partial_2)) \subset \ker h \cap \operatorname{im}(i_\#) \\ &\iff (i_\# \circ \theta)(a) + i_\#(\operatorname{im}(\partial_2)) \subset \ker h \cap \operatorname{im}(i_\#) \end{aligned}$$

Because $i_\#$ is injective, it preserves the direct sum $H \oplus \operatorname{im}(\partial_2)$. Then:

$$i_*(a) = 0 \iff (i_\# \circ \theta)(a) \in \ker h \cap \operatorname{im}(i_\#) \cap i_\#(H) \qquad \square$$

Since $(i_\# \circ \theta_i)$ is a known isomorphism, to compute $\ker i_*$ we only need to compute the space $\operatorname{im}(i_\#) \cap \ker h$. First off, $\operatorname{im} A_\rho = \ker h$ by Theorem A.3.2.

Lemma A.4.3. $\operatorname{im}(A^\rho_D(\ker A^\rho_C)) = \operatorname{im}(i_\#) \cap \ker h$ $\triangleleft$

Proof. For every vector $a \in \ker A^\rho_C$ we have

$$aA^\rho = [aA^\rho_D \mid aA^\rho_C] = [aA^\rho_D \mid 0] \in \operatorname{im}(i_\#) \cap \ker h$$

Reciprocally, let $b \in \operatorname{im}(i_\#) \cap \ker h$, and a such that $aA^\rho = b$. We know that b is only supported by the columns of $S^D_1(\rho)$, and then necessarily $a \in \ker A^\rho_C$. □

This completes the proof of the theorem. □

CONCLUSION

CONCLUSION AND FUTURE RESEARCHS

The work presented in the first part of this thesis is the continuation of a research program that started in the early 1990s to study Zariski pairs of curves and line arrangements. Our own contributions build on previous invariants that tried to properly describe the inclusion of the boundary manifold in the exterior. The induced map for the homology in $\mathbb{Z}$ was not well understood, and we hope that the graph stabiliser has settled this part of the matter. However, our long-term objective is more ambitious as we seek to study twisted homology on line arrangements. Some research has already been made in the general case of graph manifolds [ACM19b] and we intend to apply it to study $\ker i_*$ in twisted homology as we did with links and knots. For this one needs to describe characters or representations of the fundamental group of the exterior, and one must know what they induce to the boundary. This is precisely what we achieved with the homology inclusion for non-twisted integral coefficients. In addition of being an invariant of its own, it therefore also gives the framework to study the images of the cycle generators in twisted homology. The structure of $\ker i_*$ will probably require more intermediate invariants which might also be able to detect new Zariski pairs on their own.

Another axis of research is based on our construction of ordered graphed embeddings, which are the base concept behind the definition of the graph stabiliser. Theorem 1.5.11 allows to properly describe any representation of the group of the boundary induced by a representation on the exterior by removing the influence of the choice of the ordered graphed embedding. One could then use this presentation to study more general character varieties and other properties of the map i_* induced by the inclusion in twisted homology.

In addition, most of the standard tools and structures we used on line arrangements (and their computer implementations) also apply to more general type of complex algebraic curves containing non-transverse singularities. The boundary manifold has again a graph structure with weaker restricting conditions. The Zariski-van Kampen method used to obtain the fundamental group of the exterior and the braid monodromy are defined as well. Theorem 1.6.15 already established that the graph stabiliser is well defined on a wider class of graphs than the subclass corresponding to the minimal structure of line arrangements. We could therefore endeavour to extend the homology inclusion to algebraic curves whose boundary manifold has a graph structure of that type.

In the second part, we generalised the slope invariant of links. This invariant was again arising from the study of the inclusion of the boundary manifold inside the exterior, but directly in homology twisted by a character. We managed to extend the slope definition to every knot using $\mathrm{SL}_2(\mathbb{C})$-representations, and to remove some of the restrictions on the character on links using $\mathrm{SO}_2(\mathbb{R})$-representations and the characterisation of Lagrangians. This last generalisation is a concordance invariant for similar reasons as the original character slope. It is known that the character slope is related to with other known concordance invariants, mostly the Milnor linking numbers. Just like the slope, these invariants extract topological information on the links from a low-level analysis of the link group. We plan to investigate further the possible connection between the generalised slope and the Milnor linking numbers, but also with the Reidemeister torsion and twisted Alexander polynomials.

In a wider perspective, the generals theorems of Appendix A and Section 6.1 allow to define the slope using any real orthogonal representation of the link group with a non-trivial invariant space on the boundary. Finding new families of such representations could extend the scope of the generalised slope construction with the objective of detecting new non-slice links.

CONCLUSIONES E INVESTIGACIONES FUTURAS

El trabajo presentado en la primera parte de esta tesis es la continuación de un programa de investigación que comenzó a principios de la década de 1990 para estudiar los pares de Zariski de curvas y configuraciones de rectas. Nuestras contribuciones se basan en invariantes anteriores que intentaron describir adecuadamente la inclusión de la variedad borde en el exterior. La aplicación inducida para la homología en $\mathbb{Z}$ no estaba completamente desarrollada el invariante introducido, *estabilizador del gráfo*, permite entenderla mejor. Sin embargo, nuestro objetivo a largo plazo es más ambicioso a medida que buscamos estudiar cómo se aplica para calcularlo en la homología torcida, para configuraciones de rectas y su variedad de borde. Ya se han realizado algunas investigaciones en el caso general de variedades de grafos [ACM19b] y pretendemos aplicarlas para estudiar $\ker i_*$ en homología torcida como lo hicimos con enlaces y nudos. Para ello es necesario describir los caracteres o representaciones del grupo fundamental del exterior y saber qué inducen en el borde. Esto es precisamente lo que logramos para la inclusión en homología entera (con coeficientes no torcidos en $\mathbb{Z}$) con el estabilizador del grafo. Además de ser un invariante en sí mismo, también proporciona el marco para estudiar las imágenes de los generadores de ciclos en homología torcida. La estructura de $\ker i_*$ probablemente requerirá más invariantes intermedios que también podrían detectar nuevos pares de Zariski por sí solos.

Otro eje de investigación se basa en nuestra construcción de encajes ordenados del grafo, que son el concepto base detrás de la definición del estabilizador del gráficos. El Teorema 1.5.11 permite describir adecuadamente *cualquier* representación del grupo fundamental a la frontera inducida por una representación del exterior, eliminando la influencia de la elección del encaje ordenado del grafo. Luego se podría usar esta representación para estudiar variedades de caracteres más generales y otras propiedades de la aplicación i_* inducidas por la inclusión en homología torcida.

Además, la mayoría de las herramientas y estructuras estándar que utilizamos en configuraciones de rectas (y sus implementaciones informáticas) también se aplican a tipos más generales de curvas algebraicas complejas que contienen singularidades no ordinarias. La variedad límite nuevamente tiene una estructura de grafo con condiciones restrictivas más débiles. También se define el método de Zariski-van Kampen utilizado para obtener el grupo fundamental del exterior y la monodromía de trenzas. El Teorema 1.6.15 ya estableció que el estabilizador del grafos está bien definido en una clase de grafos más amplia que la subclase correspondiente a la estructura más pequeña de configuraciones de rectas. Por lo tanto, podríamos intentar extender la inclusión de homología a curvas algebraicas cuya variedad borde tenga una estructura de grafo de ese tipo.

En la segunda parte, generalizamos el invariante de *pendiente* de los enlaces. Este invariante surgió nuevamente del estudio de la inclusión de la variedad borde en el exterior del enlace, pero directamente en homología torcida por un carácter. Logramos extender su definición a cada nudo usando $\mathrm{SL}_2(\mathbb{C})$-representaciones, y eliminar algunas de las restricciones sobre el carácter en los enlaces usando $\mathrm{SO}_2(\mathbb{R})$-representaciones y la caracterización de lagrangianos. Esta última generalización es un invariante de concordancia por razones similares a la pendiente del carácter original. Se sabe que la pendiente del carácter tiene vínculos con otros invariantes de concordancia conocidos, principalmente los números de enlace de Milnor. Al igual que la pendiente, estos invariantes extraen información topológica sobre los enlaces a partir de un análisis de bajo nivel de las relaciones del grupo del enlace. Planeamos investigar más a fondo la posible conexión entre la pendiente generalizada y los números de enlace de Milnor, pero también con la torsión de Reidemeister y los polinomios torcidos de Alexander.

En una perspectiva más amplia, los teoremas generales del Apéndice A y del Apartado 6.1 permiten definir la pendiente utilizando cualquier representación ortogonal real del grupo de enlaces con un espacio invariante no trivial en el borde. Encontrar nuevas familias de tales representaciones podría ampliar el alcance de la construcción de pendientes generalizadas con el objetivo de detectar nuevos enlaces no *slice*.

CONCLUSION ET FUTURES RECHERCHES

Le travail présenté dans la première partie de cette thèse est la continuation d'un programme de recherche démarré au début des années 1990 et qui consiste à étudier les couples de Zariski de courbes et d'arrangements de droites. Notre contribution se base suer des invariants antérieurs dont l'objectif était de décrire correctement l'inclusion de la variété-bord dans l'extérieur. Le morphisme induit en homologie sur $\mathbb{Z}$ n'était pas bien compris, et nous pensons que notre définition du stabilisateur du graphe permet de régler cette partie du problème. Toutefois, notre objectif à long terme est plus ambitieux et consiste à étudier l'homologie tordue des arrangements de droites. Des recherches ont déjà été menées sur le sujet [ACM19b] et nous avons l'intention de les appliquer pour étudier $\ker i_*$ de la même manière que pour les nœuds et entrelacs. Pour cela il est nécessaire de décrire les représentations ou les caractères du groupe fondamental de l'extérieur, et de comprendre ce qu'ils induisent sur la variété-bord. C'est précisément ce que nous avons déterminé avec notre description de l'inclusion homologique pour des coefficients entiers non-tordus. En plus de représenter un invariant topologique en elle-même, l'inclusion homologique fournit donc le cadre permettant d'étudier les images dans l'extérieur des cycles de la variété-bord en homologie tordue. L'analyse complète de la structure de $\ker i_*$ va cependant probablement nécessiter la création d'autres invariants intermédiaires, qui seront peut-être à même de détecter par eux-mêmes de nouveaux couples de Zariski.

Un autre axe de recherche est basé sur les plongements ordonnés du graphe qui sont au cœur de la définition du stabilisateur du graphe. En effet, le Théorème 1.5.11 permet de maîtriser l'influence du choix du plongement du graphe lors de la description d'une représentation de la variété-bord induite par une représentation de l'extérieur. La présentation du groupe fondamental du bord ainsi obtenue peut être utilisée pour étudier les variétés caractéristiques ainsi que d'autres propriétés du morphisme i_* induit en homologie tordue par l'inclusion.

Par ailleurs, la plupart des outils et des structures standards que nous avons utilisés sur les arrangements de droites (ainsi que leurs implémentations informatiques) s'appliquent également à d'autres types de courbes algébriques complexes contenant des singularités non-transverses. Leur variété-bord a encore une structure de variété graphée mais avec des conditions plus faibles. La méthode de Zariski-van Kampen pour obtenir une présentation du groupe fondamental de l'extérieur fonctionne encore également. Le Théorème 1.6.15 a déjà établi que le stabilisateur du graphe est bien défini sur une classe plus large de graphes que ceux qui constituent les graphes minimaux d'arrangements de droites. Nous envisageons ainsi d'étendre l'inclusion homologique à des courbes algébriques dont la combinatoire appartient à cette classe plus large.

Dans la seconde partie de la thèse, nous avons généralisé l'invariant de pente (« *slope* ») sur les entrelacs. Cet invariant est également issu de l'étude de l'inclusion de la variété-bord dans l'extérieur, mais en considérant directement le morphisme induit en homologie tordue par un caractère. Nous avons étendu la définition du *slope* à n'importe quel nœud en utilisant des représentations à valeurs dans $\mathrm{SL}_2(\mathbb{C})$, et nous avons également levé une partie des restrictions du *slope* original sur les entrelacs en utilisant des représentations à valeurs dans $\mathrm{SO}_2(\mathbb{R})$ ainsi que la caractérisation des lagrangiens. Cette dernière généralisation est un invariant de concordance pour des raisons similaires au *slope* original. Il est déjà établi que celui-ci possède des liens avec d'autres invariants de concordance connus, en particulier les enlacements de Milnor. Tout comme le *slope*, ces invariants extraient des informations topologiques des entrelacs en analysant directement le groupe fondamental à bas niveau. Nous prévoyons d'étudier plus loin la possible connexion entre le *slope* généralisé et les enlacements de Milnor, mais également la torsion de Reidemeister et les polynômes d'Alexander tordus.

Dans une perspective plus large, les théorèmes généraux de l'Appendice A et de la Section 6.1 permettent de définir le *slope* avec n'importe quelle représentation réelle orthogonale du groupe de l'entrelacs ayant un espace invariant non-trivial sur la variété-bord. Trouver de telles familles de représentations permettrait d'étendre les champs d'application du *slope* généralisé dans l'objectif de découvrir de nouveaux entrelacs *non-slice*.

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